SOLVING FUZZY DIFFERENTIAL EQUATIONS BY USING PICARD METHOD

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Abstract. In this paper, The Picard method is proposed to solve the system of first-order fuzzy differential equations ($FDEs$) with fuzzy initial conditions under generalized H-differentiability. The existence and uniqueness of the solution and convergence of the proposed method are proved in details. Finally, the method is illustrated by solving some examples.

1. Introduction

First order fuzzy differential equations are one of the simplest fuzzy differential equations which may appear in many applications. Recently, some mathematicians have studied $FDEs$ [1-29].

In this work, we develope the Picard method to solve the $FDEs$ as follows:

$$\widetilde{X}'(t) = A\widetilde{X}(t) + \widetilde{f}(t), \tag{1}$$

with fuzzy initial condition:

$$\widetilde{X}(0) = \widetilde{X}_0, \tag{2}$$

where $A = (a_{ij}) \in M^n$, $X = (x_1, x_2, ..., x_n)$, $X_0 = (x_{10}, x_{20}, ..., x_{n0})$, x_{i0} are fuzzy constant values and $\widetilde{f} = (f_1, f_2, ..., f_n) : T \to E^n$ is fuzzy function.(E^n is the space of all normal, convex, upper semicontinuous and compactly supported fuzzy sets on R^n)

The structure of this paper is organized as follows: In section 2, some basic notations and definitions in fuzzy calculus are brought. In section 3, are solved Eqs.(1,2) with Picard method. The existence and uniqueness of the solution and convergence of the proposed method are proved in section 4, respectively. Finally, in section 5, the accuracy of method by solving some numerical examples are illustrated, and a brief conclusion is given in section 6.

2. Basic Concepts

Here basic definitions of a fuzzy number are given as follows , [30, 31, 32, 33, 34, 35, 36, 37, 38]

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Definition 2.1. An arbitrary fuzzy number $\widetilde{u}$ in the parametric form is represented by an ordered pair of functions $(\underline{u}, \overline{u})$ which satisfy the following requirements:

(i) $\overline{u} : r \to u_r^- \in R$ is a bounded left-continuous non-decreasing function over $[0, 1]$,
(ii) $\underline{u} : r \to u_r^+ \in R$ is a bounded left-continuous non-increasing function over $[0, 1]$,
(iii) $\underline{u} \leq \overline{u}, \quad 0 \leq r \leq 1$.

Definition 2.2. For arbitrary fuzzy numbers $\tilde{u}, \tilde{v} \in E^1$, we use the distance (Hausdorff metric). $D(u(r), v(r)) = \max\{\sup_{r\in[0,1]} |\underline{u}(r) - \underline{v}(r)|, \sup\ |\overline{u}(r) - \overline{v}(r)|\}$, and it is shown [37] that $(E^1$, D) is a complete metric space and the following properties are well known:

$$
\begin{aligned}
&D(\widetilde{u} + \widetilde{w}, \widetilde{v} + \widetilde{w}) = D(\widetilde{u}, \widetilde{v}), \forall\ \widetilde{u}, \widetilde{v} \in E^1,\\
&D(k\widetilde{u}, k\widetilde{v}) =\mid k \mid D(\widetilde{u}, \widetilde{v}), \forall\ k \in R, \widetilde{u}, \widetilde{v} \in E^1,\\
&D(\widetilde{u} + \widetilde{v}, \widetilde{w} + \widetilde{e}) \leq D(\widetilde{u}, \widetilde{w}) + D(\widetilde{v}, \widetilde{e}), \forall\ \widetilde{u}, \widetilde{v}, \widetilde{w}, \widetilde{e} \in E^1.
\end{aligned}
$$

Definition 2.3. A fuzzy number $\tilde{A}$ is of LR-type if there exist shape functions L(for left), R(for right) and scalar $\alpha \geq 0, \beta \geq 0$ with

$$
\tilde{\mu}_A(x) = \begin{cases} L(\frac{a-x}{\alpha})\ x \leq a \\ R(\frac{x-b}{\beta})\ x \geq a \end{cases} \tag{3}
$$

the mean value of $\tilde{A}, a$ is a real number, and α, β are called the left and right spreads, respectively. $\tilde{A}$ is denoted by (a, α, β).

Definition 2.4. Let $\tilde{M} = (m, \alpha, \beta)_{LR}$ and $\tilde{N} = (n, \gamma, \delta)_{LR}$ and $\lambda \in \mathbb{R}^+$. Then,
$(1) : \lambda\tilde{M} = (\lambda m, \lambda\alpha, \lambda\beta)_{LR}$
$(2) : -\lambda\tilde{M} = (-\lambda m, \lambda\beta, \lambda\alpha)_{LR}$
$(3) : \tilde{M} \oplus \tilde{N} = (m + n, \alpha + \gamma, \beta + \delta)_{LR}$

$$
(4) : \tilde{M} \odot \tilde{N} \simeq \begin{cases} (mn, m\gamma + n\alpha, m\delta + n\beta)_{LR}\ \tilde{M}, \tilde{N} > 0 \\ (mn, n\alpha - m\delta, n\beta - m\gamma)_{LR}\ \tilde{M} > 0, \tilde{N} < 0 \\ (mn, -n\beta - m\delta, -n\alpha - m\gamma)_{LR}\ \tilde{M}, \tilde{N} < 0 \end{cases} \tag{4}
$$

Definition 2.5. Consider x, $y \in E$. If there exists $z \in E$ such that $x = y + z$ then z is called the H- difference of x and y, and is denoted by $x \ominus y$. [6]

Proposition 2.6. *If $f : (a, b) \to E$ is a continuous fuzzy-valued function then $g(x) = \int_a^x f(t)\ dt$ is differentiable, with derivative $g'(x) = f(x)$.*[6]

Definition 2.7. (see [6]) Let $f : (a, b) \to E$ and $x_0 \in (a, b)$. We say that f is generalized differentiable at x_0 (Bede-Gal differentiability), if there exists an element $f'(x_0) \in E$, such that:

i) for all $h > 0$ sufficiently small, $\exists f(x_0 + h) \ominus f(x_0), \exists f(x_0) \ominus f(x_0 - h)$ and the following limits hold:

$$
\lim_{h\to 0} \frac{f(x_0 + h) \ominus f(x_0)}{h} = \lim_{h\to 0} \frac{f(x_0) \ominus f(x_0 - h)}{h} = f'(x_0)
$$

or

ii) for all $h > 0$ sufficiently small, $\exists f(x_0) \ominus f(x_0 + h), \exists f(x_0 - h) \ominus f(x_0)$ and the following limits hold:

$$\lim_{h \to 0} \frac{f(x_0) \ominus f(x_0 + h)}{-h} = \lim_{h \to 0} \frac{f(x_0 - h) \ominus f(x_0)}{-h} = f^{'}(x_0)$$

or

iii) for all $h > 0$ sufficiently small, $\exists f(x_0 + h) \ominus f(x_0), \exists f(x_0 - h) \ominus f(x_0)$ and the following limits hold:

$$\lim_{h \to 0} \frac{f(x_0 + h) \ominus f(x_0)}{h} = \lim_{h \to 0} \frac{f(x_0 - h) \ominus f(x_0)}{-h} = f^{'}(x_0)$$

or

iv) for all $h > 0$ sufficiently small, $\exists f(x_0) \ominus f(x_0 + h), \exists f(x_0) \ominus f(x_0 - h)$ and the following limits hold:

$$\lim_{h \to 0} \frac{f(x_0) \ominus f(x_0 + h)}{-h} = \lim_{h \to 0} \frac{f(x_0) \ominus f(x_0 - h)}{h} = f^{'}(x_0)$$

Definition 2.8. Let $f : (a, b) \to E$. We say f is (i)-differentiable on (a, b) if f is differentiable in the sense (i) of Definition (2.6) and similarly for (ii), (iii) and (iv) differentiability.

Lemma 2.9. *(see* [6]*) For $x_0 \in R$, the fuzzy differential equation $y^{'} = f(x, y)$, $y(x_0) = y_0 \in E$, where $f : R \times E \to E$ is supposed to be continuous, if equivalent to one of the integral equations:*

$$\widetilde{y}(x) = \widetilde{y}_0 + \int_{x_0}^{x} f(t, \widetilde{y}(t))\ dt, \quad \forall x \in [x_0, x_1],$$

or

$$\widetilde{y}(x) = \widetilde{y}_0 + (-1) \int_{x_0}^{x} f(t, \widetilde{y}(t))\ dt, \quad \forall x \in [x_0, x_1].$$

On some interval $(x_0, x_1) \subset R$, under the differentiability condition, (i) or (ii), respectively.

Remark 2.10. In the case of strongly generalized H-differentiability, to the fuzzy differential equation $y^{'} = f(x, y)$ we may attach two different integral equations, while in the case of H-differentiability we may attach only one. The second integral equation in Lemma (2.8) can be written in the form:

$$\widetilde{y}(x) = \widetilde{y}_0 \ominus (-1). \int_{x_0}^{x} f(t, \widetilde{y}(t))\ dt.$$

Theorem 2.11. *(see* [6]*) Suppose that the following conditions hold: (a) Let $R_0 = [x_0, x_0 + p] \times \overline{B}(y_0, q), p, q > 0, y_0 \in E$, where $\overline{B}(y_0, q) = y \in E : D(y, y_0) \leq q$ denotes a closed ball in E and let $f : R_0 \to E$ be a continuous function such that $d_\infty(\widetilde{0}, f(x, y) = \| f(x, y) \| \leq M$ for all $(x, y) \in R_0$. (b) Let $g : [x_0, x_0 + p] \times [0, q] \to E$, such that $g(x, 0) \equiv 0$ and $0 \leq g(x, u) \leq M_1 \forall x \in [x_0, x_0 + p], 0 \leq u \leq q$. Such that $g(x, u)$ is non-decreasing in u and the initial- value problem $u^{'} = g(x, u(x)), u(x_0) = 0$ has only the solution $u(x) \equiv 0$ on $[x_0, x_0 + p]$. (c) We* *have* $d_\infty(f(x,y),f(x,z))\forall(x,y),(x,z) \in R_0$ *and* $d_\infty(y,z) \leq q$. *(d) There exists* $d > 0$ *such that for* $x \in [x_0, x_0+d]$ *the sequence* $\overline{y}_n : [x_0, x_0+d] \to E$ *given by* $\widetilde{\widetilde{y}}_0(x) = \widetilde{y}_0, \widetilde{\widetilde{y}}_{n+1}(x) = \widetilde{y_0} \ominus (-1).\int_{x_0}^{x} f(t, \widetilde{\widetilde{y}}_n)\ dt$ *is defined for any* $n \in N$.

Then the fuzzy initial-value problem

$$y^{'} = f(x,y),$$
$$y(x_0) = y_0,$$

has two solutions (one (i)-differentiable and other one (ii)-differentiable) $y, \widehat{y} : [x_0, x_0+r] \to B(y_0, q)$ *where* $r = \min\{p, \frac{q}{M}, \frac{q}{M_1}, d\}$ *and the successive iterations*

$$\widetilde{y_0}(x) = \widetilde{y}_0, \widetilde{y}_{n+1}(x) = \widetilde{y}_0 + \int_{x_0}^{x} f(t, \widetilde{y_n}(t))\ dt,$$

or

$$\widetilde{\widetilde{y}}_0(x) = \widetilde{y}_0, \widetilde{\widetilde{y}}_{n+1}(x) = \widetilde{y}_0 \ominus (-1). \int_{x_0}^{x} f(t, \widetilde{\widetilde{y}}_n(t))\ dt,$$

converge to these two solutions, respectively.

Definition 2.12. A triangular fuzzy number is defined as a fuzzy set in E^1, that is specified by an ordered triple $u = (a,b,c) \in R^3$ with $a \leq b \leq c$ such that $[u]^r = [u_-^r, u_+^r]$ are the endpoints of r-level sets for all $r \in [0,1]$, where $u_-^r = a + (b-a)r$ and $u_+^r = c - (c-b)r$. Here, $u_-^0 = a, u_+^0 = c, u_-^1 = u_+^1 = b$, which is denoted by u^1. The set of triangular fuzzy numbers will be denoted by E^1.

Definition 2.13. (see [11]) The mapping $f : T \to E^n$ for some interval T is called a fuzzy process. Therefore, its r-level set can be written as follows:

$$[f(t)]^r = [f_-^r(t), f_+^r(t)], \quad t \in T, \quad r \in [0,1].$$

Definition 2.14. (see [11]) Let $f : T \to E^n$ be Hukuhara differentiable and denote $[f(t)]^r = [f_-^r, f_+^r]$. Then, the boundary function f_-^r and f_+^r are differentiable (or Seikkala differentiable) and

$$[f^{'}(t)]^r = [(f_-^r)^{'}(t), (f_+^r)^{'}(t)], \quad t \in T, \quad r \in [0,1].$$

3. Description of the Method

3.1. **Linear System of First-order FDEs.** We consider Eqs.(1,2), according to definition of derivative we have

$$\frac{\partial \widetilde{X}}{\partial t} = \sum_{i=1}^{n} \frac{\partial \widetilde{x}_i}{\partial t}. \tag{5}$$

Now, we consider three separate cases for elements of vector $\widetilde{X}$ as follows:

Case 1: $\widetilde{x}_i(t)$ is (1)-differentiable for any i, in this case we have,

$$\widetilde{X}(t) = \widetilde{X}(0) + \textstyle\int_0^t (\widetilde{A}\widetilde{X}(s) + \widetilde{f}(s))\ ds, \tag{6}$$

or,

$$\widetilde{x}_i(t) = \widetilde{x}_i(0) + \textstyle\int_0^t (\sum_{j=1}^{n} \widetilde{A}_{ij}\ \widetilde{x}_j(s) + \widetilde{f}_i(s))\ ds, \quad \forall\ 1 \leq i \leq n. \tag{7}$$

Case 2: $\widetilde{x}_i(t)$ is (2)-differentiable for any i, in this case we have,

$$\widetilde{X}(t) = \widetilde{X}(0) + (-1).\int_0^t (\widetilde{A}\widetilde{X}(s) + \widetilde{f}(s))\ ds, \tag{8}$$

or,

$$\widetilde{x}_i(t) = \widetilde{x}_i(0) + (-1).\int_0^t (\textstyle\sum_{j=1}^n \widetilde{A}_{ij}\ \widetilde{x}_j(s) + \widetilde{f}_i(s))\ ds, \quad \forall\ 1 \le i \le n. \tag{9}$$

Case 3: $\widetilde{x}_i(t)$ is (1)-differentiable for some i, ($1 \le i \le n$) and for another is (2)-differentiable. In this case let:

$$P = \{1 \le i \le n \mid \widetilde{x}_i(t)\ is\ (1) - differentiable\},$$

$$P^{'} = \{1 \le i \le n \mid \widetilde{x}_i(t)\ is\ (2) - differentiable\}.$$

So, we have

$$\begin{aligned} &\widetilde{x}_i(t) = \widetilde{x}_i(0) + \int_0^t (\textstyle\sum_{j=1}^n \widetilde{A}_{ij}\ \widetilde{x}_j(s) + \widetilde{f}_i(s))\ ds, \quad \forall i \in P, \\ &\widetilde{x}_i(t) = \widetilde{x}_i(0) + (-1).\int_0^t (\textstyle\sum_{j=1}^n \widetilde{A}_{ij}\ \widetilde{x}_j(s) + \widetilde{f}_i(s))\ ds, \quad \forall i \in P^{'}. \end{aligned} \tag{10}$$

Now, we can write successive iterations (by using Picard method) as follows:

Case 1: $\widetilde{x}_i(t)$ is (1)-differentiable for any i, we have,

$$\begin{aligned} &\widetilde{X}_0(t) = \widetilde{X}(0), \\ &\widetilde{X}_{n+1}(t) = \int_0^t (\widetilde{A}\widetilde{X}_n(s) + \widetilde{f}(s))\ ds, \quad n \ge 0. \end{aligned} \tag{11}$$

or,

$$\begin{aligned} &\widetilde{x}_{i0}(t) = \widetilde{x}_i(0), \\ &\widetilde{x}_{in+1}(t) = \int_0^t (\textstyle\sum_{j=1}^n \widetilde{A}_{ij}\ \widetilde{x}_{jn}(s) + \widetilde{f}_i(s))\ ds, \quad \forall\ 1 \le i \le n, \quad n \ge 0. \end{aligned} \tag{12}$$

Case 2: $\widetilde{x}_i(t)$ is (2)-differentiable for any i, we have,

$$\begin{aligned} &\widetilde{X}_0(t) = \widetilde{X}(0), \\ &\widetilde{X}_{n+1}(t) = (-1).\int_0^t (\widetilde{A}\widetilde{X}_n(s) + \widetilde{f}(s))\ ds, \quad n \ge 0. \end{aligned} \tag{13}$$

or,

$$\begin{aligned} &\widetilde{x}_{i0}(t) = \widetilde{x}_i(0), \\ &\widetilde{x}_{in+1}(t) = (-1).\int_0^t (\textstyle\sum_{j=1}^n \widetilde{A}_{ij}\ \widetilde{x}_{jn}(s) + \widetilde{f}_i(s))\ ds, \quad \forall\ 1 \le i \le n, \quad n \ge 0. \end{aligned} \tag{14}$$

Case 3: $\widetilde{x}_i(t)$ is (1)-differentiable for some i, ($1 \le i \le n$) and for another is (2)-differentiable, we have

$$\begin{aligned} &\widetilde{x}_{i0}(t) = \widetilde{x}_i(0), \\ &\widetilde{x}_{in+1}(t) = \int_0^t (\textstyle\sum_{j=1}^n \widetilde{A}_{ij}\ \widetilde{x}_{jn}(s) + \widetilde{f}_i(s))\ ds, \quad \forall i \in P, \quad n \ge 0, \\ &\widetilde{x}_{i0}(t) = \widetilde{x}_i(0), \\ &\widetilde{x}_{in+1}(t) = (-1).\int_0^t (\textstyle\sum_{j=1}^n \widetilde{A}_{ij}\ \widetilde{x}_{jn}(s) + \widetilde{f}_i(s))\ ds, \quad \forall i \in P^{'}, \quad n \ge 0. \end{aligned} \tag{15}$$

4. **Existence and Convergence Analysis**

In this section we are going to prove the existence and uniqueness of the solution and convergence of the method by using the following assumptions,

Remark 4.1.

$$D(\widetilde{A}\widetilde{X}, \widetilde{A}\widetilde{X^*}) = D(\sum_{j=1}^{n} \widetilde{a}_{ij}\widetilde{x}_j, \sum_{j=1}^{n} \widetilde{a}_{ij}\widetilde{x_j^*})$$
$$\leq D(\widetilde{a}_{i1}\widetilde{x_1}, \widetilde{a}_{i1}\widetilde{x_1^*})+$$
$$D(\widetilde{a}_{i2}\widetilde{x_2}, \widetilde{a}_{i2}\widetilde{x_2^*}) + ... + D(\widetilde{a}_{in}\widetilde{x_n}, \widetilde{a}_{in}\widetilde{x_n^*})$$
$$\leq\mid \widetilde{a}_{i1} \mid D(\widetilde{x_1}, \widetilde{x_1^*}) + ...+ \mid \widetilde{a}_{in} \mid D(\widetilde{x_n}, \widetilde{x_n^*}) =$$
$$\sum_{j=1}^{n} \mid \widetilde{a}_{ij} \mid D(\widetilde{x_j}, \widetilde{x_j^*}), \quad 1 \leq i \leq n.$$

Let,

$$\alpha = TL,$$

where,

$$\forall x \in Supp(\widetilde{a}_{ij}), \quad \widetilde{a}_{ij}(x) = \mu_{\widetilde{a}_{ij}}(x) = r_{ij},$$
$$\mid \widetilde{A} \mid := \min_x \widetilde{a}_{ij}(x) = \min_x r_{ij} = \alpha_{ij},$$
$$L = \max_{ij} \alpha_{ij}.$$

Lemma 4.2. *If $\widetilde{u}, \widetilde{v}, \widetilde{w} \in E^n$ and $\lambda \in R$, then,*

(*i*) $D(\widetilde{u} \ominus \widetilde{v}, \widetilde{u} \ominus \widetilde{w}) = D(\widetilde{v}, \widetilde{w})$,

(*ii*) $D(\ominus\lambda\widetilde{u}, \ominus\lambda\widetilde{v}) =\mid \lambda \mid D(\widetilde{u}, \widetilde{v})$.

Proof. By the definition of D, we have,

$$D(\widetilde{u} \ominus \widetilde{v}, \widetilde{u} \ominus \widetilde{w}) = \max\{\sup_{r\in[0,1]} \mid \underline{u(r) - v(r)} - \underline{u(r) - w(r)} \mid,$$
$$\sup_{r\in[0,1]} \mid \overline{u(r) - v(r)} - \overline{u(r) - w(r)} \mid\} =$$
$$\max\{\sup_{r\in[0,1]} \mid (\underline{u}(r) - \underline{v}(r)) - (\underline{u}(r) - \underline{w}(r) \mid,$$
$$\sup_{r\in[0,1]} \mid (\overline{u}(r) - \overline{v}(r)) - (\overline{u}(r) - \overline{w}(r)) \mid\} =$$
$$\max\{\sup_{r\in[0,1]} \mid \underline{w}(r) - \underline{v}(r) \mid,$$
$$\sup_{r\in[0,1]} \mid \overline{w}(r) - \overline{v}(r) \mid\} =$$
$$\max\{\sup_{r\in[0,1]} \mid \underline{v}(r) - \underline{w}(r) \mid, \sup_{r\in[0,1]} \mid \overline{v}(r) - \overline{w}(r) \mid\} = D(\widetilde{v}, \widetilde{w}).$$

□

Proof.

$$D(\ominus\lambda\widetilde{u}, \ominus\lambda\widetilde{v}) = \max\{\sup_{r\in[0,1]} \mid \underline{\lambda u(r)} - \underline{\lambda v(r)} \mid,$$
$$\sup_{r\in[0,1]} \mid \overline{\lambda u(r)} - \overline{\lambda v(r)} \mid\} =$$
$$\max\{\sup_{r\in[0,1]} \mid \underline{\lambda u(r)} - \underline{\lambda v(r)} \mid,$$
$$\sup_{r\in[0,1]} \mid \overline{\lambda u(r)} - \overline{\lambda v(r)} \mid\} = D(\lambda\widetilde{u}, \lambda\widetilde{v}) =\mid \lambda \mid D(\widetilde{u}, \widetilde{v}).$$

□

Theorem 4.3. *Let $0 < \alpha < 1$, then equations (1,2), have a unique solution when $\widetilde{x}_1$ is (1)-differentiable and $\widetilde{x}_2$ is (2)-differentiable respectively.*

Proof. Let $\widetilde{X}$ and $\widetilde{X}^*$ be two different solutions of (1,2) then

$$D(\widetilde{X}, \widetilde{X}^*) =$$
$$D(\int_0^t (\widetilde{A}\widetilde{X}(s) + \widetilde{f}(s))\ ds,$$
$$(-1).\int_0^t (\widetilde{A}\widetilde{X}^*(s) + \widetilde{f}(s))\ ds) =$$
$$D(\int_0^t \widetilde{A}\widetilde{X}(s)\ ds, (-1).\int_0^t \widetilde{A}\widetilde{X}^*(s)\ ds)$$
$$\leq \int_0^t D(\widetilde{A}\widetilde{X}(s), \widetilde{A}\widetilde{X}^*(s)) \leq TL\ D(\widetilde{X}(t), \widetilde{X}^*(t)) = \alpha D(\widetilde{X}, \widetilde{X}^*)$$

From which we get $(1-\alpha)D(\widetilde{X},\widetilde{X}^*) \leq 0$. Since $0<\alpha<1$, then $D(\widetilde{X},\widetilde{X}^*)=0$. Implies $\widetilde{X}=\widetilde{X}^*$ and completes the proof. □

Theorem 4.4. *The solution $\widetilde{X}_n(x,t)$ obtained from the relation (11) using Picard method converges to the exact solution of the problems (1,2) when $\exists\ 0<\alpha<1$.*

$$\begin{aligned}
&D(\widetilde{X}_{n+1},\widetilde{X}) = \\
&D(\textstyle\int_0^t(\widetilde{A}\widetilde{X}_n(s)+\widetilde{f}(s))\ ds, \\
&(-1).\textstyle\int_0^t(\widetilde{A}\widetilde{X}(s)+\widetilde{f}(s))\ ds) = D(\textstyle\int_0^t \widetilde{A}\widetilde{X}_n(s)\ ds, \\
&(-1).\textstyle\int_0^t \widetilde{A}\widetilde{X}(s)\ ds) \\
&\leq \textstyle\int_0^t D(\widetilde{A}\widetilde{X}_n(s),\widetilde{A}\widetilde{X}(s)) \\
&\leq TL\ D(\widetilde{X}_n(t),\widetilde{X}(t)) = \alpha\ D(\widetilde{X}_n(t),\widetilde{X}(t)).
\end{aligned}$$

Since, $0<\alpha<1$, then $D(\widetilde{X}_n(t),\widetilde{X}(t)) \to 0$ as $n\to\infty$. Therefore, $\widetilde{X}_n(t) \to \widetilde{X}(t)$.

Remark 4.5. The proof of other cases is similar to the previous theorems.

5. Numerical Examples

In this section, we solve FDEs by using the Picard method. The program has been provided with Mathematica 6 according to the following algorithm where ε is a given positive value.

Algorithm 5.1. **Step** 1**:** Set $n \leftarrow 0$.

Step 2**:** Calculate the recursive relations (12) or (14) or (15).

Step 3**:** If $D(\widetilde{X}_{n+1},\widetilde{X}_n)<\varepsilon$ then go to step 4,
else $n\leftarrow n+1$ and go to step 2.

Step 4**:** Print $\widetilde{X}_n(x)$ as the approximate of the exact solution.

Example 5.2. Consider the FDEs as follows:

$$\frac{d}{dt}\begin{pmatrix}\widetilde{x}_1(t)\\ \widetilde{x}_2(t)\end{pmatrix} = \begin{pmatrix}(0.15,0.2,0.25) & (0.2,0.25,0.3)\\ (0.17,0.2,0.28) & (0.25,0.3,0.35)\end{pmatrix}\begin{pmatrix}\widetilde{x}_1(t)\\ \widetilde{x}_2(t)\end{pmatrix} + \begin{pmatrix}(0.1,0.2,0.3)\\ (0.2,0.3,0.4)\end{pmatrix}. \tag{16}$$

With initial condition:

$$\widetilde{X}(0) = \begin{pmatrix}(0.3,0.4,0.5)\\ (0.45,0.5,0.6)\end{pmatrix}. \tag{17}$$

$$\varepsilon = 10^{-4}.$$

Case 1**:** Table 1 shows that, the approximation solution of the FDES is convergent with 18 and 21 iterations by using the Picard method when x_1' and x_2' are (1)-differentiable.

t	$\underline{x}_1, r = 0.2$	$\overline{x}_1$	$\underline{x}_2$	$\overline{x}_2$
0.1	0.3423678	0.3729607	0.2419102	0.2635704
0.2	0.4356014	0.4665317	0.3269453	0.3616734
0.3	0.5234118	0.5713507	0.4135631	0.4472026
0.4	0.6587128	0.6877915	0.5239546	0.5634894
0.5	0.7036479	0.7478216	0.6455116	0.6704903
0.6	0.7796425	0.8026508	0.7227314	0.7456752

Table 1. Numerical Results for Example 4.1

t	$\underline{x}_1$	$\overline{x}_1$	$\underline{x}_2$	$\overline{x}_2$
0.1	0.4277667	0.4651655	0.3512528	0.3729435
0.2	0.5433016	0.5735628	0.4466008	0.4688351
0.3	0.61253219	0.6648902	0.5219674	0.5426459
0.4	0.6587305	0.6810725	0.6337896	0.67006992
0.5	0.7344237	0.7612884	0.7267688	0.7455403
0.6	0.7619025	0.7938552	0.7557304	0.7853114

Table 2. Numerical Results for Example 4.1

Case 2: Table 2 shows that, the approximation solution of the FDES is convergent with 20 and 23 iterations by using the Picard method when $x_1^{'}$ and $x_2^{'}$ are (2)-differentiable.

t	$\underline{x}_1$	$\overline{x}_1$	$\underline{x}_2$	$\overline{x}_2$
0.1	0.2365629	0.2605736	0.4019667	0.4533815
0.2	0.35753615	0.3727817	0.5147205	0.5316088
0.3	0.4143672	0.4400297	0.5446551	0.5799037
0.4	0.5289415	0.5520633	0.6744309	0.7066325
0.5	0.6207885	0.65221005	0.7242068	0.7655007
0.6	0.6630891	0.6934468	0.7639107	0.7912315

Table 3. Numerical Results for Example 4.1

Case 3: Table 3 shows that, the approximation solution of the FDES is convergent with 27 and 31 iterations by using the Picard method when $x_1^{'}$ is (1)-differentiable and $x_2^{'}$ is (2)-differentiable.

Example 5.3. Consider the FDES as follows:

$$\frac{d}{dt}\begin{pmatrix} \widetilde{x}_1(t) \\ \widetilde{x}_2(t) \end{pmatrix} = \begin{pmatrix} -\widetilde{a} & \widetilde{c} \\ \widetilde{a} & \widetilde{b} \end{pmatrix}\begin{pmatrix} \widetilde{x}_1(t) \\ \widetilde{x}_2(t) \end{pmatrix} + \begin{pmatrix} \widetilde{r} \\ \widetilde{0} \end{pmatrix}, \tag{18}$$

where,

$$\widetilde{c} = (0.14, 0.17, 0.23),$$

$$\widetilde{a} = (0.02, 0.03, 0.05),$$

$$\widetilde{b} = (0.1, 0.15, 0.2).$$

With initial condition:

$$\widetilde{X}(0) = \begin{pmatrix} (0.01, 0.02, 0.03) \\ (0.04, 0.05, 0.06) \end{pmatrix}. \tag{19}$$

$$\varepsilon = 10^{-5}.$$

t	$\underline{x}_1, r = 0.3$	$\overline{x}_1$	$\underline{x}_2$	$\overline{x}_2$
0.1	0.3724619	0.4122317	0.2058809	0.2471305
0.2	0.4687421	0.4953432	0.2485677	0.2858193
0.3	0.5765658	0.6226758	0.3375223	0.3862706
0.4	0.6609846	0.6944102	0.4046552	0.4604936
0.5	0.7219055	0.7638328	0.4688113	0.5233503
0.6	0.8016442	0.8477253	0.5307664	0.5823371

Table 4. Numerical Results for Example 4.2

Case 1: Table 4 shows that, the approximation solution of the FDES is convergent with 23 and 25 iterations by using the Picard method when $x_1^{'}$ and $x_2^{'}$ are (1)-differentiable.

t	$\underline{x}_1$	$\overline{x}_1$	$\underline{x}_2$	$\overline{x}_2$
0.1	0.3327746	0.3722785	0.4237801	0.4817663
0.2	0.4137612	0.4657219	0.4719035	0.5044103
0.3	0.5388331	0.5819077	0.4988719	0.5419237
0.4	0.5724304	0.6246628	0.5324193	0.5628712
0.5	0.6209665	0.6907319	0.5813407	0.6033563
0.6	0.6944897	0.7324553	0.6275329	0.6842552

Table 5. Numerical Results for Example 4.2

Case 2: Table 5 shows that, the approximation solution of the FDES is convergent with 20 and 25 iterations by using the Picard method when $x_1^{'}$ is (2)-differentiable and $x_2^{'}$ is (2)-differentiable.

t	$\underline{x}_1$	$\overline{x}_1$	$\underline{x}_2$	$\overline{x}_2$
0.1	0.5266703	0.5746732	0.4126448	0.4352558
0.2	0.6057542	0.6461328	0.5276115	0.5717205
0.3	0.7122667	0.7489527	0.6348837	0.6609413
0.4	0.7522673	0.7953269	0.6822602	0.6953441
0.5	0.8126044	0.8404247	0.7126058	0.7504718
0.6	0.8548225	0.8832441	0.7648035	0.7985943

Table 6. Numerical Results for Example 4.2

Case 3: Table 6 shows that, the approximation solution of the FDES is convergent with 27 and 31 iterations by using the Picard method when $x_1^{'}$ is (1)-differentiable and $x_2^{'}$ is (2)-differentiable.

6. **Conclusions**

The Picard method has been shown to solve effectively, easily and accurately a large class of nonlinear problems with the approximations which convergent are rapidly to exact solutions. In this work, the Picard method has been successfully employed to obtain the approximate solution of the $FDEs$ under generalized H-differentiability.

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