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CREDIBILISTIC PARAMETER ESTIMATION AND ITS APPLICATION IN FUZZY PORTFOLIO SELECTION

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Abstract. In this paper, a maximum likelihood estimation and a minimum entropy estimation for the expected value and variance of normal fuzzy variable are discussed within the framework of credibility theory. As an application, a credibilistic portfolio selection model is proposed, which is an improvement over the traditional models as it only needs the predicted values on the security returns instead of their membership functions.

1. Introduction

Since Nahmias [15] introduced a definition of fuzzy variable, the term has appeared in several papers [1, 5, 7] with different meanings such as fuzzy measure theory [17, 18], possibility theory [4, 19] and credibility theory [8, 12]. Since a fuzzy variable is uniquely determined by its membership function, estimation of the membership function has always been a key problem for the application of fuzzy variable. In 1981, Dishkant [3] first proposed a logical approach to estimate the membership function. In 1993, Cai [2] proposed a parameter estimation approach, which provided an efficient estimations for the possibilistic expected value and variance. This paper contributes to this stream of research by presenting a credibilistic parameter estimation approach in terms of the maximum likelihood principle.

Credibilistic portfolio selection has been well developed in models such as the mean-variance model [6], mean-variance-skewness model [10] and entropy optimization model [16]. All these models assume that the security returns have known membership functions provided by the experts. However, it is clear that the experts would prefer to predict the realization values rather than the membership functions. Based on realization values, a credibilistic portfolio selection model is proposed by applying our credibilistic approach to estimate the investment return and risk.

The rest of this paper is organized as follows: in Section 2 we recall basic definitions and properties regarding fuzzy variables and credibility measures. In Section 3, we propose interval and point estimators for the expected value and variance of normal fuzzy variable and in Section 4, we introduce a credibilistic portfolio selection model as an application. The paper ends with a brief summary.

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2. **Preliminaries**

Let ξ be a fuzzy variable with membership function μ. For any $B \subset \Re$, the credibility measure of ξ taking values in B is defined by Liu and Liu [13] as

$$\mathrm{Cr}\{\xi \in B\} = \frac{1}{2}\left(\sup_{x\in B}\mu(x) + 1 - \sup_{x\in B^c}\mu(x)\right).$$

In order to rank fuzzy variables, the expected value was defined by Liu and Liu [13] as

$$E[\xi] = \int_0^{+\infty} \mathrm{Cr}\{\xi \geq r\}\mathrm{d}r - \int_{-\infty}^{0} \mathrm{Cr}\{\xi \leq r\}\mathrm{d}r$$

provided that at least one of the two integrals is finite. Furthermore, if ξ has finite expected value e, its variance is defined as $V[\xi] = E\left[(\xi - e)^2\right]$.

Let ξ be a continuous fuzzy variable with membership function μ. Then its entropy [11] is defined as

$$H[\xi] = -\int_{-\infty}^{+\infty} \frac{\mu(x)}{2}\ln\frac{\mu(x)}{2} + \left(1 - \frac{\mu(x)}{2}\right)\ln\left(1 - \frac{\mu(x)}{2}\right)\mathrm{d}x.$$

The entropy is used to measure the uncertainty associated with each fuzzy variable. In 2007, Li and Liu [9] proposed a fuzzy maximum entropy principle, which tells us that out of all the membership functions satisfying the given constraints, we should select the one which maximizes the fuzzy entropy. The reason for this principle is quite simple. If we choose a membership function with less than maximum entropy, we may have used some additional information consciously or unconsciously. However, it is not correct to use such information because this is not given to us. Furthermore, these authors proved that the maximum entropy membership function with expected value e and variance σ^2 is

$$\mu(x) = 2\left(1 + \exp\left(\frac{\pi|x-e|}{\sqrt{6}\sigma}\right)\right)^{-1}, \tag{1}$$

which they called the normal membership function. We say a fuzzy variable ξ is normal if it has the normal membership function and write $\xi \sim N(e, \sigma^2)$.

In fact, a normal fuzzy variable has been defined in different ways. For example, some authors call a fuzzy variable normal if it has the following membership function:

$$\mu(x) = \exp\left(-\frac{(x-e)^2}{\sigma^2}\right) \tag{2}$$

where $e \in \Re$ and $\sigma > 0$. The variable so (2) has properties similar to the normal random variable. For example, (i) Parameter e stands for the expected value and σ^2 stands for the variance; (ii) If $\xi_1 \sim N(e_1, \sigma_1^2)$ and $\xi_2 \sim N(e_2, \sigma_2^2)$ are independent, then $a_1\xi_1 + a_2\xi_2 \sim N(a_1e_1 + a_2e_2, |a_1|\sigma_1 + |a_2|\sigma_2)$; (iii) If $\xi \sim N(e, \sigma^2)$, then its entropy is larger than that of any fuzzy variables with expected value e and variance σ^2.

3. **Parameter Estimation**

Let $X \sim N(e, \sigma^2)$. In order to estimate parameters e and σ, we first collect a random sample $\{X_1, X_2, \cdots, X_n\}$ of size n by asking each of n experts "which is the most possible value of X?" We shall assume that $X_1, X_2, \cdots, X_n$ are independent normal fuzzy variables.

Lemma 3.1. Let $X_1, X_2, \cdots, X_n$ be a collection of independent normal fuzzy variables with expected value e and variance σ^2. Then we have $\max_{1\le i\le n} X_i \sim N(e, \sigma^2)$ and $\min_{1\le i\le n} X_i \sim N(e, \sigma^2)$.

Proof. Let μ_i be the membership function of X_i for all i, and μ the membership function of $\max_{1\le i\le n} X_i$. We will prove that μ is a normal membership function. For any $x \in \Re$, it follows from Zadeh's extension principle that

$$\mu(x) = \sup_{\max\{x_1,x_2,\cdots,x_n\}=x} \min_{1\le i\le n} \mu_i(x_i).$$

Since μ_i is increasing on the interval $(-\infty, e]$ for all i, if $x \le e$, the supremum will be obtained at $x_1 = x_2 = \cdots = x_n = x$, which implies that

$$\mu(x) = \min_{1\le i\le n} \mu_i(x) = 2\left(1 + \exp\left(\pi(e-x)/\sqrt{6}\sigma\right)\right)^{-1}.$$

On the other hand, since μ_i is decreasing on the interval $[e, +\infty)$, if $x > e$, for all i, the supremum will be obtained at $x_1 = x, x_2 = x_3 = \cdots = x_n = e$, which implies that

$$\mu(x) = \min\{\mu_1(x), \mu_2(e), \cdots, \mu_n(e)\} = 2\Big(1 + \exp\left(\pi(x-e)/\sqrt{6}\sigma\right)\Big)^{-1}.$$

In general, for any $x \in \Re$, we have

$$\mu(x) = 2\Big(1 + \exp\left(\pi|x-e|/\sqrt{6}\sigma\right)\Big)^{-1}.$$

That is, $\max_{1\le i\le n} X_i \sim N(e, \sigma^2)$.

Now, since $-X_i \sim N(-e, \sigma^2)$ for all i, we have $\max_{1\le i\le n}(-X_i)$ is a normal fuzzy variable with expected value $-e$ and variance σ^2. Hence, from

$$\min_{1\le i\le n} X_i = -\max_{1\le i\le n}(-X_i)$$

we have $\min_{1\le i\le n} X_i \sim N(e, \sigma^2)$. The proof is complete. □

3.1. **Estimation of Expected Value.**

For any $0 < \alpha < 0.5$, define $\varepsilon_\alpha = \sqrt{6}\ln(1/\alpha - 1)/\pi$. By Lemma 3.1, it is easy to prove that

$$\mathrm{Cr}\left\{\left|\max_{1\le i\le n} X_i - e\right| \ge \sigma\varepsilon_\alpha\right\} = \alpha,\ \mathrm{Cr}\left\{\left|\min_{1\le i\le n} X_i - e\right| \ge \sigma\varepsilon_\alpha\right\} = \alpha. \tag{3}$$

Since $\alpha < 0.5$, it follows from the fourth axiom of credibility measure (Li and Liu [8]) that

$$\mathrm{Cr}\left\{e < \max_{1\le i\le n} X_i - \sigma\varepsilon_\alpha \text{ or } e > \min_{1\le i\le n} X_i + \sigma\varepsilon_\alpha\right\} = \alpha.$$

Furthermore, it follows from the self-duality of credibility measure that

$$\mathrm{Cr}\left\{\max_{1\le i\le n} X_i - \sigma\varepsilon_\alpha \le e \le \min_{1\le i\le n} X_i + \sigma\varepsilon_\alpha\right\} = 1-\alpha.$$

That is, parameter e belongs to the interval

$$\left[\max_{1\le i\le n} X_i - \sigma\varepsilon_\alpha,\ \min_{1\le i\le n} X_i + \sigma\varepsilon_\alpha\right] \tag{4}$$

with credibility $1-\alpha$. In what follows, this interval will be called the $(1-\alpha)$-confidence interval of e.

In order to obtain a point estimator for parameter e, we apply the maximum likelihood principle to select one value from the confidence interval. Let $\{x_1, x_2, \cdots, x_n\}$ be the realization values of $\{X_1, X_2, \cdots, X_n\}$. Define the likelihood function as

$$L(e; x_1, x_2, \cdots, x_n) = \mathrm{Cr}\{X_1 = x_1, X_2 = x_2, \cdots, X_n = x_n\}.$$

Since $X_1, X_2, \cdots, X_n$ are mutually independent, we have

$$L = \min_{1\le i\le n} \left(1 + \exp\left(\pi|x_i - e|/\sigma\right)\right)^{-1}.$$

Then it is clear that the maximum likelihood estimator of e should minimize the function $\max_{1\le i\le n} |x_i - e|$, which implies that the maximum likelihood estimator is

$$\hat{e} = \frac{1}{2}\left(\min_{1\le i\le n} x_i + \max_{1\le i\le n} x_i\right).$$

Hence, the midpoint of the confidence interval is selected as the point estimator of e. That is,

$$\hat{e} = \frac{1}{2}\left(\max_{1\le i\le n} X_i + \min_{1\le i\le n} X_i\right). \tag{5}$$

3.2. **Estimation of Variance.** In this section, we discuss estimation for the variance parameter. First, it follows from (3) that

$$\mathrm{Cr}\left\{\frac{1}{\varepsilon_\alpha}\left|\min_{1\le i\le n} X_i - e\right| \ge \sigma\right\} = \mathrm{Cr}\left\{\frac{1}{\varepsilon_\alpha}\left|\max_{1\le i\le n} X_i - e\right| \ge \sigma\right\} = \alpha.$$

If $\alpha < 0.5$, then it follows from the fourth axiom of credibility measure (Li and Liu [8]) that

$$\mathrm{Cr}\left\{\sigma \le \frac{1}{\varepsilon_\alpha}\left|\min_{1\le i\le n} X_i - e\right| \text{ or } \sigma \le \frac{1}{\varepsilon_\alpha}\left|\max_{1\le i\le n} X_i - e\right|\right\} = \alpha.$$

Furthermore. it follows from the self-duality of credibility measure that

$$\mathrm{Cr}\left\{\sigma \ge \frac{1}{\varepsilon_\alpha}\max\left\{\left|\max_{1\le i\le n} X_i - e\right|, \left|\min_{1\le i\le n} X_i - e\right|\right\}\right\} = 1-\alpha,$$

In other words, we believe that parameter σ belongs to the interval

$$\left[\frac{1}{\varepsilon_\alpha}\max\left\{\left|\max_{1\le i\le n} X_i - e\right|, \left|\min_{1\le i\le n} X_i - e\right|\right\}, \infty\right) \tag{6}$$

with credibility $1-\alpha$. In what follows, this interval will be called the $(1-\alpha)$-confidence interval of parameter σ.

In order to obtain a point estimator, we employ the minimum entropy principle to select a value from the confidence interval with minimum entropy. It can be proved that the entropy of a normal fuzzy variable with variance σ^2, is $\sqrt{6}\pi\sigma/3$. Since the entropy is used to measure the uncertainty associated with each fuzzy variable, the minimum entropy principle selects the one which minimizes the uncertainty. Hence

$$\hat{\sigma} = \frac{1}{\varepsilon_\alpha} \max\left\{ \left| \max_{1\le i\le n} X_i - e \right|, \left| \min_{1\le i\le n} X_i - e \right| \right\}. \tag{7}$$

If parameter e is unknown, we may replace it with its maximum likelihood estimator (5) and get

$$\hat{\sigma} = \frac{1}{2\varepsilon_\alpha} \left(\max_{1\le i\le n} X_i - \min_{1\le i\le n} X_i \right). \tag{8}$$

3.3. **Summary.** We now summarize the procedure of our parameter estimation approach and present a numerical example:

Step 1: Ask each expert, which value is the most possible value of X to obtain the sample $\{x_1, x_2, \cdots, x_n\}$.

Step 2: For a predetermined confidence level $0 < \alpha < 0.5$, calculate $\varepsilon_\alpha = \sqrt{6}\ln(1/\alpha - 1)/\pi$.

Step 3: Calculate the estimators

$$\hat{e} = \frac{1}{2}\left(\max_{1\le i\le n} x_i + \min_{1\le i\le n} x_i \right) \text{ and } \hat{\sigma} = \frac{1}{2\varepsilon_\alpha}\left(\max_{1\le i\le n} x_i - \min_{1\le i\le n} x_i \right).$$

Example 3.2. *Let us consider a numerical example for the estimation of the distance between two cities. Assume that the exact value is unknown and it is estimated by a normal fuzzy variable with expected value e and variance σ^2. In order to estimate the parameters e and σ, we question six experts and obtain the following possible values $\{20, 21, 32, 18, 22, 25\}$. For $\alpha = 0.05$, it is easy to obtain $\varepsilon_{0.05} = 2.2958$. Then it follows from Step 3 that the values of the estimators values are as follows:*

$$\hat{e} = \frac{1}{2}\left(\max\{20, 21, 32, 18, 22, 25\} + \min\{20, 21, 32, 18, 22, 25\}\right) = 25,$$

$$\hat{\sigma} = \frac{1}{2\varepsilon_{0.05}}\left(\max\{20, 21, 32, 18, 22, 25\} - \min\{20, 21, 32, 18, 22, 25\}\right) = 3,$$

Hence the distance has membership function

$$\mu(x) = 2\left(1 + \exp\left(\frac{\pi|x-25|}{3\sqrt{6}}\right)\right)^{-1}.$$

4. **Credibilistic Portfolio Selection**

To evaluate the performance of credibilisitic parameter estimation we apply it to a portfolio selection problem.

Suppose that there are n securities. Let ξ_i be the fuzzy return of the ith security, and let x_i be the investment proportion on security i for all $i = 1, 2, \cdots, n$. For each portfolio $(x_1, x_2, \cdots, x_n)$, it is clear that the portfolio return $\xi = \xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n$ is also a fuzzy variable. In spirit with Markowitz's work, Huang [6] proposed a mean-variance model by maximizing the investment return of all portfolios satisfying certain risk constraint, where investment return and investment risk are respectively quantified by expected value and variance of the portfolio return ξ. The mean-variance model is formulated as follows:

$$\begin{cases} \max & E[\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n] \\ \text{s.t.} & V[\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n] \le \beta \\ & x_1 + x_2 + \cdots + x_n = 1 \\ & x_i \ge 0,\ i = 1, 2, \cdots, n \end{cases}$$

where β is the maximum risk level the investor can tolerate.

Year	1	2	3	4	5	6	7	8	9
1937	-0.305	-0.173	-0.318	-0.477	-0.457	-0.065	-0.319	-0.4	-0.435
1938	0.513	0.098	0.285	0.714	0.107	0.238	0.076	0.336	0.238
1939	0.055	0.2	-0.047	0.165	-0.424	-0.078	0.381	-0.093	-0.295
1940	-0.126	0.03	0.104	-0.043	-0.189	-0.077	-0.051	-0.09	-0.036
1941	-0.28	-0.183	-0.171	-0.277	0.637	-0.187	0.087	-0.194	-0.24
1942	-0.003	0.067	-0.039	0.476	0.865	0.156	0.262	0.113	0.126
1943	0.428	0.3	0.149	0.225	0.313	0.351	0.341	0.58	0.639
1944	0.192	0.103	0.26	0.29	0.637	0.233	0.227	0.473	0.282
1945	0.446	0.216	0.419	0.216	0.373	0.349	0.352	0.229	0.578
1946	-0.088	-0.046	-0.078	-0.272	-0.037	-0.209	0.153	-0.126	0.289
1947	-0.127	-0.071	0.169	0.144	0.026	0.355	-0.099	0.009	0.184
1948	-0.015	0.056	-0.035	0.107	0.153	-0.231	0.038	0	0.114
1949	0.305	0.038	0.133	0.321	0.067	0.246	0.273	0.223	-0.222
1950	-0.096	0.089	0.732	0.305	0.579	-0.248	0.091	0.65	0.327
1951	0.016	0.09	0.021	0.195	0.04	-0.064	0.054	-0.131	0.333
1952	0.128	0.083	0.131	0.39	0.434	0.079	0.109	0.175	0.062
1953	-0.01	0.035	0.006	-0.072	-0.027	0.067	0.21	-0.084	-0.048
1954	0.154	0.176	0.908	0.715	0.469	0.077	0.112	0.756	0.185

Table 1. Yearly Returns on Nine Securities from 1937 to 1954

In [6], Huang assumed that the membership functions for security returns may be completely provided by the experts. In practice, however, this assumption is difficult to satisfy because the experts generally prefer to predict crisp values for the returns instead of infinite values. In this section, we propose a credibilistic portfolio selection model based on the predicted values.

Let r_{ij} be the predicted return of the ith expert for the jth security, $i = 1, 2, \cdots, m, j = 1, 2, \cdots, n$. Then for each portfolio $(x_1, x_2, \cdots, x_n)$, there will be m predicted values for the portfolio return as follows:

$$\xi_i = \sum_{j=1}^{n} r_{ij} x_j, \ i = 1, 2, \cdots, m.$$

According to (5) and (8), the estimators for the expected value and variance of ξ are

$$\frac{1}{2}\left(\max_{1\le i\le m}\sum_{j=1}^{n} r_{ij}x_j + \min_{1\le i\le m}\sum_{j=1}^{n} r_{ij}x_j\right), \frac{1}{4\varepsilon_\alpha^2}\left(\max_{1\le i\le m}\sum_{j=1}^{n} r_{ij}x_j - \min_{1\le i\le m}\sum_{j=1}^{n} r_{ij}x_j\right)^2,$$

respectively. If we replace the expected value operator and variance operator in the mean-variance model with the estimated values, we get the following nonlinear programming model:

$$\begin{cases} \max & \max\limits_{1\le i\le m}\sum\limits_{j=1}^{n} r_{ij}x_j + \min\limits_{1\le i\le m}\sum\limits_{j=1}^{n} r_{ij}x_j \\ \text{s.t.} & \max\limits_{1\le i\le m}\sum\limits_{j=1}^{n} r_{ij}x_j - \min\limits_{1\le i\le m}\sum\limits_{j=1}^{n} r_{ij}x_j \le 2\sqrt{\beta}\varepsilon_\alpha \\ & x_1 + x_2 + \cdots + x_n = 1 \\ & x_i \ge 0, \ i = 1, 2, \cdots, n. \end{cases} \tag{9}$$

Compared with the traditional credibilistic portfolio selection models, model (9) has the following advantages: (i) it needs the predicted values on the security returns only, but the traditional models need the membership functions; (ii) it can be well solved by classical optimization algorithms, but the traditional ones need the fuzzy simulation technique and heuristic algorithms. In conclusion, our new model reduces the difficulty of prediction, and improves the running speed of computation.

Example 4.1. *Table 2 provides the yearly returns of nine securities from 1937 to 1954 (See* [14]*). In this example, these observation values will be employed to express the predicted values of the security returns. Take* $\alpha = 0.05$ *and* $\beta = 0.01$*. A run of the function* fmincon *in Matlab 7.1 shows that the best portfolio should invest mainly in securities 2 and 6. The details regarding the investment proportion of this portfolio for each security are shown in Table 2.*

Security	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*
Proportion	*0.000*	*0.299*	*0.085*	*0.000*	*0.054*	*0.562*	*0.000*	*0.000*	*0.000*

TABLE 2. The Best Found Portfolio

5. **Conclusion**

In this paper, maximum likelihood estimators and minimum entropy estimators for the expected value and variance of normal fuzzy variables are proposed and a credibilistic portfolio selection model is formulated as an application. This model is more advantageous than the traditional models as it only needs the predicted values about the security returns instead of their membership functions.

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