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SEMI-G-FILTERS, STONEAN FILTERS, MTL-FILTERS, DIVISIBLE FILTERS, BL-FILTERS AND REGULAR FILTERS IN RESIDUATED LATTICES

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Abstract. At present, the filter theory of BL-algebras has been widely studied, and some important results have been published (see for example [12], [13], [17], [23], [24]). In other works such as [4], [26], [19], [30] a study of a filter theory in the more general setting of residuated lattices is done, generalizing that for BL-algebras. Note that filters are also characterized by various types of fuzzy sets. Most of such characterizations is trivial but some are nontrivial, for example characterizations obtained in [20]. Both situation have revealed a rich range of classes of filters: Boolean, implicative, Heyting, positive implicative, fantastic (or MV-filter), etc. In this paper we work in the general cases of residuated lattices and put in evidence new types of filters in a residuated lattice (in the spirit of [5]): semi-G-filters, Stonean filters, divisible filters, BL-filters and regular filters.

1. Introduction

Residuation is a fundamental concept of ordered structures and categories. Ward and Dilworth ([28]) were first to introduce the concept of a *residuated lattice* as a generalization of ideal lattice of rings. The theory of residuated lattices was used to develop algebraic counterparts of fuzzy logics and substructural logics. They have been investigated by Krull ([18]), Dilworth ([11]), Ward ([27]), Balbes and Dwinger ([1]) and Pavelka ([22]).

In [15], Idziak proved that the class of residuated lattices is equational. These lattices have been known under many names: *BCK- latices* in [14], *full BCK-algebras* in [18], FL_{ew}*- algebras* in [21], and *integral, residuated, commutative l-monoids* in [2], [3].

Apart from their logical interest, residuated lattices have interesting algebraic properties (see [7], [12], [23], [25]) and include two important classes of algebras: BL*-algebras* (introduced by Hajek as the algebraic counterpart of his Basic Logic, see [12]) and MV*-algebras* (introduced by Chang in [8] to prove the completeness theorem for Łukasiewicz calculus).

An important role in the theory of lattices is played by the concept of *filter*. Afterwards, the notion of filter was defined on various algebraic structures.

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Deductive systems correspond to subsets closed with respect to Modus Ponens and they are called sometime *(implicative) filters*.

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In other works such as [4], [26], [19], [30] a study of a filter theory in the more general setting of residuated lattices is done, generalizing that for BL-algebras.

Both situation have revealed a rich range of classes of filters: Boolean, implicative, Heyting, positive implicative, fantastic (or MV-filter), etc.

Note that filters are also characterized by various types of fuzzy sets. Most of such characterizations is trivial but some are nontrivial, for example characterizations obtained in [20].

In this paper we work in the general cases of residuated lattices and put in evidence new types of filters in a residuated lattice (in the spirit of [5]): *semi-G-filters, Stonean filters, divisible filters, BL-filters* and *regular filters.*

2. Preliminaries

In this section we review some properties of residuated lattices which we need in the rest of the paper.

Definition 2.1. [2], [3], [12], [25]-[28] A *residuated lattice* is an algebra $(L, \vee, \wedge, \odot, \to, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ equipped with an order $\leq$ satisfying the following:

(LR_1) $(L, \vee, \wedge, 0, 1)$ is a bounded lattice relative to the order $\leq$;
(LR_2) $(L, \odot, 1)$ is a commutative monoid;
(LR_3) The operations $\odot$ and $\to$ form an *adjoint pair*, i.e. $a \odot x \leq y$ iff $a \leq x \to y$ for every $a, x, y \in L$.

In what follows, by L we denote the universe of a residuated lattice.

For $x \in L$ and a natural number n we define $x^* = x \to 0, x^{**} = (x^*)^*, x^0 = 1$ and $x^n = x^{n-1} \odot x$ for $n \geq 1$.

The following rules of calculus in L can be found, for example, in [2], [3], [7], [12], [13], [23], [24], [28]:

(c_1) $1 \to x = x, x \to x = 1, x \to 1 = 1$;
(c_2) $x \leq y$ iff $x \to y = 1$;
(c_3) If $x \leq y$, then $x \odot z \leq y \odot z, z \to x \leq z \to y$ and $y \to z \leq x \to z$ (so, $y^* \leq x^*$ and $x^{**} \leq y^{**}$);
(c_4) $x \to y \leq (y \to z) \to (x \to z)$ (so, $x \to y \leq y^* \to x^*$ and $(x \to y) \odot (y \to z) \leq (x \to z)$);
(c_5) $x \odot x^* = 0$, $x \odot y = 0$ iff $x \leq y^*$;
(c_6) $x \odot (x \to y) \leq y$ (so, $x \odot (x \to y) \leq x \wedge y$);
(c_7) $x \to (y \to z) = (x \odot y) \to z = y \to (x \to z)$ (so, $x \to y^* = y \to x^* = x^{**} \to y^* = (x \odot y)^*$);
(c_8) $x \to (y \wedge z) = (x \to y) \wedge (x \to z), (y \vee z) \to x = (y \to x) \wedge (z \to x)$;
(c_9) $x \leq x^{**}, x^{***} = x^*, x \leq x^* \to y$;
(c_{10}) $(x \vee y)^* = x^* \wedge y^*$.

Proposition 2.2. *If* $x \in L$, *then:*

(c_{11}) $x \vee x^* \leq x^* \vee x^{**} \leq (x \wedge x^*)^*$;
(c_{12}) *If consider* $a = (x \wedge x^*)^*$, *then* $a^* \leq a$;
(c_{13}) *If consider* $b = (x^* \to x) \to x$, *then* $b^* \leq b$.

Proof. (c_{11}). From $x \leq x^{**}$ we deduce that $x \vee x^* \leq x^* \vee x^{**}$. From $x \wedge x^* \leq x, x^*$ we deduce that $x^*, x^{**} \leq (x \wedge x^*)^*$, so $x^* \vee x^{**} \leq (x \wedge x^*)^*$.

(c_{12}). We have $a^* = (x \wedge x^*)^{**} \leq x^{**} \wedge x^{***} = x^* \wedge x^{**}$. Since $(x \wedge x^*) \odot (x^* \wedge x^{**}) \leq x \odot x^* = 0$, we deduce that $x^* \wedge x^{**} \leq (x \wedge x^*)^* = a$, hence $a^* \leq a$.

(c_{13}). From $x, x^* \leq b$ we deduce $b^* \leq x^*$ and $1 = b^* \to x^* \leq b^* \to b$, so $b^* \to b = 1$, that is, $b^* \leq b$. □

Definition 2.3. [23], [25] A nonempty subset D of L is called a *deductive system* (**ds** for short) if :

(D_1) $1 \in D$;
(D_2) If $x, x \to y \in D$, then $y \in D$.

An equivalent definition for **ds** is ([23], [25]):

(D_1') If $x \leq y$ and $x \in D$, then $y \in D$;
(D_2') If x, $y \in D$, then $x \odot y \in D$.

A **ds** D of L is called *proper* if $D \neq L$ (that is, $0 \notin D$).

We denote by $\mathbf{Ds(L)}$ the set of all deductive systems of L.

Remark 2.4. Following above equivalence, in the most papers dedicated to the study of residuated lattices, for the notion of *deductive system* is used the notion of filter (or *implication filter*, *i-filter* for short). If there is danger of confusion with the notion of filter in the underlying lattice of a residuated lattice, we call these *latticeal filters*. Every **ds** is a latticeal filter in the lattice $(L, \vee, \wedge, 0, 1)$ but the converse is not true ([25]).

If $D \in \mathbf{Ds(L)}$, then the relation $\sim_D$ defined on L by $(x, y) \in \sim_D$ iff $x \to y, y \to x \in D$ iff $(x \to y) \odot (y \to x) \in D$ is a congruence relation on L. For $x \in L$ we denote by x/D the congruence class of x modulo $\sim_D$, $\mathbf{1} = 1/D$ and $\mathbf{0} = 0/D$.

The quotient set $L/\sim_D$ denoted by L/D becomes a residuated lattice in a natural way, with the operations induced from those of L. So, the order relation on L/D is given by $x/D \leq y/D$ iff $x \to y \in D$. Clearly, $x/D = \mathbf{1}$ iff $x \in D$ and $x/D = \mathbf{0}$ iff $x^* \in D$.

Since in the most papers the notion of **filter** is used instead of that of deductive system, we do the same.

So, if we denote by $\mathbf{F(L)}$ the set of all filters of L, then clearly, $\mathbf{F(L)} = \mathbf{Ds(L)}$.

In L we consider the following identities:

(LR_4) $x \odot (x \to y) = x \wedge y$ (*divisibility*);
(LR_5) $(x \to y) \vee (y \to x) = 1$ (*preliniarity*).

Definition 2.5. [7], [12], [25] A residuated lattice L is called

(i) *divisible* if L verifies (LR_4);
(ii) an MTL-*algebra* if L verifies (LR_5);

(*iii*) a *BL-algebra* if L verifies (LR_4) and (LR_5), that is, L is a divisible MTL-algebra.

A residuated lattice L has the *Double Negation condition* ([16]) if it satisfies the equation

(DN) $x^{**} = x$.

In [9], a residuated lattice verifying (DN) condition is called a *regular residuated lattice*; in [16], it is called a *Girard monoid*.

Remark 2.6. If $(L, \wedge, \vee, \rceil, 0, 1)$ is a Boolean algebra, then $(L, \wedge, \vee, \rceil, 0, 1)$ is a regular BL-algebra, where for $x, y \in L, x \odot y = x \wedge y$ and $x \to y = \rceil x \vee y$.

3. Boolean Filters in a Residuated Lattice

We recall some preliminaries of Boolean filters in a residuated lattice because we use these results in the rest of the paper.

We recall ([12], [23]) that an element $x \in L$ is *complemented* iff $x \vee x^* = 1$. We denote by $B(L)$ the Boolean algebra of all complemented elements of L .

We say ([24]) that a filter F of L is a *Boolean filter* iff L/F is a Boolean algebra.

Remark 3.1. F is a Boolean filter iff $x \vee x^* \in F$ for every $x \in L$.

We denote by $\mathbf{BF(L)}$ the set of all Boolean filters of L.

For Boolean filters we have the following characterization:

Proposition 3.2. *For a filter F of L, the following conditions are equivalent:*

(*i*) *F is a Boolean filter;*

(*ii*) *$x \vee x^* \in F$ for every $x \in L$.*

(*iii*) *If $x^* \to x \in F$, then $x \in F$.*

Proof. $(i) \Rightarrow (ii)$. Follows from Remark 3.1.

$(ii) \Rightarrow (iii)$. For $x \in L$, we have $(x \vee x^*) \to x \overset{(c_8)}{=} (x \to x) \wedge (x^* \to x) = 1 \wedge (x^* \to x) = x^* \to x$.

So, if suppose $x^* \to x \in F$, since $x \vee x^* \in F$, then $x \in F$.

$(iii) \Rightarrow (ii)$. Consider $x \in L$ and denote $b = (x^* \to x) \to x$. By (c_{13}), $b^* \le b$, hence $b^* \to b = 1 \in F$. By hypothesis, $b \in F$, that is, $(x^* \to x) \to x \in F$, for every $x \in L$. If $a = x \vee x^*$, then $a^* = x^* \wedge x^{**} \le x^* \le a$.

Then from $(a^* \to a) \to a \in F$ and $a^* \to a = 1$, we deduce that $a \in F$. □

Example 3.3. In [23] it is proved that $L = \{0, a, b, c, 1\}$ with $0 < a, b < c < 1$, and a, b incomparable, is a residuated lattice relative to the following operations:

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	b	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

,

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	0	a	a
b	0	0	b	b	b
c	0	a	b	c	c
1	0	a	b	c	1

.

It is easy to prove that $F = \{a, c, 1\} \in \mathbf{F(L)}$. Since $0 \vee 0^* = 1, a \vee a^* = c, b \vee b^* = c, c \vee c^* = c$ and $1 \vee 1^* = 1$ we deduce that $x \vee x^* \in F$, for every $x \in L$, hence $F \in \mathbf{BF(L)}$.

4. Semi-G-filters in a Residuated Lattice

Using the model of BL-algebras ([12]), we say that a residuated lattice L is called a *Gödel algebra* (*G-algebra* for short or *G(RL)-algebra* as in ([30])) if $x^2 = x$ for every $x \in L$.

L will be called a *semi-G-algebra* ([7]) if $(x^2)^* = x^*$ for every $x \in L$.

Clearly, every G-algebra is a semi-G-algebra, but the converse is not true ([7]).

In the spirit of [5] we have the following:

Definition 4.1. We say that a filter F of L is called a *G-filter* iff L/F is a G-algebra.

Remark 4.2. Using the model of BL-algebras (see [13], [17]), G-filters are called in [4] *implicative filters.*

We denote by $\mathbf{GF(L)}$ the set of all G-filters of L.

Proposition 4.3. [4] *For a residuated lattice L, the following conditions are equivalent:*

(*i*) $F \in \mathbf{GF(L)}$;
(*ii*) L/F *is a Hilbert algebra.*

Since $F \in \mathbf{GF(L)}$ iff L/F is a Hilbert algebra, in the spirit of [5], then F must be called a *Hilbert filter* ! So, F is a G-filter iff F is a Hilbert filter.

Proposition 4.4. [6] *For a residuated lattice L, the following conditions are equivalent:*

(*i*) L *is a semi-G-algebra;*
(*ii*) $x \wedge x^* = 0$ *for every* $x \in L$.

In the spirit of [5] we have the following:

Definition 4.5. We say that a filter F of L is called *semi-G- filter* iff L/F is a semi-G-algebra.

We denote by $\mathbf{SgF(L)}$ the set of all semi-G-filters of L.

Remark 4.6. $\mathbf{BF(L)} \subseteq \mathbf{GF(L)} \subseteq \mathbf{SgF(L)}$.

Remark 4.7. F is a semi-G- filter of L iff $(x \wedge x^*)^* \in F$ for every $x \in L$. Indeed, if F is a semi-G-filter, then L/F is a semi-G-algebra, hence, from Proposition 4.4, $x/F \wedge (x/F)^* = \mathbf{0}$, for every $x \in L$. But the condition $(x/F) \wedge (x/F)^* = \mathbf{0}$ is equivalent with $(x \wedge x^*)/F = 0/F$, hence $(x \wedge x^*) \to 0 \in F$, that is, $(x \wedge x^*)^* \in F$.

Conversely, if $(x \wedge x^*)^* \in F$ for every $x \in L$, then $(x \wedge x^*)^*/F = \mathbf{1}$, hence $(x/F \wedge (x/F)^*)^* = \mathbf{1}$. We deduce that $[x/F \wedge (x/F)^*]^{**} = \mathbf{0}$, hence $x/F \wedge (x/F)^* = \mathbf{0}$. From Proposition 4.4, L/F is a semi-G-algebra, that is, F is a a semi-G- filter of L.

Proposition 4.8. *For a filter F of L the following assertions are equivalent:*

(i) *F is a semi-G-filter;*

(ii) $(x \wedge x^*)^* \in F$ *for every* $x \in L$;

(iii) *If* $x \to x^* \in F$, *then* $x^* \in F$.

Proof. $(i) \Leftrightarrow (ii)$. It follows from Remark 4.7.

$(ii) \Rightarrow (iii)$. Consider $x \in L$ such that $x \to x^* \in F$. We have $x \to x^* = 1 \wedge (x \to x^*) = (x \to x) \wedge (x \to x^*) = x \to (x \wedge x^*) \leq (x \wedge x^*)^* \to x^*$.

Then $(x \wedge x^*)^* \to x^* \in F$. Since $(x \wedge x^*)^* \in F$, then $x^* \in F$.

$(iii) \Rightarrow (ii)$. For $x \in L$, we have $(x^{**} \to x^*) \to x^* = [x^{**} \odot (x^{**} \to x^*)]^*$ (see (c_{42}) from [7]). If denote $b = (x^{**} \to x^*) \to x^*$, then by (c_{13}), we deduce that $b^* \leq b$, hence $b^* \to b^{**} = 1 \in F$. By hypothesis, $b^{**} \in F$. But $b^{**} = [(x^{**} \to x^*) \to x^*]^{**} = [x^{**} \odot (x^{**} \to x^*)]^{***} = [x^{**} \odot (x^{**} \to x^*)]^* = b$, hence $b \in F$, that is, $(x^{**} \to x^*) \to x^* \in F$, for every $x \in L$.

If we replace x with x^* we obtain that $(x^* \to x^{**}) \to x^{**} \in F$, for every $x \in L$. For $x \in L$, we denote $a = (x \wedge x^*)^*$. By (c_{12}), $a^* \leq a$, hence $a^* \leq a^{**}$. Then $(a^* \to a^{**}) \to a^{**} \in F$, hence $a^{**} \in F$. But $a^{**} = a$, so we deduce that $a \in F$. □

5. Stonean Filters in a Residuated Lattice

We recall ([6]) that a residuated lattice L is called *Stonean* if $x^* \vee x^{**} = 1$, for every $x \in L$.

In the spirit of [5] we have the following:

Definition 5.1. We say that a filter F of L is *Stonean* iff L/F is a Stonean residuated lattice.

We denote by $\mathbf{StF(L)}$ the set of all Stonean filters of L.

Remark 5.2. F is a Stonean filter of L iff $x^* \vee x^{**} \in F$, for every $x \in L$. Clearly, every Boolean filter is Stonean, hence $\mathbf{BF(L)} \subseteq \mathbf{StF(L)}$.

Example 5.3. Let $L = \{0, a, b, c, 1\}$ the residuated lattice from Example 3.3. Then $F = \{1, c\}$ is a Stonean filter of L because $0^* \vee 0^{**} = 1 \in F, a^* \vee a^{**} = b \vee a = c \in F, b^* \vee b^{**} = a \vee b = c \in F, c^* \vee c^{**} = 0 \vee 1 = 1 \in F$ and $1^* \vee 1^{**} = 1 \in F$.

Theorem 5.4. *Let L be a residuated lattice. We consider the following assertions:*

(i) *L is a Stonean residuated lattice;*

(ii) *L is a semi-G-algebra.*

Then $(i) \Rightarrow (ii)$.

If L is an $MTL-$algebra, then $(i) \Leftrightarrow (ii)$.

Proof. $(i) \Rightarrow (ii)$. If L is a Stonean residuated lattice, then $x^* \vee x^{**} = 1$, for every $x \in L$. We have that $0 = (x^* \vee x^{**})^* = x^{**} \wedge x^*$, for every $x \in L$. Since $x \leq x^{**}$, we obtain that $x \wedge x^* = 0$, for every $x \in L$, that is, L is a semi-G-algebra.

$(ii) \Rightarrow (i)$. Let L be an $MTL-$algebra and suppose that $x \wedge x^* = 0$, for every $x \in L$. Then $1 = (x \wedge x^*)^* = x^* \vee x^{**}$, (see [6]) so, L is a Stonean residuated lattice. □

Corollary 5.5. *Let L be a residuated lattice. Then* $\mathbf{StF}(\mathbf{L}) \subseteq \mathbf{SgF}(\mathbf{L})$.

Proof. Let $F \in \mathbf{StF}(\mathbf{L})$. Then L/F is a Stonean residuated lattice. Using Theorem 5.4, L/F is a semi-G-algebra, so $F \in \mathbf{SgF}(\mathbf{L})$. □

Using Theorem 5.4, we deduce that

Corollary 5.6. *If L is an $MTL-$algebra, then* $\mathbf{StF}(\mathbf{L}) = \mathbf{SgF}(\mathbf{L})$.

Lemma 5.7. *For a filter F of L, the following assertions are equivalent:*

(i) *If $x^{**} \to x^* \in F$, then $x^* \in F$;*

(ii) *If $x^* \to x^{**} \in F$, then $x^{**} \in F$.*

Proof. $(i) \Rightarrow (ii)$. Let $x \in L$ such that $x^* \to x^{**} \in F \Leftrightarrow x^{***} \to x^{**} \in F \Leftrightarrow (x^*)^{**} \to (x^*)^* \in F$. Using (i), we deduce that $(x^*)^* \in F$, so $x^{**} \in F$.

$(ii) \Rightarrow (i)$. Let $x \in L$ such that $x^{**} \to x^* \in F \Leftrightarrow x^{**} \to x^{***} \in F \Leftrightarrow (x^*)^* \to (x^*)^{**} \in F$. Using (ii), we deduce that $(x^*)^{**} \in F$, so $x^* \in F$. □

Proposition 5.8. *For a filter F of L consider the following assertions:*

(i) *F is a Stonean filter ;*

(ii) *If $x^{**} \to x^* \in F$, then $x^* \in F$.*

We have that: $(i) \Rightarrow (ii)$ but $(ii) \not\Rightarrow (i)$.

Proof. $(i) \Rightarrow (ii)$. For $x \in L$ we have $(x^* \vee x^{**}) \to x^* = (x^* \to x^*) \wedge (x^{**} \to x^*) = 1 \wedge (x^{**} \to x^*) = x^{**} \to x^*$. So, if $x^{**} \to x^* \in F$, then $(x^* \vee x^{**}) \to x^* \in F$.

Since $x^* \vee x^{**} \in F$, then $x^* \in F$.

Using the equivalent conditions from Lemma 5.7, we deduce the conclusion.

$(ii) \not\Rightarrow (i)$. We consider the residuated lattice from Example 3.3 which is not an $MTL-$algebra because $(a \to b) \vee (b \to a) = b \vee a = c \neq 1$. In this algebra, $F = \{1\} \in \mathbf{F}(\mathbf{L})$.

Obviously, $0^{**} \to 0^* = 0 \to 1 = 1 \in F$ and $0^* = 1 \in F$. Since $a^{**} \to a^* = a \to b = b \notin F$, $b^{**} \to b^* = b \to a = a \notin F$, $c^{**} \to c^* = 1 \to 0 = 0 \notin F$, $1^{**} \to 1^* = 1 \to 0 = 0 \notin F$, we deduce that $F = \{1\}$ verifies the condition (ii).

Clearly, $a^* \vee a^{**} = b \vee a = c \notin F$, so F is not a Stonean filter.

Since $(0 \wedge 0^*)^* = (a \wedge a^*)^* = (b \wedge b^*)^* = (c \wedge c^*)^* = (1 \wedge 1^*)^* = 1 \in F$, we deduce that $F \in \mathbf{SgF}(\mathbf{L})$. □

Using Propositions 4.8, 5.8 and (c_7) we deduce that:

Corollary 5.9. *If L is a residuated lattice, then* $\mathbf{StF}(\mathbf{L}) \subsetneqq \mathbf{SgF}(\mathbf{L})$.

Suppose that L verifies the condition

$(\mathcal{C})$ $\quad [(x^{**} \to x^*) \to x^*] \wedge [(x^* \to x^{**}) \to x^{**}] = x^* \vee x^{**}$, for every $x \in L$.

Remark 5.10. There are residuated lattices which don't verify the condition $(\mathcal{C})$. For example, the residuated lattice from Example 3.3. We have $[(a^{**} \to a^*) \to a^*] \wedge [(a^* \to a^{**}) \to a^{**}] = [(a \to b) \to b] \wedge [(b \to a) \to a] = (b \to b) \wedge (a \to a) = 1$, $a^* \vee a^{**} = b \vee a = c$, but $c \neq 1$.

Proposition 5.11. *If L verify condition $(\mathcal{C})$, then in Proposition 5.8, $(i) \Leftrightarrow (ii)$, i.e.* $\mathbf{StF}(\mathbf{L}) = \mathbf{SgF}(\mathbf{L})$.

Proof. $(i) \Rightarrow (ii)$. Following from Proposition 5.8.

$(ii) \Rightarrow (i)$. For $x \in L$ we denote $b = (x^{**} \to x^*) \to x^*$. By (c_{13}), we deduce that $b^* \leq b$, hence $b^* \to b^{**} = 1 \in F$, so $(b^*)^{**} \to (b^*)^* \in F$. By hypothesis, $b^{**} \in F$. As in the case of Proposition 4.8, we deduce that $b \in F$, hence $(x^{**} \to x^*) \to x^* \in F$, for every $x \in L$. If we replace x with x^* we deduce that $(x^* \to x^{**}) \to x^{**} \in F$, for every $x \in L$. Then $[(x^{**} \to x^*) \to x^*] \odot [(x^* \to x^{**}) \to x^{**}] \in F$, hence $[(x^{**} \to x^*) \to x^*] \wedge [(x^* \to x^{**}) \to x^{**}] = x^* \vee x^{**} \in F$. □

Open problem. Clearly, if L is an $MTL-$algebra, then L verify condition $(\mathcal{C})$, hence in Proposition 5.8, we have $(i) \Leftrightarrow (ii)$. Find other classes of residuated lattices which verify the equivalence $(i) \Leftrightarrow (ii)$ of Proposition 5.8.

Remark 5.12. Since $x^{**} \to x^* \overset{(c_7)}{=} x \to x^*$, we deduce in particular, from Propositions 4.8 and 5.8, that every Stonean filter is a semi-G-filter.

6. MTL - filters, Divisible Filters and BL - filters in a Residuated lattice

In the spirit of [5] we have the following:

Definition 6.1. A filter F is called an *MTL-filter* iff L/F is an MTL-algebra (see also [29]).

We denote by $\mathbf{MTLF(L)}$ the set of all MTL-filters of L.

Proposition 6.2. [29] *For a filter F of a residuated lattice L, the following conditions are equivalent:*

(i) $F \in \mathbf{MTLF(L)}$;
(ii) *If* $x \to (y \vee z) \in F$, *then* $(x \to y) \vee (x \to z) \in F$;
(iii) $(x \to (y \vee z)) \to ((x \to y) \vee (x \to z)) \in F$, *for every* $x, y, z \in L$;
(iv) *If* $(y \wedge z) \to x \in F$, *then* $(y \to x) \vee (z \to x) \in F$;
(v) $((y \wedge z) \to x) \to ((y \to x) \vee (z \to x)) \in F$, *for every* $x, y, z \in L$;
(vi) *If* $x \to z \in F$, *then* $(x \to y) \vee (y \to z) \in F$;
(vii) $(x \to z) \to ((x \to y) \vee (y \to z)) \in F$, *for every* $x, y, z \in L$;
$(viii)$ *If* $(x \to y) \to z \in F$, *then* $((y \to x) \to z) \to z \in F$;
(ix) $((x \to y) \to z) \to (((y \to x) \to z) \to z) \in F$, *for every* $x, y, z \in L$.

Remark 6.3. If the residuated lattice L is an MTL-algebra, then every filter is an MTL-filter.

Following Corollary 5.6, we deduce that

Corollary 6.4. *Let L be a residuated lattice. Then*

$$\mathbf{MTLF(L)} \cap \mathbf{StF(L)} = \mathbf{MTLF(L)} \cap \mathbf{SgF(L)}.$$

In the spirit of [5], we have the following:

Definition 6.5. A filter F is called *divisible filter* iff L/F is a divisible residuated lattice.

We denote by $\mathbf{DivF(L)}$ the set of all divisible filters of L.

Remark 6.6. If L is a Boolean filter, then L/F is a Boolean algebra, hence L/F is a divisible residuated lattice (by Remark 2.6), so every Boolean filter is a divisible filter (that is, $\mathbf{BF(L)} \subseteq \mathbf{DivF(L)}$).

Proposition 6.7. *For a filter F of a residuated lattice L, the following conditions are equivalent:*

(i) $F \in \mathbf{DivF(L)}$;
(ii) $(x \wedge y) \to [x \odot (x \to y)] \in F$, *for every* $x, y \in L$;
(iii) *If* $z \to (x \wedge y) \in F$, *then* $z \to [x \odot (x \to y)] \in F$;
(iv) *If* $(x \to y) \to (x \to z) \in F$, *then* $(x \wedge y) \to z \in F$.

Proof. $(i) \Leftrightarrow (ii)$. Clearly.

$(ii) \Rightarrow (iii)$. Suppose $z \to (x \wedge y) \in F$. Since by hypothesis, $(x \wedge y) \to [x \odot (x \to y)] \in F$, then $z \to [x \odot (x \to y)] \in F$.

$(iii) \Rightarrow (ii)$. Since $(x \wedge y) \to (x \wedge y) = 1 \in F$, by hypothesis, $(x \wedge y) \to [x \odot (x \to y)] \in F$.

$(ii) \Rightarrow (iv)$. Suppose $(x \to y) \to (x \to z) \in F$, that is, $(x \odot (x \to y)) \to z \in F$. We have $(x \wedge y) \to (x \odot (x \to y)) \overset{(c_4)}{\leq} ((x \odot (x \to y)) \to z) \to ((x \wedge y) \to z)$.

By hypothesis, $(x \wedge y) \to (x \odot (x \to y)) \in F$, hence $((x \odot (x \to y)) \to z) \to ((x \wedge y) \to z) \in F$, hence $(x \wedge y) \to z \in F$.

$(iv) \Rightarrow (ii)$. Since $(x \to y) \to (x \to (x \odot (x \to y))) = [x \odot (x \to y)] \to [x \odot (x \to y)] = 1 \in F$, by hypothesis, we deduce that $(x \wedge y) \to [x \odot (x \to y)] \in F$. □

Proposition 6.8. *Let L be a residuated lattice and $F \in \mathbf{DivF(L)}$. Then $[x \wedge (y \vee z)] \to [(x \wedge y) \vee (x \wedge z)] \in F$, for every $x, y, z \in L$.*

Proof. Since L/F is a divisible residuated lattice, then by Corollary 1 of [7] we deduce that $[x/F \wedge (y/F \vee z/F)] \leq [(x/F \wedge y/F) \vee (x/F \wedge z/F)]$, that is, $[x \wedge (y \vee z)]/F \leq [(x \wedge y) \vee (x \wedge z)]/F$, so we conclude that, $[x \wedge (y \vee z)] \to [(x \wedge y) \vee (x \wedge z)] \in F$, for every $x, y, z \in L$. □

In the spirit of [5] we have the following:

Definition 6.9. A filter F of L is called *BL-filter* iff L/F is a BL-algebra.

If we denote by $\mathbf{BLF(L)}$ the set of all BL-filters of L, then

$$\mathbf{BLF(L)} = \mathbf{MTLF(L)} \cap \mathbf{DivF(L)}.$$

Remark 6.10. Since a G-algebra L is a BL-algebra with $x \odot x = x$, for every $x \in L$ (see [16]), we deduce that $\mathbf{GF(L)} \subseteq \mathbf{BLF(L)} = \mathbf{MTLF(L)} \cap \mathbf{DivF(L)}$.

Remark 6.11. If the residuated lattice L is a BL-algebra, then every filter of L is a BL-filter.

Proposition 6.12. *For a filter F of a residuated lattice L the following conditions are equivalent:*

(i) $F \in \mathbf{BLF(L)}$;
(ii) *If* $(x \to y) \to (x \to z) \in F$, *then* $(x \to z) \vee (y \to z) \in F$;
(iii) $((x \to y) \to (x \to z)) \to ((x \to z) \vee (y \to z)) \in F$, *for every* $x, y, z \in L$.

Proof. $(i) \Rightarrow (ii)$. Suppose $(x \to y) \to (x \to z) \in F$. Since every BL-filter is a divisible filter, by Proposition 6.7, (iv), we deduce that $(x \wedge y) \to z \in F$. Since every BL-filter is an MTL-filter, following Proposition 6.2, (iv), we deduce that $(x \to z) \vee (y \to z) \in F$.

$(ii) \Rightarrow (i)$. To prove that F is a BL-filter it suffices to prove that F is an MTL-filter and a divisible filter. Following Proposition 6.2, to prove that F is an MTL-filter it is suffices to prove condition (iv) from this proposition.

So, suppose $(x \wedge y) \to z \in F$. Since $x \odot (x \to y) \leq x \wedge y$, then by (c_3) we deduce that $(x \wedge y) \to z \leq (x \odot (x \to y)) \to z$.

Then $(x \odot (x \to y)) \to z \in F$, that is, $(x \to y) \to (x \to z) \in F$. By hypothesis, we deduce that, $(x \to z) \vee (y \to z) \in F$, that is, F is an MTL-filter. To prove that F is a divisible filter, it is suffices to prove that F verifies condition (iv) of Proposition 6.7.

So, suppose $(x \to y) \to (x \to z) \in F$. By hypothesis, we deduce that, $(x \to z) \vee (y \to z) \in F$. Since $(x \to z) \vee (y \to z) \leq (x \wedge y) \to z$ we deduce that $(x \wedge y) \to z \in F$, that is, F is a BL-filter.

$(iii) \Rightarrow (ii)$. Clearly.

$(ii) \Rightarrow (iii)$. We denote $a = (x \to y) \to (x \to z) = (x \odot (x \to y)) \to z$. We have $1 = a \to a = a \to [(x \odot (x \to y)) \to z] = (x \odot (x \to y)) \to (a \to z) =$
$(x \to y) \to [x \to (a \to z)]$, hence, $(x \to y) \to [x \to (a \to z)] \in F$.

By hypothesis, we deduce that, $(x \to (a \to z)) \vee (y \to (a \to z)) \in F$.

We have $(x \to (a \to z)) \vee (y \to (a \to z)) = (a \to (x \to z)) \vee (a \to (y \to z)) \leq a \to [(x \to z) \vee (y \to z)]$.

Then $a \to [(x \to z) \vee (y \to z)] \in F$, that is, $((x \to y) \to (x \to z)) \to ((x \to z) \vee (y \to z)) \in F$. □

7. MV- filters and Regular Filters in a Residuated Lattice

Definition 7.1. [10] An *MV- algebra* is an algebra $(M, \oplus, ^*, 0)$ of type $(2,1,0)$ such that the following axioms are verified:

(MV_1) $(M, \oplus, 0)$ is a commutative monoid;
(MV_2) $x^{**} = x$, for every $x \in M$;
(MV_3) $x \oplus 1 = 1$, for every $x \in M$ (where $1 = 0^*$);
(MV_4) $(x \to y) \to y = (y \to x) \to x$, for every $x, y \in M$ (where $x \to y = x^* \oplus y$).

Remark 7.2. [12], [25]

1. It is not hard to see that an equivalent presentation of $MV-$ algebras can be given as $BL-$ algebras plus condition (MV_2).
2. Let L be a residuated lattice. If for $x, y \in L$ we denote $x \oplus y = x^* \to y$, then $(L, \oplus, ^*, 0)$ is an MV algebra iff $(x \to y) \to y = (y \to x) \to x$, for every $x, y \in L$.

We recall that in [30] a filter F of L will be called an *MV- filter* iff L/F is an MV-algebra.

We denote by $\mathbf{MVF(L)}$ the set of all MV-filters of L.

Clearly, if L is *commutative* (i.e., $(x \to y) \to y = (y \to x) \to x$, for every $x, y \in L$), then every filter of L is an MV-filter.

Lemma 7.3. [4] *If* $F \in \mathbf{MVF(L)}$, *then* $x^{**} \to x \in F$, *for every* $x \in L$.

In [4] is proved the following result : $\mathbf{GF(L)} \cap \mathbf{MVF(L)} = \mathbf{BF(L)}$.
In the spirit of [5] we have the following:

Definition 7.4. A filter F of a residuated lattice L will be called a *regular filter* iff L/F is a regular residuated lattice.

In [26], a *regular filter* is called an *involution filter*.
We denote by $\mathbf{RF(L)}$ the set of all regular filters of L.

Remark 7.5. From Lemma 7.3, we deduce that $\mathbf{MVF(L)} \subseteq \mathbf{RF(L)}$.

Proposition 7.6. [16] *A divisible residuated lattice satisfies* (DN) *condition iff it is an* $MV-$*algebra.*

Using this result we obtain:

Corollary 7.7. *Let* $F \in \mathbf{DivF(L)}$. *Then* $F \in \mathbf{RF(L)}$ *iff* $F \in \mathbf{MVF(L)}$.

Proposition 7.8. *For a filter* F *of a residuated lattice* L *the following conditions are equivalent:*

(i) $F \in \mathbf{RF(L)}$;
(ii) $x^{**} \to x \in F$, *for every* $x \in L$;
(iii) $(y^* \to x^*) \to (x \to y) \in F$, *for every* $x, y \in L$;
(iv) $(x^* \to y) \to (y^* \to x) \in F$, *for every* $x, y \in L$.

Proof. $(i) \Leftrightarrow (ii)$. $x^{**} \to x \in F$, for every $x \in L$ iff $(x/F)^{**} \leq x/F$, for every $x \in L$ iff $(x/F)^{**} = x/F$, for every $x \in L$ iff L/F has (DN) condition iff $F \in \mathbf{RF(L)}$.

$(ii) \Rightarrow (iii)$. Suppose that $x^{**} \to x \in F$, for every $x \in L$. Then L/F has (DN) condition, i.e., $(x/F)^{**} = x/F$, for every $x \in L$.

To prove that $(y^* \to x^*) \to (x \to y) \in F$, for every $x, y \in L$, we will prove that $((y/F)^* \to (x/F)^*) \to (x/F \to y/F) = \mathbf{1}$ in L/F.

Indeed, for every $x, y \in L$, $x/F \to y/F \overset{(DN)}{=} x/F \to (y/F)^{**} \overset{c_7}{=} (y/F)^* \to (x/F)^*$, hence $((y/F)^* \to (x/F)^*) \to (x/F \to y/F) = \mathbf{1}$.

$(iii) \Rightarrow (iv)$. By hypothesis and c_4 we deduce that for every $x, y \in L$ we have $(y/F)^* \to (x/F)^* = x/F \to y/F$.

Replacing x with x^* (respective y with x), we obtain $(x/F)^* \to (y/F) = (y/F)^* \to (x/F)^{**}$ and $(y/F)^* \to (x/F) = (x/F)^* \to (y/F)^{**}$.

Using c_7 we deduce that $(y/F)^* \to (x/F)^{**} = (x/F)^* \to (y/F)^{**}$.

So, we obtain that $(x/F)^* \to (y/F) = (y/F)^* \to (x/F)$, hence $(x^* \to y) \to (y^* \to x) \in F$, for every $x, y \in L$.

$(iv) \Rightarrow (ii)$. For $y = 0$ we obtain $x^{**} \to x \in F$, for every $x \in L$. □

Proposition 7.9. *Let* $F \in \mathbf{RF(L)}$. *Then* $(x \wedge y)^* \to (x^* \vee y^*) \in F$, *for every* $x, y \in L$.

Proof. If $F \in \mathbf{RF}(\mathbf{L})$, then L/F has (DN) condition.

To prove that $(x \wedge y)^* \to (x^* \vee y^*) \in F$, for every $x, y \in L$, we will prove in L/F that $(x/F \wedge y/F)^* = (x/F)^* \vee (y/F)^*$, for every $x, y \in L$.

Obviously, $(x/F)^*, (y/F)^* \leq (x/F \wedge y/F)^*$. Let $t \in L$ such that $(x/F)^*, (y/F)^* \leq t/F$. Then $(t/F)^* \leq (x/F)^{**} = x/F$ and $(t/F)^* \leq (y/F)^{**} = y/F$.

We deduce $(t/F)^* \leq x/F \wedge y/F$, so $(x/F \wedge y/F)^* \leq (t/F)^{**} = t/F$.

We have $(x/F \wedge y/F)^* = (x/F)^* \vee (y/F)^*$, for every $x, y \in L$, hence $(x \wedge y)^* \to (x^* \vee y^*) \in F$, for every $x, y \in L$. $\square$

Remark 7.10. The converse of Proposition 7.9 is not true. Indeed, consider $L = \{0, a, b, c, 1\}$ such that $0 < c < a, b < 1$, but a, b are incomparable. L becomes ([16]) a $BL-$algebra relative to the operations:

$\to$	0	c	a	b	1
0	1	1	1	1	1
c	0	1	1	1	1
a	0	b	1	b	1
b	0	a	a	1	1
1	0	c	a	b	1

,

$\odot$	0	c	a	b	1
0	0	0	0	0	0
c	0	c	c	c	c
a	0	c	a	c	a
b	0	c	c	b	b
1	0	c	a	b	1

.

Then $F = \{1, a\}$ is a filter of L. Since L is a $BL-$algebra, then $(x \wedge y)^* = x^* \vee y^*$, for every $x, y \in L$, so $(x \wedge y)^* \to (x^* \vee y^*) = 1 \in F$, for every $x, y \in L$, but $F \notin \mathbf{RF}(\mathbf{L})$ because $c^{**} \to c = 1 \to c = c \notin F$.

Remark 7.11. $\mathbf{BF}(\mathbf{L}) \subseteq \mathbf{RF}(\mathbf{L})$. Indeed, if $F \in \mathbf{BF}(\mathbf{L})$, then $x \vee x^* \in F$, for every $x \in L$. We have $x \vee x^* \leq [(x \to x^*) \to x^*] \wedge [(x^* \to x) \to x]$, so $(x^* \to x) \to x \in F$. Since $x^{**} \leq x^* \to x$ we deduce that $(x^* \to x) \to x \leq x^{**} \to x$, so $x^{**} \to x \in F$, for every $x \in L$, that is, $F \in \mathbf{RF}(\mathbf{L})$.

Proposition 7.12. $\mathbf{BLF}(\mathbf{L}) \cap \mathbf{RF}(\mathbf{L}) = \mathbf{MVF}(\mathbf{L})$.

Proof. If consider $F \in \mathbf{BLF}(\mathbf{L}) \cap \mathbf{RF}(\mathbf{L})$, then L/F is a BL-algebra and since $F \in \mathbf{RF}(\mathbf{L})$, $x^{**} \to x \in F$, for every $x \in L$. We deduce that $(x/F)^{**} \to x/F = 1/F \Leftrightarrow (x/F)^{**} \leq x/F \Leftrightarrow (x/F)^{**} = x/F$. We conclude that L/F is an MV-algebra, so $F \in \mathbf{MVF}(\mathbf{L})$. For the converse inclusion, consider $F \in \mathbf{MVF}(\mathbf{L})$. Then L/F is an MV-algebra, so L/F is also a BL-algebra, and $F \in \mathbf{BLF}(\mathbf{L})$. Using Lemma 7.3, we deduce that $x^{**} \to x \in F$, for every $x \in L$, so $F \in \mathbf{RF}(\mathbf{L})$. We deduce that $F \in \mathbf{BLF}(\mathbf{L}) \cap \mathbf{RF}(\mathbf{L})$. $\square$

Proposition 7.13. $\mathbf{GF}(\mathbf{L}) \cap \mathbf{RF}(\mathbf{L}) = \mathbf{SgF}(\mathbf{L}) \cap \mathbf{RF}(\mathbf{L}) = \mathbf{BF}(\mathbf{L})$.

Proof. Let $F \in \mathbf{SgF}(\mathbf{L}) \cap \mathbf{RF}(\mathbf{L})$. Then L/F is a semi-G-algebra with (DN) condition. We have $(x/F) \wedge (x/F)^* = 0/F$. Using Proposition 7.9, $(x/F) \vee (x/F)^* = (x/F)^{**} \vee (x/F)^* = [(x/F) \wedge (x/F)^*]^* = (0/F)^* = 1/F$. We deduce that L/F is a Boolean algebra, so $F \in \mathbf{BF}(\mathbf{L})$. For the converse inclusion, consider $F \in \mathbf{BF}(\mathbf{L})$. From Remarks 4.6 and 7.11, we deduce that $F \in \mathbf{SgF}(\mathbf{L}) \cap \mathbf{RF}(\mathbf{L})$. We conclude that $\mathbf{SgF}(\mathbf{L}) \cap \mathbf{RF}(\mathbf{L}) = \mathbf{BF}(\mathbf{L})$.

From Remark 4.6 and Remark 7.11 we deduce $\mathbf{GF}(\mathbf{L}) \cap \mathbf{RF}(\mathbf{L}) = \mathbf{BF}(\mathbf{L})$. $\square$

Corollary 7.14. $\mathbf{BF}(\mathbf{L}) \subsetneqq \mathbf{MVF}(\mathbf{L}) \subsetneqq \mathbf{BLF}(\mathbf{L}) \subsetneqq \mathbf{MTLF}(\mathbf{L})$.

Proof. Obviously, $\mathbf{BF(L)} \subseteq \mathbf{MVF(L)} \subseteq \mathbf{BLF(L)} \subseteq \mathbf{MTLF(L)}$.

To prove that $\mathbf{BF(L)} \subsetneqq \mathbf{MVF(L)}$ we consider the following example ([16]) of finite residuated lattice which is an MV-algebra. Let $L = \{0, a, b, c, d, 1\}$, with $0 < a, b < c < 1, 0 < b < d < 1$, but a, b and, respective c, d are incomparable. We have in L the following operations:

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	d	1	d	1
b	c	c	1	1	1	1
c	b	c	d	1	d	1
d	a	a	c	c	1	1
1	0	a	b	c	d	1

,

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	a	0	a
b	0	0	0	0	b	b
c	0	a	0	a	b	c
d	0	0	b	b	d	d
1	0	a	b	c	d	1

It is easy to see that L is not a Boolean algebra because $b^* \vee b = b \vee c = c \neq 1$. Since L is an MV-algebra, every filter of L is an $MV-$filter. In particular $\{1, d\}$ is an $MV-$filter.

Since $b^* \vee b = c \notin \{1, d\}$, we deduce that $\{1, d\}$ is not a Boolean filter of L, so $\mathbf{BF(L)} \subsetneqq \mathbf{MVF(L)}$.

To prove that $\mathbf{MVF(L)} \subsetneqq \mathbf{BLF(L)}$ we consider the following example ([16]) of a finite BL-algebra which is not an MV-algebra. Let $L = \{0, a, b, c, 1\}$, with $0 < c < a, b < 1$, but a, b are incomparable.

Define on L the following operations:

$\rightarrow$	0	c	a	b	1
0	1	1	1	1	1
c	0	1	1	1	1
a	0	b	1	b	1
b	0	a	a	1	1
1	0	c	a	b	1

,

$\odot$	0	c	a	b	1
0	0	0	0	0	0
c	0	c	c	c	c
a	0	c	a	c	a
b	0	c	c	b	b
1	0	c	a	b	1

.

It is easy to see that $F = \{1, a\}$ is a BL-filter of L. Using Proposition 7.12, a BL-filter F is an MV-filter iff $x^{**} \rightarrow x \in F$, for every $x \in L$. In this case, $c^{**} \rightarrow c = 1 \rightarrow c = c \notin F$, so, F is not an $MV-$ filter. We deduce that $\mathbf{MVF(L)} \subsetneqq \mathbf{BLF(L)}$.

To prove that $\mathbf{BLF(L)} \subsetneqq \mathbf{MTLF(L)}$ we consider the following example of $MTL-$algebra which is not a $BL-$ algebra. Let the residuated lattice be defined on the unit interval $L = [0, 1]$, for all $x, y \in L$, such that

$$x \odot y = 0 \text{ if } x + y \leq \frac{1}{2} \text{ and } x \wedge y \text{ elsewhere,}$$

$$x \rightarrow y = 1 \text{ if } x \leq y \text{ and } \max\{\frac{1}{2} - x, y\} \text{ elsewhere (see [25], p.16).}$$

This residuated lattice is a chain, so it is an $MTL-$algebra, but the divisibility condition not hold. Since L is an $MTL-$algebra, then every filter of L is an $MTL-$filter. In particular, $F = (\frac{1}{2}, 1]$ is an $MTL-$filter of L. We say that F is a $BL-$filter iff $(x \wedge y) \rightarrow (x \odot (x \rightarrow y)) \in F$, for every $x, y \in L$.

In particular, consider $x, y \in L$ such that $0 < y < x$, $x + y < \frac{1}{2}$. Then $y < \frac{1}{2} - x$ and $0 \neq y = x \wedge y$, $x \odot (x \to y) = x \odot (\frac{1}{2} - x) = 0$, so $x \wedge y \to [x \odot (x \to y)] = y \to 0 = \max\{\frac{1}{2} - y, 0\} = \frac{1}{2} - y \notin F$.

We deduce that F is not a $BL-$filter, so $\mathbf{BLF(L)} \subsetneqq \mathbf{MTLF(L)}$. $\square$

Finally, the connections between filters are resumed in the Figures 1, 2, 3, 4 and 5.

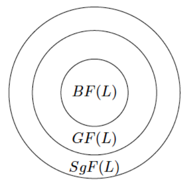

FIGURE 1. Remark 5

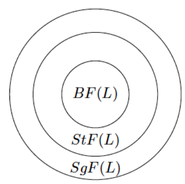

FIGURE 2. Remark 7 + Corollary 7

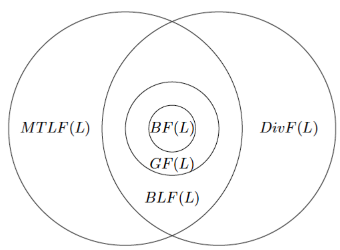

FIGURE 3. Remark 12

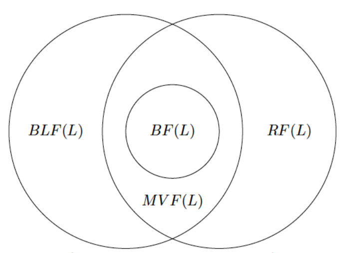

FIGURE 4. Proposition 23 + Corollary 25

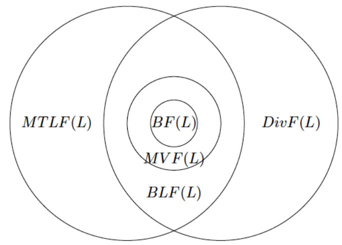

FIGURE 5. Corollary 25

8. Conclusions and Future Research

In this paper we have introduced new types of filters in residuated lattices in the spirit of [5]: semi-G-filters, Stonean filters, MTL-filters, divisible filters, BL-filters and regular filters. In Propositions 5, 10, 15, 17 and 21 we have established new characterizations and connections between these types of filters. In our future work, we will consider the non-commutative cases of residuated lattices and try to define other types of filters in residuated lattices (in commutative and non-commutative cases).

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