

## ON THE BICOMPLETION OF INTUITIONISTIC FUZZY QUASI-METRIC SPACES

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ABSTRACT. Based on previous results that study the completion of fuzzy metric spaces, we show that every intuitionistic fuzzy quasi-metric space, using the notion of fuzzy metric space in the sense of Kramosil and Michalek to obtain a generalization to the quasi-metric setting, has a bicompletion which is unique up to isometry.

### 1. Introduction

The core concepts we will use to study on the bicompletion of fuzzy constructions are based on the Kramosil and Michalek notion of a fuzzy metric space [11] (a KM-fuzzy metric space from now on) and on the definition of an intuitionistic fuzzy metric space [1, 14].

Indeed the KM-fuzzy metric notion has a close relationship with probabilistic spaces. As a matter of fact, Kramosil and Michalek observed that the class of fuzzy metric spaces in their sense, is “equivalent” to the class of Menger spaces having a continuous t-norm. Adapting Sherwood’s construction, who proved in [19] that every Menger space belonging to this class has a completion which is unique up to isometry, to the fuzzy metric context, it can be deduced that every KM-fuzzy metric space has a completion which is unique up to isometry.

In addition, the notion of a KM-fuzzy metric space was generalized to the quasi-metric setting in [4, 8], where several properties of these structures were discussed. Recently, there were given in [15, 16, 17], applications of fixed point theorems, in the realm of fuzzy quasi-metric spaces, to deduce the existence and uniqueness of solution for the recurrence equations associated with some types of algorithms.

Both paths lead to the notion of the intuitionistic fuzzy quasi-metric space. In [16] the authors generalize the notion of intuitionistic fuzzy metric space by Alaca et al [1] to the quasi-metric setting and obtain an intuitionistic fuzzy quasi-metric version of the Banach contraction principle which is applied to deduce the existence of solution for the recurrence equation which is typically associated with the complexity analysis of Quicksort.

In this context, the completion of fuzzy quasi-metric spaces appears as a natural and attractive question. After Section 2, where some pertinent definitions in order to introduce these constructs will be exposed, in Section 3 we will recall that every

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KM-fuzzy quasi-metric space has a (KM-fuzzy quasi-metric) bicompletion which is unique up to isometry [3]. This fact allows us to restate the completion of a KM-fuzzy metric space as a particular case.

Here we shall extend our results announced in [2] by applying these constructions in Section 4 to study the bicompletion of intuitionistic fuzzy quasi-metric spaces.

## 2. Preliminary Concepts

Our basic references for quasi-uniform and quasi-metric spaces are [6, 12].

**Definition 2.1.** A quasi-metric on a set  $X$  is a nonnegative real valued function defined on  $X \times X$  such that for all  $x, y, z \in X$ :

- (i)  $d(x, y) = d(y, x) = 0 \iff x = y$ ;
- (ii)  $d(x, y) \leq d(x, z) + d(y, z)$ .

**Definition 2.2.** A quasi-metric space is a pair  $(X, d)$  such that  $X$  is a set and  $d$  is a quasi-metric on  $X$ .

It is known that each quasi-metric  $d$  on  $X$  induces a  $T_0$  topology  $\tau_d$  on  $X$  which has as a base the family of open balls  $\{B_d(x, r) : x \in X, r > 0\}$ , where  $B_d(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$  for all  $x \in X$  and  $\varepsilon > 0$ .

Given a quasi-metric  $d$  on  $X$ , then the function  $d^{-1}$  defined by  $d^{-1}(x, y) = d(y, x)$ , is also a quasi-metric on  $X$ , called the conjugate of  $d$ , and the function  $d^s$  defined by  $d^s(x, y) = \max\{d(x, y), d^{-1}(x, y)\}$  is a metric on  $X$ .

A quasi-metric space  $(X, d)$  is said to be bicomplete if  $(X, d^s)$  is a complete metric space (all Cauchy sequences converge). In this case we say that  $d$  is a bicomplete quasi-metric on  $X$ .

A topological space  $(X, \tau)$  is called quasi-metrizable if there is a quasi-metric  $d$  on  $X$  such that  $\tau = \tau_d$ .

**Definition 2.3.** [18] A binary operation  $*$  :  $[0, 1] \times [0, 1] \rightarrow [0, 1]$  is a continuous t-norm if  $*$  satisfies:

- (i)  $*$  is associative and commutative;
- (ii)  $*$  is continuous;
- (iii)  $a * 1 = a$  for every  $a \in [0, 1]$ ;
- (iv)  $a * b \leq c * d$  whenever  $a \leq c$  and  $b \leq d$ , with  $a, b, c, d \in [0, 1]$ .

**Definition 2.4.** [13] A continuous t-conorm is a binary operation  $\diamond$  :  $[0, 1] \times [0, 1] \rightarrow [0, 1]$  such that:

- (i)  $\diamond$  is commutative and associative;
- (ii)  $\diamond$  is continuous;
- (iii)  $a \diamond 0 = a$  for all  $a \in [0, 1]$ ;
- (iv)  $a \diamond b \leq c \diamond d$  when  $a \leq c$  and  $b \leq d$  ( $a, b, c, d \in [0, 1]$ ).

**Definition 2.5.** [5] For a given t-norm  $*$  and  $x, y \in [0, 1]$ , the t-conorm  $*'$  defined as:

$$x *' y = 1 - ((1 - x) * (1 - y))$$

is called the dual t-conorm of  $*$ . Similarly, for any given t-conorm  $\diamond$ , the t-norm  $\diamond'$  defined as:

$$x \diamond' y = 1 - ((1 - x) \diamond (1 - y))$$

is called the dual t-norm of  $\diamond$ .

**Definition 2.6.** [8] A KM-fuzzy quasi-metric on a set  $X$  is a pair  $(M, *)$  such that  $*$  is a continuous t-norm and  $M$  is a fuzzy set in  $X \times X \times [0, \infty)$  such that for all  $x, y, z \in X$ :

- (KM1)  $M(x, y, 0) = 0$ ;
- (KM2)  $x = y$  if and only if  $M(x, y, t) = M(y, x, t) = 1$  for all  $t > 0$ ;
- (KM3)  $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$  for all  $t, s \geq 0$ ;
- (KM4)  $M(x, y, -) : [0, \infty) \rightarrow [0, 1]$  is left continuous;

**Definition 2.7.** [11] A KM-fuzzy metric on a set  $X$  is a KM-fuzzy quasi-metric  $(M, *)$  on  $X$  such that for each  $x, y \in X$ :

- (KM5)  $M(x, y, t) = M(y, x, t)$  for all  $t > 0$ .

**Definition 2.8.** [4, 8] A KM-fuzzy quasi-metric space is a triple  $(X, M, *)$  such that  $X$  is a nonempty set and  $(M, *)$  is a KM-fuzzy quasi-metric on  $X$ .

**Definition 2.9.** [16] An intuitionistic fuzzy quasi-metric on a set  $X$  is a 4-tuple  $(M, N, *, \diamond)$  such that  $*$  is a continuous t-norm,  $\diamond$  is a continuous t-conorm and  $M, N$  are fuzzy sets in  $X \times X \times [0, \infty)$  such that for all  $x, y, z \in X$ :

- (1)  $M(x, y, t) + N(x, y, t) \leq 1$  for all  $t \geq 0$ ;
- (2)  $M(x, y, 0) = 0$ ;
- (3)  $x = y$  if and only if  $M(x, y, t) = M(y, x, t) = 1$  for all  $t > 0$ ;
- (4)  $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$  for all  $t, s \geq 0$ ;
- (5)  $M(x, y, -) : [0, \infty) \rightarrow [0, 1]$  is left continuous;
- (6)  $N(x, y, 0) = 1$ ;
- (7)  $x = y$  if and only if  $N(x, y, t) = N(y, x, t) = 0$  for all  $t > 0$ ;
- (8)  $N(x, y, t) \diamond N(y, z, s) \geq N(x, z, t + s)$  for all  $t, s \geq 0$ ;
- (9)  $N(x, y, -) : [0, \infty) \rightarrow [0, 1]$  is left continuous.

**Definition 2.10.** [16] An intuitionistic fuzzy quasi-metric space is a 5-tuple  $(X, M, N, *, \diamond)$  such that  $(M, N, *, \diamond)$  is an intuitionistic fuzzy quasi-metric on a set  $X$ .

**Remark 2.11.** It is clear that if  $(X, M, N, *, \diamond)$  is an intuitionistic fuzzy quasi-metric space, then  $(X, M, *)$  is a KM-fuzzy quasi-metric space.

Similarly to the fuzzy metric case, each fuzzy quasi-metric  $(M, *)$  on a set  $X$  induces a  $T_0$  topology  $\tau_M$  on  $X$  which has as a base the family of open balls  $\{B_M(x, \varepsilon, t) : x \in X, 0 < \varepsilon < 1, t > 0\}$ , where  $B_M(x, \varepsilon, t) = \{y \in X : M(x, y, t) > 1 - \varepsilon\}$  (see [8, 9]).

If  $(M, N, *, \diamond)$  is an intuitionistic fuzzy quasi-metric on  $X$ , then one has that  $(M^{-1}, N^{-1}, *, \diamond)$  is also an intuitionistic fuzzy quasi-metric on  $X$ , where  $M^{-1}$  is the fuzzy set in  $X \times X \times [0, \infty)$  defined by  $M^{-1}(x, y, t) = M(y, x, t)$  and  $N^{-1}$  is the fuzzy set in  $X \times X \times [0, \infty)$  defined by  $N^{-1}(x, y, t) = N(y, x, t)$ .

Moreover, if we define  $M^i$  as the fuzzy set in  $X \times X \times [0, \infty)$  given by  $M^i(x, y, t) = \min\{M(x, y, t), M^{-1}(x, y, t)\}$  and denote by  $N^s$  the fuzzy set in  $X \times X \times [0, \infty)$  given by  $N^s(x, y, t) = \max\{N(x, y, t), N^{-1}(x, y, t)\}$  then  $(M^i, N^s, *, \diamond)$  is an intuitionistic fuzzy metric in the sense of [1].

A KM-fuzzy metric space  $(X, M, *)$  is complete [7, 20] provided that each Cauchy sequence in  $X$  is convergent with respect to  $\tau_M$ , where a sequence  $(x_n)_n$  in  $X$  is said to be a Cauchy sequence if for each  $\varepsilon \in (0, 1)$  and each  $t > 0$  there is  $n_0 \in \mathbb{N}$  such that  $M(x_n, x_m, t) > 1 - \varepsilon$  for all  $n, m \geq n_0$ .

According to [8, 9], a KM-fuzzy quasi-metric space  $(X, M, *)$  is called bicomplete if  $(X, M^i, *)$  is a complete fuzzy metric space. We say that  $(M, *)$  is a bicomplete fuzzy quasi-metric on  $X$ .

In order to construct a suitable topology on an intuitionistic fuzzy quasi-metric space  $(X, M, N, *, \diamond)$ , Romaguera and Tirado considered in [16] the natural “balls”  $B(x, \varepsilon, t)$  defined, similarly to [14] and [1], by:

$$B(x, \varepsilon, t) = \{y \in X : M(x, y, t) > 1 - \varepsilon, N(x, y, t) < \varepsilon\}$$

for all  $x \in X$ ,  $0 < \varepsilon < 1$ , and  $t > 0$ .

Then, they proved that  $B(x, \varepsilon, t) = B_M(x, \varepsilon, t)$  (compare [10] for the metric case), and thus the topology induced by  $(M, N, *, \diamond)$  coincides with the topology  $\tau_M$  induced by  $(M, *)$ .

In [14], Park introduced the notion of a complete intuitionistic fuzzy metric space. It is proved in [10] that an intuitionistic fuzzy metric space  $(X, M, N, *, \diamond)$  is complete if and only if  $(X, M, *)$  is complete.

For the quasi-metric case we have the following definition and lemma according to [16] which will be useful in section 4.

**Definition 2.12.** (a) A sequence  $(x_n)_n$  in an intuitionistic fuzzy quasi-metric space  $(X, M, N, *, \diamond)$  is called Cauchy if for each  $\varepsilon \in (0, 1)$ ,  $t > 0$ , there exists  $n_0 \in \mathbb{N}$  such that  $M(x_n, x_m, t) > 1 - \varepsilon$ , and  $N(x_n, x_m, t) < \varepsilon$ , for all  $n, m \geq n_0$ .

(b) An intuitionistic fuzzy quasi-metric space  $(X, M, N, *, \diamond)$  is called bicomplete if  $(X, M^i, N^s, *, \diamond)$  is a complete intuitionistic fuzzy metric space.

**Lemma 2.13.** (a) A sequence in an intuitionistic fuzzy quasi-metric space  $(X, M, N, *, \diamond)$  is a Cauchy sequence if and only if it is a Cauchy sequence in the fuzzy metric space  $(X, M^i, *)$ .

(b) An intuitionistic fuzzy quasi-metric space  $(X, M, N, *, \diamond)$  is bicomplete if and only if the fuzzy quasi-metric space  $(X, M, *)$  is bicomplete.

### 3. The Completion of a Fuzzy Metric Space and the bicompletion of a Fuzzy Quasi-Metric Space

In this section we will recall some known and crucial results on the completion of fuzzy metric spaces and the bicompletion of fuzzy quasi-metric spaces in the sense of Kramosil-Michalek which should be useful to provide a better understanding of the constructions made in the rest of the paper.

It was shown in [3] that the bicompletion of KM-fuzzy quasi-metric spaces can be achieved using the suprema of subsets of  $[0, 1]$  and lower limits of sequences in  $[0, 1]$  as it is sketched below:

Let  $(X, M, *)$  be a KM-fuzzy quasi-metric space.

Denote by  $S$  the collection of all Cauchy sequences in  $(X, M^i, *)$ .

**Definition 3.1.** Define a relation  $\sim$  on  $S$  by

$$(x_n)_n \sim (y_n)_n \iff \sup_{0 < s < t} \underline{\lim} M^i(x_n, y_n, s) = 1 \text{ for all } t > 0,$$

where by  $\underline{\lim} M^i(x_n, y_n, s)$  we denote, as usual, the lower limit of the sequence  $(M^i(x_n, y_n, s))_n$ , i.e.,

$$\underline{\lim} M^i(x_n, y_n, s) = \sup_k \inf_{n \geq k} M^i(x_n, y_n, s).$$

**Lemma 3.2.** [3, Lemma 1]  $\sim$  is an equivalence relation on  $S$ .

Now denote by  $\tilde{X}$  the quotient  $S/\sim$ , and by  $[(x_n)_n]$  the class of the element  $(x_n)_n$  of  $S$ .

**Lemma 3.3.** [3, Lemma 3] For each  $(x_n)_n, (y_n)_n \in S$  and each  $(a_n)_n \in [(x_n)_n]$ ,  $(b_n)_n \in [(y_n)_n]$ , one has

$$\sup_{0 < s < t} \underline{\lim} M((x_n)_n, (y_n)_n, t) = \sup_{0 < s < t} \underline{\lim} M((a_n)_n, (b_n)_n, t),$$

for all  $t > 0$ .

**Lemma 3.4.** [8, Proposition 4.5] Let  $(X, M, *)$  be a fuzzy quasi-metric space and  $(Y, N, \star)$  a bicomplete fuzzy quasi-metric space. If there is a  $\tau_{M^i}$ -dense subset  $A$  of  $X$  and an isometry  $f : (A, M, *) \rightarrow (Y, N, \star)$ , then there exists a unique isometry  $F : (X, M, *) \rightarrow (Y, N, \star)$  such that  $F|_A = f$ .

**Definition 3.5.** For each  $[(x_n)_n], [(y_n)_n] \in \tilde{X}$ , define

$$\tilde{M}([(x_n)_n], [(y_n)_n], 0) = 0,$$

and

$$\tilde{M}([(x_n)_n], [(y_n)_n], t) = \sup_{0 < s < t} \underline{\lim} M(x_n, y_n, t),$$

for  $t > 0$ .

Then  $\tilde{M}$  is a function from  $\tilde{X} \times \tilde{X} \times [0, \infty)$  to  $[0, 1]$ .

**Definition 3.6.** Define  $i : X \rightarrow \tilde{X}$  such that, for each  $x \in X$ ,  $i(x)$  is the class of the constant sequence  $x, x, \dots$

And from the above constructions we obtain:

**Theorem 3.7.** [3, Theorem 1] Let  $(X, M, *)$  be a fuzzy quasi-metric space. Then:

- (a)  $(\tilde{M}, *)$  is a fuzzy quasi-metric on  $\tilde{X}$ .
- (b)  $i(X)$  is dense in  $(\tilde{X}, \tilde{M}^i, *)$ .
- (c)  $(X, M, *)$  is isometric to  $(i(X), \tilde{M}, *)$ .

(d)  $(\widetilde{M}, *)$  is bicomplete.

(e) If  $(Y, N, *)$  is a bicomplete fuzzy quasi-metric space such that  $(X, M, *)$  is isometric to a  $\tau_{N^i}$ -dense subspace of  $Y$ , then  $(Y, N, *)$  and  $(\widetilde{X}, \widetilde{M}, *)$  are isometric.

**Remark 3.8.** The preceding theorem implies that every fuzzy quasi-metric space  $(X, M, *)$  has a bicompletion which is unique up to isometry. We refer to  $(\widetilde{X}, \widetilde{M}, *)$  as the bicompletion of  $(X, M, *)$ .

#### 4. The Bicompletion of an Intuitionistic Fuzzy Quasi-Metric Space

Now we shall show, with the help of the construction of the bicompletion of a KM-fuzzy quasi-metric space recalled above, that every intuitionistic fuzzy quasi-metric space in the sense of Kramosil and Michalek has a bicompletion which is unique up to isometry.

To this end we shall use the following two natural definitions.

**Definition 4.1.** A mapping  $f$  from an intuitionistic fuzzy quasi-metric space  $(X, M, N, *, \diamond)$  to an intuitionistic fuzzy quasi-metric space  $(Y, M_Y, N_Y, *_Y, \diamond_Y)$  is called an isometry if for each  $x, y \in X$  and each  $t > 0$ ,

$$M_Y(f(x), f(y), t) = M(x, y, t) \quad \text{and} \quad N_Y(f(x), f(y), t) = N(x, y, t).$$

It is clear that every isometry is a one-to-one mapping.

Two intuitionistic fuzzy quasi-metric spaces  $(X, M, N, *, \diamond)$  and  $(Y, M_Y, N_Y, *_Y, \diamond_Y)$  are called isometric if there is an isometry from  $X$  onto  $Y$ .

**Definition 4.2.** Let  $(X, M, N, *, \diamond)$  be an intuitionistic fuzzy quasi-metric space. A bicompletion of  $(X, M, N, *, \diamond)$  is a bicomplete fuzzy quasi-metric space  $(Y, M_Y, N_Y, *_Y, \diamond_Y)$  such that  $(X, M, N, *, \diamond)$  is isometric to a  $\tau_{(M_Y)^i}$ -dense subspace of  $Y$ .

Let  $(X, M, N, *, \diamond)$  be an intuitionistic fuzzy quasi-metric space. Consider the fuzzy quasi-metric space  $(X, M, *)$ , and let  $(\widetilde{X}_M, \widetilde{M}, *)$  be its bicompletion as constructed in Theorem 3.7.

**Remark 4.3.** Recall that, in particular, from Theorem 3.7 the mapping  $i : X \rightarrow \widetilde{X}_M$  such that, for each  $x \in X$ ,  $i(x)$  is the class of the constant sequence  $x, x, \dots$ , is an isometry between  $(X, M, *)$  and the dense subspace  $(i(X), \widetilde{M}, *)$  of  $(\widetilde{X}_M, \widetilde{M}, *)$ .

In the following we will refer to  $i$  as  $i_M$ .

**Lemma 4.4.** [16]  $(X, 1 - N, \diamond')$ , where  $\diamond'$  is the dual  $t$ -norm of  $\diamond$ , is a fuzzy quasi-metric space.

Now we construct a fuzzy set  $\widetilde{1 - N}$  in  $\widetilde{X}_M \times \widetilde{X}_M \times [0, \infty)$  by

$$\widetilde{1 - N}([(x_n)_n], [(y_n)_n], 0) = 0,$$

and

$$\widetilde{1 - N}([(x_n)_n], [(y_n)_n], t) = \sup_{0 < s < t} \underline{\lim} (1 - N)(x_n, y_n, s),$$

whenever  $t > 0$ .

**Remark 4.5.** From the fact that  $M + N \leq 1$ , it obviously follows that every Cauchy sequence in the fuzzy metric space  $(X, M^i, *)$  is a Cauchy sequence in the fuzzy metric space  $(X, (1 - N)^i, \diamond')$ .

We shall prove the following theorem that shows that each intuitionistic fuzzy quasi-metric space has a bicompletion which is unique up to isometry.

**Theorem 4.6.** *Let  $(X, M, N, *, \diamond)$  be an intuitionistic fuzzy quasi-metric space. Then:*

- (a)  $(\widetilde{M}, 1 - (1 - N), *, \diamond)$  is an intuitionistic fuzzy quasi-metric on  $\widetilde{X}_M$ .
- (b)  $i_M(X)$  is dense in  $(\widetilde{X}_M, \widetilde{M}^i, *)$ .
- (c)  $(X, M, N, *, \diamond)$  is isometric to  $(i_M(X), \widetilde{M}, 1 - (1 - N), *, \diamond)$ .
- (d)  $(\widetilde{M}, 1 - (1 - N), *, \diamond)$  is bicomplete.
- (e) If  $(Y, M_Y, N_Y, *_Y, \diamond_Y)$  is a bicomplete intuitionistic fuzzy quasi-metric space such that  $(X, M, N, *, \diamond)$  is isometric to a  $\tau_{(M_Y)^i}$ -dense subspace of  $Y$ , then  $(Y, M_Y, N_Y, *_Y, \diamond_Y)$  and  $(\widetilde{X}_M, \widetilde{M}, 1 - (1 - N), *, \diamond)$  are isometric.

*Proof.* (a) Since  $(X, M, *)$  is a fuzzy quasi-metric space, we know that  $(\widetilde{M}, *)$  is a fuzzy quasi-metric on  $\widetilde{X}$  according to Theorem 3.7 (a).

Next we prove that  $(\widetilde{X}_M, 1 - N, \diamond')$  is a fuzzy quasi-metric space.

Indeed  $(1 - N, \diamond')$  satisfies property (KM1) of Definition 2.6.

Now let  $(x_n)_n, (y_n)_n \in S$  such that  $1 - N([(x_n)_n], [(y_n)_n], t) = 1$  for all  $t > 0$ . If  $(z_n)_n \in [(y_n)_n]$ , it follows from Lemma 3.3 that

$$\sup_{0 < s < t} \liminf (1 - N)((z_n)_n, (y_n)_n, t) = 1$$

for all  $t > 0$ , i.e.,  $(z_n)_n \in [(y_n)_n]$ . The same argument shows that  $(z_n)_n \in [(x_n)_n]$  whenever  $(z_n)_n \in [(y_n)_n]$ . We conclude that  $1 - N([(x_n)_n], [(y_n)_n], t) = 1$  for all  $t > 0$ , if and only if  $[(x_n)_n] = [(y_n)_n]$  (KM2).

For (KM3) to hold we need to prove that  $(1 - N)([(x_n)_n], [(y_n)_n], t + s) \geq (1 - N)([(x_n)_n], [(z_n)_n], t) \diamond' (1 - N)([(z_n)_n], [(y_n)_n], s)$ .

Indeed, since  $(1 - N)(x_n, y_n, t + s) \geq (1 - N)(x_n, z_n, t) \diamond' (1 - N)(z_n, y_n, s)$  then

$$\begin{aligned} \sup_{0 < r < t+s} \liminf (1 - N)(x_n, y_n, t + s) &\geq \sup_{0 < r < t} \liminf (1 - N)(x_n, z_n, t) \diamond' \\ &\quad \sup_{0 < r < s} \liminf (1 - N)(z_n, y_n, s) \end{aligned}$$

and hence the desired conclusion.

Similarly  $(1 - N)([(x_n)_n], [(y_n)_n], -) : [0, \infty] \rightarrow [0, 1]$  (KM4) is left continuous as  $(1 - N)(x_n, y_n, -)$  is left continuous as both the supremum and the inferior limit maintain the continuity.

Consequently  $(\widetilde{M}, 1 - (1 - N), *, \diamond)$  is a fuzzy quasi-metric on  $\widetilde{X}$  and then  $(\widetilde{M}, 1 - (1 - N), *, \diamond)$  is an intuitionistic fuzzy quasi-metric space.

(b) As shown in Theorem 3.7 (b) and due to the way that  $i_M(X)$  is constructed in Remark 4.3 we immediately have that  $i_M(X)$  is dense in  $(\widetilde{X}_M, \widetilde{M}^i, *)$ .

(c) The mapping  $i_M$  satisfies for each  $x, y \in X$  and  $t > 0$ ,

$$\widetilde{M}(i_M(x), i_M(y), t) = M(x, y, t),$$

and, also using Definitions 3.5 and 4.1,

$$(1 - (\widetilde{1 - N}))(i_M(x), i_M(y), t) = (1 - (1 - N))(x, y, t) = N(x, y, t).$$

Thus  $i_M$  is an isometry between  $(X, M, N, *, \diamond)$  and the subspace  $i_M(X)$  of  $(\widetilde{X}_M, \widetilde{M}, 1 - (\widetilde{1 - N}), *, \diamond)$ , which is dense with respect to  $\tau_{M^i}$ .

(d) Since  $M \leq 1 - N$ , we deduce that

$$\sup_{0 < s < t} \underline{\lim} M(x_n, y_n, s) \leq \sup_{0 < s < t} \underline{\lim} (1 - N)(x_n, y_n, s)$$

for each  $t > 0$  and each pair  $(x_n)_n, (y_n)_n$ , of Cauchy sequences in  $(X, M^i, *)$ . Thus

$$\widetilde{M} \leq \widetilde{1 - N}.$$

Since, according to Theorem 3.7,  $(\widetilde{X}_M, \widetilde{M}, *)$  is a bicomplete fuzzy quasi-metric space, we conclude using Lemma 2.13 (b) that  $(\widetilde{X}_M, \widetilde{M}, 1 - (\widetilde{1 - N}), *, \diamond)$  is a bicomplete intuitionistic fuzzy quasi-metric space.

(e) Let  $(Y, M_Y, N_Y, *_Y, \diamond_Y)$  be any bicompletion of  $(X, M, N, *, \diamond)$ . Then, there is an isometry  $j$  from  $(X, M, N, *, \diamond)$  to  $(Y, M_Y, N_Y, *_Y, \diamond_Y)$  as defined in 4.1. On the other hand, since the fuzzy quasi-metric space  $(Y, M_Y, *_Y)$  is a bicompletion of  $(X, M, *)$ , following Lemma 3.4, there is a unique isometry  $F$  from  $(\widetilde{X}_M, \widetilde{M}, *)$  onto  $(Y, M_Y, *_Y)$  such that  $F(i_M) = j$ . Taking into account Remark 4.5 we deduce from standard arguments (see for instance the proof of [8, Proposition 4.5]) that

$$N_Y(F(\tilde{x}), F(\tilde{y}), t) = (1 - (\widetilde{1 - N}))(\tilde{x}, \tilde{y}, t),$$

whenever  $\tilde{x}, \tilde{y} \in \widetilde{X}_M$  and  $t > 0$ .

Then  $F$  is an isometry from  $(\widetilde{X}_M, \widetilde{M}, 1 - (\widetilde{1 - N}), *, \diamond)$  onto  $(Y, M_Y, N_Y, *_Y, \diamond_Y)$ .  $\square$

**Remark 4.7.** The preceding theorem implies that every intuitionistic fuzzy quasi-metric space  $(X, M, N, *, \diamond)$  has a bicompletion which is unique up to isometry. We refer to  $(\widetilde{X}_M, \widetilde{M}, 1 - (\widetilde{1 - N}), *, \diamond)$  as the bicompletion of  $(X, M, N, *, \diamond)$ .

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