

COINCIDENCE POINT THEOREM IN ORDERED FUZZY METRIC SPACES AND ITS APPLICATION IN INTEGRAL INCLUSIONS

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ABSTRACT. The purpose of this paper is to present some coincidence point and common fixed point theorems for multivalued contraction maps in complete fuzzy metric spaces endowed with a partial order. As an application, we give an existence theorem of solution for general classes of integral inclusions by the coincidence point theorem.

1. Introduction

We here use the concept of fuzzy metric space that George and Veeramani [4] introduced and studied with the help of continuous t -norm. Recently, many authors stated the various contraction mappings in fuzzy or probabilistic metric spaces (see [1]-[3],[6]-[16],[21]-[23]). Mihet in [15] introduced a larger class of contractions and proved the existence of a fixed point for weaker contractions under geometrically convergent t -norms. The aim of the present article, is to introduce the new notion of contractions for multivalued mappings and to prove some new coincidence point theorems for such mappings. As application, we achieve an existence theorem of solution for classes of Volterra integral inclusions by the coincidence point theorem. We organize this paper as follows. In Section 2, we introduce some preliminary definitions and the results in theory of fuzzy metric spaces. Section 3 is devoted to our main results in partially ordered fuzzy metric space: apart from Theorem 3.4, Corollaries 3.6-3.11 can be seen as generalizations of some known results. In Section 4, we let (X, N, T) be a complete fuzzy normed space and consider the class of the Volterra integral inclusion:

$$fu(t) \in \int_0^t K(t, s)G(s, u(s))ds + g(t),$$

for all $t \in I := [0, 1]$, where $f : X \rightarrow X$, $K \in C(I \times I, \mathbb{R})$, $g \in C(I, X)$, and $G : I \times X \rightarrow 2^X$ is a multivalued mapping satisfying some assumptions that will be specified later, and next by using Theorem 3.4, we prove the existence of at least a solution for the above problem.

Received: December 2013; Revised: January 2015; Accepted: October 2015

Key words and phrases: Coincidence point, Fixed point, Multivalued mapping, Ordered fuzzy metric space, Volterra integral inclusion.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts from fuzzy metric spaces that are used throughout this paper.

A triangular norm, briefly t -norm, is a binary operation T on $[0, 1]$ (see [19]) which is commutative, associative, non-decreasing at both places and $T(x, 1) = x$ for every $x \in [0, 1]$. The $T_L(a, b) = \max\{a + b - 1, 0\}$ (Lukasiewicz t -norm), and the t -norms $T_p(a, b) = a \cdot b$, and $T_M(a, b) = \min\{a, b\}$ on $[0, 1]$ are basic t -norms. It is clear that $T(a, b) \leq \min\{a, b\}$, for all $a, b \in [0, 1]$, i.e., T_M is the largest t -norm on $[0, 1]$.

If T be a t -norm then $x_T^{(n)}$ is defined for every $x \in [0, 1]$ and $n \in \mathbb{N} \cup \{0\}$ as 1, if $n = 0$, and $T(x_T^{(n-1)}, x)$ if $n \geq 1$. A t -norm T is said to be Hadžić-type (denoted as $T \in \mathcal{H}$) if the family $(x_T^{(n)})_{n \in \mathbb{N}}$ is equicontinuous at $x = 1$.

A t -norm T can be extended to a n -ary operation taking for $(x_1, \dots, x_n) \in [0, 1]^n$ ($n \geq 3$), the value $T(x_1, \dots, x_n)$ defined recurrently by

$$T_{i=1}^n x_i = T(T_{i=1}^{n-1} x_i, x_n), \quad \forall n \geq 3.$$

$T_{i=1}^\infty x_i$ is defined as $\lim_{n \rightarrow \infty} T_{i=1}^n x_i$ and $T_{i=n}^\infty x_i$ as $T_{i=1}^\infty x_{n+i}$. The sequence $(T_{i=1}^n x_i)_{n \in \mathbb{N}}$ is non-increasing and bounded from below, thus the limit $T_{i=1}^\infty x_i$ exists.

If $q \in (0, 1)$ is given, we say that the t -norm T is geometrically convergent (for short g -convergent) if $\lim_{n \rightarrow \infty} T_{i=n}^\infty (1 - q^i) = 1$.

The t -norm T_M is a trivial example of t -norm of Hadžić-type, but there are other t -norms of Hadžić-type (see [10]).

The Lukasiewicz t -norm and t -norms of Hadžić-type and some subclasses of Dombi, Aczel-Alsina and Sugeno-Weber families of t -norms [8] are examples of g -convergent t -norms.

Proposition 2.1. [8] For Lukasiewicz t -norm the following implication holds:

$$\lim_{n \rightarrow \infty} T_{i=1}^\infty x_{n+i} = 1 \Leftrightarrow \sum_{n=1}^{\infty} (1 - x_n) < \infty.$$

A fuzzy metric space in the sense of George and Veeramani [4] is defined as follows:

Definition 2.2. The 3-tuple (X, M, T) is said to be a (GV-fuzzy) fuzzy metric space if X is an arbitrary nonempty set, T is a continuous t -norm and M is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions for all $x, y, z \in X$ and $t, s > 0$:

- (FM-1) $M(x, y, t) > 0$;
- (FM-2) $M(x, y, t) = 1$ for all $t > 0$ iff $x = y$;
- (FM-3) $M(x, y, t) = M(y, x, t)$;
- (FM-4) $M(x, z, t + s) \geq T(M(x, y, t), M(y, z, s))$;
- (FM-5) $M(x, y, \cdot) : (0, \infty) \rightarrow [0, 1]$ is continuous.

If (X, M, T) be a fuzzy metric space, we will say that (M, T) is a fuzzy metric on X . Every metric space is a fuzzy metric space and is called a standard fuzzy metric space (cf. [4]).

George and Veeramani proved that every fuzzy metric (M, T) on a set X induces

a topology τ_M on X which has as a base the family of sets of the form $\{U_x(\epsilon, \lambda) : x \in X, \epsilon > 0, \lambda \in (0, 1)\}$ is defined by

$$U_x(\epsilon, \lambda) = \{y \in X : M(x, y, \epsilon) > 1 - \lambda\}.$$

Lemma 2.3. [5, 17] (i) For every $x, y \in X$, the mapping $M(x, y, \cdot)$ is nondecreasing on $(0, \infty)$.

(ii) M is a continuous function on $X^2 \times (0, \infty)$.

Definition 2.4. [4] Let (X, M, T) be a fuzzy metric space.

(1) A sequence $\{x_n\}$ in X is said to be convergent to some $x \in X$, if for all $t > 0$,

$$\lim_{n \rightarrow \infty} M(x_n, x, t) = 1.$$

(2) The sequence $\{x_n\}$ is said to be Cauchy if for all $\epsilon > 0$ and $\lambda \in (0, 1)$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_{n+p}, \epsilon) > 1 - \lambda$, for all $n \geq n_0$ and every $p \in \mathbb{N}$.

(3) A fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.

Definition 2.5. [15] Let (X, M, T) be a fuzzy metric space and $\psi : (0, 1) \rightarrow (0, 1)$ be a mapping. A mapping $f : X \rightarrow X$ is called a fuzzy ψ -contraction of (ϵ, λ) -type if the following implication holds for every $\epsilon > 0$, $\lambda \in (0, 1)$ and $x, y \in X$:

$$M(x, y, \epsilon) > 1 - \lambda \Rightarrow M(f(x), f(y), \epsilon) > 1 - \psi(\lambda).$$

Next we recall the Hausdorff GV-fuzzy metric of a given fuzzy metric space (X, M, T) .

Throughout the article, we denote by $\mathcal{P}_0(X)$, $\mathcal{C}_0(X)$ and $\mathcal{K}_0(X)$, the collection of all nonempty subsets of X , the collection of all nonempty closed subsets of X and the collection of all nonempty compact subsets of X , respectively.

Given $x \in X$, $A \in \mathcal{P}_0(X)$ and $t > 0$, set $M(x, A, t) := \sup_{a \in A} M(x, a, t)$. Now, for each $A, B \in \mathcal{K}_0(X)$ and $t > 0$ define

$$H_M(A, B, t) = \min\{\inf_{a \in A} M(a, B, t), \inf_{b \in B} M(A, b, t)\}.$$

It is shown in [17] that the above formula provides a suitable Hausdorff GV-fuzzy metric on $\mathcal{K}_0(X)$ for any GV-fuzzy metric space (X, M, T) , i.e., 3-tuple $(\mathcal{K}_0(X), H_M, T)$ is a GV-fuzzy metric space.

Lemma 2.6. [17] If $B \in \mathcal{K}_0(X)$ and $a \in X$, then for each $t > 0$, there exists $b_0 \in B$ such that $M(a, B, t) = M(a, b_0, t)$.

Let $f : X \rightarrow X$ be a single-valued mapping and $F : X \rightarrow 2^X$ be a multivalued mapping.

(i) A point $x \in X$ is a fixed point of f (resp. F) if $fx = x$ (resp. $x \in Fx$).

(ii) A point $x \in X$ is a coincidence point of f and F if $fx \in Fx$. The set of all coincidence points of f and F is denoted by $C(f, F)$.

(iii) A point $x \in X$ is a common fixed point of f and F if $x = fx \in Fx$.

Definition 2.7. [20] The maps $f : X \rightarrow X$ and $F : X \rightarrow 2^X$ are weakly compatible if they commute at their coincidence points, i.e., $fFx = Ffx$ whenever $fx \in Fx$ for some $x \in X$.

Finally, let us recall that if (X, \preceq) be a partial ordered set, then two elements x, y of X are called comparable if $x \preceq y$ or $y \preceq x$ holds.

3. Main Results

We begin this section with introducing the notion of a multivalued fuzzy (h, ψ) -contraction of (ϵ, λ) -type mapping and then we state and prove our theorems for these mappings in complete partially ordered fuzzy metric spaces. Our results generalize or extend several results from the existing literature (see [15], [21] and [23]).

Definition 3.1. Let (X, M, T) be a fuzzy metric space, and $f : X \rightarrow X$. Let $h : [0, 1) \rightarrow (0, 1]$ and $\psi : (0, 1) \rightarrow (0, 1)$ be two mappings. The mapping $F : X \rightarrow \mathcal{K}_0(X)$ is called a multivalued fuzzy (h, ψ) -contraction of (ϵ, λ) -type with respect to f if, for every $\epsilon > 0$ and $0 < \lambda < 1$,

$$M(fx, fy, \epsilon) > h(\lambda) \Rightarrow H_M(Fx, Fy, \epsilon) > h(\psi(\lambda)). \quad (1)$$

Remark 3.2. Obviously, a mapping $F : X \rightarrow \mathcal{K}_0(X)$ is a multivalued fuzzy (h, ψ) -contraction of (ϵ, λ) -type with respect to f if and only if for all $\epsilon > 0$ and $\lambda \in (0, 1)$, the following implication holds:

$$M(fx, fy, \epsilon) > h(\lambda) \Rightarrow \forall a \in Fx \exists b \in Fy; \quad M(a, b, \epsilon) > h(\psi(\lambda)). \quad (2)$$

Therefore, in Definition 3.1, if F is a single-valued mapping and $f = I$, $h(t) = 1 - t$, then this notation deduces to the contraction given in Definition 2.5.

Remark 3.3. Observe that if F is multivalued fuzzy (h, ψ) -contraction of (ϵ, λ) -type with respect to f with $\psi(t) < t$ for all $t \in (0, 1)$ and $h(t) = 1 - t$, then we can deduce that $H_M(Fx, Fy, t) \geq M(fx, fy, t)$, for all elements $x, y \in X$ and $t > 0$. To see this, similar to proof in [15], let for some $x, y \in X$ and $t > 0$, we have $H_M(Fx, Fy, t) < M(fx, fy, t)$. Then there exists $\lambda \in (0, 1)$ such that $H_M(Fx, Fy, t) < 1 - \lambda < M(fx, fy, t)$, therefore $M(fx, fy, t) > 1 - \lambda$ but $H_M(Fx, Fy, t) < 1 - \lambda < 1 - \psi(\lambda)$, and it is a contradiction.

Now we discuss our main results.

Theorem 3.4. Let (X, M, T) be a complete fuzzy metric space with a partial order “ \preceq ” defined on X and $f : X \rightarrow X$, $F : X \rightarrow \mathcal{K}_0(X)$ mappings with $FX \subseteq fX$ such that fX be closed. Suppose that $\lim_{n \rightarrow \infty} T_{i=n}^\infty h(\psi^{(i)}(t)) = 1$ for all $t \in (0, 1)$ and that F be a multivalued fuzzy (h, ψ) -contraction of (ϵ, λ) -type with respect to f for every two comparable elements $x, y \in X$. Furthermore $\lim_{n \rightarrow \infty} \psi^{(n)}(t) = 0$ for every $t \in (0, 1)$ and h is continuous and strictly decreasing function with $h(0) = 1$. Assume that X satisfies:

- (i) If $fy \in Fx$ then $x \preceq y$,
- (ii) For a sequence (x_n) with $y_{n+1} \in Fx_n$ for all n , $y_n \rightarrow y = fx$ implies that $x_n \preceq x$.

If, there exist $x_0 \in X$ and $b \in Fx_0$ such that $M(fx_0, b, 0^+) > 0$, then there exists $x \in X$ such that $fx \in Fx$. Moreover, if f and F are weakly compatible and for $x \in C(f, F)$, $ffx = fx$ then f and F have a common fixed point.

Proof. If $M(fx_0, b, 0^+) = 1$, then $M(fx_0, b, \epsilon) = 1$ for each $\epsilon > 0$. It follows that $fx_0 \in Fx_0$. Now assume that $M(fx_0, b, 0^+) < 1$. We will prove our result on three steps:

Step 1: There exist two sequences $\{x_n\}$ and $\{y_n\}$ with $y_{n+1} \in Fx_n$ such that

$$\lim_{n \rightarrow \infty} M(y_n, y_{n+1}, \epsilon) = 1.$$

In fact, there is $t_0 \in (0, 1)$ such that

$$M(fx_0, b, \epsilon) > M(fx_0, b, 0^+) = h(t_0) > h(t_1),$$

for each $\epsilon > 0$ and $t_1 \in (t_0, 1)$. On other hand since $b \in Fx_0 \subseteq f(X)$, we can find an $x_1 \in X$ such that $b = fx_1 \in Fx_0$ with $x_0 \preceq x_1$. By putting $y_0 = fx_0$ and $y_1 = b$, we get

$$M(y_0, y_1, \epsilon) = M(fx_0, b, \epsilon) > h(t_1).$$

Since x_0 and x_1 are comparable, then by implication (1), we have

$$H_M(F(x_0), F(x_1), \epsilon) > h(\psi(t_1)).$$

Now from $y_1 \in F(x_0)$ and by definition of Hausdorff GV-fuzzy metric, there is $y_2 \in F(x_1)$ such that

$$H_M(F(x_0), F(x_1), \epsilon) \leq \sup_{y_1 \in F(x_0)} M(y_1, y_2, \epsilon) = M(y_1, F(x_1), \epsilon) = M(y_1, y_2, \epsilon),$$

where, end equality satisfies by Lemma 2.6. Therefore

$$M(y_1, y_2, \epsilon) > h(\psi(t_1)).$$

Again by $FX \subseteq fX$, there exists an $x_2 \in X$ such that $x_1 \preceq x_2$ with $y_2 = fx_2 \in Fx_1$ that satisfies

$$H_M(F(x_1), F(x_2), \epsilon) > h(\psi^2(t_1)).$$

In general, $x_n \in X$ is chosen such that $fx_{n+1} \in Fx_n$ with $x_n \preceq x_{n+1}$, we obtain a sequence $\{y_n\}$ in X such that $y_{n+1} = fx_{n+1} \in Fx_n$, $n > 0$, and that

$$M(y_n, y_{n+1}, \epsilon) > h(\psi^n(t_1)).$$

Now by given assumptions, $\lim_{n \rightarrow \infty} h(\psi^n(t_1)) = 1$, then for all $\epsilon > 0$ we obtain

$$\lim_{n \rightarrow \infty} M(y_n, y_{n+1}, \epsilon) = 1. \quad (3)$$

Step 2: We claim that $\{y_n\}$ is a Cauchy sequence.

Let $\epsilon_1 > 0$ and $\lambda_1 \in (0, 1)$ be given. Since $\lim_{n \rightarrow \infty} T_{i=n}^{\infty} h(\psi^{(i)}(t)) = 1$, there is $n_0(\lambda_1) \in \mathbb{N}$ such that $T_{i=n}^{\infty} h(\psi^{(i)}(t_1)) > 1 - \lambda_1 \forall n \geq n_0(\lambda_1)$. Now, for all $p > 0$ we have

$$\begin{aligned} M(y_n, y_{n+p}, \epsilon_1) &\geq T(T^{n+p-2}(M(y_n, y_{n+1}, \frac{\epsilon_1}{p}), \dots, M(y_{n+p-2}, y_{n+p-1}, \frac{\epsilon_1}{p}))) \\ &\quad , M(y_{n+p-1}, y_{n+p}, \frac{\epsilon_1}{p})) \\ &\geq T^{n+p-1} (h(\psi^n(t_1)), h(\psi^{n+1}(t_1)), \dots, h(\psi^{n+p-1}(t_1))) \\ &\geq T_{i=n}^{\infty} h(\psi^i(t_1)), \end{aligned}$$

thus $M(y_n, y_{n+p}, \epsilon_1) > 1 - \lambda_1$ for all $n \geq n_0$ and $p \in \mathbb{N}$. Therefore $\{y_n\} = \{f(x_n)\}_{n \in \mathbb{N}}$ is a Cauchy sequence in X .

Step 3: Existence of a Coincidence point for f and F .

From the completeness of X , one can deduce existence of some $y \in X$ such that

$$\lim_{n \rightarrow \infty} y_n = \lim_{n \rightarrow \infty} f(x_n) = y. \quad (4)$$

Now, since $f(X)$ is closed, thus there exists an $x \in X$ such that $y = fx \in f(X)$, it follows that $x_n \preceq x$ for every n . But since Fx is closed, it suffices to get $fx \in \overline{Fx}$. In fact, we prove that for every $\epsilon > 0$ and $\lambda \in (0, 1)$ there exists an $r(\epsilon, \lambda) \in Fx$ such that $r(\epsilon, \lambda) \in U_{f(x)}(\epsilon, \lambda)$.

Since T is continuous, there exists $\delta_1(\lambda) \in (0, 1)$ such that

$$T(1 - \delta_1(\lambda), 1 - \delta_1(\lambda)) > 1 - \lambda.$$

For δ_1 there is a $\delta_2 \in (0, 1)$ such that

$$T(1 - \delta_2(\lambda), 1 - \delta_2(\lambda)) > 1 - \delta_1(\lambda),$$

therefore by putting $\delta(\lambda) = \min\{\delta_1, \delta_2\}$, we have

$$\begin{aligned} &T(1 - \delta(\lambda), T(1 - \delta(\lambda), 1 - \delta(\lambda))) \\ &\geq T(1 - \delta_1(\lambda), T(1 - \delta_2(\lambda), 1 - \delta_2(\lambda))) > 1 - \lambda. \end{aligned}$$

Now, for fixed $t > 0$ there exists an integer $k > 0$ such that $h(\psi^{(n)}(t)) > 1 - \delta(\lambda)$ for all $n > k$. If $s_0 = \psi^k(t)$, one can easily obtains that $h(s_0) < 1$ and $h(\psi(s_0)) > 1 - \delta(\lambda)$. Therefore for $h(s_0)$, equality (4) shows the existence of a $k_1 \in \mathbb{N}$ such that for all $k \geq k_1$,

$$M(fx_k, fx, \frac{\epsilon}{3}) > h(s_0).$$

Since x_k and x are comparable, then by implication (1) we get

$$H_M(Fx_k, Fx, \frac{\epsilon}{3}) > h(\psi(s_0)), \quad (5)$$

which implies that $H_M(Fx_k, Fx, \frac{\epsilon}{3}) > 1 - \delta(\lambda)$. On the other hand again by the definition of Hausdorff GV-fuzzy metric and Lemma 2.6, we get an $r(\epsilon, \lambda) \in Fx$ such that

$$H_M(Fx_k, Fx, \frac{\epsilon}{3}) \leq M(fx_{k+1}, Fx, \frac{\epsilon}{3}) = M(fx_{k+1}, r, \frac{\epsilon}{3}),$$

thus for all $k \geq k_1$, we obtain

$$M(fx_{k+1}, r, \frac{\epsilon}{3}) > 1 - \delta(\lambda).$$

Next, it follows from (3) that there is a $k_2 \in \mathbb{N}$ such that for all $k \geq k_2$,

$$M(fx_k, fx_{k+1}, \frac{\epsilon}{3}) > 1 - \delta(\lambda).$$

Finally from (4), one can conclude that there exists a positive integer k_3 such that for every $k \geq k_3$,

$$M(fx_k, fx, \frac{\epsilon}{3}) > 1 - \delta(\lambda).$$

Hence, for a $k \geq \max\{k_1, k_2, k_3\}$, we obtain

$$M(fx, r, \epsilon) \geq T(M(fx, fx_k, \frac{\epsilon}{3}), T(M(fx_k, fx_{k+1}, \frac{\epsilon}{3}), M(fx_{k+1}, r, \frac{\epsilon}{3}))) > 1 - \lambda,$$

what means that $r(\epsilon, \lambda) \in U_{f(x)}(\epsilon, \lambda)$. Thus we proved that x is a coincidence point of F and f .

Suppose now that $ffx = fx$. Putting $y = fx$ and that f and F are weakly compatible, we have

$$y = fy \in f(Fx) = F(fx) = Fy.$$

Therefore, y is the common fixed point of f and F . The proof of theorem is complete. \square

Definition 3.5. If in implication (1) put $h(t) = 1 - t$, that is, for every $\epsilon > 0$ and $0 < \lambda < 1$,

$$M(fx, fy, \epsilon) > 1 - \lambda \Rightarrow H_M(Fx, Fy, \epsilon) > 1 - \psi(\lambda),$$

then we say F is a multivalued fuzzy ψ -contraction of (ϵ, λ) -type with respect to f , and if also $\psi(t) = qt$, then F is called a multivalued fuzzy q -contraction of (ϵ, λ) -type with respect to f .

Let $h(t) = 1 - t$ in Theorem 3.4, we obtain the following corollary:

Corollary 3.6. Let (X, M, T) be a complete fuzzy metric space with a partial order “ \preceq ” defined on X and $f : X \rightarrow X, F : X \rightarrow \mathcal{K}_0(X)$ mappings with $FX \subseteq fX$ such that fX be closed. Suppose that $\lim_{n \rightarrow \infty} T_{i=n}^\infty(1 - \psi^{(i)}(t)) = 1$ for all $t \in (0, 1)$ and that F be a multivalued fuzzy ψ -contraction of (ϵ, λ) -type with respect to f for every two comparable elements $x, y \in X$ with $\lim_{n \rightarrow \infty} \psi^{(n)}(t) = 0$. Assume that X satisfies:

(i) If $fy \in Fx$ then $x \preceq y$,

(ii) For a sequence (x_n) with $y_{n+1} \in Fx_n$ for all $n, y_n \rightarrow y = fx$ implies that $x_n \preceq x$.

If, there exist $x_0 \in X$ and $b \in Fx_0$ such that $M(fx_0, b, 0^+) > 0$, then there exists $x \in X$ such that $fx \in Fx$. Moreover, if f and F are weakly compatible and for $x \in C(f, F), ffx = fx$ then f and F have a common fixed point.

Corollary 3.7. Let (X, M, T_L) be a complete fuzzy metric space with a partial order “ \preceq ” defined on X and $f : X \rightarrow X$, $F : X \rightarrow \mathcal{K}_0(X)$ mappings with the property that $FX \subseteq fX$ such that fX be closed and there is $q \in (0, 1)$ such that

$$1 - H_M(F(x), F(y), t) \leq q(1 - M(fx, fy, t)),$$

for every two comparable elements $x, y \in X$ and every $t > 0$. Assume that X satisfies:

(i) If $fy \in Fx$ then $x \preceq y$,

(ii) For a sequence (x_n) with $y_{n+1} \in Fx_n$ for all n , $y_n \rightarrow y = fx$ implies that $x_n \preceq x$.

Then there exists $x \in X$ such that $fx \in Fx$. Moreover, if f and F are weakly compatible and for $x \in C(f, F)$, $ffx = fx$ then f and F have a common fixed point.

Proof. First we show that F is a multivalued fuzzy q -contraction of (ϵ, λ) -type with respect to f for every two comparable elements $x, y \in X$. If $M(fx, fy, \epsilon) > 1 - \lambda$, then for every two comparable elements $x, y \in X$,

$$1 - H_M(F(x), F(y), \epsilon) \leq q(1 - M(fx, fy, \epsilon)) < q\lambda,$$

thus, we get $H_M(F(x), F(y), \epsilon) > 1 - q\lambda$. Also, according to Proposition 2.1, as $\sum_{i=1}^{\infty} \psi^{(i)}(\lambda) = \sum_{i=1}^{\infty} q^i \lambda < \infty$, then $\lim_{n \rightarrow \infty} T_{i=n}^{\infty}(1 - q^i \lambda) = 1$ for all $\lambda \in (0, 1)$. Furthermore

$$H_M(F(x), F(y), t) \geq 1 - q + qM(fx, fy, t) \geq 1 - q > 0,$$

thus, for all $y_1 \in F(x)$ there is $y_2 \in F(y)$ such that for every two comparable elements $x, y \in X$ and $t > 0$, we have

$$M(y_1, y_2, t) = M(y_1, F(y), t) \geq H_M(F(x), F(y), t) > 0.$$

Therefore, by supposing $x = x_0$, $y_1 = fx_1$ and $y = x_1$ with $x_0 \preceq x_1$, one can see that there is $y_2 \in Fx_1$ such that $M(fx_1, y_2, t) > 0$ for all $t > 0$. Consequently $M(fx_1, y_2, 0^+) > 0$. Finally, Corollary 3.6 gives the result. \square

Remark 3.8. Suppose that F be the single-valued mapping. Taking $f = I$ in Corollary 3.6, we get a generalization of Theorem 3.7 in [15] to fuzzy metric spaces endowed with a partial order.

Remark 3.9. Corollary 3.7 is an extension and generalization of corresponding theorem of Tirado in [21] to multivalued case in fuzzy metric spaces endowed with a partial order. Because, we have obtained the coincidence and common fixed points for two mappings in Corollary 3.7.

Note that if $\psi(t) = qt$, then in Theorem 3.4 one can replace the assumption “ $\lim_{n \rightarrow \infty} T_{i=n}^{\infty} h(q^i t) = 1$ for all $t \in (0, 1)$ ” with “ $\lim_{n \rightarrow \infty} T_{i=n}^{\infty} h(q^i) = 1$ ”. This is because, let t_1 and $\{y_n\}$ be the same with what defined in the Theorem 3.4. We show that $\{y_n\}$ is a Cauchy sequence. In fact, for $\epsilon_1 > 0$ and $\lambda_1 \in (0, 1)$ given, since h is strictly decreasing, we have

$$\begin{aligned} M(y_n, y_{n+p}, \epsilon_1) &\geq T_{i=n}^{n+p-1} h(q^i t_1) \\ &\geq T_{i=n}^{n+p-1} h(q^i) \geq T_{i=n}^{\infty} h(q^i), \end{aligned}$$

thus $M(y_n, y_{n+p}, \epsilon_1) > 1 - \lambda_1$ for every $n \geq n_0$ and $p > 0$. Then, using the same arguments as in the proof of Theorem 3.4, we obtain the following corollary:

Corollary 3.10. Let (X, M, T) be a complete fuzzy metric space with a partial order “ \preceq ” defined on X and $f : X \rightarrow X, F : X \rightarrow \mathcal{K}_0(X)$ mappings with $FX \subseteq fX$ such that fX be closed. Suppose that $\lim_{n \rightarrow \infty} T_{i=n}^\infty h(q^i) = 1$ and that F be a multivalued fuzzy (h, ψ) -contraction of (ϵ, λ) -type with respect to f for every two comparable elements $x, y \in X$ with $\psi(t) = qt$ and that h is continuous and strictly decreasing function with $h(0) = 1$. Assume that X satisfies:

- (i) If $fy \in Fx$ then $x \preceq y$,
- (ii) For a sequence (x_n) with $y_{n+1} \in Fx_n$ for all $n, y_n \rightarrow y = fx$ implies that $x_n \preceq x$.

If, there exist $x_0 \in X$ and $b \in Fx_0$ such that $M(fx_0, b, 0^+) > 0$, then there exists $x \in X$ such that $fx \in Fx$. Moreover, if f and F are weakly compatible and for $x \in C(f, F), ffx = fx$, then f and F have a common fixed point.

Corollary 3.11. Let (X, M, T) be a complete fuzzy metric space with the g -convergent t -norm T and partial order “ \preceq ” defined on X . Let $f : X \rightarrow X, F : X \rightarrow \mathcal{K}_0(X)$ mappings with $FX \subseteq fX$ such that fX be closed. Suppose that F be a multivalued fuzzy q -contraction of (ϵ, λ) -type with respect to f for every two comparable elements $x, y \in X$. Assume that X satisfies:

- (i) If $fy \in Fx$ then $x \preceq y$,
- (ii) For a sequence (x_n) with $y_{n+1} \in Fx_n$ for all $n, y_n \rightarrow y = fx$ implies that $x_n \preceq x$.

If, there exist $x_0 \in X$ and $b \in Fx_0$ such that $M(fx_0, b, 0^+) > 0$, then there exists $x \in X$ such that $fx \in Fx$. Moreover, if f and F are weakly compatible and for $x \in C(f, F), ffx = fx$ then f and F have a common fixed point.

Proof. By defining the function $h(t) = 1 - t$ in Corollary 3.10, it follows that this conclusion holds. □

Remark 3.12. Corollaries 3.10 and 3.11 are extensions and generalizations of Theorem 2 and Corollary 1 of [23], because, every single-valued mapping can be considered as a multivalued mapping and we here investigated the existence of coincidence and common fixed points for multivalued mappings and in partially ordered fuzzy metric spaces.

4. Application to Integral Inclusion

In this section, at first we recall the definition of fuzzy normed spaces and some facts which were stated in [18].

Definition 4.1. The 3-tuple (X, N, T) is said to be a fuzzy normed space if X be a vector space, T be a continuous t -norm and N a fuzzy set on $X \times (0, \infty)$ satisfying the following conditions for every $x, y \in X$ and $t, s > 0$:

- (i) $N(x, t) > 0$;
- (ii) $N(x, t) = 1$ if and only if $x = 0$;

- (iii) $N(\alpha x, t) = N(x, \frac{t}{|\alpha|})$ for all real number $\alpha \neq 0$;
- (iv) $T(N(x, t), N(y, s)) \leq N(x + y, t + s)$;
- (v) $N(x, \cdot) : (0, \infty) \rightarrow [0, 1]$ is continuous;
- (vi) $\lim_{t \rightarrow \infty} N(x, t) = 1$.

Lemma 4.2. [18] Let N be a fuzzy norm. Then

- (i) $N(x, t)$ is nondecreasing with respect to t for each $x \in X$.
- (ii) $N(x - y, t) = N(y - x, t)$.

Lemma 4.3. [18] Let (X, N, T) be a fuzzy normed space. If we define

$$M(x, y, t) = N(x - y, t),$$

then M is a fuzzy metric on X , which is called the fuzzy metric induced by the fuzzy norm N .

Above lemma shows that every fuzzy norm induces a fuzzy metric, thus we can present our results of previous section for every fuzzy normed space.

Now, let $C([0, a], X)$ be a set of mappings from $[0, a]$ into X which are continuous in the topology τ on (X, N, T) . For $u, v \in C([0, a], X)$ we define that $u \preceq v$ if and only if $u(t) \leq v(t)$ for each $t \in [0, a]$. Let $\tilde{N} : C([0, a], X) \times (0, \infty) \rightarrow [0, 1]$ are defined by

$$\tilde{N}(u(\cdot), t) = \lim_{k \rightarrow t} \inf_{s \in [0, a]} N(u(s), k).$$

Since every fuzzy normed space is a Menger normed space, we can conclude that if (X, N, T) with a continuous t -norm T be a complete fuzzy normed space then $(C([0, a], X), \tilde{N}, T)$ is also a complete fuzzy normed space (see [2], Propositions 3 and 4).

In what follows, we study the existence of solutions for a class of the Volterra integral inclusion. Theorem 3.4 coupled with the space $(C([0, a], X))$ are used to establish our result. Let (X, N, T_M) be a complete fuzzy normed space and consider the Volterra integral inclusion of the form:

$$fu(t) \in \int_0^t K(t, s)G(s, u(s))ds + g(t), \quad (6)$$

for every $t \in I := [0, 1]$, where $f : X \rightarrow X$, $K \in C(I^2, \mathbb{R})$, $g \in C(I, X)$, and $G : I \times X \rightarrow 2^X$ is a multivalued mapping. In order to prove the existence of solutions of (6), we assume the following:

- (S1) $G(t, u)$ is compact subset for all $(t, u) \in I \times C(I, X)$, and fX is closed.
- (S2) For every $u, v \in C(I, X)$, if u, v are comparable then for each $t \in I$ and $k > 0$,

$$1 - H_M(G(t, u(t)), G(t, v(t)), k) \leq q(1 - N(fu - fv, k)),$$

for some $q \in (0, 1)$.

- (S3) There exists real number $c \leq 1$ such that for all $t, s \in I$, $|K(t, s)| \leq c$, and for every $u \in C(I, X)$, there exists $w \in C(I, X)$ such that

$$(fw)(t) \in \int_0^t K(t, s)G(s, u(s))ds + g(t).$$

(S4) If for each $u, y \in C(I, X)$, one has $fy(t) = \int_0^t K(t, s)v(s)ds + g(t)$ for $t \in I$ and $v \in G(t, u(t))$, then u and y are comparable.

(S5) There exist $u \in C(I, X)$ and $v_0 \in G(t, u(t))$ such that

$$N(fu(t) - h_0(t), 0^+) > 0,$$

where $h_0(t) = \int_0^t K(t, s)v_0(s)ds + g(t)$.

(S6) Every term of the sequence $\{u_n\}$ with $fu_{n+1} = \int_0^t K(t, s)v_n(s)ds + g(t)$ for every $v_n \in G(t, u_n(t))$ that $fu_n \rightarrow y = fx(n \rightarrow \infty)$, is comparable with x .

Theorem 4.4. With the conditions (S1) – (S6) on the multivalued function $G : I \times X \rightarrow 2^X$, the problem (6) has at least a solution in $C(I, X)$.

Proof. Let $Y = C(I, X)$, and the multivalued map $F : Y \rightarrow 2^Y$ be defined as

$$(Fu)(t) = \int_0^t K(t, s)G(s, u(s))ds + g(t). \quad (7)$$

Note that (Y, N, T_M) is a complete partially ordered fuzzy normed space. Next we shall show that F satisfies all conditions of Theorem 3.4. We first show that F has compact values. Let $u \in Y$ be arbitrary and h_n be a sequence in Fu , then there exists a sequence $u_n(t) \in G(t, u(t))$ such that $h_n(t) = \int_0^t K(t, s)u_n(s)ds + g(t)$ for all $t \in I$. Since $G(t, u(t))$ is compact, there exists a convergent subsequence of $u_n(t)$, say $u_{n_k}(t)$ that convergent to some $v \in G(t, u(t))$ for all $t \in I$, i.e. $\lim_{k \rightarrow \infty} N(u_{n_k}(t) - v(t), r) = 1$ for all $r > 0$. Now, we have

$$\begin{aligned} & N\left(\int_0^t K(t, s)(u_{n_k}(s) - v(s))ds, r\right) \\ &= \lim_{\max_{1 \leq i \leq N}(\Delta t_i) \rightarrow 0} N\left(\sum_{i=1}^n K(t, \xi)(u_{n_k}(\xi) - v(\xi))\Delta t_i, r\right), \end{aligned}$$

where

$$0 = \xi_0 < \xi_1 < \xi_2 < \dots < \xi_N = t \quad \Delta t_i = \xi_i - \xi_{i-1}, \quad i = 1, 2, \dots, N.$$

For any given $t \in I$ and $r > 0$, there exists a sequence $\{k_t^{(n)}\} \subset I$ which for all the continuous points of $N(\int_0^t K(t, s)G(s, u(s))ds, r)$ such that $\{k_t^{(n)}\} \rightarrow r$ provided $n \rightarrow \infty$. Then, we have

$$\begin{aligned} & N\left(\int_0^t K(t, s)(u_{n_k}(s) - v(s))ds, r\right) \\ & \geq \lim_{\max_{1 \leq i \leq N}(\Delta t_i) \rightarrow 0} \min\left\{N(K(t, \xi)(u_{n_k}(\xi) - v(\xi)), \frac{k_t^{(n)}}{t})\right\} \\ & \geq \lim_{\max_{1 \leq i \leq N}(\Delta t_i) \rightarrow 0} \min\left\{N(u_{n_k}(\xi) - v(\xi), \frac{k_t^{(n)}}{ct})\right\} \\ & \geq \inf_{s \in I} \left\{N(u_{n_k}(s) - v(s), \frac{k_t^{(n)}}{ct})\right\} \geq \inf_{s \in I} \left\{N(u_{n_k}(s) - v(s), k_t^{(n)})\right\}. \end{aligned}$$

By putting $h_{n_k}(t) = \int_0^t K(t, s)u_{n_k}(s)ds + g(t)$ and $h(t) = \int_0^t K(t, s)v(s)ds + g(t)$, we can conclude that $\lim_{k \rightarrow \infty} \tilde{N}(h_{n_k}(t) - h(t), r) = 1$ for all $r > 0$ and this means that F has compact values. Next if $u^* \in FY$, then $u^* \in Fu$ for $u \in Y$. By (S3) there exists $w \in Y$ such that

$$(fw)(t) \in \int_0^t K(t, s)G(s, u(s))ds + g(t) = (Fu)(t),$$

hence $FY \subseteq fY$. Also F is a multivalued fuzzy (h, ψ) -contraction of (ϵ, λ) -type with respect to f with $h(x) = 1 - x$ and $\psi(x) = qx$ for every two comparable elements $x, y \in X$ and some $q \in (0, 1)$. Indeed, Let $u, v \in Y$ that $u \preceq v$ and $h_1 \in Fu$. From (7), there exists $v_1 \in G(t, u(t))$ such that

$$h_1(t) = \int_0^t K(t, s)v_1(s)ds + g(t),$$

for $t \in I$. The condition (S2) implies that there exists $v_2 \in G(t, v(t))$ such that

$$1 - N(v_1(t) - v_2(t), k) \leq q(1 - N(fu(t) - fv(t), k)).$$

Now if define $h_2(t) = \int_0^t K(t, s)v_2(s)ds + g(t)$, for each $t \in I$, then $h_2 \in Fv$. Therefore, we have

$$\begin{aligned} N(h_1(t) - h_2(t), k) &\geq N\left(\int_0^t K(t, s)(v_1(s) - v_2(s))ds, k_t^{(n)}\right) \\ &= \lim_{\max_{1 \leq i \leq n} (\Delta t_i) \rightarrow 0} N\left(\sum_{i=1}^n K(t, \xi_i)(v_1(\xi_i) - v_2(\xi_i))\Delta t_i, k_t^{(n)}\right), \end{aligned}$$

thus, it follows that

$$\begin{aligned} N(h_1(t) - h_2(t), k) &\geq \lim_{\max_{1 \leq i \leq n} (\Delta t_i) \rightarrow 0} \min\left\{N(K(t, \xi_i)(v_1(\xi_i) - v_2(\xi_i)), \frac{k_t^{(n)}}{t})\right\} \\ &\geq \lim_{\max_{1 \leq i \leq n} (\Delta t_i) \rightarrow 0} \min\left\{N(v_1(\xi_i) - v_2(\xi_i), \frac{k_t^{(n)}}{ct})\right\} \\ &\geq \inf_{s \in I} N(v_1(s) - v_2(s), \frac{k_t^{(n)}}{ct}) \geq \inf_{s \in I} N(v_1(s) - v_2(s), k_t^{(n)}). \end{aligned}$$

From this inequality and (S2) we obtain

$$N(h_1(t) - h_2(t), k) \geq 1 - q + q \inf_{s \in I} N(fu(s) - fv(s), k_t^{(n)}),$$

for some $q \in (0, 1)$. By letting n go to infinity,

$$\begin{aligned} N(h_1(t) - h_2(t), k) &\geq 1 - q + q \lim_{k_t^{(n)} \rightarrow k} \left(\inf_{s \in I} N(fu(s) - fv(s), k_t^{(n)})\right) \\ &= 1 - q + q \tilde{N}(fu(\cdot) - fv(\cdot), k). \end{aligned}$$

Therefore

$$\begin{aligned} \tilde{N}(h_1(\cdot) - h_2(\cdot), k) &= \lim_{k \rightarrow \epsilon} \left(\inf_{t \in I} N(h_1(t) - h_2(t), k)\right) \\ &\geq 1 - q + q \lim_{k \rightarrow \epsilon} \tilde{N}(fu(\cdot) - fv(\cdot), k), \end{aligned}$$

and so

$$1 - \tilde{N}(h_1(\cdot) - h_2(\cdot), \epsilon) \leq q(1 - \tilde{N}(fu(\cdot) - fv(\cdot), \epsilon)),$$

for every $\epsilon > 0$. This implies that if $\tilde{N}(fu(\cdot) - fv(\cdot), \epsilon) > 1 - \lambda$, then we get

$$\tilde{N}(h_1(\cdot) - h_2(\cdot), \epsilon) > 1 - q\lambda.$$

Thus by implication (1) and Remark 3.2, F is a multivalued fuzzy (h, ψ) -contraction of (ϵ, λ) -type with respect to f with $h(x) = 1 - x$ and $\psi(x) = qx$ on (Y, N, T_M) . Finally, Theorem 3.4 gives that there exists $x \in X$ such that $fx \in Fx$, and this means the problem (6) has at least a solution in $C(I, X)$. \square

5. Conclusion

The new notion of contractions for multivalued mappings in partially ordered fuzzy metric spaces has been presented and so, the coincidence point theorems have been established which extended the results of Mihet [15], Tirado [21] and Žikić-Došenović [23]. Moreover, as an application of our results, an existence theorem of solution has been proved for classes of Volterra integral inclusions.

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