$L-$ORDERED FUZZIFYING CONVERGENCE SPACES

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Abstract. Based on a complete Heyting algebra, we modify the definition of lattice-valued fuzzifying convergence space using fuzzy inclusion order and construct in this way a Cartesian-closed category, called the category of $L-$ordered fuzzifying convergence spaces, in which the category of $L-$fuzzifying topological spaces can be embedded. In addition, two new categories are introduced, which are called the category of principal $L-$ordered fuzzifying convergence spaces and that of topological $L-$ordered fuzzifying convergence spaces, and it is shown that they are isomorphic to the category of $L-$fuzzifying neighborhood spaces and that of $L-$fuzzifying topological spaces respectively.

1. Introduction

Convergence structures are more general than topological structures. If a convergence structure additionally satisfies proper conditions, it is equivalent to a topological structure. Lowen [12] constructed convergence systems using prefilters, through which Min [13] proposed fuzzy limit structures. Xu [14] proved that topological $L-$fuzzifying convergence structures and $L-$fuzzifying topologies [17] are equivalent, where classical filters play a crucial role. By stratified $L-$filters [7], Jäger [8] introduced stratified $L-$fuzzy convergence spaces in the many-valued case. The category of these spaces was developed to a significant extent in the recent years [1,2,4,5,9-11,14,15].

In 2009, Yao [16] defined $L-$fuzzifying convergence spaces, and showed the category of $L-$fuzzifying topological spaces [17] could be embedded in the category of $L-$fuzzifying convergence spaces as a reflective subcategory and the latter is Cartesian-closed. $L-$fuzzifying convergence spaces were based on $L-$filters of ordinary subsets.

This paper can be seen as a further step towards [16]. It proposes a new lattice-valued fuzzifying convergence structure, called $L-$ordered fuzzifying convergence structure, which is compatible with the fuzzy inclusion order of $L-$filters of ordinary subsets. The category of $L-$fuzzifying topological spaces [17] can be embedded in the resulting category. As a matter of fact, it is easier for a bigger category to be Cartesian-closed, and it makes sense to establish a smaller Cartesian-closed category. Note that the category of $L-$ordered fuzzifying convergence spaces is

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"smaller" than that of $L-$fuzzifying convergence spaces [16], and it is Cartesian-closed. In addition, two new categories are introduced, which are called the category of principal $L-$ordered fuzzifying convergence spaces and that of topological $L-$ordered fuzzifying convergence spaces, and it is shown that they are isomorphic to the category of $L-$fuzzifying neighborhood spaces and that of $L-$fuzzifying topological spaces respectively.

2. **Preliminaries**

Let $(L, \vee, \wedge)$ be a complete lattice. If the finite meets are distributive over arbitrary joins, i.e. for all $a, b_i \in L, (i \in J)$

$$a \wedge (\bigvee_{i \in J} b_i) = \bigvee_{i \in J} (a \wedge b_i),$$

L is called a complete Heyting algebra. For L, we define an implication operator $\rightarrow: L \times L \rightarrow L$ as follows:

$$\forall a, b \in L, a \rightarrow b = \bigvee \{c \in L | a \wedge c \leq b\}.$$

Then it is the right adjoint for $\wedge$, i.e.,

$$\forall a, b, c \in L, a \wedge c \leq b \Leftrightarrow c \leq a \rightarrow b.$$

Theorem 2.1. [7] *Let L be a complete Heyting algebra. For all $a, b, c, d, a_i, b_i \in L, (i \in J)$, the following holds:*

$(H1)$ $a \leq (b \rightarrow c) \Leftrightarrow a \wedge b \leq c$, *and* $a \leq b \Leftrightarrow (a \rightarrow b) = 1$,

$(H2)$ $a \rightarrow (\bigwedge_{i \in J} b_i) = \bigwedge_{i \in J} (a \rightarrow b_i)$, $(\bigvee_{i \in J} b_i) \rightarrow a = \bigwedge_{i \in J} (b_i \rightarrow a)$,

$(H3)$ $(b \rightarrow c) \leq (a \rightarrow b) \rightarrow (a \rightarrow c)$, $(a \rightarrow c) \wedge (b \rightarrow d) \leq (a \wedge b) \rightarrow (c \wedge d)$,

$(H4)$ $a \rightarrow b \geq b$, $a \leq (a \rightarrow b) \rightarrow b$,

$(H5)$ $a \wedge b = a \wedge (a \rightarrow b)$, *therefore,* $b = 1 \rightarrow b$,

$(H6)$ $a \rightarrow (b \rightarrow c) = (a \wedge b) \rightarrow c$,

$(H7)$ $\bigwedge_{i \in J} (a_i \rightarrow b_i) \leq (\bigwedge_{i \in J} a_i) \rightarrow (\bigwedge_{i \in J} b_i)$.

In what follows, we consider X a nonempty set and L a complete Heyting algebra unless otherwise stated.

For a given set X, L^X denotes the set of all $L-$subsets on X. Define a binary mapping $S(-,-) : L^X \times L^X \rightarrow L$ by $S(U, V) = \bigwedge_{x \in X} (U(x) \rightarrow V(x))$ for each pair $(U, V) \in L^X \times L^X$.

Definition 2.2. [6] A map $\mathscr{F} : 2^X \rightarrow L$ is called an $L-$filter of ordinary subsets of X if it satisfies $\forall x \in X, A, B \in 2^X$,

(F1) $\mathscr{F}(\emptyset) = 0, \mathscr{F}(X) = 1$,

(F2) $A \subseteq B \Rightarrow \mathscr{F}(A) \leq \mathscr{F}(B)$,

(F3) $\mathscr{F}(A \cap B) \geq \mathscr{F}(A) \wedge \mathscr{F}(B)$.

The family of all $L-$filters of ordinary subsets on X will be denoted by $\mathscr{F}_L(X)$. An order on $\mathscr{F}_L(X)$ is defined as follows: $\forall \mathscr{F}, \mathscr{G} \in \mathscr{F}_L(X), \mathscr{F} \leq \mathscr{G} \Leftrightarrow \forall\ U \in 2^X, \mathscr{F}(U) \leq \mathscr{G}(U)$.

For every $x \in X,\ [x] \in \mathscr{F}_L(X)$ is defined by $\forall A \in 2^X$,

$$[x](A) = \begin{cases} 1, & x \in A, \\ 0, & otherwise. \end{cases}$$

Let $\mathscr{F}$ be a filter of ordinary subsets on X and $f : X \to Y$ be a mapping. Then the mapping $f^{\Rightarrow}(\mathscr{F}) : 2^Y \to L$, where $\forall B \in 2^Y, f^{\Rightarrow}(\mathscr{F})(B) = \mathscr{F}(f^{\leftarrow}(B))$, is an $L-$filter of ordinary subsets on Y and is called the image of $\mathscr{F}$ under f.

For every $\mathscr{F} \in \mathscr{F}_L(X),\ \mathscr{G} \in \mathscr{F}_L(Y),\ \mathscr{F} \times \mathscr{G} \in \mathscr{F}_L(X \times Y)$ is defined as follows: $\forall C \in 2^{X\times Y},\ (\mathscr{F} \times \mathscr{G})(C) = \bigvee_{A\times B \subseteq C} \mathscr{F}(A) \wedge \mathscr{G}(B)$.

Definition 2.3. [18] An $L-$fuzzifying neighborhood structure on a set X is a family of functions $N = \{N_x : 2^X \to L \mid x \in X\}$ with the following conditions: For all $x \in X,\ U,\ V \in 2^X$,

(LN1) $N_x(X) = 1$,

(LN2) $N_x(U) > 0$ *implies* $x \in U$,

(LN3) $N_x(U \cap V) = N_x(U) \wedge N_x(V)$.

The pair (X, N) is called an $L-$fuzzifying neighborhood space, and it will be called topological if it satisfies moreover: For all $x \in X,\ U \in 2^X$,

(LN4) $N_x(U) = \bigvee_{x\in V\subseteq U} \bigwedge_{y\in V} N_y(V)$.

A continuous function between $L-$fuzzifying neighborhood spaces (X, N^1) and (Y, N^2) is a map $f : X \to Y$ such that for all $x \in X,\ U \in 2^Y,\ N^1_x(f^{\leftarrow}(U)) \geq N^2_{f(x)}(U)$.

Let $L-$**NGH** denote the category of $L-$fuzzifying neighborhood spaces with continuous maps, and $L-$**TNGH** the full subcategory of $L-$**NGH** consisting of topological $L-$fuzzifying neighborhood spaces.

Definition 2.4. [17] An $L-$fuzzifying topology on X is a function $\tau : 2^X \to L$ which satisfies

(FO1) $\tau(\emptyset) = \tau(X) = 1$,

(FO2) $\tau(A \cap B) \geq \tau(A) \wedge \tau(B)$,

(FO3) $\tau(\bigcup_{j\in J} A_j) \geq \bigwedge_{j\in J} \tau(A_j)$.

For an $L-$fuzzifying topology τ on X, the pair (X, τ) is called an $L-$fuzzifying topological space. A map $f : X \to Y$ is called continuous with respect to the given $L-$fuzzifying topological spaces (X, τ_1) and (Y, τ_2) iff $\forall B \in 2^Y,\ \tau_1(f^{\leftarrow}(B)) \geq \tau_2(B)$. The category of $L-$fuzzifying topological spaces with continuous maps as morphisms will be denoted by $L-$**FYS**.

It was proved in [20] that for any completely distributive lattice L, topological $L-$fuzzifying neighborhood systems and $L-$fuzzifying topologies are conceptually equivalent with transferring process $N_x(U) = \bigvee_{x\in V\subseteq U} \tau(V)$ and $\tau(U) = \bigwedge_{x\in U} N_x(U)$.

Theorem 2.5. [19] *Let* $\varphi : (X, \tau_1) \to (Y, \tau_2)$ *be a mapping. If* L *is a completely distributive lattice, then* φ *is continuous iff* $N_x^{\tau_1}(\varphi^{\leftarrow}(U)) \geq N_{\varphi(x)}^{\tau_2}(U), \forall x \in X, U \in 2^Y$.

3. $L-$ordered Fuzzifying Convergence Structure

In [16], the author developed lattice-valued convergence structure $\lim : \mathscr{F}_L(X) \to L^X$ as follows:

Definition 3.1. [16] A mapping $\lim : \mathscr{F}_L(X) \to L^X$, subject to the conditions

(LY1) $\forall x \in X,\ \lim[x](x) = 1$,

(LY2) $\forall \mathscr{F}, \mathscr{G} \in \mathscr{F}_L(X),\ \mathscr{F} \leq \mathscr{G} \Rightarrow \forall x \in X, \lim \mathscr{F}(x) \leq \lim \mathscr{G}(x)$,

is called an $L-$fuzzifying convergence structure on X, and $(X, \lim)$ an $L-$fuzzifying convergence space.

The set of all $L-$fuzzifying convergence structures on X is denoted by $\lim_{ly}(X)$. An order on $\lim_{ly}(X)$ can be defined by $\lim_1 \leq \lim_2$ iff for all $\mathscr{F} \in \mathscr{F}_L(X),\ x \in X,\ \lim_2 \mathscr{F}(x) \leq \lim_1 \mathscr{F}(x)$.

In Definition 3.1, the $L-$filters in the axiom (LY2) are in nature $L-$sets on 2^X. We use the method in [3] and define an $L-$partial order $S_F(-,-)$ on $\mathscr{F}_L(X)$ as follows: $S_F(-,-) : \mathscr{F}_L(X) \times \mathscr{F}_L(X) \to L$

$$\forall \mathscr{F}, \mathscr{G} \in \mathscr{F}_L(X), S_F(\mathscr{F}, \mathscr{G}) = \bigwedge_{U \in 2^X} (\mathscr{F}(U) \to \mathscr{G}(U)).$$

In this case, we can redefine the axiom (LY2) in Definition 3.1, proposing the following new lattice-valued convergence structure.

Definition 3.2. An $L-$fuzzifying convergence structure $\lim : \mathscr{F}_L(X) \to L^X$, satisfying the following condition:

$$\text{(OLY2)}\ \ \forall \mathscr{F}, \mathscr{G} \in \mathscr{F}_L(X),\ S_F(\mathscr{F}, \mathscr{G}) \leq S(\lim \mathscr{F}, \lim \mathscr{G}),$$

is called an $L-$ordered fuzzifying convergence structure, and the pair $(X, \lim)$ an $L-$ordered fuzzifying convergence space.

A function $\varphi : (X, \lim^X) \to (Y, \lim^Y)$, $(X, \lim^X), (Y, \lim^Y)$ $L-$ordered fuzzifying convergence spaces, is called continuous iff for all $\mathscr{F} \in \mathscr{F}_L(X),\ x \in X, \lim^X \mathscr{F}(x) \leq \lim^Y \varphi^{\Rightarrow}(\mathscr{F})(\varphi(x))$.

We do not go into details here, but only remark that (OLY2) implies (LY2).

The next example shows there exists an $L-$fuzzifying convergence structure lim which is not an $L-$ordered fuzzifying convergence structure.

Example 3.3. Let $X = \{x, y\},\ L = \{0, \alpha, 1\}$ be a chain. Define a map $\lim : \mathscr{F}_L(X) \to L^X$, $\forall \mathscr{F} \in \mathscr{F}_L(X),\ z \in X$,

$$\lim \mathscr{F}(z) = \begin{cases} 1, & \mathscr{F} \geq [z], \\ 0, & otherwise. \end{cases}$$

It is obvious that lim is an $L-$fuzzifying convergence structure. Define a mapping $\mathscr{F}^* : 2^X \to L$ as follows: $\forall A \in 2^X$,

$$\mathscr{F}^*(A)=\begin{cases}1, & A=X,\\ \alpha, & A=\{x\},\\ 0, & A=\{y\} \text{ or } A=\emptyset.\end{cases}$$

It can be verified that $\mathscr{F}^*$ is an $L-$filter of ordinary subsets on X. Then

$$\begin{aligned} S_F([x],\mathscr{F}^*) &= \bigwedge_{A\in 2^X}([x](A)\to \mathscr{F}^*(A))\\ &= ([x](\emptyset)\to\mathscr{F}^*(\emptyset))\bigwedge([x](\{x\})\to\mathscr{F}^*(\{x\}))\\ &\quad \bigwedge([x](\{y\})\to\mathscr{F}^*(\{y\}))\bigwedge([x](X)\to\mathscr{F}^*(X))\\ &= 1\wedge\alpha\wedge 1\wedge 1\\ &= \alpha \end{aligned}$$

And

$$\begin{aligned} S(\lim[x],\lim\mathscr{F}^*) &= \bigwedge_{z\in X}\big(\lim[x](z)\to\lim\mathscr{F}^*(z)\big)\\ &= \big(\lim[x](x)\to\lim\mathscr{F}^*(x)\big)\bigwedge\big(\lim[x](y)\to\lim\mathscr{F}^*(y)\big)\\ &= (1\to 0)\wedge(0\to 0)\\ &= 0 \end{aligned}$$

We can see that $S_F([x],\mathscr{F}^*)\nleq S(\lim[x],\lim\mathscr{F}^*)$, hence lim is not an $L-$ordered fuzzifying convergence structure.

Example 3.4. Let $(X,\tau)\in L-$**FYS** and define a mapping $\lim_\tau$: $\mathscr{F}_L(X)\to L^X$, $\forall\mathscr{F}\in\mathscr{F}_L(X),\ x\in X, \lim_\tau\mathscr{F}(x)=S_F(N_\tau^x,\mathscr{F})$. Here, the $L-$fuzzifying neighborhood system N_τ^x of $x\in X$ is defined by $N_\tau^x(A)=\bigvee_{x\in B\subseteq A}\tau(B)$. Then $\lim_\tau$ is an $L-$ordered fuzzifying convergence structure.

From Example 3.4, we see that an $L-$ fuzzifying topology can induce an $L-$ordered fuzzifying convergence structure. The following theorem shows that the induced $L-$ordered fuzzifying convergence structure from the $L-$fuzzifying topology can determine the induced $L-$fuzzifying neighborhood structure from the $L-$fuzzifying topology. This idea has been presented in [8].

Theorem 3.5. *Let* $(X,\tau)\in L-$**FYS**. *Then the following holds:*

$$N_\tau^x(U)=\bigwedge_{\mathscr{F}\in\mathscr{F}_L(X)}(\lim_\tau\mathscr{F}(x)\to\mathscr{F}(U)),\forall x\in X, U\in 2^X.$$

Let $L-$**FYCS** [16] denote the category of $L-$fuzzifying convergence spaces with continuous maps and $L-$**OFYC** the full subcategory of $L-$**FYCS** formed by all $L-$ordered fuzzifying convergence spaces.

The set of all $L-$ordered fuzzifying convergence structures on X is denoted by $\lim_{loy}(X)$. An order on $\lim_{loy}(X)$ can be defined by $\lim_1\leq\lim_2$ iff for all $\mathscr{F}\in\mathscr{F}_L(X),\ x\in X,\ \lim_2\mathscr{F}(x)\leq\lim_1\mathscr{F}(x)$. For $\lim_{loy}(X)$ here, we immediately obtain that there are a maximum element and a minimum element in $(\lim_{loy}(X), \leq)$, denoted by $\lim_{sm}$ and $\lim_m$ respectively: $\forall \mathscr{F} \in \mathscr{F}_L(X),\ x \in X,\ \lim_{sm} \mathscr{F} = 1_X$; $\lim_m \mathscr{F}(x) = S_F([x], \mathscr{F})$. The supremum element of a family of $L-$ordered fuzzifying convergence structures $(\lim_j)_{j\in J} \subseteq \lim_{loy}(X)$ is defined by $(\sup\limits_{j\in J} \lim_j)\mathscr{F}(x) = \bigwedge\limits_{j\in J} \lim_j \mathscr{F}(x), \forall \mathscr{F} \in \mathscr{F}_L(X),\ x \in X$. Obviously, $\sup\limits_{j\in J} \lim_j \in \lim_{loy}(X)$. Therefore, the following proposition holds.

Proposition 3.6. $(\lim_{loy}(X), \leq)$ *is a complete lattice.*

We will next address the result that the category of $L-$ordered fuzzifying convergence spaces is a topological category. To this end, we note the following proposition.

Proposition 3.7. *The category* $L-$**OFYC** *is a full reflective subcategory in the category* $L-$**FYCS**.

Proof. Let $(X, \overline{\lim}) \in L-$**FYCS** and $E_{\overline{\lim}} = \{ \lim \mid (X, \lim) \in L-$**OFYC** $, \lim \leq \overline{\lim}\}$. Note that $E_{\overline{\lim}}$ is not empty because it always contains $\lim_{sm}$. Then with Proposition 3.6, we can construct an $L-$ordered fuzzifying convergence structure $\overline{\lim}_* : \mathscr{F}_L(X) \to L^X$ as follows: For all $\mathscr{F} \in \mathscr{F}_L(X),\ x \in X,\ \overline{\lim}_* \mathscr{F}(x) = \bigwedge_{\lim \in E_{\overline{\lim}}} \lim \mathscr{F}(x)$. From this, we have

(1) $id_X : (X, \overline{\lim}) \to (X, \overline{\lim}_*)$ is trivially continuous;

(2) For an $L-$ordered fuzzifying convergence space $(Y, \lim^Y)$, if $f : (X, \overline{\lim}) \to (Y, \lim^Y)$ is a continuous mapping, then $f : (X, \overline{\lim}_*) \to (Y, \lim^Y)$ is also continuous. We leave the above check to the reader.

From the above facts, we immediately obtain that $L-$**OFYC** is a full reflective subcategory in $L-$**FYCS**. □

In [16] Yao proved that the category $L-$**FYCS** is topological. By Proposition 3.7, we have the following main result.

Theorem 3.8. *The category of $L-$ordered fuzzifying convergence spaces* $L-$**OFYC** *is topological.*

4. The Relations Between Categories of $L-$FYS and $L-$OFYC

This section is motivated by reference [8]. In this section, we will resolve the embedding of $L-$**FYS** into $L-$**OFYC**. By Example 3.4 and Theorem 3.5, we see that $L-$ordered convergence structures can be induced from $L-$fuzzifying topologies. Moreover, they are unique. In order to show that $L-$**FYS** can be embedded in the category of $L-$**OFYC**, the following theorem is necessary.

Theorem 4.1. *Let L be a completely distributive lattice. Then the map $f : (X, \tau_1) \to (Y, \tau_2)$ between two $L-$fuzzifying topological spaces is continuous iff $f : (X, \lim_{\tau_1}) \to (Y, \lim_{\tau_2})$ is continuous.*

Proof. Suppose that $f:(X,\tau_1)\to(Y,\tau_2)$ is continuous, by Theorem 2.5, we have for all $\mathscr{F}\in\mathscr{F}_L(X),\ x\in X$,

$$\begin{aligned}\lim_{\tau_2}\varphi^{\Rightarrow}(\mathscr{F})(\varphi(x)) &= \bigwedge_{V\in 2^Y}(N_{\tau_2}^{\varphi(x)}(V)\to\varphi^{\Rightarrow}(\mathscr{F})(V))\\ &\geq \bigwedge_{V\in 2^Y}(N_{\tau_1}^{x}(\varphi^{\leftarrow}(V))\to\mathscr{F}(\varphi^{\leftarrow}(V)))\\ &\geq \bigwedge_{U\in 2^X}(N_{\tau_1}^{x}(U)\to\mathscr{F}(U))\\ &= \lim_{\tau_1}\mathscr{F}(x).\end{aligned}$$

Hence, $f:(X,\lim_{\tau_1})\to(Y,\lim_{\tau_2})$ is continuous.

Conversely, if $f:(X,\lim_{\tau_1})\to(Y,\lim_{\tau_2})$ is continuous, by Theorem 3.5, we have $\forall x\in X, U\in 2^Y$,

$$\begin{aligned}N_{\tau_1}^{x}(\varphi^{\leftarrow}(U)) &= \bigwedge_{\mathscr{F}\in\mathscr{F}_L(X)}(\lim_{\tau_1}\mathscr{F}(x)\to\mathscr{F}(\varphi^{\leftarrow}(U)))\\ &\geq \bigwedge_{\mathscr{F}\in\mathscr{F}_L(X)}(\lim_{\tau_2}(\varphi^{\Rightarrow}(\mathscr{F}))(\varphi(x))\to(\varphi^{\Rightarrow}(\mathscr{F}))(U))\\ &\geq \bigwedge_{\mathscr{G}\in\mathscr{F}_L(Y)}(\lim_{\tau_2}\mathscr{G}(\varphi(x))\to\mathscr{G}(U))\\ &= N_{\tau_2}^{\varphi(x)}(U).\end{aligned}$$

Therefore, by Theorem 2.5, $f:(X,\tau_1)\to(Y,\tau_2)$ is continuous. □

As a consequence of the above theorems, we have the following result.

Theorem 4.2. *Let L be a completely distributive lattice. $L-$**FYS** can be embedded in the category of $L-$**OFYC**.*

In Theorem 3.8 we know that $L-$**OFYC** is topological. So, in order to show that it is Cartesian-closed, the following results are necessary. Similar to the definition of product spaces in $L-$**FYCS**, it can be shown that there are also product spaces in $L-$**OFYC**. We refer the reader to [16]. Here, we only present the main results. Note that for two $L-$ordered fuzzifying convergence spaces $(X,\lim_X),(Y,\lim_Y)$, let $[X\to Y]$ denote the set of all continuous maps from $(X,\lim_X)$ to $(Y,\lim_Y)$.

Lemma 4.3. [16] *Let $g:X\to Y$ and $\mathscr{G}\in\mathscr{F}_L(X)$, then $g^{\Rightarrow}(\mathscr{G})\leq ev^{\Rightarrow}([g]\times\mathscr{G})$, where $ev:[X\to Y]\times X\to Y$ is the evaluation map.*

Theorem 4.4. *Let $(X,\lim_X),(Y,\lim_Y)$ be $L-$ordered fuzzifying convergence spaces, then $\lim_{[X\to Y]}:F_L([X\to Y])\to L^{[X\to Y]}$, $\forall\mathscr{F}\in F_L([X\to Y])$, $\forall f\in[X\to Y]$, $\lim_{[X\to Y]}\mathscr{F}(f)=\bigwedge_{(\mathscr{G},x)\in\mathscr{F}_L(X)\times X}(\lim_X\mathscr{G}(x)\to\lim_Y ev^{\Rightarrow}(\mathscr{F}\times\mathscr{G})(f(x)))$ is an $L-$ordered fuzzifying convergence structure on $[X\to Y]$.*

Proof. For (LY1), $\forall g \in [X \to Y]$,

$$\begin{aligned}
\lim_{[X\to Y]}[g](g) &= \bigwedge_{(\mathscr{G},x)\in \mathscr{F}_L(X)\times X} \lim_X \mathscr{G}(x) \to \lim_Y (ev^{\Rightarrow}([g]\times \mathscr{G}))(g(x)) \\
&\geq \bigwedge_{(\mathscr{G},x)\in \mathscr{F}_L(X)\times X} \lim_X \mathscr{G}(x) \to \lim_Y (g^{\Rightarrow}(\mathscr{G}))(g(x)) \\
&= 1.
\end{aligned}$$

For (OLY2), $\forall \mathscr{F}, \mathscr{G} \in \mathscr{F}_L([X \to Y])$,

$$\begin{aligned}
& S(\lim_{[X\to Y]} \mathscr{F}, \lim_{[X\to Y]} \mathscr{G}) \\
= & \bigwedge_{g\in[X\to Y]} \Big(\Big(\bigwedge_{(\mathscr{E},x)\in \mathscr{F}_L(X)\times X} \lim_X \mathscr{E}(x) \to \lim_Y (ev^{\Rightarrow}(\mathscr{F}\times\mathscr{E}))(g(x))\Big) \\
& \quad \to \Big(\bigwedge_{(\mathscr{H},x)\in \mathscr{F}_L(X)\times X} \lim_X \mathscr{H}(x) \to \lim_Y (ev^{\Rightarrow}(\mathscr{G}\times\mathscr{H}))(g(x))\Big)\Big) \\
\geq & \bigwedge_{g\in[X\to Y]} \bigwedge_{(\mathscr{H},x)\in \mathscr{F}_L(X)\times X} \Big(\Big(\lim_X \mathscr{H}(x) \to \lim_Y (ev^{\Rightarrow}(\mathscr{F}\times\mathscr{H}))(g(x))\Big) \\
& \quad \to \Big(\lim_X \mathscr{H}(x) \to \lim_Y (ev^{\Rightarrow}(\mathscr{G}\times\mathscr{H}))(g(x))\Big)\Big) \\
\geq & \bigwedge_{\mathscr{H}\in \mathscr{F}_L(X)} S(\lim_Y (ev^{\Rightarrow}(\mathscr{F}\times\mathscr{H})), \lim_Y (ev^{\Rightarrow}(\mathscr{G}\times\mathscr{H}))) \\
\geq & \bigwedge_{\mathscr{H}\in \mathscr{F}_L(X)} S_F(ev^{\Rightarrow}(\mathscr{F}\times\mathscr{H}), ev^{\Rightarrow}(\mathscr{G}\times\mathscr{H})).
\end{aligned}$$

$\forall \mathscr{H} \in \mathscr{F}_L(X)$,

$$\begin{aligned}
& S_F(ev^{\Rightarrow}(\mathscr{F}\times\mathscr{H}), ev^{\Rightarrow}(\mathscr{G}\times\mathscr{H})) \\
= & \bigwedge_{U\in 2^Y} \Big((\mathscr{F}\times\mathscr{H})(ev^{\Leftarrow}(U)) \to (\mathscr{G}\times\mathscr{H})(ev^{\Leftarrow}(U))\Big) \\
= & \bigwedge_{U\in 2^Y} \Big(\Big(\bigvee_{A\times B\subseteq ev^{\Leftarrow}(U)} \mathscr{F}(A)\wedge \mathscr{H}(B)\Big) \to \Big(\bigvee_{C\times D\subseteq ev^{\Leftarrow}(U)} \mathscr{G}(C)\wedge \mathscr{H}(D)\Big)\Big) \\
\geq & \bigwedge_{U\in 2^Y} \bigwedge_{A\times B\subseteq ev^{\Leftarrow}(U)} \Big((\mathscr{F}(A)\wedge \mathscr{H}(B)) \to (\mathscr{G}(A)\wedge \mathscr{H}(B))\Big) \\
\geq & \bigwedge_{U\in 2^Y} \bigwedge_{A\times B\subseteq ev^{\Leftarrow}(U)} \Big(\mathscr{F}(A) \to \mathscr{G}(A)\Big) \\
\geq & \bigwedge_{C\in 2^{[X\to Y]}} (\mathscr{F}(C) \to \mathscr{G}(C)) \\
= & S_F(\mathscr{F}, \mathscr{G}).
\end{aligned}$$

Therefore, the above completes the proof. In other words, $\lim_{[X\to Y]}$ is an $L-$ordered fuzzifying convergence structure on $[X \to Y]$. □

Remark 4.5. The evaluation map $ev : [X \to Y] \times X \to Y$ mentioned above is continuous. Let $f : X \times Y \to Z$ be a map, $\forall x \in X$, define a map $f_x : Y \to Z$, $\forall y \in Y$, $f_x(y) = f(x, y)$, $f^* : X \to Z^Y, \forall x \in X$, $f^*(x) = f_x$, and $\varphi : Z^{(X \to Y)} \to (Z^Y)^X, \forall f \in Z^{(X \to Y)}, \varphi(f) = f^*$. Then it can be proved that

(1) If $f : (X, \lim_X) \times (Y, \lim_Y) \to (Z, \lim_Z)$ is continuous, then for each $x \in X$, $f_x : (Y, \lim_Y) \to (Z, \lim_Z)$ is continuous.

(2) For all $\mathscr{F} \in \mathscr{F}_L(X)$, $\mathscr{G} \in \mathscr{F}_L(Y)$, $ev^{\Rightarrow}(\varphi(f)^{\Rightarrow}(\mathscr{F}) \times \mathscr{G}) = f^{\Rightarrow}(\mathscr{F} \times \mathscr{G})$.

(3) If $f : X \times Y \to Z$ is continuous, then $\varphi(f) : X \to [Y \to Z]$ is continuous. (We refer to [16] for a detail proof of the above results.)

We collect our findings in the following theorem.

Theorem 4.6. *$L-$**OFYC** is a Cartesian-closed category.*

5. The Relations Between $L-$fuzzifying Neighborhood Spaces and Principle $L-$ordered Fuzzifying Convergence Spaces

In this section, we define a subcategory of the category of $L-$ordered fuzzifying convergence spaces: the category of principle $L-$ordered fuzzifying convergence spaces and show that the new category and that of $L-$fuzzifying neighborhood spaces are isomorphic. Furthermore, each fibre on a fixed set of the category of $L-$fuzzifying neighborhood spaces and that of the category of principal $L-$ordered fuzzifying convergence spaces are isomorphic. At the end of the section, we propose that the category of principle $L-$ordered fuzzifying convergence spaces is a reflective subcategory of $L-$**OFYC** and it is a topological category. Again, this section is mostly motivated by reference [8].

Proposition 5.1. *Let $(X, \lim) \in L-$**OFYC**. The structure $\{N^x_{\lim} : 2^X \to L\}_{x \in X}$ defined by: For $x \in X$, $\forall U \in 2^X, N^x_{\lim}(U) = \bigwedge_{\mathscr{F} \in \mathscr{F}_L(X)}(\lim \mathscr{F}(x) \to \mathscr{F}(U))$ is an $L-$fuzzifying neighborhood structure. We call it the induced $L-$fuzzifying neighborhood structure of $(X, \lim)$.*

Theorem 3.5 suggests for $(X, \lim) \in L-$**OFYC** the following definition.

Definition 5.2. Let lim be an $L-$ordered fuzzifying convergence structure. If in addition the following condition (LYP) holds,

$$(LYP)\ \forall \mathscr{F} \in \mathscr{F}_L(X),\ x \in X,\ \lim \mathscr{F}(x) = S_F(N^x_{\lim}, \mathscr{F}),$$

then lim is called a principal $L-$ordered fuzzifying convergence structure, and the pair $(X, \lim)$ is called a principle $L-$ordered fuzzifying convergence space.

The full subcategory of $L-$**OFYC** consisting of all principle $L-$ordered fuzzifying convergence spaces is denoted by $L-$**POFYC**.

If an $L-$ordered fuzzifying convergence spaces satisfies (LYP), then a nice characterization of principle $L-$ordered convergence spaces in terms of $L-$fuzzifying neighborhood spaces is possible. We first need three theorems for preparation.

Theorem 5.3. *Let (X, N) be an $L-$fuzzifying neighborhood space. Then there exists a principle $L-$ordered fuzzifying convergence structure* lim *on X satisfying* $\forall x \in X, N^x_{\lim} = N^x$.

Proof. For the $L-$fuzzifying neighborhood space (X, N), define $\lim_N : \mathscr{F}_L(X) \to L^X$

$$\forall \mathscr{F} \in \mathscr{F}_L(X),\ x \in X,\ \lim_N \mathscr{F}(x) = \bigwedge_{A \in 2^X} (N^x(A) \to \mathscr{F}(A)) = S_F(N^x, \mathscr{F}).$$

It is then readily checked that for $(X, \lim_N)$, the axiom (LY1), (OLY2), (LYP) hold. The properties of the residual implication of Theorem 2.1 are used.

(LY1): $\forall x \in X,\ \lim_N [x](x) = \bigwedge_{A \in 2^X} (N^x(A) \to [x](A)) = 1.$

(OLY2): In fact,

$$\begin{aligned} S(\lim_N \mathscr{F}, \lim_N \mathscr{G}) &= \bigwedge_{x \in X} (S_F(N^x, \mathscr{F}) \to S_F(N^x, \mathscr{G})) \\ &= \bigwedge_{x \in X} (\bigwedge_{A \in 2^X} (N^x(A) \to \mathscr{F}(A)) \to \bigwedge_{B \in 2^X} (N^x(B) \to \mathscr{G}(B))) \\ &= \bigwedge_{x \in X} \bigwedge_{B \in 2^X} (\bigwedge_{A \in 2^X} (N^x(A) \to \mathscr{F}(A)) \to (N^x(B) \to \mathscr{G}(B))) \\ &\geq \bigwedge_{x \in X} \bigwedge_{B \in 2^X} ((N^x(B) \to \mathscr{F}(B)) \to (N^x(B) \to \mathscr{G}(B))) \\ &\geq \bigwedge_{x \in X} \bigwedge_{B \in 2^X} ((\mathscr{F}(B) \to \mathscr{G}(B)) \\ &= S_F(\mathscr{F}, \mathscr{G}). \end{aligned}$$

(LYP): For all $\mathscr{F} \in \mathscr{F}_L(X)$, we prove $\lim_N \mathscr{F}(x) = S_F(N^x_{\lim_N}, \mathscr{F})$. By the definition of $\lim_N$, $\lim_N \mathscr{F}(x) = S_F(N^x, \mathscr{F})$. It remains to verify that $N^x_{\lim_N} = N^x$. On one hand, for all $A \in 2^X$,

$$\begin{aligned} N^x_{\lim_N}(A) &= \bigwedge_{\mathscr{F} \in \mathscr{F}_L(X)} (\lim_N \mathscr{F}(x) \to \mathscr{F}(A)) \\ &\leq \lim_N N^x(x) \to N^x(A) \\ &= N^x(A). \end{aligned}$$

On the other hand,

$$\begin{aligned} N^x_{\lim_N}(A) &= \bigwedge_{\mathscr{F} \in \mathscr{F}_L(X)} (\lim_N \mathscr{F}(x) \to \mathscr{F}(A)) \\ &= \bigwedge_{\mathscr{F} \in \mathscr{F}_L(X)} \Big(\bigwedge_{B \in 2^X} \Big(N^x(B) \to \mathscr{F}(B)\Big) \to \mathscr{F}(A)\Big) \\ &\geq \bigwedge_{\mathscr{F} \in \mathscr{F}_L(X)} (N^x(A) \to \mathscr{F}(A) \to \mathscr{F}(A)) \\ &\geq N^x(A). \end{aligned}$$

From this, the result follows by a standard argument. □

In view of the above theorem, if N is an $L-$fuzzifying neighborhood structure, then there exists a principle $L-$ordered fuzzifying convergence structure $\lim_N$ on X. Moreover, $N_{\lim_N}$ is also an $L-$fuzzifying neighborhood structure and $N_{\lim_N} = N$. Conversely, we have the following theorem.

Theorem 5.4. *If* $\lim$ *is a principle* $L-$*ordered fuzzifying convergence structure on* X*, then* $\lim_{N_{\lim}} = \lim$.

Proof. For all $\mathscr{F} \in \mathscr{F}_L(X), x \in X$, by (LYP), we have,

$\lim_{N_{\lim}} \mathscr{F}(x) = \bigwedge_{A\in 2^X} \Big(N^x_{\lim}(A) \to \mathscr{F}(A)\Big) = S_F(N^x_{\lim}, \mathscr{F}) = \lim \mathscr{F}(x).$ □

With respect to Theorem 5.3 and Theorem 5.4, we have a one-one correspondence between the objects of $L-$**NGH** and $L-$**POFYC**. The following theorem is about the relation between morphisms of them.

Theorem 5.5. *Let* $(X, \lim^X)$*,*$(Y, \lim^Y)$ *be principle* $L-$*ordered fuzzifying convergence spaces,* (X, N_1)*,*(Y, N_2) *be* $L-$*fuzzifying neighborhood spaces, then we have*

(1) *If* $f : (X, \lim^X) \to (Y, \lim^Y)$ *is continuous, then* $f : (X, N_{\lim^X}) \to (Y, N_{\lim^Y})$ *is also continuous;*

(2) *If* $f : (X, N_1) \to (Y, N_2)$ *is continuous, then* $f : (X, \lim^X_{N_1}) \to (Y, \lim^Y_{N_2})$ *is also continuous.*

Proof. (1) By the fact that $f : (X, \lim^X) \to (Y, \lim^Y)$ is continuous, we have $\forall x \in X$, $U \in 2^Y$,

$$\begin{aligned}
N^x_{\lim^X}(f^{\leftarrow}(U)) &= \bigwedge_{\mathscr{F}\in\mathscr{F}_L(X)} \Big(\lim^X \mathscr{F}(x) \to \mathscr{F}(f^{\leftarrow}(U))\Big) \\
&\geq \bigwedge_{\mathscr{F}\in\mathscr{F}_L(X)} \Big(\lim^Y f^{\Rightarrow}(\mathscr{F})(f(x)) \to f^{\Rightarrow}(\mathscr{F})(U)\Big) \\
&\geq \bigwedge_{\mathscr{G}\in\mathscr{F}_L(Y)} \Big(\lim^Y \mathscr{G}(f(x)) \to \mathscr{G}(U)\Big) \\
&= N^{f(x)}_{\lim^Y}(U),
\end{aligned}$$

as desired.

(2) Conversely, by the fact that $f : (X, N_1) \to (Y, N_2)$ is continuous, we have $\forall \mathscr{F} \in \mathscr{F}_L(X), x \in X$,

$$\begin{aligned}
\lim^Y_{N_2} f^{\Rightarrow}(\mathscr{F})(f(x)) &= \bigwedge_{B\in 2^Y} \Big(N^{f(x)}_2(B) \to f^{\Rightarrow}(\mathscr{F})(B)\Big) \\
&\geq \bigwedge_{B\in 2^Y} \Big(N^x_1(f^{\leftarrow}(B)) \to \mathscr{F}(f^{\leftarrow}(B))\Big) \\
&\geq \bigwedge_{A\in 2^X} \Big(N^x_1(A) \to \mathscr{F}(A)\Big) \\
&= \lim^X_{N_1} \mathscr{F}(x),
\end{aligned}$$

as desired. □

By Theorems 5.3, 5.4 and 5.5, we actually have proved the following comprehensive theorem.

Theorem 5.6. *$L-$**NGH** is isomorphic to $L-$**POFYC**.*

Let X be a set. A fibre on X of the category of $L-$fuzzifying neighborhood spaces is denoted by $\mathbf{PrFN_L(X)}$. An order " $\leq$ " on $\mathbf{PrFN_L(X)}$ can be defined by

$$N^1 \leq N^2 \Leftrightarrow \forall x \in X, A \in 2^X, N_x^1(A) \leq N_x^2(A).$$

A fibre on X of the category of principle $L-$ordered fuzzifying convergence spaces is denoted by $\mathbf{PFYC_L(X)}$. An order " $\leq$ " on $\mathbf{PFYC_L(X)}$ can be defined as follows:

$$\lim_1 \leq \lim_2 \Leftrightarrow \forall \mathscr{F} \in \mathscr{F}_L(X), x \in X, \lim_2 \mathscr{F}(x) \leq \lim_1 \mathscr{F}(x).$$

Then we have the following result.

Theorem 5.7. *$(\mathbf{PFYC_L(X)}, \leq)$ and $(\mathbf{PrFN_L(X)}, \leq)$ are isomorphic.*

Proof. Define a mapping: $h: \mathbf{PFYC_L(X)} \to \mathbf{PrFN_L(X)}$, $\forall \lim \in \mathbf{PFYC_L(X)}$, $h(\lim) = N_{\lim}$, and a mapping: $k: \mathbf{PrFN_L(X)} \to \mathbf{PFYC_L(X)}$, $\forall N \in \mathbf{PrFN_L(X)}$, $k(N) = \lim_N$. It has been verified in Theorems 5.3 and 5.4 that $h \circ k = id_{\mathbf{PrFN_L(X)}}$, $k \circ h = id_{\mathbf{PFYC_L(X)}}$. So h and k are both bijective. Furthermore, $k = h^{-1}$.

(1) For all $\lim_1, \lim_2 \in \mathbf{PFYC_L(X)}$, if $\lim_1 \leq \lim_2$, then for all $x \in X, A \in 2^X$,

$$\begin{aligned} N_{\lim_1}^x(A) &= \bigwedge_{\mathscr{F} \in \mathscr{F}_L(X)} \Big(\lim_1 \mathscr{F}(x) \to \mathscr{F}(A)\Big) \\ &\leq \bigwedge_{\mathscr{F} \in \mathscr{F}_L(X)} \Big(\lim_2 \mathscr{F}(x) \to \mathscr{F}(A)\Big) \\ &= N_{\lim_2}^x(A). \end{aligned}$$

So, $N_{\lim_1} \leq N_{\lim_2}$. i.e. $h(\lim_1) \leq h(\lim_2)$. Therefore, h is an order preserving map.

(2) For all $N_1, N_2 \in \mathbf{PrFN_L(X)}$, if $N_1 \leq N_2$, then for all $\mathscr{F} \in \mathscr{F}_L(X), x \in X$,

$$\begin{aligned} \lim_{N_1} \mathscr{F}(x) &= \bigwedge_{A \in 2^X} \Big(N_1^x(A) \to \mathscr{F}(A)\Big) \\ &\geq \bigwedge_{A \in 2^X} \Big(N_2^x(A) \to \mathscr{F}(A)\Big) \\ &= \lim_{N_2} \mathscr{F}(x). \end{aligned}$$

Hence, $\lim_{N_1} \leq \lim_{N_2}$, i.e. $h^{-1}(N_1) \leq h^{-1}(N_2)$. So h^{-1} is also an order preserving mapping.

From the above proof, we conclude that $(\mathbf{PFYC_L(X)}, \leq)$ and $(\mathbf{PrFN_L(X)}, \leq)$ are isomorphic. □

At the end of this section, we propose the following results.

Theorem 5.8. *The category $L-$**POFYC** is a reflective subcategory of $L-$**OFYC**.*

Proof. Let $(X, \overline{\lim}) \in L-$**OFYC** and $E_{\overline{\lim}} = \{ \lim \mid (X, \lim) \in L-$ **POFYC** , $\lim \leq \overline{\lim}\}$. Note that $E_{\overline{\lim}}$ is not empty because it always contains $\lim_{sm}$. Then we can construct a principal $L-$ordered fuzzifying convergence structure $\overline{\lim}_* : \mathscr{F}_L(X) \to L^X$ as follows: For all $\mathscr{F} \in \mathscr{F}_L(X),\ x \in X,\ \overline{\lim}_* \mathscr{F}(x) = \bigwedge_{\lim \in E_{\overline{\lim}}} \lim \mathscr{F}(x)$. From this, we have

(1) $id_X : (X, \overline{\lim}) \to (X, \overline{\lim}_*)$ is trivially continuous;

(2) For a principal $L-$ordered fuzzifying convergence space $(Y, \lim^Y)$, if $f : (X, \overline{\lim}) \to (Y, \lim^Y)$ is a continuous mapping, then $f : (X, \overline{\lim}_*) \to (Y, \lim^Y)$ is also continuous.

From the above facts, we immediately obtain that $L-$**POFYC** is a full reflective subcategory in $L-$**OFYC**. □

Corollary 5.9. *The category $L-$**POFYC** is topological.*

6. The Relations Between $L-$fuzzifying Topological Spaces and Topological $L-$ordered Fuzzifying Convergence Spaces

In this section, we define another important subcategory of $L-$**OFYC**: the category of topological $L-$ordered fuzzifying convergence spaces. We will find out that the category mentioned above is isomorphic to $L-$**FYS** and to $L-$**TNGH** in case of a completely distributive lattice L. Furthermore, each fibre on X of the category of topological $L-$ordered fuzzifying convergence spaces is isomorphic to that of $L-$fuzzifying topological spaces and that of topological $L-$fuzzifying neighborhood spaces.

Definition 6.1. Let $(X, \lim) \in L-$**POFYC**, if in addition the mapping $\lim : \mathscr{F}_L(X) \to L^X$ satisfies the following axiom:

$$(LYT)\ \forall U \in 2^X,\ N^x_{\lim}(U) \leq \bigvee_{x \in V \subseteq U} \bigwedge_{y \in V} N^y_{\lim}(V),$$

then lim is called a topological $L-$ordered fuzzifying convergence structure, and $(X, \lim)$ is a topological $L-$ordered fuzzifying convergence space. The full subcategory of $L-$**OFYC** consisting of all topological $L-$ordered fuzzifying convergence spaces is denoted by $L-$**TOFYC**.

If lim is a topological $L-$ordered fuzzifying convergence structure, then a nice characterization of $L-$fuzzifying topologies is possible. We need two lemmas for preparation.

Lemma 6.2. *Let (X, N) be a topological $L-$fuzzifying neighborhood space, then $(X, \lim_N)$ is a topological $L-$ordered fuzzifying convergence space.*

Proof. As for (X, N) is a topological $L-$fuzzifying neighborhood space, (X, N) is an $L-$fuzzifying neighborhood space. By Theorem 5.3, we see $N = N_{\lim_N}$. For N is a topological $L-$fuzzifying neighborhood structure, we know for all $x \in$

X, $\forall U \in 2^X$, $N^x(U) \leq \bigvee_{x\in V\subseteq U} \bigwedge_{y\in V} N^y(V)$. Therefore, $N_{\lim_N}$ satisfies (LYT). With Definition 6.1, the lemma holds. □

Lemma 6.3. *Let* $(X, \lim) \in L-$**TOFYC**, *then there exists an* $L-$*fuzzifying topology* τ *on* X *such that* $\lim_\tau = \lim$.

Proof. Firstly, by Definition 6.1, $(X, \lim) \in L-$**TOFYC** implies that $N_{\lim}$ satisfies $(N1) - (N4)$.

Secondly, let $\tau: \ 2^X \to L$, $\forall A \in 2^X, \tau(A) = \bigwedge_{x\in A} N^x_{\lim}(A)$. It can be easily proved that τ is an $L-$fuzzifying topology on X. Moreover, $N_\tau = N_{\lim}$. In fact, for all $x \in X, A \in 2^X$, we have by the axiom (LYT),

$$\begin{aligned} N^x_\tau(A) &= \bigvee_{x\in B\subseteq A} \tau(B) \\ &= \bigvee_{x\in B\subseteq A} \bigwedge_{y\in B} N^y_{\lim}(B) \\ &= N^x_{\lim}(A). \quad \text{(LYT)} \end{aligned}$$

With this and (LYP), we obtain for all $\mathscr{F} \in \mathscr{F}_L(X)$, $x \in X$, $\lim_\tau \mathscr{F}(x) = S_F(N^x_\tau, \mathscr{F}) = S_F(N^x_{\lim}, \mathscr{F}) = \lim \mathscr{F}(x)$.

Therefore, $\lim_\tau = \lim$ holds. □

Lemmas 6.2, 6.3 together with the relations between topological $L-$fuzzifying neighborhood spaces and $L-$fuzzifying topological spaces in case that L is a completely distributive lattice show the following result.

Theorem 6.4. *If* L *is a completely distributive lattice, then* $L-$**FYS**, $L-$**TOFYC** *and* $L-$**TNGH** *are isomorphic to each other.*

We denote a fibre on X of the category of $L-$fuzzifying topological spaces by $\mathbf{FY_L(X)}$, and an order " $\leq$ " on it is defined as follows: $\forall\ \tau_1, \tau_2 \in \mathbf{FY_L(X)}$,

$$\tau_1 \leq \tau_2 \Leftrightarrow \forall A \in 2^X, \tau_1(A) \leq \tau_2(A).$$

Denote a fibre on X of the category of topological $L-$fuzzifying neighborhood spaces by $\mathbf{FN_L(X)}$ and a fibre on X of the category of topological $L-$ordered fuzzifying convergence spaces by $\mathbf{TFYC_L(X)}$. In the same way as in the proof of Theorem 5.7, we obtain the following theorem trivially, and leave the straightforward proof for the interested reader.

Theorem 6.5. $(\mathbf{FN_L(X)}, \leq)$, $(\mathbf{FY_L(X)}, \leq)$, $(\mathbf{TFYC_L(X)}, \leq)$ *are isomorphic.*

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