ON THE MULTIVARIATE PROCESS CAPABILITY VECTOR IN FUZZY ENVIRONMENT

Z. ABBASI GANJI AND B. SADEGHPOUR GILDEH

Abstract. The production of a process is expected to meet customer demands, specifications or engineering tolerances. The ability of a process to meet these expectations is expresed as a single number using a process capability index. When the quality of the products relates to more than one characteristic, multivariate process capability indices are applied. As it is known, in some circumstances we are faced with imprecise data. So, fuzzy logic is engaged to deal with them. In this article, the specification limits and the target value of each characteristic and also, the data gathered from the process are assumed to be imprecise and a new fuzzy multivariate capability vector is introduced. As a whole, the present article provides a research of the application of fuzzy logic in multivariate capability vector.

1. Introduction

To evaluate process performance, one can use capability indices which are numerical quantities that show whether or not the process products meet the required specifications considered for them. When the products quality depends on one characteristic, we use univariate process capability indices (UPCIs) and when it relates to more than one characteristic, we utilize multivariate process capability indices (MPCIs). For more information, one can see [2, 12, 17, 19].

Shahriari and Abdollahzadeh [16] proposed a multivariate process capability vector with three components to assess the performance of the multivariate normal process. The first component of the vector is related to the process variability, the second one is based on the hypothesis testing about the process mean and the third component is concerned to the process shape in analogy with the tolerance region.

In real world, there are many cases in which the data are not precise. So, one can describe them based on fuzzy logic. UPCIs have been extensively aggregated with the fuzzy set theory in the literature. (see [6, 7, 8, 9, 10, 11, 13, 14, 15]). But MPCIs have received a lot less attention in this field comparatively than UPCIs. So, in this paper, we consider the situation in which the specification limits and

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the target value of each characteristic and also the data taken from the process are fuzzy numbers. Then, we propose fuzzy multivariate capability vector.

The organization of the rest of this paper is as follows. In the subsequent section, we review the multivariate capability vector introduced by Shahriari and Abdollahzadeh [16]. In section 3, we present fuzzy logic and basic definitions. We fuzzify the vector in section 4. Section 5 presents a numerical example to demonstrate the applicability of the proposed index and finally, conclusions are given in section 6.

2. Multivariate Process Capability Vector

Suppose the quality of the process products counts on p correlated characteristics $X_1, X_2, ..., X_p$ which are distributed as normal. So, $\mathbf{X} = (X_1, X_2, ..., X_p)'$ has multivariate normal distribution as $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ where, $\boldsymbol{\mu} = (\mu_1, \mu_2, ..., \mu_p)'$ is the mean vector and $\boldsymbol{\Sigma}$ is the variance-covariance matrix of $\mathbf{X}$ given as

$$\boldsymbol{\Sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \dots & \sigma_{1p} \\ \sigma_{21} & \sigma_{22} & \dots & \sigma_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{p1} & \sigma_{p2} & \dots & \sigma_{pp} \end{pmatrix}.$$

Let $[LSL_i, USL_i], i = 1, 2, ..., p$ be the tolerance interval for X_i. In two-dimension cases, these tolerance ranges compose a rectangular tolerance region. In higher dimensions, they form a hypercube. Also, suppose T_i be the target value for the i^{th} characteristic.

Shahriari and Abdollahzadeh [16] introduced a multivariate process capability vector with three components as $NMPCV = [NMC_{pm},\ PV,\ LI]$. The first componente, NMC_{pm}, is the p^{th} root of the fraction of the volume of the modified tolerance region (R_1) on the volume of the region of multivariate normal distribution which covers 99.73% of the process (R_2). They considered the modified tolerance region as the largest ellipsoid centered at the target with its axes parallel to the axes of the process ellipsoid and completely within the actual tolerance region, which is defined as $(\mathbf{X}-\mathbf{T})'\boldsymbol{\Sigma}^{-1}(\mathbf{X}-\mathbf{T}) \leq c'^2$, where c' is constant. Furthermore, the region that covers 99.73% of the process is defined as $(\mathbf{X}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{X}-\boldsymbol{\mu}) \leq \chi^2_{p,0.0027}$. So, the volumes are as what follows;

$$vol(R_1) = \mid \boldsymbol{\Sigma} \mid^{\frac{1}{2}} (\pi c'^2)^{\frac{p}{2}} [\Gamma(\frac{p}{2}+1)]^{-1}, \tag{1}$$

$$vol(R_2) = \mid \boldsymbol{\Sigma} \mid^{\frac{1}{2}} (\pi \chi^2_{p,0.0027})^{\frac{p}{2}} [\Gamma(\frac{p}{2}+1)]^{-1}. \tag{2}$$

Therefore, NMC_{pm} is defined as the p^{th} root of the two volumes as the following;

$$NMC_{pm} = \left[\frac{vol(R_1)}{vol(R_2)}\right]^{\frac{1}{p}} = \frac{c'}{\sqrt{\chi^2_{p,0.0027}}}, \tag{3}$$

where

$$c' = \min\big\{ \min\{\frac{USL_i - T_i}{\sqrt{\sigma_{ii}}},\ \frac{T_i - LSL_i}{\sqrt{\sigma_{ii}}}\};\ i = 1, 2, ..., p\big\}, \tag{4}$$

and $\chi^2_{p,0.0027}$ is the upper 0.27 percentile of chi-square distribution with p degrees of freedom.

Now, let $\mathbf{X}_1, \mathbf{X}_2, ..., \mathbf{X}_n$ be a random sample vector of the process $\mathbf{X}$. The sample mean vector is $\bar{\mathbf{X}}$ which its elements are the sample mean of each characteristic and sample variance-covariance matrix is as

$$\mathbf{S} = \begin{pmatrix} s_{11} & s_{12} & \dots & s_{1p} \\ s_{21} & s_{22} & \dots & s_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ s_{p1} & s_{p2} & \dots & s_{pp} \end{pmatrix},$$

where s_{ij} is the sample covariance of the characteristics i and j for $i, j = 1, 2, ..., p$. Therefore, the index NMC_{pm} is estimated as the following;

$$\widehat{NMC}_{pm} = \frac{\widehat{c'}}{\sqrt{\chi^2_{p,0.0027}}}, \tag{5}$$

where

$$\widehat{c'} = \min\big\{ \min\{\frac{USL_i - T_i}{\sqrt{s_{ii}}},\ \frac{T_i - LSL_i}{\sqrt{s_{ii}}}\};\ i = 1, 2, ..., p\big\}. \tag{6}$$

The second component of the vector is based on the hypothesis testing on the assumption of equality of the process mean vector and the target vector, and is defined the significant level of the observed value as what follows;

$$PV = P_r\bigg(F_{p,n-p} > \frac{n-p}{p(n-1)}t^2\bigg), \tag{7}$$

where

$$t^2 = n\big(\bar{\mathbf{X}} - \boldsymbol{T}\big)'\mathbf{S}^{-1}\big(\bar{\mathbf{X}} - \boldsymbol{T}\big), \tag{8}$$

and $F_{p,n-p}$, denotes a random variable having F distribution with p, $n-p$ degrees of freedom. When the process mean is far from the target, PV is close to zero and moving it towards the target makes PV larger to be near one.

The third component (LI), taking values 0 or 1, compares the location of the region that covers 99.73% of the process with the tolerance region. It takes the value 1 if the whole region that covers 99.73% of the process is contained within the tolerance region and takes the value 0 otherwise.

A process is said to be “capable”if NMC_{pm} exceeds one, PV exceeds 0.05 and LI is equal to one.

3. **Fuzzy Logic and Basic Definitions**

Fuzzy logic is a form of many-valued logic which deals with reasoning that is approximate rather than fixed and exact. This concept which was introduced by Zadeh [18] to manipulate the data possesing nonstatistical uncertainties, has many succesful applications. Here, we present some definitions.

Definition 3.1. LR-fuzzy Number (Quantity). A fuzzy number (quantity) with the following membership function is named LR-fuzzy number (quantity)

$$\tilde{A}(t) = \begin{cases} L(\frac{a-t}{\alpha}) & t \in [a-\alpha, a], \\ 1 & t \in [a, b], \\ R(\frac{t-b}{\beta}) & t \in [b, b+\beta], \\ 0 & otherwise. \end{cases}$$

where $L : [0, 1] \to [0, 1]$ and $R : [0, 1] \to [0, 1]$, are continuous and non-increasing shape functions with $L(0) = R(0) = 1$ and $L(1) = R(1) = 0$. We represent these numbers (quantities) as $\tilde{A} =< a, b, \alpha, \beta >_{LR}$.

Definition 3.2. Trapezoidal Fuzzy Quantity. An LR-fuzzy quantity by the case $L(x) = R(x) = 1 - x$ is trapezoidal fuzzy quantity and is denoted by $\tilde{A} =< a, b, \alpha, \beta >$. We note that a, b are called center parts and α and β are called spread parts.

Definition 3.3. Triangular Fuzzy Number. An LR-fuzzy number by the case $a = b$ and $L(x) = R(x) = 1 - x$ is triangular fuzzy number and is denoted by $\tilde{A} =< a, \alpha, \beta >$, a is called center part and α and β are called spread parts. A triangular fuzzy number is said to be symmetric if $\alpha = \beta$ and is denoted by $\tilde{A} =< a, \alpha >$.

Because of the wide field of applications of triangular fuzzy numbers, our research in this article is based on these numbers. To simplify caculations, we work with symmetric triangular fuzzy numbers.

Definition 3.4. Addition and Subtraction. Addition of two triangular fuzzy numbers $\tilde{A} =< a_1, \alpha_1, \beta_1 >$ and $\tilde{B} =< a_2, \alpha_2, \beta_2 >$ is triangular fuzzy number $\tilde{A} \oplus \tilde{B} =< a_1 + a_2, \alpha_1 + \alpha_2, \beta_1 + \beta_2 >$. Furthermore, subtraction of these numbers is also fuzzy triangular fuzzy number as $\tilde{A} \ominus \tilde{B} =< a_1 - a_2, \alpha_1 + \beta_2, \beta_1 + \alpha_2 >$.

Definition 3.5. Scalar Multipication. Scalar multipication for a triangular fuzzy number $\tilde{A} =< a, \alpha, \beta >$ is triangular fuzzy number, which depends on the sign of the scalar λ as the following;

$$\lambda \otimes \tilde{A} = \begin{cases} < \lambda a, \lambda\alpha, \lambda\beta > & \lambda \geq 0, \\ < \lambda a, -\lambda\beta, -\lambda\alpha > & \lambda < 0. \end{cases}$$

Definition 3.6. Triangular Fuzzy One Number. The triangular fuzzy number with $a = 1$ and $0 < \alpha,\ \beta < 1$ is called triangular fuzzy one number and is denoted by $\tilde{1} =< 1, \alpha, \beta >$. In symmetric case, it is shown by $\tilde{1} =< 1, \alpha >$.

Definition 3.7. Triangular Fuzzy Zero Number. The triangular fuzzy number with $a = 0$ and $0 < \alpha,\ \beta < 1$ is called triangular fuzzy zero number and is denoted by $\tilde{0} =< 0, \alpha, \beta >$ and in symmetric case, is shown by $\tilde{0} =< 0, \alpha >$.

Definition 3.8. Triangular Fuzzy Matrix. A triangular fuzzy matrix of order $m \times n$ is a matrix of order $m \times n$ with fuzzy triangular numbers entries, that is, $\tilde{M} = (\tilde{m}_{ij})_{m\times n}$, where $\tilde{m}_{ij} =< a_{ij}, \alpha_{ij}, \beta_{ij} >$.

In this paper, we focus our attention on the case in which matrix elements are symmetric fuzzy triangular numbers, i.e., $\tilde{m}_{ij} =< a_{ij}, \alpha_{ij} >$. When $m = n$, the matrix is called square.

Definition 3.9. Unit Triangular Fuzzy Matrix. A square triangular fuzzy matrix is called unit triangular fuzzy matrix (triangular fuzzy identity matrix) and is denoted by $\tilde{I}$, if its main diagonal elements are triangular fuzzy one numbers and the others are triangular fuzzy zero numbers. In other words, $\tilde{m}_{ii} = \tilde{1}$ and $\tilde{m}_{ij} = \tilde{0}$ for $i \neq j$, and for all i, j.

Definition 3.10. Inverse of Fuzzy Matrix. The inverse of the fuzzy matrix $\tilde{M}$ is fuzzy matrix $\tilde{M}^{-1}$, such that $\tilde{M} \otimes \tilde{M}^{-1} \approx \tilde{I}$, where $\tilde{I}$ is fuzzy identity matrix.

To obtain $\tilde{M}^{-1}$, some methods have been proposed. For example, Dehghan, et. al [3] proposed two techniques, scenario-based and arithmetic-based. Basaran [1] pursued fuzzy linear equation system which we employed it here.

Definition 3.11. Ranking Fuzzy Numbers. The fuzzy number $\tilde{A}$ is greater than or equal to the fuzzy number $\tilde{B}$, noted $\tilde{A} \geq_R \tilde{B}$, if and only if $C(\tilde{A} \geq \tilde{B}) \geq 0$, or equivalently, $C(\tilde{B} \geq \tilde{A}) \leq 0$. Similarly, $\tilde{A}$ is greater than $\tilde{B}$, noted $\tilde{A} >_R \tilde{B}$, if and only if $C(\tilde{A} \geq \tilde{B}) > 0$, equivalently, $C(\tilde{B} \geq \tilde{A}) < 0$, where

$$C(\tilde{A} \geq \tilde{B}) = R(\tilde{A}) - R(\tilde{B}),$$

and

$$R(\tilde{A}) = \frac{1}{2} \int_0^1 \big(\tilde{A}_l(\alpha) + \tilde{A}_r(\alpha)\big) d\alpha.$$

This ranking method proposed by Fortemps and Roubens [4], which induced a complete ranking of all fuzzy numbers and corresponds to the defuzzification function R.

4. Fuzzy Multivariate Capability Vector

In this section, we consider the specification limits and the target value of each characteristic and the data taken from the process are fuzzy numbers and then, fuzzify the multivariate capability vectore mentioned earlier.

4.1. Fuzzy Specification Limits And Target Value.

Suppose fuzzy lower and upper specification limits are as $\widetilde{LSL} =< l, \beta >$, $\widetilde{USL} =< u, \gamma >$. Furthermore, fuzzy target value is $\widetilde{T} =< t, \delta >$. So, the α-cut sets are as follows;

$$\begin{aligned} \widetilde{LSL}(\alpha) &= \big[L_l(\alpha),\ \ L_r(\alpha)\big] = \big[l - (1-\alpha)\beta,\ \ l + (1-\alpha)\beta\big], \\ \widetilde{USL}(\alpha) &= \big[U_l(\alpha),\ \ U_r(\alpha)\big] = \big[u - (1-\alpha)\gamma,\ \ u + (1-\alpha)\gamma\big], \\ \widetilde{T}(\alpha) &= \big[T_l(\alpha),\ \ T_r(\alpha)\big] = \big[t - (1-\alpha)\delta,\ \ t + (1-\alpha)\delta\big]. \end{aligned} \tag{9}$$

4.2. Fuzzy Mean and Fuzzy Variance-covariance Matrix.

Let fuzzy mean vector is as $\tilde{\boldsymbol{\mu}} = \big(\tilde{\mu}_1, \tilde{\mu}_2, ..., \tilde{\mu}_p\big)'$, where $\tilde{\mu}_i =< \mu_i, \eta_i >$, for $i = 1, 2, ..., p$. Also, fuzzy variance-covariance matrix is as the following;

$$\tilde{\boldsymbol{\Sigma}} = \begin{pmatrix} \tilde{\sigma}_{11} & \tilde{\sigma}_{12} & \dots & \tilde{\sigma}_{1p} \\ \tilde{\sigma}_{21} & \tilde{\sigma}_{22} & \dots & \tilde{\sigma}_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{\sigma}_{p1} & \tilde{\sigma}_{p2} & \dots & \tilde{\sigma}_{pp} \end{pmatrix}, \tag{10}$$

where $\tilde{\sigma}_{ij} =< \sigma_{ij}, \vartheta_{ij} >$, for $i, j = 1, 2, ..., p$. So, α-cut sets of each element of the mean vector and similarly, of the variance-covariance matrix is as

$$\begin{aligned} \tilde{\mu}_i(\alpha) &= \big[\mu_{il}(\alpha), \ \ \mu_{ir}(\alpha)\big] = \big[\mu_i - (1-\alpha)\eta_i, \ \ \mu_i + (1-\alpha)\eta_i\big], \\ \tilde{\sigma}_{ij}(\alpha) &= \big[\sigma_{ijl}(\alpha), \ \ \sigma_{ijr}(\alpha)\big] = \big[\sigma_{ij} - (1-\alpha)\vartheta_{ij}, \ \ \sigma_{ij} + (1-\alpha)\vartheta_i\big]. \end{aligned} \quad (11)$$

Based on the sample data $\tilde{x}_{11}, \tilde{x}_{12}, ..., \tilde{x}_{1n}, \tilde{x}_{21}, \tilde{x}_{22}, ..., \tilde{x}_{2n}, ..., \tilde{x}_{p1}, \tilde{x}_{p2}, ..., \tilde{x}_{pn}$, fuzzy sample mean vector is as $\tilde{\bar{\mathbf{X}}} = \big(\tilde{\bar{x}}_1, \tilde{\bar{x}}_2, ..., \tilde{\bar{x}}_p\big)'$, where $\tilde{\bar{x}}_i =< \bar{x}_i, \epsilon_i >$, for $i = 1, 2, ..., p$. Moreover, the fuzzy sample variance-covariance matrix is as follows;

$$\tilde{\mathbf{S}} = \begin{pmatrix} \tilde{s}_{11} & \tilde{s}_{12} & \dots & \tilde{s}_{1p} \\ \tilde{s}_{21} & \tilde{s}_{22} & \dots & \tilde{s}_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{s}_{p1} & \tilde{s}_{p2} & \dots & \tilde{s}_{pp} \end{pmatrix}, \quad (12)$$

where $\tilde{s}_{ij} =< s_{ij}, r_{ij} >$, for $i, j = 1, 2, ..., p$. Therefore, α-cut sets of each element of the sample mean vector and similarly, of the sample variance-covariance matrix is as the following;

$$\begin{aligned} \tilde{\bar{x}}_i(\alpha) &= \big[\bar{x}_{il}(\alpha), \ \ \bar{x}_{ir}(\alpha)\big] = \big[\bar{x}_i - (1-\alpha)\epsilon_i, \ \ \bar{x}_i + (1-\alpha)\epsilon_i\big], \\ \tilde{s}_{ij}(\alpha) &= \big[s_{ijl}(\alpha), \ \ s_{ijr}(\alpha)\big] = \big[s_{ij} - (1-\alpha)r_{ij}, \ \ s_{ij} + (1-\alpha)r_i\big]. \end{aligned} \quad (13)$$

To compute the sample mean of each characteristic, we use the following equations;

$$\tilde{\bar{x}}_i = \frac{1}{n} \otimes \big(\tilde{x}_{i1} \oplus \tilde{x}_{i2} \oplus ... \oplus \tilde{x}_{in}\big) \ ; \ i = 1, 2, ..., p. \quad (14)$$

Since $\tilde{x}_{ik}(\alpha) = \big(x_{ikl}(\alpha), x_{ikr}(\alpha)\big)$; $k = 1, 2, ..., n$, the α-cut sets of the sample mean of i^{th} characteristic is obtained as

$$\begin{aligned} \bar{x}_{il}(\alpha) &= \frac{\sum_{k=1}^n x_{ikl}(\alpha)}{n}, \\ \bar{x}_{ir}(\alpha) &= \frac{\sum_{k=1}^n x_{ikr}(\alpha)}{n}. \end{aligned} \quad (15)$$

Furthermore, each element of the sample variance-covariance matrix is gain as what follows;

$$\begin{aligned} \tilde{s}_{ij} &= \frac{\bigoplus_{k=1}^n \big((\tilde{x}_{ik} \ominus \tilde{\bar{x}}_i) \otimes (\tilde{x}_{jk} \ominus \tilde{\bar{x}}_j)\big)}{n-1}; \qquad i, j = 1, 2, ..., p \\ &= \frac{1}{n-1} \otimes \big((\tilde{x}_{i1} \ominus \tilde{\bar{x}}_i) \otimes (\tilde{x}_{j1} \ominus \tilde{\bar{x}}_j) \oplus ... \oplus ((\tilde{x}_{in} \ominus \tilde{\bar{x}}_i) \otimes (\tilde{x}_{jn} \ominus \tilde{\bar{x}}_j)\big). \end{aligned} \quad (16)$$

The α-cut set of the difference between each member of the sample of the characteristic i and its sample mean is as

$$\big(\tilde{x}_{ik} \ominus \tilde{\bar{x}}_i\big)(\alpha) = \big(\tilde{x}_{ikl}(\alpha) - \tilde{x}_{ir}(\alpha), \ \tilde{x}_{irl}(\alpha) - \tilde{x}_{il}(\alpha)\big); \ k = 1, 2, ..., n. \quad (17)$$

Hence, the α-cut sets of the elements of the the sample variance-covariance matrix depend on the sign of the lower and upper bounds of the above equation.

4.3. **Fuzzy Estimation of** $NMPCV$**.** In order to estimate $NMPCV$ according to the fuzzy set theory, it is needed to estimate NMC_{pm}, PV and LI. The factor c' in the first component is estimated as

$$\widetilde{c'} = \min\big\{ \min\{\frac{\widetilde{USL_i} \ominus \widetilde{T}_i}{\sqrt{\widetilde{\sigma}_{ii}}}, \ \frac{\widetilde{T}_i \ominus \widetilde{LSL_i}}{\sqrt{\widetilde{\sigma}_{ii}}}\}; \ i = 1, 2, ..., p\big\}, \tag{18}$$

where $\widetilde{\sigma}_{ii}$ is the fuzzy variance of the characteristic i. Based on the sample data taken from the process, we have

$$\widehat{\widetilde{c'}} = \min\big\{ \min\{\frac{\widetilde{USL_i} \ominus \widetilde{T}_i}{\sqrt{\widetilde{s}_{ii}}}, \ \frac{\widetilde{T}_i \ominus \widetilde{LSL_i}}{\sqrt{\widetilde{s}_{ii}}}\}; \ i = 1, 2, ..., p\big\}, \tag{19}$$

where $\tilde{s}_{ii}$ is the sample fuzzy variance of the characteristic i.

To obtain $\widehat{\widetilde{c'}}$, first we get α-cut intervals of each factor in equation (19) as follows;

$$\begin{aligned} \frac{\widetilde{USL_i} \ominus \widetilde{T}_i}{\sqrt{\tilde{s}_{ii}}}(\alpha) &= \Big[\frac{U_{il}(\alpha) - T_{ir}(\alpha)}{\sqrt{s_{iir}(\alpha)}}, \ \frac{U_{ir}(\alpha) - T_{il}(\alpha)}{\sqrt{s_{iil}(\alpha)}}\Big], \\ \frac{\widetilde{T}_i \ominus \widetilde{LSL_i}}{\sqrt{\tilde{s}_{ii}}}(\alpha) &= \Big[\frac{T_{il}(\alpha) - L_{ir}(\alpha)}{\sqrt{s_{iir}(\alpha)}}, \ \frac{T_{ir}(\alpha) - L_{il}(\alpha)}{\sqrt{s_{iil}(\alpha)}}\Big]. \end{aligned} \tag{20}$$

Then, we apply ranking function to obtain $\widehat{\widetilde{c'}}(\alpha)$. Now place these α-cut intervals, one on top of the other to produce fuzzy number $\widehat{\widetilde{c'}}$, and dividing by $\sqrt{\chi^2_{p,0.0027}}$, we get fuzzy estimation of NMC_{pm} as the following;

$$\widehat{\widetilde{NMC}}_{pm} = \frac{\widehat{\widetilde{c'}}}{\sqrt{\chi^2_{p,0.0027}}}. \tag{21}$$

The $\widetilde{NMC}_{pm}$ would be at least approximately one to assess whether the process variation is running stable. To make a comparison between $\widetilde{NMC}_{pm}$ and $\tilde{1}$ we apply ranking function.

Now we fuzzify the second component, PV. To do this, first we should obtain inverse of the fuzzy sample variance-covariance matrix.

As it was mentioned earlier, the inverse of $\tilde{\mathbf{S}}$ is said to be $\tilde{\mathbf{S}}^{-1}$ as $\tilde{\mathbf{S}} \otimes \tilde{\mathbf{S}}^{-1} \approx \tilde{I}$. Set $\tilde{\mathbf{S}}^{-1}$ as the following;

$$\tilde{\mathbf{S}}^{-1} = \begin{pmatrix} \tilde{s}'_{11} & \tilde{s}'_{12} & \dots & \tilde{s}'_{1p} \\ \tilde{s}'_{21} & \tilde{s}'_{22} & \dots & \tilde{s}'_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{s}'_{p1} & \tilde{s}'_{p2} & \dots & \tilde{s}'_{pp} \end{pmatrix}. \tag{22}$$

In order to compute $\tilde{\mathbf{S}}^{-1}$, we place the equations (12) and (22) in the relation as follows;

$$\begin{pmatrix} \tilde{s}_{11} & \tilde{s}_{12} & \dots & \tilde{s}_{1p} \\ \tilde{s}_{21} & \tilde{s}_{22} & \dots & \tilde{s}_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{s}_{p1} & \tilde{s}_{p2} & \dots & \tilde{s}_{pp} \end{pmatrix} \otimes \begin{pmatrix} \tilde{s}'_{11} & \tilde{s}'_{12} & \dots & \tilde{s}'_{1p} \\ \tilde{s}'_{21} & \tilde{s}'_{22} & \dots & \tilde{s}'_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{s}'_{p1} & \tilde{s}'_{p2} & \dots & \tilde{s}'_{pp} \end{pmatrix} \approx \begin{pmatrix} \tilde{1} & \tilde{0} & \dots & \tilde{0} \\ \tilde{0} & \tilde{1} & \dots & \tilde{0} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{0} & \tilde{0} & \dots & \tilde{1} \end{pmatrix}. \tag{23}$$

Then, we have;

$$\begin{aligned}
(\tilde{s}_{11}\otimes\tilde{s}'_{11})\oplus(\tilde{s}_{12}\otimes\tilde{s}'_{21})\oplus\ldots\oplus(\tilde{s}_{1p}\otimes\tilde{s}'_{p1}) &= \tilde{1}\\
(\tilde{s}_{21}\otimes\tilde{s}'_{11})\oplus(\tilde{s}_{22}\otimes\tilde{s}'_{21})\oplus\ldots\oplus(\tilde{s}_{2p}\otimes\tilde{s}'_{p1}) &= \tilde{0}\\
\vdots\\
(\tilde{s}_{p1}\otimes\tilde{s}'_{1p})\oplus(\tilde{s}_{p2}\otimes\tilde{s}'_{2p})\oplus\ldots\oplus(\tilde{s}_{pp}\otimes\tilde{s}'_{pp}) &= \tilde{1}.
\end{aligned} \tag{24}$$

By setting $\tilde{s}_{ij} =< s_{ij}, r_{ij} >$, $\tilde{s}'_{ij} =< s'_{ij}, w_{ij} >$, $\tilde{1} =< 1, \delta >$ and $\tilde{0} =< 0, \gamma >$, we have the following equations;

$$\begin{aligned}
< s_{11}, r_{11} >< s'_{11}, w_{11} > \;\oplus\; & < s_{12}, r_{12} >< s'_{21}, w_{21} > \oplus\ldots\oplus\\
& < s_{1p}, r_{1p} >< s'_{p1}, w_{p1} > =< 1, \delta >\\
< s_{21}, r_{21} >< s'_{11}, w_{11} > \;\oplus\; & < s_{22}, r_{22} >< s'_{21}, w_{21} > \oplus\ldots\oplus\\
& < s_{2p}, r_{2p} >< s'_{p1}, w_{p1} > =< 0, \gamma >\\
& \vdots\\
< s_{p1}, r_{p1} >< s'_{1p}, w_{1p} > \;\oplus\; & < s_{p2}, r_{p2} >< s'_{2p}, w_{2p} > \oplus\ldots\oplus\\
& < s_{pp}, r_{pp} >< s'_{pp}, w_{pp} > =< 1, \delta > .
\end{aligned} \tag{25}$$

By Using the multiplication of fuzzy systems approximation, the above equations are divided into two systems of equations which one is related to the center part and the other one relates to the spread part. The equations related to the center part is written as follows;

$$\begin{aligned}
s_{11}s'_{11}+s_{12}s'_{21}+\ldots+s_{1p}s'_{p1} &= 1\\
s_{21}s'_{11}+s_{22}s'_{21}+\ldots+s_{2p}s'_{p1} &= 0\\
\vdots\\
s_{p1}s'_{1p}+s_{p2}s'_{2p}+\ldots+s_{pp}s'_{pp} &= 1.
\end{aligned} \tag{26}$$

The following equations are related to the spread part

$$\begin{aligned}
s_{11}w_{11}+s'_{11}r_{11}+s_{12}w_{21}+s'_{21}r_{12}+\ldots+s_{1p}w_{p1}+s'_{p1}r_{1p} &= \delta\\
s_{21}w_{11}+s'_{11}r_{21}+s_{22}w_{21}+s'_{21}r_{22}+\ldots+s_{2p}w_{p1}+s'_{p1}r_{2p} &= \gamma\\
\vdots\\
s_{p1}w_{1p}+s'_{1p}r_{p1}+s_{p2}w_{2p}+s'_{2p}r_{p2}+\ldots+s_{pp}w_{pp}+s'_{pp}r_{pp} &= \delta.
\end{aligned} \tag{27}$$

By solution the above equations, we obtain s'_{ij} and w_{ij}, for $i, j = 1, 2, ..., p$ and then get $\tilde{\mathbf{S}}^{-1}$.

Now we obtain $\widetilde{PV}$. First, we get its α-cut intervals and then by placing these intervals, one on top of the other, obtain $\widetilde{PV}$. So,

$$\widetilde{PV}(\alpha) = \big[PV_l(\alpha), \quad PV_r(\alpha)\big]. \tag{28}$$

Based on the equations (7) and (8), the right and the left end points of the α-cut interval of $\widetilde{PV}$ is as what follows;

$$PV_l(\alpha) = \min \quad \Bigg\{ \quad P_r\Bigg(F_{p,n-p} > \frac{n(n-p)}{p(n-1)} \begin{pmatrix} m_1 - n_1 \\ m_2 - n_2 \\ \vdots \\ m_p - n_p \end{pmatrix}' \begin{pmatrix} q_{11} & q_{12} & \dots & q_{1p} \\ q_{p1} & q_{22} & \dots & q_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ q_{p1} & q_{p2} & \dots & q_{pp} \end{pmatrix}$$

$$\times \quad \begin{pmatrix} m_1 - n_1 \\ m_2 - n_2 \\ \vdots \\ m_p - n_p \end{pmatrix} \Bigg); \quad m_i \in \tilde{\bar{x}}_{ik}(\alpha),\ n_i \in \widetilde{T}_{ik}(\alpha),\ q_{ij} \in \tilde{s}'_{ijk}(\alpha),$$

$$k = \{l,\ r\},\ i, j = \{1, 2, ..., p\}\Bigg\}, \tag{29}$$

and

$$PV_r(\alpha) = \max \quad \Bigg\{ \quad P_r\Bigg(F_{p,n-p} > \frac{n(n-p)}{p(n-1)} \begin{pmatrix} m_1 - n_1 \\ m_2 - n_2 \\ \vdots \\ m_p - n_p \end{pmatrix}' \begin{pmatrix} q_{11} & q_{12} & \dots & q_{1n} \\ q_{21} & q_{22} & \dots & q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ q_{p1} & q_{p2} & \dots & q_{nn} \end{pmatrix}$$

$$\times \quad \begin{pmatrix} m_1 - n_1 \\ m_2 - n_2 \\ \vdots \\ m_p - n_p \end{pmatrix} \Bigg); \quad m_i \in \tilde{\bar{x}}_{ik}(\alpha)\ n_i \in \widetilde{T}_{ik}(\alpha),\ q_{ij} \in \tilde{s}'_{ijk}(\alpha),$$

$$k = \{l,\ r\},\ i, j = \{1, 2, ..., p\}\Bigg\}. \tag{30}$$

Several answers are obtained from the above equations but we choose the ones satisfy in the following condition;

$$PV_l(\beta) \leq PV_l(\gamma) \leq PV_l(1) = PV_r(1) \leq PV_r(\gamma) \leq PV_r(\beta), \tag{31}$$

for all $\beta, \gamma \in [0, 1],\ \beta < \gamma$.

Hence, $\widetilde{PV}$ is constructed by its α-cut intervals. To make a decision, first we detect a degree of uncertainty such $\lambda \in (0, 1)$. Therefore, we have a three decision problem as follows

- If $PV_l(\lambda) > 0.05$, then the process mean vector is not far from the target vector.
- If $PV_r(\lambda) < 0.05$, then the process mean vector is far from the target vector.
- If $PV_l(\lambda) \leq 0.05 \leq PV_r(\lambda)$, then we can not come to a clear decision. In such cases, one may take more samples and follow the procedure until making a decision.

For the third component, we defuzzify specification limits, target value, sample mean and variance for each characteristic and all covariances between characteristics. To do this, we utilize the ranking function mentioned earlier. So, by the geometry of the tolerance and the process ellipses we can obtain this component.

It should be noted that for the circumstances in which the data are not vague, i.e., the data are crisp, $\epsilon_i = r_{ij} = 0$ for $i, j = 1, 2, ..., p$. Then, we have the crisp sample mean vector and sample variance-covariance matrix as $\bar{X}$ and S. Hence, we use 1 instead of $\tilde{1}$ and 0 instead of $\tilde{0}$. That is, $\delta = \gamma = 0$ and so, we employ I instead of $\tilde{I}$. Therefore, S^{-1} is obtained instead of $\widetilde{\tilde{S}^{-1}}$.

Furthermore, in all fuzzy numbers if $\alpha = 1$, $\widetilde{\widehat{NMC}}_{pm}$ transforms to $\widehat{NMC}_{pm}$, as equation (5). For $\widetilde{PV}$ based on equations (29) and (30), we use $\bar{x}_i$ instead of m_i, T_i instead of n_i and s_{ij} instead of q_{ij}. Therefore, this index transforms to PV as equation (7). As a whole, the new index reduces to $NMPCV$.

5. A Numerical Example

In this section, we apply our proposed index to assess the capability of a film-developing solution process discussed by Jackson [5]. Quality of this process products depends on two characteristics, Elon and Hydroguinone. Suppose the specification limits and the target value of each characteristic are symmetric triangular fuzzy numbers as shown in Table 1.

Based on a random sample of size 75, fuzzy sample mean vector and fuzzy sample variance-covariance matrix are as follows;

$$\tilde{\tilde{\mathbf{X}}} = \begin{pmatrix} < 264.32, 1 > \\ < 471.48, 1 > \end{pmatrix}, \qquad \tilde{\mathbf{S}} = \begin{pmatrix} < 102.65, 1 > & < 68.87, 1 > \\ < 68.87, 1 > & < 107.96, 1 > \end{pmatrix}.$$

	Elon	Hydroguinone
$\widetilde{USL}$	$< 295, 1 >$	$< 500, 1 >$
$\widetilde{LSL}$	$< 235, 1 >$	$< 440, 1 >$
$\widetilde{T}$	$< 265, 1 >$	$< 470, 1 >$

Table 1. The Data of Film-developing Solution Measurements

It was noted in [5] that the observations are from a bivariate normal distribution. So, we obtain $\widetilde{\widehat{NMC}}_{pm} =< 0.8395, 0.0596, 0.0601 >$, i.e., as "approximately 0.8395". By comparing this fuzzy number with triangular fuzzy one number, it is concluded that this fuzzy number is less than $\tilde{1}$, that is, the process has unallowable variation.

In addition, we obtain $\widetilde{PV} =< 0.0979, 0.3763, 0.0227, 0.0873 >$, i.e., as "approximately between 0.0979 and 0.3763". Figure 1 shows the membership function of $\widetilde{\widehat{NMC}}_{pm}$ and $\widetilde{PV}$. By selecting $\lambda = 0.85$, we have $\widetilde{PV}(0.85) = [0.0941, 0.3882]$, then $PV_l(0.85) > 0.05$. So, the mean vector is not far from the target vector.

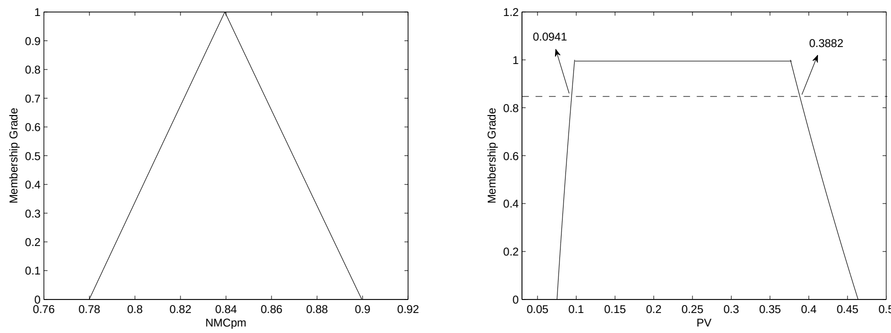

Figure 1. The Membership Functions of $\widetilde{\widetilde{NMC}}_{pm}$ (the Left One) and $\widetilde{PV}$ (the Right One)

To defuzzify the specification limits and the target value of two characteristics, we applied the ranking function and results are shown in Table 2.

Also, we get defuzzified sample mean vector and sample variance-covariance matrix as

$$\bar{\mathbf{X}} = \begin{pmatrix} 264.32 \\ 471.48 \end{pmatrix}, \qquad \mathbf{S} = \begin{pmatrix} 102.65 & 68.87 \\ 68.87 & 107.96 \end{pmatrix}.$$

	Elon	Hydroguinone
$R(\widetilde{USL})$	295	500
$R(\widetilde{LSL})$	235	440
$R(\widetilde{T})$	265	470

Table 2. The Data of Defuzzification of the Fuzzy Specification Limits and the Fuzzy Target Value of Each Characteristic

Figure 2 displays these crisp specification limits, modified tolerance region and 99.73% of the process region. Intuitively, it is seen that the region which covers 99.73% of the process products exceeds the tolerance region, so $LI = 0$. Hence, the process is incapable because of unallowable variability.

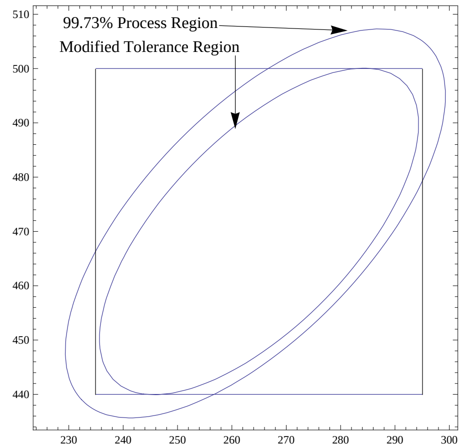

FIGURE 2. Geometry of the Modified Tolerance Region and the Process Ellipse

6. Conclusion

Multivariate process capability indices are utilized to measure the performance of the manufacturing process that its products' quality depends on more than one characteristic. In this paper, we employed fuzzy logic to deal with the circumstances in which not only the specification limits and the target value of each characteristic but also the data taken from the process products are imprecise. Then, we introduced fuzzy multivariate capability vector that provides much more information about the process performance than the classical one.

The new fuzzy MPC vector has three components. The first component counts the fuzzy variability of the process comparised to the fuzzy tolerance region. The second one is fuzzy significant level of hypothesis testing about fuzzy mean vector compared to the fuzzy target vector. The third one summarizes a comparison of the location of the process region and the tolerance region. In addition, we employed the proposed index to calculate the capability of a process in an example.

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Zainab Abbasi Ganji, Department of Statistics, Faculty of Mathematical Sciences, Ferdowsi University of Mashhad, Mashhad, Iran
E-mail address: abbasiganji@stu.um.ac.ir

Bahram Sadeghpour Gildeh*, Department of Statistics, Faculty of Mathematical Sciences, Ferdowsi University of Mashhad, Mashhad, Iran
E-mail address: sadeghpour@um.ac.ir

*Corresponding author