ACCEPTANCE SINGLE SAMPLING PLAN WITH FUZZY PARAMETER

E. BALOUI JAMKHANEH, B. SADEGHPOUR GILDEH AND G. YARI

Abstract. The acceptance sampling plan problem is an important topic in quality control and both the theory of probability and theory of fuzzy sets may be used to solve it. In this paper, we discuss the single acceptance sampling plan, when the proportion of nonconforming products is a fuzzy number. We show that the operating characteristic (OC) curve of the plan is a band having high and low bounds and that for fixed sample size and acceptance number, the width of the band depends on the ambiguity proportion parameter in the lot. When the acceptance number equals zero, this band is convex and the convexity increases with n Finally, we compare the OC bands for a given value of c.

1. Introduction

Single sampling plans for acceptance or rejection of a lot are important in statistical quality control. Such plana are characterized by the sample size n and an acceptance number c. If the number of defective items is less than or equal to c, the lot is accepted. Traditional sampling plans have crisp parameters. However, the proportion of defective items (p in a lot is often not precisely known in real decision making problems, while there also exists some uncertainty in the value of p obtained from experiment, personal judgment or estimation. The theory of fuzzy sets is widely used in solving such problems. This theory is a powerful and well-known tool to formulate and analyze uncertainty resulting from ambiguity and personal judgment. In this paper, we try to restore the uncertainty existing in the problem by assuming that the parameter is a fuzzy number, and hence achieve a result with a higher degree of certainty.

Classical acceptance sampling plans have been studied by many researchers. They are thoroughly discussed in Schilling (1982). Single sampling by attributes with relaxed requirements was discussed by Ohta and Ichihashi (1988) Kanagawa and Ohta (1990), Tamaki, Kanagawa and Ohta (1991) and Grzegorzewski (1998, 2001). Grzegorzewski (2001, 2002) also considered sampling plans by variables with fuzzy requirements. Sampling plans by attributes for vague data were considered by Hrniewicz (1992, 1994). Chakraborty (1992, 1994b) addresses the problem of designing a single sampling plan when the lot tolerance percent defective, consumer's risk and incoming quality level are triangular fuzzy numbers. Grzegrozewski (2002) also considered sampling plans by variables with fuzzy requirements. Hrniewicz

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(2008) provided a short overview of basic problems of statistical quality control that have been solved by probability theory and fuzzy set theory. Finally the properties of a sampling plan under situations involving both impreciseness and randomness using the theory of chance was studied by Sampath (2009). It is well known that the probability distribution plays a crucial role in acceptance sampling plans. Since in traditional probability theory, parameters are assumed to be precise values, difficulties arise when they become imprecise. Hence a new approach is necessary in the designing of sampling plans. Here, fuzzy probability theory introduced by Buckley (2003) is used to explore the possibility of introducing a suitable sampling plan for a situation involving impreciseness, and the operating characteristic of a single sampling plan is calculated using the concept of fuzzy probability. According to Buckley's definition, the number of defective items in the sample with fuzzy parameter $\widetilde{p}$ has a fuzzy binomial probability mass function. The organization of the paper is as follows.

We recall some definitions of fuzzy sets theory and fuzzy probability in section 2. In section 3 we consider the fuzzy probability of lot acceptance and compute its value for a special case. In section 4

2. **Preliminaries**

Let $X = \{x_1, ..., x_n\}$ be a finite set and P be a probability function defined on all subsets of X with $P(\{x_i\}) = k_i,\ 0 < k_i < 1,\ 1 \leq i \leq n$, and $\sum_{i=1}^{n} k_i = 1$. X together with P is a discrete (finite) probability function. If B is a subset of X, we have $P(B) = \sum_{x_i \in B} P(\{x_i\})$. In practice, all the k_is must be known exactly. Often these values are estimated, or they are provided by experts. We now assume that the values of the k_is are uncertain and model this uncertainty using fuzzy set theory (see Buckley (2003)).

Definition 2.1. [5] The fuzzy subset $\widetilde{A}$, of real line, with the membership function $\mu_{\widetilde{A}} : \mathbb{R} \to [0, 1]$ is a fuzzy number if and only if (a) $\widetilde{A}$ is normal , (b) $\widetilde{A}$ is fuzzy convex, (c) $\mu_{\widetilde{A}}$ is upper semicontinuous, and (d) $\mathrm{supp}(\widetilde{A})$ is bounded.

Definition 2.2. [5] A trapezoidal fuzzy number $\widetilde{A}$ is a fuzzy number whose membership function is defined by four values, $a_1 \leq a_2 \leq a_3 \leq a_4$. The base of the trapezoid is the interval $[a_1, a_4]$ and its top boundary (where the membership equals one) is the line segment $[a_2, a_3]$.Hence we may write its membership function $\mu_{\widetilde{A}}$ as follows:

$$\mu_{\widetilde{A}}(x) = \begin{cases} 0 & , \quad x < a_1, \\ \frac{x-a_1}{a_2-a_1} & , \quad a_1 \leq x < a_2, \\ 1 & , \quad a_2 \leq x \leq a_3, \\ \frac{a_4-x}{a_4-a_3} & , \quad a_3 < x \leq a_4, \\ 0 & , \quad a_4 < x. \end{cases} \tag{1}$$

Trapezoidal fuzzy numbers with $a_2 = a_3$ are called triangular fuzzy numbers.

Definition 2.3. [5] The α-cut of a fuzzy number $\widetilde{A}$ is a non-fuzzy set defined as $\widetilde{A}[\alpha] = \{x \in \mathbb{R},\, \mu_A(x) \geqq \alpha\}$. Hence, we have $\widetilde{A}[\alpha] = [A^L[\alpha]\ ,A^U[\alpha]]$, where $A^L[\alpha] = \inf\{x \in \mathbb{R},\, \mu_A(x) \geqq \alpha\}$ and $A^U[\alpha] = \sup\{x \in \mathbb{R},\, \mu_A(x) \geqq \alpha\}$.

Definition 2.4. [1] Suppose that $\widetilde{k}_i$, $i = 1, ..., k$ are fuzzy numbers and that the probability of the event $X = x_i$ equals $\widetilde{k}_i$. We say that the random variable X has a discrete fuzzy probability function. We write $\widetilde{P}$ for fuzzy P and $\widetilde{P}(\{x_i\}) = \widetilde{k}_i$,where the support of $\widetilde{k}_i$ is $[0, 1]$, and there are $k_i \in \widetilde{k}_i$ [1] such that $\sum_{i=1}^{n} k_i = 1$. That is, we can choose k_i in $\widetilde{k}_i[\alpha]$, all α, so that we get a discrete probability distribution. Let $B = \{x_1, ..., x_k\}$ be a subset of X and, for $0 \le \alpha \le 1$, define:

$$\widetilde{P}(B)[\alpha] = \{\sum_{i=1}^{k} k_i|S\}, \tag{2}$$

where S stands for the statement "$k_i \in \widetilde{k}_i[\alpha],\ 1 \le i \le n,\ \sum_{i=1}^{n} k_i = 1$," This is our restricted fuzzy arithmetic.

Definition 2.5. [2,19] Let p be the probability of a "success" in each Bernoulli trial; p is not known precisely. We substitute $\widetilde{p}$ instead of p and $\widetilde{q}$ for q, so that there is a $p \in \widetilde{p}[1]$ and a $q \in \widetilde{q}[1]$ with $p + q = 1$. Now let $\widetilde{P}(r)$ be the fuzzy probability of r successes in m independent trials of the experiment. Under our restricted fuzzy algebra the fuzzy binomial probability mass function is defined as:

$$\widetilde{P}(r)[\alpha] = \{C_m^r p^r q^{m-r}|S\}, \qquad 0 \le \alpha \le 1, \tag{3}$$

where now S is the statement "$p \in \widetilde{p}[\alpha],\ q \in \widetilde{q}[\alpha],\ p+q = 1$" and $C_m^r = \frac{m!}{r!(m-r)!}$. If $\widetilde{P}(r)[\alpha] = [P^L[\alpha], P^U[\alpha]]$ then

$$P^L[\alpha] = min\{C_m^r p^r q^{m-r}|S\} \text{ and } P^U[\alpha] = max\{C_m^r p^r q^{m-r}|S\}. \tag{4}$$

If the membership function of $\widetilde{p}$ is as given in (1) then the membership function of $\widetilde{q}$ is as follows:

$$\mu_{\widetilde{q}}(x) = \begin{cases} 0 & , \quad x < 1 - a_4, \\ \frac{x-(1-a_4)}{a_4-a_3} & , \quad 1 - a_4 \le x < 1 - a_3, \\ 1 & , \quad 1 - a_3 \le x \le 1 - a_2, \\ \frac{(1-a_1)-x}{a_2-a_1} & , \quad 1 - a_2 < x \le 1 - a_1, \\ 0 & , \quad 1 - a_1 < x. \end{cases} \tag{5}$$

Example 2.6. Suppose that $m = 3$, $\widetilde{p} = (0.2, 0.3, 0.3, 0.4)$("about 0.3") for p and $\widetilde{q} = (0.6, 0.7, 0.7, 0.8)$ for q. We will calculate the fuzzy probabilities $\widetilde{P}(3)$ and $\widetilde{P}(B)$, where $B = \{0, 1\}$. By Definition 2.5, we have

$$\widetilde{P}(3)[\alpha] = \{p^3|S\} = [P^L[\alpha], P^U[\alpha]],$$

where

$$P^L[\alpha] = min\{p^3|S\}, P^U[\alpha] = max\{p^3|S\},$$

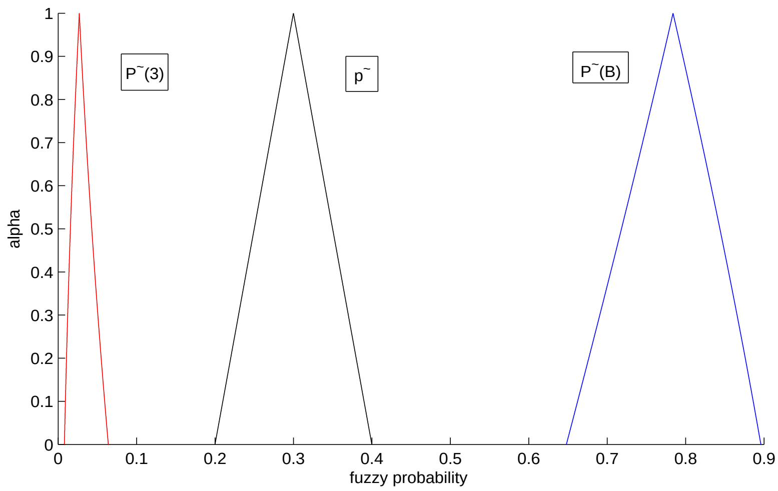

FIGURE 1. The Fuzzy Probabilities $\widetilde{P}(3), \widetilde{P}(B)$ and $\widetilde{p}$

since $\frac{d(p^3)}{dp} > 0$ on $\widetilde{p}[\alpha] = [p^L[\alpha], p^U[\alpha]] = [0.2 + 0.1\alpha, 0.4 - 0.1\alpha]$, we obtain

$$\widetilde{P}(3)[\alpha] = [(0.2 + 0.1\alpha)^3, (0.4 - 0.1\alpha)^3],$$

when $\alpha = 0$, $\widetilde{P}(3)[0] = [0.008, 0.064]$. Hence

$$P(B)[\alpha] = [P^L[\alpha], P^U[\alpha]],$$

where

$P^L[\alpha] = min\{(1-p)^3 + 3p(1-p)^2|S\}$ and $P^U[\alpha] = max\{(1-p)^3 + 3p(1-p)^2|S\}$.

Since $\frac{\partial((1-p)^3+3p(1-p)^2)}{\partial p} < 0$ on $\widetilde{p}[\alpha]$,

$\widetilde{P}(B)[\alpha] = [(1 - p^U[\alpha])^3 + 3p^U[\alpha](1 - p^U[\alpha])^2, (1 - p^L[\alpha])^3 + 3p^L[\alpha](1 - p^L[\alpha])^2]$.

when $\alpha = 0$, $\widetilde{P}(B)[0] = [0.648, 0.896]$.

3. An Acceptance Single Sampling Plan with Fuzzy Parameter

In this section, first we introduce the single sampling plan for classical attributes characteristics. Suppose that we want to inspect a lot of size N. First, we choose and inspect a random sample of size n, and count the number of defective items or damaged units (D). If the number of observed defective items (d) is less than or equal to the acceptance number c, the lot is accepted and otherwise it is rejected. If the size of the lot is very large, the random variable D has a binomial distribution with parameters n and p, where p is the proportion of the defective items in the lot. So, the probability for the number of defective items to exactly equal d is

$$P(D = d) = C_n^d p^d (1 - p)^{n-d}, \tag{6}$$

and hence the probability of acceptance of the lot is:

$$p_a = P(D \leq c) = \sum_{d=0}^{c} C_n^d p^d (1 - p)^{n-d}. \tag{7}$$

Suppose that we want to inspect a lot of size of N, where the proportion of defective items is not known precisely. We suppose that this parameter is the fuzzy number $\widetilde{p}$ as follows:

$$\widetilde{p} = (a_1, a_2, a_3, a_4) \quad , \quad \widetilde{p}[\alpha] = [a_1 + (a_2 - a_1)\alpha, a_4 - (a_4 - a_3)\alpha]. \tag{8}$$

A single sampling plan with a fuzzy parameter is defined by the sample size n, and acceptance number c, and if the number of observed defective items (D) is less than or equal to c, the lot will be accepted. If N is a large number, then the number of defective items in this sample has a fuzzy binomial probability distribution [2]. So for $0 \leq \alpha \leq 1$, the fuzzy probability that there will be exactly d defective items in a sample of size n, is

$$\widetilde{P}(D = d)[\alpha] = \{C_n^d p^d q^{n-d} | S\} = [P^L[\alpha], P^U[\alpha]], \tag{9}$$

$$P^L[\alpha] = \min\{C_n^d p^d q^{n-d} | S\} \text{ and } P^U[\alpha] = \max\{C_n^d p^d q^{n-d} | S\}, \tag{10}$$

where S stands for the statement "$p \in \widetilde{p}[\alpha],\ q \in \widetilde{q}[\alpha],\ p + q = 1$".

Example 3.1. Kaleh Company of Mazandaran produces and packages dairy productions**s**that approximately "between 1 and 2 percent" of the products have packaging problems. A customer who wants to buy one of these lots of boxes, must choose one box randomly. He will buy the lot, only if all the product units in that box are good. Since the proportion of defective products is described linguistically, we can consider it to be a fuzzy number $\tilde{p} = (0, 0.01, 0.02, 0.03)$. Accordingly, we have $\widetilde{q} = (0.97, 0.98, 0.99, 1)$. Then the α-cut of the fuzzy probability of lot acceptance is:

$$\widetilde{p}[\alpha] = [0.01\alpha, 0.03 - 0.01\alpha],$$
$$\widetilde{p}_a[\alpha] = \widetilde{P}(0)[\alpha] = \{(1 - p)^{10} | S\} = [(0.97 + 0.01\alpha)^{10}, (1 - 0.01\alpha)^{10}],$$

4. **FOC Band**

One important criterion for assessing a sampling plan, is its operating characteristic curve. This curve indicates the probability of a lot acceptance in terms of different values of the proportion of defective items. In other words, the *OC* curve is obtained according to the relation (8) and for different values of p. Using this curve, one may determine the probability of acceptance or rejection of a lot having a specific number defective items. *OC* curves are also used to compare the efficiency of different plans. Critical points, as well as the producer and customer risks may also be determined by the *OC* curve (see Montgomery (1991)). Suppose that the event B is the event of lot acceptance. Then the fuzzy probability of lot acceptance in terms of fuzzy proportion of defective items is a band with upper and lower bounds. We shall call this the fuzzy operating characteristic (*FOC*) band.

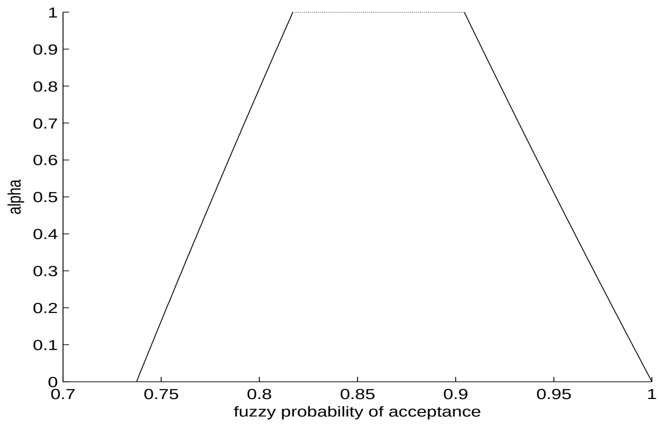

FIGURE 2. The Fuzzy Probability of Lot Acceptance for a Single Sampling Plan with Fuzzy Parameter of $c = 0, \widetilde{p} = (0, 0.01, 0.02, 0.03)$

One of the factors that the bandwidth depends on is the uncertainty value of the proportion parameter. A small uncertainty value results in a small bandwidth, and if the proportion parameter is crisp, the lower and upper bounds will become equal, the bandwidth is zero and we get the classical OC curve. Knowing the uncertainty value of the proportion parameter (given a_1, a_2, a_3, a_4) and the variation of its position on the horizontal axis, we have a fuzzy number ($\widetilde{p}$) for which the FOC band. To achieve this aim we consider the structure of $\widetilde{p}$ as follows:

$$\widetilde{p}_k = (k, b_2 + k, b_3 + k, b_4 + k),\ p_k \in \widetilde{p}_k[\alpha], q_k \in \widetilde{q}_k[\alpha],\ p_k + q_k = 1, \tag{11}$$

where $b_i = a_i - a_1, i = 2, 3, 4$, and $k \in [0, 1 - b_4]$. The α-cut of FOC band is plotted according to the values of the following fuzzy probability:

$$\widetilde{p}_k[\alpha] = [p_k^L[\alpha], p_k^U[\alpha]] = [k + b_2\alpha, b_4 + k - (b_4 - b_3)\alpha], \tag{12}$$

$$\widetilde{p}_{a,k} = \widetilde{P}_k(B)[\alpha] = [P_k^L[\alpha], P_k^U[\alpha]], \tag{13}$$

$$P_k^L[\alpha] = min\{\sum_B C_n^d p_k^d q_k^{n-d} | S\} \text{ and } P_k^U[\alpha] = max\{\sum_B C_n^d p_k^d q_k^{n-d} | S\}. \tag{14}$$

Example 4.1. For the Kaleh Company of Mazandaran in Example 3.1, we have assumed that $\widetilde{p} = (0.004, 0.01, 0.018, 0.024)$ then

$$\widetilde{p}_k[0] = [k, 0.02 + k],\ 0 \le k \le 0.98,$$

and

$$\widetilde{p}_{a,k} = \widetilde{P}_k(0)[0] = [(1 - p_k^U[\alpha])^n, (1 - p_k^L[\alpha])^n] = [(0.98 - k)^n, (1 - k)^n].$$

k	$\widetilde{p}_k[0]$	$\widetilde{P}_k(0)[0]$
0	$[0, 0.02]$	$[0.9039, 1]$
0.01	$[0.01, 0.03]$	$[0.8587, 0.951]$
0.02	$[0.02, 0.04]$	$[0.8154, 0.9039]$
0.03	$[0.03, 0.05]$	$[0.7738, 0.8587]$
0.04	$[0.04, 0.06]$	$[0.7339, 0.8154]$
0.05	$[0.05, 0.07]$	$[0.6957, 0.7738]$
0.06	$[0.06, 0.08]$	$[0.6591, 0.7339]$

Table 1. Fuzzy Probability of Acceptance with c=0 and n=5

Figure 3 shows two FOC bands for $n = 5$ and $n = 10$, indicating that FOC bands are convex for an acceptance number of zero. This leads to a quick reduction of the fuzzy probability of acceptance for the proportion of defective items with small fuzzy numbers, which increases with n.

Example 4.2. Assume that $c = 1$ in Example 4.1 In this case, the fuzzy probability of lot acceptance in terms of n and k is given by

$$\widetilde{p}_{a,k} = \widetilde{P}_k(D \leq 1)[0] = [(0.98-k)^n + n(0.02+k)(0.98-k)^{n-1}, (1-k)^n + nk(1-k)^{n-1}].$$

Figure 4 shows the FOC band of Example 4.2 for $n = 5$ and $n = 10$.

5. Conclusion

In this paper, we use fuzzy probability theory to solve problems of impreciseness arising in statistical quality control. It is clear that sampling plans are important in statistical quality control and we propose a method for designing acceptance single sampling plans with fuzzy parameters using fuzzy probability theory. These plans are well defined since, if the proportion of defective items is crisp, the results agree with classical plans. We also calculate the operating characteristic curve of a single sampling plan and its acceptance probability and finally, show that in our plan, the α-cut of the FOC band is such that for $c = 0$ when the value of $\widetilde{p}$ increases, the width of the band narrows down. When the acceptance number equals zero, this band is convex for different values of n, and the convexity increases with n. As a continuation of this research, we are at present studying the performance characteristics of rectifying inspection and other sampling plans in fuzzy environment.

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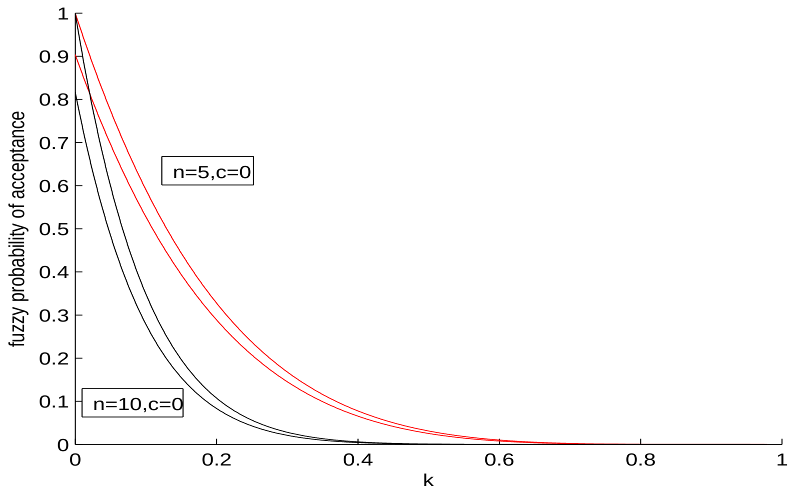

Figure 3. FOC Band for a Single Sampling Plan with Fuzzy Parameter of $c = 0$

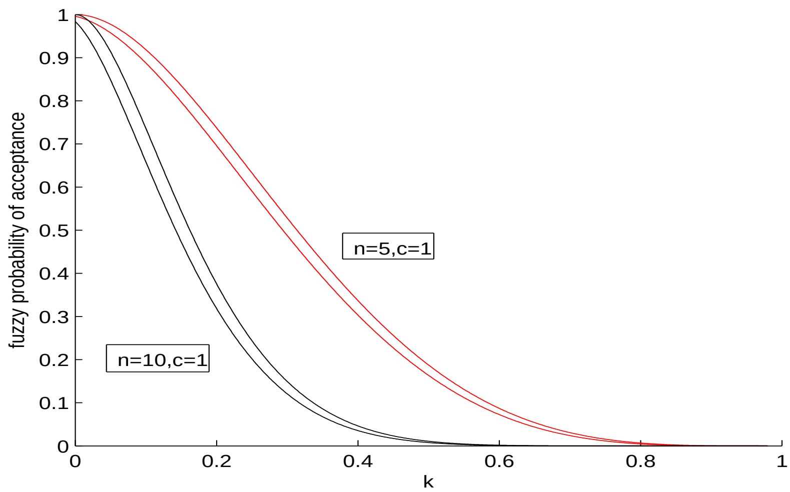

Figure 4. FOC Band for a Single Sampling Plan with Fuzzy Parameter of $c = 1$

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