

## LATTICE-VALUED CATEGORIES OF LATTICE-VALUED CONVERGENCE SPACES

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ABSTRACT. We study  $L$ -categories of lattice-valued convergence spaces. Such categories are obtained by “fuzzifying” the axioms of a lattice-valued convergence space. We give a natural example, study initial constructions and function spaces. Further we look into some  $L$ -subcategories. Finally we use this approach to quantify how close certain lattice-valued convergence spaces are to being lattice-valued topological spaces.

### 1. Introduction

The classical theory of (topological or convergence) spaces is based on two-valued logic. Properties or axioms of a space are either satisfied or not. So, e.g., either a space is a topological space or not or either a mapping is continuous or it is not. If we replace the lattice  $L = \{0, 1\}$  by more general lattices  $L$ , as it is done in lattice-valued (or “fuzzy”) mathematics, we can use the order of  $L$  to “measure” grades that a property holds. We then end with a finer classification and it makes sense e.g. to say that “a space has a certain grade of being an  $L$ -topological space” or that “one space is more topological than another”. Likewise for mappings it makes sense to say that “a certain mapping is more continuous than another”. This was observed quite some time ago ([17, 15, 25]). In order to define such grades to which a property holds, the lattice should have a certain structure. For the purpose of this paper, frames seem appropriate. They allow the definition of a residual implication and with the help of this operation we can define grades of being a “space” or being “continuous” in a systematic way.

We thus end in a new situation: we have “potential objects” which qualify to a certain grade for a “proper space” (e.g. a stratified  $L$ -convergence space) and “potential morphisms” which qualify to a certain grade for being a “proper morphism” (e.g. a continuous map). This observation led Šostak [18, 19, 20, 21, 22] to the concept of an  $L$ -category (or fuzzy category – as opposed to a (classical) category of “fuzzily structured sets”).

The starting point of this paper is the structure of continuous convergence,  $c$ -lim. This function space structure (defined on the sets  $C(X, Y)$  of continuous mappings between spaces  $X, Y$ ) makes the categories  $CONV$  of (classical) convergence spaces

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and *SL-GCS* of lattice-valued convergence spaces cartesian closed. The definition of this structure, however, can be naturally extended to an arbitrary set of mappings between convergence spaces. As Poppe [16] points out, however, continuous convergence may then fail to be a convergence structure. Poppe was working with classical convergence spaces, i.e. with  $L = \{0, 1\}$ , and he encountered the following: If the function space contains a non-convergent mapping, then the axiom (L1) is not satisfied by *c-lim*. (The axiom (L1) means essentially that constant sequences converge.) In the lattice-valued case, however, we can ask the question: “How far is the function space then away from satisfying (L1)?”

After a preliminary section, where we collect all the necessary background, we will show in Section 3 that, depending on how far the mappings are away from being continuous, the axiom (L1) will be satisfied by *c-lim* to a certain grade. Having observed this natural example of a space satisfying an axiom only to a certain degree, we then consider in Section 4 Šostak’s general situation and define an *L*-category of lattice-valued convergence spaces. We study initial constructions and function spaces in this *L*-category. Further, in Section 5, we discuss two general ways of obtaining *L*-subcategories. One way is to enforce certain axioms and the other is to add further axioms. In Section 6 we look into the property of a stratified *L*-convergence space being *L*-topological. We give three different methods of assigning a grade of being *L*-topological, all coming from different diagonal axioms. Finally, we draw some conclusions.

## 2. Preliminaries

We consider in this paper *frames*, i.e. complete lattices,  $L$ , where finite meets distribute over arbitrary joins. That means that  $\alpha \wedge \bigvee_{\iota \in I} \beta_\iota = \bigvee_{\iota \in I} (\alpha \wedge \beta_\iota)$  is true for all  $\alpha, \beta_\iota$  ( $\iota \in I$ ) and all index sets  $I$ . The bottom (resp. top) element of  $L$  is denoted by  $\perp$  (resp.  $\top$ ). The distributivity of finite meets over arbitrary joins allows to define a *residual implication* by  $\alpha \rightarrow \beta = \bigvee \{ \lambda \in L : \alpha \wedge \lambda \leq \beta \}$  (where  $\alpha, \beta \in L$ ). This operation is characterized by  $\delta \leq \alpha \rightarrow \beta \iff \delta \wedge \alpha \leq \beta$ . For this reason, a frame is often called a *complete Heyting algebra*. In the following lemma we collect several properties of the residual implication that we are going to use later. The proofs follow easily from the characterization above. Many of them may also be found in [4, 6].

**Lemma 2.1.** *Let  $\alpha, \beta, \alpha_i, \beta_i, \gamma \in L$ . Then*

- (1)  $\alpha \rightarrow \beta = \top$  if and only if  $\alpha \leq \beta$ ;
- (2) If  $\alpha \leq \beta$ , then  $\alpha \rightarrow \gamma \geq \beta \rightarrow \gamma$ ;
- (3) If  $\alpha \leq \beta$ , then  $\gamma \rightarrow \alpha \leq \gamma \rightarrow \beta$ ;
- (4)  $\alpha \rightarrow \bigwedge_{i \in J} \beta_i = \bigwedge_{i \in J} (\alpha \rightarrow \beta_i)$ ;
- (5)  $\alpha \wedge (\alpha \rightarrow \beta) \leq \beta$ ;
- (6)  $\top \rightarrow \alpha = \alpha$ ;
- (7)  $(\alpha \rightarrow \beta) \wedge (\beta \rightarrow \gamma) \leq \alpha \rightarrow \gamma$ ;
- (8)  $\beta \leq (\alpha \rightarrow (\alpha \wedge \beta))$ ;
- (9)  $\beta \rightarrow \gamma \leq (\alpha \wedge \beta) \rightarrow (\alpha \wedge \gamma)$ ;
- (10)  $\alpha \rightarrow \delta \leq (\delta \rightarrow \beta) \rightarrow (\alpha \rightarrow \beta)$ .

We extend the lattice operations pointwise from  $L$  to  $L^X = \{a : X \rightarrow L\}$ , the set of all  $L$ -sets on  $X$ . We denote especially for  $A \subseteq X$  and  $\alpha \in L$ ,  $\alpha_A : X \rightarrow L$ ,  $x \mapsto \begin{cases} \alpha & \text{if } x \in A \\ \perp & \text{else} \end{cases}$ . So e.g.,  $\alpha_X$  is the constant  $L$ -set with value  $\alpha$  and  $\top_A$  is the characteristic function of  $A$ . For  $a \in L^X$  and  $b \in L^Y$  we define their cartesian product,  $a \times b \in L^{X \times Y}$ , by  $(a \times b)(x, y) = a(x) \wedge b(y)$  for  $(x, y) \in X \times Y$ .

For notions from category theory we refer to [1].

For a category  $\mathcal{C}$  we denote  $|\mathcal{C}|$  its class of objects and  $\text{Mor}(\mathcal{C}) = \bigcup \{\text{Mor}(X, Y) \mid X, Y \in |\mathcal{C}|\}$  (with the sets  $\text{Mor}(X, Y)$  pairwise disjoint) its class of morphisms. As usual, we write  $\varphi : X \rightarrow Y$  for  $\varphi \in \text{Mor}(X, Y)$ . We usually do not mention the identities, but denote them by  $id_X, id_Y$  etc. when they occur. We then state the category as  $\mathcal{C} = (|\mathcal{C}|, \text{Mor}(\mathcal{C}), \circ)$  with the composition  $\circ$ .

An  $L$ -category or *lattice-valued category*  $\mathcal{C} = (|\mathcal{C}|, \omega, \text{Mor}(\mathcal{C}), \mu, \circ)$  [18, 19, 20, 21, 22] is a category  $\mathcal{C} = (|\mathcal{C}|, \text{Mor}(\mathcal{C}), \circ)$  together with two  $L$ -classes  $\omega : |\mathcal{C}| \rightarrow L$ ,  $\mu : \text{Mor}(\mathcal{C}) \rightarrow L$  such that the following axioms are satisfied:

- (LCat1)  $\varphi \in \text{Mor}(X, Y)$  implies  $\mu(\varphi) \leq \omega(X) \wedge \omega(Y)$ ;
- (LCat2)  $\varphi \in \text{Mor}(X, Y), \psi \in \text{Mor}(Y, Z)$  implies  $\mu(\varphi \circ \psi) \geq \mu(\varphi) \wedge \mu(\psi)$ ;
- (LCat3)  $\mu(id_X) = \omega(X)$  for each identity  $id_X \in \text{Mor}(X, X)$ .

The class  $|\mathcal{C}|$  is interpreted as the *class of potential objects*, where the grade of  $X \in |\mathcal{C}|$  being an object is evaluated by  $\omega(X)$ . The class  $\text{Mor}(\mathcal{C})$  is interpreted as the *class of potential morphisms* and  $\mu(\varphi)$  gives the grade that  $\varphi \in \text{Mor}(\mathcal{C})$  is a morphism in  $\mathcal{C}$ .

Any (usual) category  $\mathcal{C} = (|\mathcal{C}|, \text{Mor}(\mathcal{C}), \circ)$  can be considered as an  $L$ -category if we define  $\omega(X) = \top$  for each  $X \in |\mathcal{C}|$  and  $\mu(\varphi) = \top$  for any  $\varphi \in \text{Mor}(\mathcal{C})$ .

Let  $\mathcal{C} = (|\mathcal{C}|, \omega, \text{Mor}(\mathcal{C}), \mu, \circ)$  and  $\mathcal{D} = (|\mathcal{D}|, \omega', \text{Mor}(\mathcal{D}), \mu', *)$  be two  $L$ -categories. Then  $\mathcal{D}$  is called an  $L$ -subcategory of  $\mathcal{C}$  ([18]), in symbols  $\mathcal{D} \leq \mathcal{C}$ , if  $|\mathcal{D}| = |\mathcal{C}|$ ,  $\text{Mor}(\mathcal{D}) = \text{Mor}(\mathcal{C})$ ,  $\circ = *$  and  $\omega' \leq \omega$  and  $\mu' \leq \mu$ . If  $\mathcal{C} = (|\mathcal{C}|, \omega, \text{Mor}(\mathcal{C}), \mu, \circ)$  is an  $L$ -category and if  $\mathcal{D} = (|\mathcal{D}|, \text{Mor}(\mathcal{D}), \circ)$  is a (usual) subcategory of the category  $\mathcal{C} = (|\mathcal{C}|, \text{Mor}(\mathcal{C}), \circ)$ , then  $\mathcal{D}$  can be identified with the following  $L$ -subcategory  $\mathcal{D}' = (|\mathcal{C}|, \omega', \text{Mor}(\mathcal{C}), \mu', \circ)$  of  $\mathcal{C}$ , where  $\omega' = \top_{|\mathcal{D}|}$  and  $\mu' = \top_{\text{Mor}(\mathcal{D})}$ .

A mapping  $F : \mathcal{C} = (|\mathcal{C}|, \omega, \text{Mor}(\mathcal{C}), \mu, \circ) \rightarrow \mathcal{D} = (|\mathcal{D}|, \omega', \text{Mor}(\mathcal{D}), \mu', *)$  between two  $L$ -categories is called a *functor* [18] if it is a functor from the category  $\mathcal{C} = (|\mathcal{C}|, \text{Mor}(\mathcal{C}), \circ)$  to the category  $\mathcal{D} = (|\mathcal{D}|, \text{Mor}(\mathcal{D}), *)$  and if  $\mu(\varphi) \leq \mu'(F(\varphi))$  for each  $\varphi \in \text{Mor}(\mathcal{C})$ . (This concept was originally called a  $\top$ -functor in [18]. We will only encounter functors of this kind.)

For further results on  $L$ -categories we refer to the papers [14, 18, 19, 20, 21, 22].

A *stratified  $L$ -filter*  $\mathcal{F}$  on  $X$  [5, 6] is a mapping  $\mathcal{F} : L^X \rightarrow L$  with the properties

- (F1)  $\mathcal{F}(\top_X) = \top$ ,  $\mathcal{F}(\perp_X) = \perp$ ,
- (F2)  $a \leq b \implies \mathcal{F}(a) \leq \mathcal{F}(b)$ ,
- (F3)  $\mathcal{F}(a) \wedge \mathcal{F}(b) \leq \mathcal{F}(a \wedge b)$  and
- (Fs)  $\alpha \wedge \mathcal{F}(a) \leq \mathcal{F}(\alpha_X \wedge a) \quad \forall \alpha \in L, a \in L^x$

(where  $a, b \in L^X$ ). The set of all stratified  $L$ -filters on  $X$  is denoted by  $\mathbb{F}_L^s(X)$ . An example of a stratified  $L$ -filter is the *point  $L$ -filter*,  $[x]$ , defined by  $[x](a) = a(x)$

(see e.g. [5]). An order on  $\mathbb{F}_L^s(X)$  can be defined by  $\mathcal{F} \leq \mathcal{G}$  iff for all  $a \in L^X$ ,  $\mathcal{F}(a) \leq \mathcal{G}(a)$ . The meet of a family of stratified  $L$ -filters,  $(\mathcal{F}_\lambda)_{\lambda \in \Lambda}$ , is given by  $(\bigwedge_{\lambda \in \Lambda} \mathcal{F}_\lambda)(a) = \bigwedge_{\lambda \in \Lambda} \mathcal{F}_\lambda(a)$  for all  $a \in L^X$ . Obviously  $\bigwedge_{\lambda \in \Lambda} \mathcal{F}_\lambda \in \mathbb{F}_L^s(X)$ .

Two stratified  $L$ -filters  $\mathcal{F}, \mathcal{G} \in \mathbb{F}_L^s(X)$  have an upper bound if and only if  $\mathcal{F}(a) \wedge \mathcal{G}(a) = \perp$  whenever  $a \wedge b = \perp_X$ . The least upper bound of  $\mathcal{F}$  and  $\mathcal{G}$  is given in this case by  $(\mathcal{F} \vee \mathcal{G})(a) = \bigvee \{\mathcal{F}(f) \wedge \mathcal{G}(g) : f \wedge g \leq a\}$  (see [6]).

For a mapping  $\varphi : X \rightarrow Y$  and  $\mathcal{F} \in \mathbb{F}_L^s(X)$  we define  $\varphi(\mathcal{F}) \in \mathbb{F}_L^s(Y)$  by  $\varphi(\mathcal{F})(b) = \mathcal{F}(\varphi^\leftarrow(b))$  where  $\varphi^\leftarrow(b) = b \circ \varphi$  for  $b \in L^Y$  (see e.g. [5]). Clearly, then  $\varphi([x]) = [\varphi(x)]$  for every  $x \in X$  and if  $\mathcal{F} \leq \mathcal{G}$  then also  $\varphi(\mathcal{F}) \leq \varphi(\mathcal{G})$ . Moreover, for a further mapping  $\psi : Y \rightarrow Z$  we have  $\psi \circ \varphi(\mathcal{F}) = \psi(\varphi(\mathcal{F}))$ . For  $\mathcal{G} \in \mathbb{F}_L^s(Y)$  we define  $\varphi^\leftarrow(\mathcal{G}) : L^X \rightarrow L$  by  $\varphi^\leftarrow(\mathcal{G})(a) = \bigvee \{\mathcal{G}(b) : \varphi^\leftarrow(b) \leq a\}$ . Then  $\varphi^\leftarrow(\mathcal{G}) \in \mathbb{F}_L^s(X)$  if and only if  $\mathcal{G}(b) = \perp$  whenever  $\varphi^\leftarrow(b) = \perp_X$  (see [8]). If  $Y \subseteq X$  and  $\iota_Y = \iota : Y \hookrightarrow X, x \mapsto x$  is the inclusion mapping, then we denote for  $\mathcal{F} \in \mathbb{F}_L^s(X)$ ,  $\mathcal{F}_Y = \iota^\leftarrow(\mathcal{F})$ , in case this is a stratified  $L$ -filter on  $Y$ . For  $\mathcal{F} \in \mathbb{F}_L^s(Y)$  we further denote  $[\mathcal{F}]_X = [\mathcal{F}] = \iota_Y(\mathcal{F})$  with  $[\mathcal{F}](a) = \mathcal{F}(a|_Y)$  for  $a \in L^X$ . If  $Z \subseteq Y \subseteq X$  and  $\mathcal{F} \in \mathbb{F}_L^s(Z)$  then  $[[\mathcal{F}]_Y]_X = [\mathcal{F}]_X$ .

For two stratified  $L$ -filters,  $\mathcal{F} \in \mathbb{F}_L^s(X)$  and  $\mathcal{G} \in \mathbb{F}_L^s(Y)$ , we define their product,  $\mathcal{F} \times \mathcal{G} \in \mathbb{F}_L^s(X \times Y)$ , by  $\mathcal{F} \times \mathcal{G} = pr_X^\leftarrow(\mathcal{F}) \vee pr_Y^\leftarrow(\mathcal{G})$  (see [3, 8]). Here  $pr_X$  resp.  $pr_Y$  are the projections from  $X \times Y$  onto  $X$ , resp. onto  $Y$ . Apparently, for  $a \in L^{X \times Y}$  we have  $\mathcal{F} \times \mathcal{G}(a) = \bigvee \{\mathcal{F}(f) \wedge \mathcal{G}(g) : f \times g \leq a\}$  (see [8]). It is shown in [8] that  $pr_X(\mathcal{F} \times \mathcal{G}) \leq \mathcal{F}$  for every  $\mathcal{F} \in \mathbb{F}_L^s(X \times Y)$  and that  $pr_X(\mathcal{F} \times \mathcal{G}) \geq \mathcal{F}$  and  $pr_Y(\mathcal{F} \times \mathcal{G}) \geq \mathcal{G}$ . Moreover, for  $\mathcal{F} \in \mathbb{F}_L^s(X), \mathcal{G} \in \mathbb{F}_L^s(Y)$  and two mapping  $\varphi : X \rightarrow U, \psi : Y \rightarrow V$ , with the *product mapping*  $\varphi \times \psi : X \times Y \rightarrow U \times V, (x, y) \mapsto (\varphi(x), \psi(y))$  we have  $(\varphi \times \psi)(\mathcal{F} \times \mathcal{G}) = \varphi(\mathcal{F}) \times \psi(\mathcal{G})$  ([8]).

If  $J$  is a set and  $\mathcal{G} \in \mathbb{F}_L^s(J)$  and, for each  $i \in J$ ,  $\mathcal{F}_i \in \mathbb{F}_L^s(X)$  is given, then the *stratified  $L$ -diagonal filter*  $\mathcal{G}(\mathcal{F}_{(\cdot)}) \in \mathbb{F}_L^s(X)$  [10, 11] is defined by  $\mathcal{G}(\mathcal{F}_{(\cdot)})(a) = \mathcal{G}(\mathcal{F}_{(\cdot)}(a))$  for  $a \in L^X$ . Here,  $\mathcal{F}_{(\cdot)}(a) : J \rightarrow L, i \mapsto \mathcal{F}_i(a)$ .

A *stratified  $L$ -generalized convergence space* [8, 9] is a set  $X$  together with a limit map  $\lim : \mathbb{F}_L^s(X) \rightarrow L$  which satisfies the axioms

- (L1)  $\lim[x](x) = \top$  for all  $x \in X$ ,
- (L2)  $\lim \mathcal{F} \leq \lim \mathcal{G}$  whenever  $\mathcal{F} \leq \mathcal{G}, \mathcal{F}, \mathcal{G} \in \mathbb{F}_L^s(X)$ .

A mapping  $\varphi : (X, \lim) \rightarrow (X', \lim')$  between two stratified  $L$ -generalized convergence spaces is called *continuous* if for all  $\mathcal{F} \in \mathbb{F}_L^s(X)$  and all  $x \in X$  we have  $\lim \mathcal{F}(x) \leq \lim' \varphi(\mathcal{F})(\varphi(x))$ . The category with objects all stratified  $L$ -generalized convergence spaces and the continuous mappings as morphisms is denoted by *SL-GCS*. This category is topological over *SET* ([8, 9]). Especially, for a source  $(\varphi_i : X \rightarrow (X_i, \lim_i))_{i \in I}$  the initial limit map on  $X$  is given by

$$\lim \mathcal{F}(x) = \bigwedge_{i \in I} \lim_i \varphi_i(\mathcal{F})(\varphi_i(x)), \quad (\mathcal{F} \in \mathbb{F}_L^s(X)).$$

In particular, we endow the product of two stratified  $L$ -generalized convergence spaces  $(X, \lim_X), (Y, \lim_Y)$  with the product structure

$$\lim_{X \times Y} \mathcal{F}(x, y) = \lim_X pr_X(\mathcal{F})(x) \wedge \lim_Y pr_Y(\mathcal{F})(y)$$

where  $pr_X : X \times Y \rightarrow X$  and  $pr_Y : X \times Y \rightarrow Y$  are the projections and  $\mathcal{F} \in \mathbb{F}_L^s(X \times Y)$ . Moreover,  $SL-GCS$  has “natural” function spaces, i.e. it is cartesian closed ([8]). The function space structure is the structure of continuous convergence and is described in the next section. For  $(X, \lim) \in |SL-GCS|$  we can define a *neighbourhood L-filter*  $\mathcal{U}^x = \mathcal{U}_{\lim}^x \in \mathbb{F}_L^s(X)$  [8] by

$$\mathcal{U}^x(a) = \bigwedge_{\mathcal{F} \in \mathbb{F}_L^s(X)} (\lim \mathcal{F}(x) \rightarrow \mathcal{F}(a)) \quad \text{for } a \in L^X.$$

Often, further axioms are required, leading to subcategories. The following is a list of such axioms.

- (L3)  $\lim \mathcal{F} \wedge \lim \mathcal{G} \leq \lim(\mathcal{F} \wedge \mathcal{G})$  for all  $\mathcal{F}, \mathcal{G} \in \mathbb{F}_L^s(X)$ ;
- (Lp)  $\lim \mathcal{F}(x) = \bigwedge_{a \in L^X} (\mathcal{U}^x(a) \rightarrow \mathcal{F}(a))$ ;
- (Lt)  $\mathcal{U}^x \leq \mathcal{U}^x(\mathcal{U}^{(\cdot)})$ ;
- (LK) For all  $\mathcal{G} \in \mathbb{F}_L^s(X)$ ,  $\mathcal{F}_y \in \mathbb{F}_L^s(X)$  ( $y \in X$ ),  $x \in X$  :  
 $\lim \mathcal{G}(x) \wedge \bigwedge_{y \in X} \lim \mathcal{F}_y(y) \leq \lim \mathcal{G}(\mathcal{F}_{(\cdot)})(x)$ ;
- (LF) For all  $J$ ,  $\psi : J \rightarrow X$ ,  $\mathcal{G} \in \mathbb{F}_L^s(J)$ ,  $\mathcal{F}_i \in \mathbb{F}_L^s(X)$  ( $i \in J$ ),  $x \in X$  :  
 $\lim \psi(\mathcal{G})(x) \wedge \bigwedge_{i \in J} \lim \mathcal{F}_i(\psi(i)) \leq \lim \mathcal{G}(\mathcal{F}_{(\cdot)})(x)$ .

Note that  $\mathcal{U}^x(\mathcal{U}^{(\cdot)})(a) = \bigvee \{\mathcal{U}^x(b) \mid b(y) \leq \mathcal{U}^y(a) \text{ for all } y \in X\}$  for  $a \in L^X$ .

We call a space  $(X, \lim) \in |SL-GCS|$

- a *stratified L-limit space* if it satisfies the axiom (L3);
- a *stratified L-pretopological convergence space* if it satisfies the axiom (Lp);
- a *stratified L-topological convergence space* if it satisfies the axioms (Lp) and (Lt).

The subcategories of  $SL-GCS$  with the stratified  $L$ -limit spaces (resp. the stratified  $L$ -pretopological spaces, the stratified  $L$ -topological convergence spaces) as objects and the continuous mappings between them as morphisms are denoted by  $SL-LIM$  (resp.  $SL-PCS$ ,  $SL-TCS$ ). For more information on these categories we refer to [9]. It is shown in [8, 9] that the categories of stratified  $L$ -topological spaces [6] and of stratified  $L$ -topological convergence spaces are isomorphic.

### 3. Continuous Convergence

Let  $(X, \lim_X), (Y, \lim_Y) \in |SL-GCS|$  and  $H \subseteq M(X, Y) = \{\varphi \mid \varphi : X \rightarrow Y\}$  be a set of mappings from  $X$  to  $Y$ . Define for  $\mathcal{F} \in \mathbb{F}_L^s(H)$  and  $\varphi \in H$

$$c\text{-}\lim_H \mathcal{F}(\varphi) = \bigwedge_{\mathcal{G} \in \mathbb{F}_L^s(X)} \bigwedge_{x \in X} (\lim_X \mathcal{G}(x) \rightarrow \lim_Y \text{ev}_H(\mathcal{F} \times \mathcal{G})(\varphi(x)))$$

with the evaluation mapping  $\text{ev}_H : \begin{cases} H \times X & \rightarrow & Y \\ (\varphi, x) & \mapsto & \varphi(x) \end{cases}$ . Then  $c\text{-}\lim$  is called the *stratified L-convergence structure of continuous convergence on H*. It was shown in [8] that if  $H = C(X, Y) = \{\varphi : X \rightarrow Y : \varphi \text{ continuous}\}$ , then  $(H, c\text{-}\lim_H) \in |SL-GCS|$  and with this structure,  $SL-GCS$  has function spaces, i.e.  $SL-GCS$  is cartesian closed. We will show in the sequel that whenever  $H \not\subseteq C(X, Y)$ , then the axiom (L1) will not be satisfied for  $c\text{-}\lim_H$ . First we will show that  $c\text{-}\lim_H$  appears as a subspace structure. To this end, we denote  $c\text{-}\lim_{M(X, Y)}$  by  $c\text{-}\lim$  and  $\text{ev}_{M(X, Y)}$  simply by  $\text{ev}$ . We need a lemma for preparation.

**Lemma 3.1.** *Let  $(X, \lim_X), (Y, \lim_Y) \in |SL\text{-}GCS|$  and let  $H \subseteq M(X, Y)$ . If  $\mathcal{F} \in \mathbb{F}_L^s(H)$  and  $\mathcal{G} \in \mathbb{F}_L^s(X)$ , then  $\text{ev}([\mathcal{F}] \times \mathcal{G}) = \text{ev}_H(\mathcal{F} \times \mathcal{G})$ .*

*Proof.* Let  $b \in L^Y$ . Then

$$\text{ev}([\mathcal{F}] \times \mathcal{G})(b) = \bigvee \{ \mathcal{F}(f|_H) \wedge \mathcal{G}(g) \mid f \in L^{M(X, Y)}, g \in L^X \text{ s.t. } f \times g \leq \text{ev}^{\leftarrow}(b) \}.$$

Clearly, for  $f \in L^{M(X, Y)}$  and  $g \in L^X$  with  $f \times g \leq \text{ev}^{\leftarrow}(b)$  we have for  $\varphi \in H$  and  $x \in X$ ,  $(f|_H \times g)(\varphi, x) = f|_H(\varphi) \wedge g(x) = f(\varphi) \wedge g(x) \leq \text{ev}^{\leftarrow}(b)(\varphi, x) = b(\varphi(x)) = \text{ev}_H^{\leftarrow}(b)(\varphi, x)$ . If, further,  $\bar{f} \in L^H$ ,  $g \in L^X$  such that  $\bar{f} \times g \leq \text{ev}_H^{\leftarrow}(b)$ , then we define  $f^* \in L^{M(X, Y)}$  by  $f^*(\varphi) = \begin{cases} \bar{f}(\varphi) & \text{if } \varphi \in H \\ \perp & \text{else.} \end{cases}$ . Then  $f^*|_H = \bar{f}$

and  $(f^* \times g)(\varphi, x) = \begin{cases} (\bar{f} \times g)(\varphi, x) & \text{if } \varphi \in H \\ \perp & \text{if } \varphi \notin H \end{cases} \leq \text{ev}^{\leftarrow}(b)(\varphi, x)$ . Hence

$$\begin{aligned} \text{ev}([\mathcal{F}] \times \mathcal{G})(b) &\leq \bigvee \{ \mathcal{F}(f|_H) \wedge \mathcal{G}(g) \mid f|_H \times g \leq \text{ev}^{\leftarrow}(b) \} \\ &\leq \bigvee \{ \mathcal{F}(\bar{f}) \wedge \mathcal{G}(g) \mid \bar{f} \in L^H, g \in L^X \text{ s.t. } \bar{f} \times g \leq \text{ev}^{\leftarrow}(b) \} \\ &= \text{ev}_H(\mathcal{F} \times \mathcal{G})(b) \\ &\leq \bigvee \{ \mathcal{F}(f^*|_H) \wedge \mathcal{G}(g) \mid f^* \times g \leq \text{ev}^{\leftarrow}(b) \} \leq \text{ev}([\mathcal{F}] \times \mathcal{G})(b). \end{aligned}$$

□

We obtain as a direct consequence the following result.

**Corollary 3.2.** *Let  $(X, \lim_X), (Y, \lim_Y) \in |SL\text{-}GCS|$  and let  $H \subseteq M(X, Y)$ . Then  $\text{c-lim}_H = \text{c-lim}|_H$ , i.e. for  $\mathcal{F} \in \mathbb{F}_L^s(H)$  and  $\varphi \in H$  we have  $\text{c-lim}_H \mathcal{F}(\varphi) = \text{c-lim}[\mathcal{F}](\varphi)$  (with  $[\mathcal{F}](f) = \mathcal{F}(f|_H)$ , where  $f \in L^{M(X, Y)}$  and  $f|_H(\psi) = f(\psi)$  for  $\psi \in H$ ).*

**Lemma 3.3.** *Let  $\varphi : X \rightarrow Y$  and  $\mathcal{F} \in \mathbb{F}_L^s(X)$ . Then  $\varphi(\mathcal{F}) = \text{ev}([\varphi] \times \mathcal{F})$ .*

*Proof.* It was shown in [8] that  $\varphi(\mathcal{F}) \leq \text{ev}([\varphi] \times \mathcal{F})$ . To prove the converse inequality, we take  $b \in L^Y$ . Then

$$\begin{aligned} \text{ev}([\varphi] \times \mathcal{F})(b) &= ([\varphi] \times \mathcal{F})(\text{ev}^{\leftarrow}(b)) \\ &= \bigvee \{ [\varphi](f) \wedge \mathcal{F}(a) \mid f \times a \leq \text{ev}^{\leftarrow}(b) \} \\ &= \bigvee \{ f(\varphi) \wedge \mathcal{F}(a) \mid f \times a \leq \text{ev}^{\leftarrow}(b) \} \\ &\leq \bigvee \{ \mathcal{F}((f(\varphi))_X \wedge a) \mid f \times a \leq \text{ev}^{\leftarrow}(b) \}. \end{aligned}$$

If  $f \times a \leq \text{ev}^{\leftarrow}(b)$  then for all  $\varphi \in M(X, Y)$  and  $x \in X$  we have  $f(\varphi) \wedge a(x) \leq b(\varphi(x)) = \varphi^{\leftarrow}(b)(x)$ . Hence

$$\text{ev}([\varphi] \times \mathcal{F})(b) \leq \bigvee \{ \mathcal{F}((f(\varphi))_X \wedge a) \mid (f(\varphi))_X \wedge a \leq \varphi^{\leftarrow}(b) \} \leq \mathcal{F}(\varphi^{\leftarrow}(b)) = \varphi(\mathcal{F})(b)$$

□

Again, application of Lemma 3.1 and observing that for  $\varphi \in H \subseteq M(X, Y)$  we have  $[[\varphi]]_{M(X, Y)} = [\varphi]$ , yields the following Corollary.

**Corollary 3.4.** *For any  $H \subseteq M(X, Y)$ ,  $\varphi \in H$  we have  $\varphi(\mathcal{F}) = \text{ev}_H([\varphi] \times \mathcal{F})$ .*

We come to the main result of this section.

**Lemma 3.5.** *Let  $(X, \lim_X), (Y, \lim_Y) \in |SL\text{-}GCS|$  and let  $H \not\subseteq C(X, Y)$ . Then  $(H, \text{c-lim}_H)$  does not satisfy the axiom (L1).*

*Proof.* We choose  $\varphi \in H$  not continuous. Then there is  $\mathcal{F} \in \mathbb{F}_L^s(X)$  and  $x \in X$  such that

$$\top > \lim_X \mathcal{F}(x) \rightarrow \lim_Y \varphi(\mathcal{F})(\varphi(x)) = \lim_X \mathcal{F}(x) \rightarrow \lim_Y \text{ev}([\varphi] \times \mathcal{F})(\varphi(x)).$$

Hence  $\text{c-lim}_H[\varphi](\varphi) \leq \lim_X \mathcal{F}(x) \rightarrow \lim_Y \text{ev}([\varphi] \times \mathcal{F})(\varphi(x)) < \top$   $\square$

This result was already observed for  $L = \{0, 1\}$  by Poppe [16]. For  $L \neq \{0, 1\}$  it makes sense to ask “how close is  $(H, \text{c-lim})$  to satisfy (L1)”? We will measure this “closeness” by attaching a grade  $\alpha \in L$ . This leads in a natural way to the concept of a lattice-valued category in the sense of Šostak [18].

#### 4. A Lattice-valued Category of Stratified $L$ -convergence Spaces

In order to develop a theory for such an  $L$ -category of  $L$ -convergence spaces, we consider *stratified  $L$ -preconvergence spaces*  $(X, \lim)$ , where  $X$  is a set and  $\lim : \mathbb{F}_L^s(X) \rightarrow L^X$  is a mapping. Note that we do not require any of the axioms (L1), (L2) etc. for this mapping. We denote by  $|SL\text{-}PreCS|$  the class of all these stratified  $L$ -preconvergence spaces. Together with  $\text{Mor}(SL\text{-}PreCS)$ , the class of mappings between the underlying sets of objects in  $|SL\text{-}PreCS|$ , then obviously  $(|SL\text{-}PreCS|, \text{Mor}(SL\text{-}PreCS), \circ)$  forms a category which we shall denote  $SL\text{-}PreCS$ . We define now the following  $L$ -classes  $\omega_{L1}, \omega_{L2}, \omega : |SL\text{-}PreCS| \rightarrow L$ :

$$\begin{aligned} \omega_{L1}(X, \lim) &= \bigwedge_{x \in X} \lim[x](x); \\ \omega_{L2}(X, \lim) &= \bigwedge \{\lim \mathcal{F}(x) \rightarrow \lim \mathcal{G}(x) \mid \mathcal{F}, \mathcal{G} \in \mathbb{F}_L^s(X) \text{ s.t. } \mathcal{F} \leq \mathcal{G}, x \in X\}; \\ \omega(X, \lim) &= \omega_{L1}(X, \lim) \wedge \omega_{L2}(X, \lim). \end{aligned}$$

Apparently, if  $\omega_{L1}(X, \lim) = \top$ , then  $(X, \lim)$  satisfies the axiom (L1) and if  $\omega_{L2}(X, \lim) = \top$  then  $(X, \lim)$  satisfies the axiom (L2). We therefore interpret  $\omega_{L1}(X, \lim)$  as *grade to which  $(X, \lim)$  satisfies the axiom (L1)* and  $\omega_{L2}(X, \lim)$  as *grade to which  $(X, \lim)$  satisfies the axiom (L2)*. The interpretation of  $\omega(X, \lim)$  is therefore the *grade to which  $(X, \lim)$  is a stratified  $L$ -generalized convergence space*. Further, we define the following two mappings  $\mu_1, \mu : \text{Mor}(SL\text{-}PreCS) \rightarrow L$ . For  $\varphi : (X, \lim_X) \rightarrow (Y, \lim_Y)$  we define

$$\begin{aligned} \mu_1(\varphi) &= \bigwedge \{\lim_X \mathcal{F}(x) \rightarrow \lim_Y \varphi(\mathcal{F})(\varphi(x)) \mid \mathcal{F} \in \mathbb{F}_L^s(X), x \in X\}; \\ \mu(\varphi) &= \mu_1(\varphi) \wedge \omega(X, \lim_X) \wedge \omega(Y, \lim_Y). \end{aligned}$$

Clearly, if  $\mu(\varphi) = \top$ , then  $\varphi$  is a continuous mapping between the stratified  $L$ -generalized convergence spaces  $(X, \lim_X)$  and  $(Y, \lim_Y)$ . We interpret  $\mu(\varphi)$  (and, by lack of a better word, also  $\mu_1(\varphi)$ ) as the *grade of continuity of the mapping  $\varphi$* . Similar concepts have been studied before, in the realm of  $L$ -topological spaces see

e.g. [17, 25], in the realm of fuzzy convergence spaces see [7] and for  $L$ -convergence spaces see [24].

There is an interesting connection between  $\mu_1$  and the structure of continuous convergence.

**Lemma 4.1.** *Let  $\varphi \in \text{Mor}(SL\text{-PreCS})$ , i.e.  $\varphi : (X, \lim_X) \longrightarrow (Y, \lim_Y)$ . Then*

$$\mu_1(\varphi) = \text{c-lim}[\varphi](\varphi).$$

Note that according to Lemma 3.1 it is not important which  $H \subseteq M(X, Y)$  we choose (as long as  $\varphi \in H$ ).

*Proof.* We have with Lemma 3.2

$$\begin{aligned} \mu_1(\varphi) &= \bigwedge_{\mathcal{F}, x} (\lim_X \mathcal{F}(x) \rightarrow \lim_Y \varphi(\mathcal{F})(\varphi(x))) \\ &= \bigwedge_{\mathcal{F}, x} (\lim_X \mathcal{F}(x) \rightarrow \lim_Y \text{ev}([\varphi] \times \mathcal{F})(\varphi(x))) = \text{c-lim}[\varphi](\varphi). \end{aligned}$$

□

In this way, the structure of continuous convergence can be used as a “measure of continuity” of a mapping. It also reveals a connection between the axiom (L1) and the continuity of mappings.

**Example 4.2.** We look at two stratified  $L$ -generalized convergence spaces  $(X, \lim_X)$  and  $(Y, \lim_Y) \in |SL\text{-GCS}|$ . We endow  $H \subseteq M(X, Y)$  with the structure of continuous convergence. Note that  $(H, \text{c-lim})$  always satisfies the axiom (L2). From Lemma 3.2, however, it is clear that if  $H \not\subseteq C(X, Y)$ , then the axiom (L1) is not satisfied. For the grade of (L1) of  $(H, \text{c-lim})$  we find

$$\omega_{L1}(H, \text{c-lim}) = \bigwedge_{\varphi \in H} \text{c-lim}[\varphi](\varphi) = \bigwedge_{\varphi \in H} \mu_1(\varphi) = \bigwedge_{\varphi \in H} \mu(\varphi).$$

(The last equality is true because  $(X, \lim_X)$  and  $(Y, \lim_Y)$  satisfy both (L1) and (L2).) This result can be interpreted as follows: The grade to which  $(H, \text{c-lim})$  satisfies the axiom (L1) is the infimum of the grades of continuity of all mappings  $\varphi \in H$ . This shows again that only if all mappings  $\varphi \in H$  are continuous, then  $(H, \text{c-lim})$  satisfies (L1).

We look next at some special cases which simplify the evaluation of  $\mu(\varphi)$ .

**Lemma 4.3.** *Let  $\varphi : (X, \lim_X) \longrightarrow (Y, \lim_Y)$  be a morphism in  $L\text{-SL-GCS}$ .*

- (1) *If  $\omega_{L1}(Y, \lim_Y) = \top$  then  $\mu_1(\varphi) \leq \omega_{L1}(X, \lim_X)$  and, consequently,  $\mu(\varphi) = \mu_1(\varphi) \wedge \omega_{L2}(X, \lim_X) \wedge \omega_{L2}(Y, \lim_Y)$ .*
- (2) *If  $\varphi$  is surjective, then  $\mu_1(\varphi) \leq \omega_{L1}(X, \lim_X) \rightarrow \omega_{L1}(Y, \lim_Y)$ .*

*Proof.* (1) We have  $\top = \bigwedge_{y \in Y} \lim_Y [y](y) \leq \bigwedge_{x \in X} \lim_Y [\varphi(x)](\varphi(x))$ . Consequently,

$$\begin{aligned} \mu_1(\varphi) &= \bigwedge_{\mathcal{F}, x} (\lim_X \mathcal{F}(x) \rightarrow \lim_Y \varphi(\mathcal{F})(\varphi(x))) \\ &\leq \bigwedge_{x \in X} (\lim_X [x](x) \rightarrow \lim_Y [\varphi(x)](\varphi(x))) \\ &\leq \bigwedge_{x \in X} \left( \left( \bigwedge_{z \in X} \lim_X [z](z) \right) \rightarrow \lim_Y [\varphi(x)](\varphi(x)) \right) \\ &= \omega_{L1}(X, \lim_X) \rightarrow \bigwedge_{x \in X} \lim_Y [\varphi(x)](\varphi(x)) \\ &= \omega_{L1}(X, \lim_X) \rightarrow \top = \omega_{L1}(X, \lim_X). \end{aligned}$$

(2) Follows in a similar way using  $\bigwedge_{x \in X} \lim_Y [\varphi(x)](\varphi(x)) = \bigwedge_{y \in Y} \lim_Y [y](y)$  for a surjective mapping  $\varphi$ .  $\square$

**Lemma 4.4.**  $(|SL-PreCS|, \omega, \text{Mor}(SL-PreCS), \mu, \circ)$  is an  $L$ -category in Šostak's sense.

*Proof.* It has already been mentioned that  $(|SL-PreCS|, \text{Mor}(SL-PreCS), \circ)$  is a category. So we only need to check the axioms (LCat1), (LCat2) and (LCat3). (LCat1) is satisfied by the definition of  $\mu$ . For (LCat2) it is sufficient to show that  $\mu_1(\varphi) \wedge \mu_1(\psi) \leq \mu_1(\psi \circ \varphi)$  for two mappings  $\varphi : (X, \lim_X) \rightarrow (Y, \lim_Y)$  and  $\psi : (Y, \lim_Y) \rightarrow (Z, \lim_Z)$ . We have, using Lemma 1.1(4),

$$\begin{aligned} &\mu_1(\varphi) \wedge \mu_1(\psi) \\ &\leq \bigwedge_{\mathcal{F}, x} [(\lim_X \mathcal{F}(x) \rightarrow \lim_Y \varphi(\mathcal{F})(\varphi(x))) \wedge (\lim_Y \varphi(\mathcal{F})(\varphi(x)) \\ &\hspace{15em} \rightarrow \lim_Z \psi(\varphi(\mathcal{F}))(\psi(\varphi(x))))] \\ &\leq \bigwedge_{\mathcal{F}, x} [\lim_X \mathcal{F}(x) \rightarrow \lim_Z \psi \circ \varphi(\mathcal{F})(\psi \circ \varphi(x))] = \mu_1(\psi \circ \varphi). \end{aligned}$$

(LCat3) is obvious as  $\mu_1(id_X) = \top$ .  $\square$

We call  $(|SL-PreCS|, \omega, \text{Mor}(SL-PreCS), \mu, \circ)$  the  $L$ -category of stratified  $L$ -generalized convergence spaces and denote it by  $L\text{-}SL\text{-}GCS$ .

The  $L$ -category  $L\text{-}SL\text{-}GCS$  allows initial constructions in the following sense.

**Lemma 4.5.** Let for each  $i \in J$ ,  $(X_i, \lim_i) \in |SL-PreCS|$  and let  $X$  be a set. Let further  $\varphi_i : X \rightarrow X_i$  be mappings ( $i \in J$ ). Then there is exactly one structure  $\lim : \mathbb{F}_L^s(X) \rightarrow L^X$  with the following properties.

- (1)  $\omega(X, \lim) \geq \bigwedge_{i \in J} \omega(X_i, \lim_i)$ ;
- (2) for any  $\psi : (Y, \lim_Y) \rightarrow (X, \lim)$  (so that  $\psi \in \text{Mor}(SL-PreCS)$ ) we have  $\mu_1(\psi) = \bigwedge_{i \in J} \mu_1(\varphi_i \circ \psi)$ .

Note that the property (2) entails  $\mu(\psi) \geq \bigwedge_{i \in J} \mu(\varphi_i \circ \psi)$  but that this latter condition does not seem to guarantee the uniqueness of the structure  $\lim$ .

*Proof.* We define for  $\mathcal{F} \in \mathbb{F}_L^s(X)$  and  $x \in X$ ,

$$\lim \mathcal{F}(x) = \bigwedge_{i \in J} \lim_i \varphi_i(\mathcal{F})(\varphi_i(x)).$$

Then we have

$$\begin{aligned} \omega_{L1}(X, \lim) &= \bigwedge_{x \in X} \bigwedge_{i \in J} \lim_i [\varphi_i(x)](\varphi_i(x)) \\ &= \bigwedge_{i \in J} \bigwedge_{x \in X} \lim_i [\varphi_i(x)](\varphi_i(x)) \geq \bigwedge_{i \in J} \omega_{L1}(X_i, \lim_i). \end{aligned}$$

Further we find

$$\begin{aligned} \omega_{L2}(X, \lim) &= \bigwedge_{\mathcal{F} \leq \mathcal{G}, x} \left[ \left( \bigwedge_{i \in J} \lim_i \varphi_i(\mathcal{F})(\varphi_i(x)) \right) \rightarrow \left( \bigwedge_{j \in J} \lim_j \varphi_j(\mathcal{G})(\varphi_j(x)) \right) \right] \\ &= \bigwedge_{j \in J} \bigwedge_{\mathcal{F} \leq \mathcal{G}, x} \left[ \left( \bigwedge_{i \in J} \lim_i \varphi_i(\mathcal{F})(\varphi_i(x)) \right) \rightarrow \lim_j \varphi_j(\mathcal{G})(\varphi_j(x)) \right] \\ &\geq \bigwedge_{j \in J} \bigwedge_{\mathcal{F} \leq \mathcal{G}, x} [\lim_j \varphi_j(\mathcal{F})(\varphi_j(x)) \rightarrow \lim_j \varphi_j(\mathcal{G})(\varphi_j(x))] \\ &\geq \bigwedge_{j \in J} \bigwedge_{\mathcal{F}_j \leq \mathcal{G}_j, x_j} [\lim_j \mathcal{F}_j(x_j) \rightarrow \lim_j \mathcal{G}_j(x_j)] \\ &= \bigwedge_{j \in J} \omega_{L2}(X_j, \lim_j). \end{aligned}$$

From this we obtain (1) as

$$\begin{aligned} \omega(X, \lim) &\geq \bigwedge_{i \in J} \omega_{L1}(X_i, \lim_i) \wedge \bigwedge_{i \in J} \omega_{L2}(X_i, \lim_i) \\ &= \bigwedge_{i \in J} (\omega_{L1}(X_i, \lim_i) \wedge \omega_{L2}(X_i, \lim_i)) = \bigwedge_{i \in J} \omega(X_i, \lim_i). \end{aligned}$$

For a mapping  $\psi : (Y, \lim_Y) \rightarrow (X, \lim_X)$  we have further

$$\begin{aligned} \mu_1(\psi) &= \bigwedge_{\mathcal{F} \in \mathbb{F}_L^s(Y), y \in Y} [\lim_Y \mathcal{F}(y) \rightarrow \lim \psi(\mathcal{F})(\psi(y))] \\ &= \bigwedge_{\mathcal{F} \in \mathbb{F}_L^s(Y), y \in Y} \left[ \lim_Y \mathcal{F}(y) \rightarrow \bigwedge_{i \in J} \lim_i \varphi_i(\psi(\mathcal{F}))(\varphi_i(\psi(y))) \right] \\ &= \bigwedge_{i \in J} \bigwedge_{\mathcal{F} \in \mathbb{F}_L^s(Y), y \in Y} [\lim_Y \mathcal{F}(y) \rightarrow \lim_i \varphi_i \circ \psi(\mathcal{F})(\varphi_i \circ \psi(y))] \\ &= \bigwedge_{i \in J} \mu_1(\varphi_i \circ \psi). \end{aligned}$$

To show the uniqueness of  $\lim$ , we assume that there is a second structure,  $\overline{\lim}$ , on  $X$  with the properties (1) and (2). From the following diagram (where  $id_1 :$

$$(X, \lim) \longrightarrow (X, \overline{\lim}), x \longmapsto x$$

$$\begin{array}{ccc} (X, \lim) & \xrightarrow{id_1} & (X, \overline{\lim}) \\ & \searrow & \downarrow \varphi_i \\ & & (X_i, \lim_i) \end{array}$$

we see that  $\varphi_i = \varphi_i \circ id_1$  and hence  $\mu_1(\varphi_i \circ id_1) = \top$  for every  $i \in J$ . But this means that  $\mu_1(id_1) = \bigwedge_{i \in J} \mu_1(\varphi_i \circ id_1) = \top$  and hence for every  $\mathcal{F} \in \mathbb{F}_L^s(X)$  and every  $x \in X$ ,  $\lim \mathcal{F}(x) \leq \overline{\lim} \mathcal{F}(x)$ . On the other hand, the following diagram (with  $id_2 : (X, \overline{\lim}) \longrightarrow (X, \overline{\lim}), x \longmapsto x$ )

$$\begin{array}{ccc} (X, \overline{\lim}) & \xrightarrow{id_2} & (X, \overline{\lim}) \\ & \searrow & \downarrow \varphi_i \\ & & (X_i, \lim_i) \end{array}$$

shows that  $\varphi_i \circ id_2 = \varphi_i$  and hence  $\top = \mu_1(id_2) = \bigwedge_{i \in J} \mu_1(\varphi_i \circ id_2) = \bigwedge_{i \in J} \mu_1(\varphi_i)$ . This implies  $\mu_1(\varphi_i) = \top$  for every  $i \in J$ . But this means that for all  $\mathcal{F} \in \mathbb{F}_L^s(X)$  and all  $x \in X$  we have  $\overline{\lim} \mathcal{F}(x) \leq \lim_{i \in J} \varphi_i(\mathcal{F})(\varphi_i(x))$  and therefore also  $\overline{\lim} \mathcal{F}(x) \leq \lim \mathcal{F}(x)$ . This completes the proof.  $\square$

Note that if all  $\varphi_i$  are surjective, the proof of (1) shows that  $\omega_{L1}(X, \lim) = \bigwedge_{i \in J} \omega_{L1}(X_i, \lim_i)$ . This applies especially to product spaces where the  $\varphi_i$  are the projection mappings.

We describe the properties of function spaces in the  $L$ -category  $L\text{-}SL\text{-}GCS$  using the structure of continuous convergence. Let  $(X, \lim_X), (Y, \lim_Y) \in |L\text{-}SL\text{-}GCS|$ . For  $H = M(X, Y)$  we define, as before,

$$c\text{-}\lim \mathcal{F}(\varphi) = \bigwedge_{\mathcal{G} \in \mathbb{F}_L^s(X)} \bigwedge_{x \in X} (\lim_X \mathcal{G}(x) \rightarrow \lim_Y \text{ev}(\mathcal{F} \times \mathcal{G})(\varphi(x)))$$

where  $\mathcal{F} \in \mathbb{F}_L^s(H)$  and  $\varphi \in H$  and  $\text{ev} : H \times X \longrightarrow Y$ ,  $(\varphi, x) \longmapsto \varphi(x)$  is the evaluation mapping.

**Lemma 4.6.** *For all  $(X, \lim_X), (Y, \lim_Y) \in |L\text{-}SL\text{-}GCS|$  and  $H = M(X, Y)$  we have*

- (1)  $\omega_{L1}(H, c\text{-}\lim) = \bigwedge_{\varphi \in H} \mu_1(\varphi)$  and  $\omega_{L2}(H, c\text{-}\lim) \geq \omega_{L2}(Y, \lim_Y)$ .  
Consequently,  $\omega(H, c\text{-}\lim) \geq \bigwedge_{\varphi \in H} \mu(\varphi)$ .
- (2)  $\mu_1(\text{ev}) \geq \omega_{L2}(Y, \lim_Y)$  and, consequently,  $\mu(\text{ev}) = \omega(H \times X, c\text{-}\lim \times \lim_X) \wedge \omega(Y, \lim_Y)$ .
- (3) For  $\varphi : (X, \lim_X) \times (Y, \lim_Y) \longrightarrow (Z, \lim_Z)$  there exists exactly one mapping  $\widehat{\varphi} : (X, \lim_X) \longrightarrow (\text{Mor}(Y, Z), c\text{-}\lim)$  with
  - (i)  $\text{ev} \circ (\widehat{\varphi} \times id_Y) = \varphi$  and
  - (ii)  $\mu_1(\widehat{\varphi}) \geq \mu_1(\varphi) \wedge \omega_{L2}(X, \lim_X) \wedge \omega_{L2}(Y, \lim_Y)$ .

*Proof.* (1) As in Example 4.2, we see that  $\omega_{L1}(H, \text{c-lim}) = \bigwedge_{\varphi \in H} \mu_1(\varphi)$ . Further we have

$$\begin{aligned}
\omega_{L2}(H, \text{c-lim}) &= \bigwedge_{\mathcal{F} \leq \mathcal{G}, \varphi} [\text{c-lim} \mathcal{F}(\varphi) \rightarrow \text{c-lim} \mathcal{G}(\varphi)] \\
&\geq \bigwedge_{\mathcal{F} \leq \mathcal{G}, \varphi} \bigwedge_{\mathcal{H}, x} [(\lim_X \mathcal{H}(x) \rightarrow \lim_Y \text{ev}(\mathcal{H} \times \mathcal{F})(\varphi(x))) \\
&\quad \rightarrow (\lim_X \mathcal{H}(x) \rightarrow \lim_Y \text{ev}(\mathcal{H} \times \mathcal{G})(\varphi(x)))] \\
&\geq \bigwedge_{\mathcal{F} \leq \mathcal{G}, \varphi} \bigwedge_{\mathcal{H}, x} [\lim_Y \text{ev}(\mathcal{H} \times \mathcal{F})(\varphi(x)) \rightarrow \lim_Y \text{ev}(\mathcal{H} \times \mathcal{G})(\varphi(x))] \\
&\geq \bigwedge_{\mathcal{F}' \leq \mathcal{G}', y} [\lim_Y \mathcal{F}'(y) \rightarrow \lim_Y \mathcal{G}'(y)] \\
&= \omega_{L2}(Y, \lim_Y).
\end{aligned}$$

Consequently,

$$\omega(H, \text{c-lim}) = \omega_{L1}(H, \text{c-lim}) \wedge \omega_{L2}(H, \text{c-lim}) \geq \bigwedge_{\varphi \in H} \mu_1(\varphi) \wedge \omega_{L2}(Y, \lim_Y) \geq \bigwedge_{\varphi \in H} \mu(\varphi).$$

(2) We have

$$\begin{aligned}
\mu_1(\text{ev}) &= \bigwedge_{\mathcal{H}, (\varphi, x)} [(\text{c-lim} \times \lim_X) \mathcal{H}(\varphi, x) \rightarrow \lim_Y \text{ev}(\mathcal{H})(\varphi(x))] \\
&= \bigwedge_{\mathcal{H}, (\varphi, x)} [(\text{c-lim} \text{pr}_H(\mathcal{H})(\varphi) \wedge \lim_X \text{pr}_X(\mathcal{H})(x)) \rightarrow \lim_Y \text{ev}(\mathcal{H})(\varphi(x))] \\
&\geq \bigwedge_{\mathcal{H}, (\varphi, x)} [((\lim_X \text{pr}_X(\mathcal{H})(x) \rightarrow \lim_Y \text{ev}(\text{pr}_H(\mathcal{H}) \times \text{pr}_X(\mathcal{H}))(\varphi(x))) \\
&\quad \wedge \lim_X \text{pr}_X(\mathcal{H})(x) \rightarrow \lim_Y \text{ev}(\mathcal{H})(\varphi(x))] \\
&\geq \bigwedge_{\mathcal{H}, (\varphi, x)} \left[ \lim_Y \text{ev}(\underbrace{\text{pr}_H(\mathcal{H}) \times \text{pr}_X(\mathcal{H})}_{\leq \mathcal{H}})(\varphi(x)) \rightarrow \lim_Y \text{ev}(\mathcal{H})(\varphi(x)) \right] \\
&\geq \bigwedge_{\mathcal{F}' \leq \mathcal{G}', y} [\lim_Y \mathcal{F}'(y) \rightarrow \lim_Y \mathcal{G}'(y)] \\
&= \omega_{L2}(Y, \lim_Y).
\end{aligned}$$

As a consequence we obtain  $\mu(\text{ev}) = \omega(H \times X, \text{c-lim} \times \lim_X) \wedge \omega(Y, \lim_Y)$ , as  $\mu_1(\text{ev}) \geq \omega_{L2}(Y, \lim_Y) \geq \omega(Y, \lim_Y)$ .

(3) We define for  $\varphi : X \times Y \rightarrow Z$  and for  $x \in X$  the mapping  $\varphi_x : Y \rightarrow Z, y \mapsto \varphi(x, y)$  and with this the mapping  $\widehat{\varphi} : X \rightarrow \text{Mor}(Y, Z), x \mapsto \varphi_x$ . Clearly then  $\text{ev} \circ (\widehat{\varphi} \times \text{id}_Y)(x, y) = \text{ev}(\widehat{\varphi}(x), y) = \widehat{\varphi}(x)(y) = \varphi(x, y)$ . From this also the

uniqueness of  $\widehat{\varphi}$  follows. Moreover, we have

$$\begin{aligned}\mu_1(\widehat{\varphi}) &= \bigwedge_{\mathcal{F}, x} [\lim_X \mathcal{F}(x) \rightarrow \text{c-lim} \widehat{\varphi}(\mathcal{F})(\widehat{\varphi}(x))] \\ &= \bigwedge_{\mathcal{F}, x} \left[ \lim_X \mathcal{F}(x) \rightarrow \left( \bigwedge_{\mathcal{G}, y} (\lim_Y \mathcal{G}(y) \rightarrow \lim_Z \text{ev}(\widehat{\varphi}(\mathcal{F}) \times \mathcal{G})(\widehat{\varphi}(x)(y))) \right) \right] \\ &= \bigwedge_{\mathcal{G}, y} \bigwedge_{\mathcal{F}, x} [\lim_X \mathcal{F}(x) \rightarrow (\lim_Y \mathcal{G}(y) \rightarrow \lim_Z \varphi(\mathcal{F} \times \mathcal{G})(\varphi(x, y)))] .\end{aligned}$$

From

$$\begin{aligned}\mu_1(\varphi) &= \bigwedge_{\mathcal{H} \in \mathbb{F}_L^s(X \times Y), (x, y)} [(\lim_X \times \lim_Y) \mathcal{H}(x, y) \rightarrow \lim_Z \varphi(\mathcal{H})(\varphi(x, y))] \\ &\leq \bigwedge_{\mathcal{F}, \mathcal{G}, x, y} [(\lim_X \times \lim_Y) \mathcal{F} \times \mathcal{G}(x, y) \rightarrow \lim_Z \varphi(\mathcal{F} \times \mathcal{G})(\varphi(x, y))]\end{aligned}$$

we conclude

$$\lim_Z \varphi(\mathcal{F} \times \mathcal{G})(\varphi(x, y)) \geq \mu_1(\varphi) \wedge ((\lim_X \times \lim_Y) \mathcal{F} \times \mathcal{G}(x, y))$$

for all  $\mathcal{F} \in \mathbb{F}_L^s(X)$ ,  $\mathcal{G} \in \mathbb{F}_L^s(Y)$  and  $x \in X$ ,  $y \in Y$ . Further, from  $\mathcal{F} \leq \text{pr}_X(\mathcal{F} \times \mathcal{G})$  and  $\mathcal{G} \leq \text{pr}_Y(\mathcal{F} \times \mathcal{G})$  we conclude

$$\omega_{L2}(X, \lim_X) \leq \lim_X \mathcal{F}(x) \rightarrow \lim_X \text{pr}_X(\mathcal{F} \times \mathcal{G})(x)$$

and

$$\omega_{L2}(Y, \lim_Y) \leq \lim_Y \mathcal{G}(y) \rightarrow \lim_Y \text{pr}_Y(\mathcal{F} \times \mathcal{G})(y),$$

and hence

$$\omega_{L2}(X, \lim_X) \wedge \lim_X \mathcal{F}(x) \leq \lim_X \text{pr}_X(\mathcal{F} \times \mathcal{G})(x)$$

and

$$\omega_{L2}(Y, \lim_Y) \wedge \lim_Y \mathcal{G}(y) \leq \lim_Y \text{pr}_Y(\mathcal{F} \times \mathcal{G})(y).$$

Together with

$$(\lim_X \times \lim_Y)(\mathcal{F} \times \mathcal{G})(x, y) = \lim_X \text{pr}_X(\mathcal{F} \times \mathcal{G})(x) \wedge \lim_Y \text{pr}_Y(\mathcal{F} \times \mathcal{G})(y)$$

this leads to

$$\begin{aligned}&\mu_1(\widehat{\varphi}) \\ &\geq \bigwedge_{\mathcal{F} \in \mathbb{F}_L^s(X), x \in X} \bigwedge_{\mathcal{G} \in \mathbb{F}_L^s(Y), y \in Y} [\lim_X \mathcal{F}(x) \rightarrow (\lim_Y \mathcal{G}(y) \rightarrow \lim_Z \text{ev}(\widehat{\varphi}(\mathcal{F}) \times \mathcal{G})(\widehat{\varphi}(x)(y)))] \\ &\quad \rightarrow [\mu_1(\varphi) \wedge \omega_{L2}(X, \lim_X) \wedge \lim_X \mathcal{F}(x) \wedge \omega_{L2}(Y, \lim_Y) \wedge \lim_Y \mathcal{G}(y)] \\ &\geq \bigwedge_{\mathcal{F} \in \mathbb{F}_L^s(X), x \in X} \bigwedge_{\mathcal{G} \in \mathbb{F}_L^s(Y), y \in Y} [\lim_X \mathcal{F}(x) \rightarrow \lim_Z \text{ev}(\widehat{\varphi}(\mathcal{F}) \times \mathcal{G})(\widehat{\varphi}(x)(y))] \\ &\quad \rightarrow [\mu_1(\varphi) \wedge \omega_{L2}(X, \lim_X) \wedge \lim_X \mathcal{F}(x) \wedge \omega_{L2}(Y, \lim_Y)] \\ &\geq \mu_1(\varphi) \wedge \omega_{L2}(X, \lim_X) \wedge \omega_{L2}(Y, \lim_Y).\end{aligned}$$

□

### 5. $L$ -subcategories of $L$ - $SL$ - $GCS$

There are at least two natural ways for obtaining  $L$ -subcategories of  $L$ - $SL$ - $GCS$  (and combinations of these). The first way is to *enforce axioms*. We will discuss this for the (L2)-axiom. (The requirement of this axiom for a convergence space seems more natural than the requirement of (L1) as was shown in Section 3. In fact, we do not know of any meaningful example of a ( $L$ -)convergence structure where the axiom (L2) is not satisfied.) We define

$$\bar{\omega}(X, \lim) = \begin{cases} \omega_{L1}(X, \lim) & \text{if } \omega_{L2}(X, \lim) = \top \\ \perp & \text{else.} \end{cases}$$

Then clearly  $\bar{\omega}(X, \lim) \leq \omega(X, \lim)$  and hence we obtain the  $L$ -subcategory

$$L\text{-}SL\text{-}L2CS = (|SL\text{-}PreCS|, \bar{\omega}, \text{Mor}(SL\text{-}PreCS), \mu)$$

of  $L$ - $SL$ - $GCS$ . If we denote by  $|SL\text{-}L2CS|$  the class of all stratified  $L$ -preconvergence spaces which satisfy (L2), then we can identify  $L\text{-}SL\text{-}L2CS$  with the following  $L$ -category,  $(|SL\text{-}L2CS|, \omega_{L1}, \text{Mor}(SL\text{-}L2CS), \mu)$ , where  $\text{Mor}(SL\text{-}L2CS)$  is the class of mappings between the underlying sets of the spaces in  $|SL\text{-}L2CS|$ . More precisely, we have  $\bar{\omega} = \top_{SL\text{-}L2CS} \wedge \omega_{L1}$ . For this subcategory, some results from the previous section can be stated in a nicer form.

**Lemma 5.1.** *Let  $\varphi : (X, \lim_X) \rightarrow (Y, \lim_Y)$  be a morphism in  $L\text{-}SL\text{-}L2CS$ . If  $\omega_{L1}(Y, \lim_Y) = \top$  then  $\mu(\varphi) = \mu_1(\varphi)$ .*

**Lemma 5.2.** *For all  $(X, \lim_X), (Y, \lim_Y) \in |L\text{-}SL\text{-}L2CS|$  and  $H = M(X, Y)$  we have*

- (1)  $\omega_{L1}(H, c\text{-}\lim) = \bigwedge_{\varphi \in H} \mu_1(\varphi)$  and, consequently,  $\omega(H, c\text{-}\lim) \geq \bigwedge_{\varphi \in H} \mu(\varphi)$ .
- (2)  $\mu_1(\text{ev}) = \top$  and, consequently,  $\mu(\text{ev}) = \omega(H \times X, c\text{-}\lim \times \lim_X) \wedge \omega(Y, \lim_Y)$ .
- (3) For  $\varphi : (X, \lim_X) \times (Y, \lim_Y) \rightarrow (Z, \lim_Z)$  there exists exactly one mapping  $\hat{\varphi} : (X, \lim_X) \rightarrow (\text{Mor}(Y, Z), c\text{-}\lim)$  with  $\text{ev} \circ (\hat{\varphi} \times \text{id}_Y) = \varphi$  and  $\mu_1(\hat{\varphi}) \geq \mu_1(\varphi)$ .

Note here, that  $(H, c\text{-}\lim)$  satisfies the axiom (L2) whenever  $(Y, \lim_Y)$  satisfies this axiom.

If we additionally enforce the axiom (L1) by defining  $\bar{\bar{\omega}} = \top_{SL\text{-}GCS}$ , then we end up in a fuzzy subcategory, where we consider only grades of continuity of potential morphisms between the underlying sets of stratified  $L$ -generalized convergence spaces. Note, however, that with such an approach, we can no longer formulate Lemma 4.6 as  $\omega_{L1}(H, c\text{-}\lim) \neq \top$  for  $H = M(X, Y)$ .

Another possibility for obtaining  $L$ -subcategories in the spirit of this approach is to demand  $\mu_1(\varphi) = \top$ , i.e. to consider as morphisms only those which satisfy the continuity definition. We then consider only the ‘‘fuzziness’’ of the spaces.

Another, general approach for obtaining  $L$ -subcategories of  $L$ - $SL$ - $GCS$  is to *add further axioms*. We consider the following  $L$ -classes.

$$\begin{aligned} \omega_{L3}(X, \lim) &= \bigwedge_{n \in \mathbf{N}} \bigwedge_{\mathcal{F}_1, \dots, \mathcal{F}_n \in \mathbb{F}_L^s(X)} \bigwedge_{x \in X} [(\lim \mathcal{F}_1(x) \wedge \dots \wedge \lim \mathcal{F}_n(x)) \\ &\quad \rightarrow \lim(\mathcal{F}_1 \wedge \dots \wedge \mathcal{F}_n)(x)] \\ \omega_{Lp}(X, \lim) &= \bigwedge_{\mathcal{F} \in \mathbb{F}_L^s(X)} \bigwedge_{x \in X} [\overline{\lim} \mathcal{F}(x) \rightarrow \lim \mathcal{F}(x)], \end{aligned}$$

with

$$\overline{\lim}\mathcal{F}(x) = \bigwedge_{a \in L^X} \left[ \left( \bigwedge_{\mathcal{G} \in \mathbb{F}_L^s(X)} (\lim \mathcal{G}(x) \rightarrow \mathcal{G}(a)) \right) \rightarrow \mathcal{F}(a) \right].$$

We note that a space  $(X, \lim) \in |SL\text{-}PreCS|$  satisfies the axiom (L3) (resp. (Lp)) if and only if  $\omega_{L3}(X, \lim) = \top$  (resp.  $\omega_{Lp}(X, \lim) = \top$ ). With these we define

$$\begin{aligned} \omega_{LIM}(X, \lim) &= \omega(X, \lim) \wedge \omega_{L3}(X, \lim); \\ \omega_{PCS}(X, \lim) &= \omega(X, \lim) \wedge \omega_{Lp}(X, \lim). \end{aligned}$$

Then we obtain the following  $L$ -subcategories of  $L\text{-}SL\text{-}GCS$ :

$$\begin{aligned} L\text{-}SL\text{-}LIM &= (SL\text{-}PreCS, \omega_{LIM}, \text{Mor}(SL\text{-}PreCS, \mu)); \\ L\text{-}SL\text{-}PCS &= (SL\text{-}PreCS, \omega_{PCS}, \text{Mor}(SL\text{-}PreCS, \mu)). \end{aligned}$$

In the sequel, we want to show that  $L\text{-}SL\text{-}PCS$  is also an  $L$ -subcategory of  $L\text{-}SL\text{-}LIM$ . We therefore need to show that  $\omega_{Lp} \leq \omega_{L3}$ . To this end, we note that for  $(X, \lim) \in |SL\text{-}GCS|$ , the space  $(X, \overline{\lim})$  satisfies (L3). We have

$$\begin{aligned} \overline{\lim}(\mathcal{F} \wedge \mathcal{G})(x) &= \bigwedge_{a \in L^X} \left[ \left( \bigwedge_{\mathcal{H} \in \mathbb{F}_L^s(X)} \lim \mathcal{H}(x) \rightarrow \mathcal{H}(a) \right) \rightarrow (\mathcal{F} \wedge \mathcal{G})(a) \right] \\ &= \bigwedge_{a \in L^X} \left[ \left\{ \left( \bigwedge_{\mathcal{H} \in \mathbb{F}_L^s(X)} \lim \mathcal{H}(x) \rightarrow \mathcal{H}(a) \right) \rightarrow \mathcal{F}(a) \right\} \right. \\ &\quad \left. \wedge \left\{ \left( \bigwedge_{\mathcal{H} \in \mathbb{F}_L^s(X)} \lim \mathcal{H}(x) \rightarrow \mathcal{H}(a) \right) \rightarrow \mathcal{G}(a) \right\} \right] \\ &\geq \overline{\lim}\mathcal{F}(x) \wedge \overline{\lim}\mathcal{G}(x). \end{aligned}$$

Moreover,  $\overline{\lim}\mathcal{F}(x) \geq \bigwedge_{a \in L^X} [(\lim \mathcal{F}(x) \rightarrow \mathcal{F}(a)) \rightarrow \mathcal{F}(a)] \geq \lim \mathcal{F}(x)$ . We conclude from this

$$\begin{aligned} &\omega_{L3}(X, \lim) \\ &= \bigwedge_{\mathcal{F}_1, \dots, \mathcal{F}_n, x, n} [(\lim \mathcal{F}_1(x) \wedge \dots \wedge \lim \mathcal{F}_n(x)) \rightarrow \lim(\mathcal{F}_1 \wedge \dots \wedge \mathcal{F}_n)(x)] \\ &\geq \bigwedge_{\mathcal{F}_1, \dots, \mathcal{F}_n, x, n} [(\overline{\lim}\mathcal{F}_1(x) \wedge \dots \wedge \overline{\lim}\mathcal{F}_n(x)) \rightarrow \lim(\mathcal{F}_1 \wedge \dots \wedge \mathcal{F}_n)(x)] \\ &\geq \bigwedge_{\mathcal{F}_1, \dots, \mathcal{F}_n, x, n} [\overline{\lim}(\mathcal{F}_1 \wedge \dots \wedge \mathcal{F}_n)(x) \rightarrow \lim(\mathcal{F}_1 \wedge \dots \wedge \mathcal{F}_n)(x)] \\ &= \bigwedge_{\mathcal{F}, x} [\overline{\lim}\mathcal{F}(x) \rightarrow \lim \mathcal{F}(x)] \\ &= \omega_{Lp}(X, \lim). \end{aligned}$$

We write  $\mathcal{C} \leq \mathcal{D}$  if the  $L$ -category  $\mathcal{C}$  is an  $L$ -subcategory of the  $L$ -category  $\mathcal{D}$ . We collect the above results as follows.

**Lemma 5.3.** *We have  $L\text{-}SL\text{-}PCS \leq L\text{-}SL\text{-}LIM$ .*

We note further that  $\omega_{PCS}(X, \text{lim}) = \omega_{L1}(X, \text{lim}) \wedge \omega_{LP}(X, \text{lim})$ , as  $\omega_{LP} \leq \omega_{L2}$ . To see this, we note that for  $\mathcal{F} \leq \mathcal{G}$  we have  $\overline{\text{lim}}\mathcal{F}(x) \leq \overline{\text{lim}}\mathcal{G}(x)$ . Hence we have

$$\text{lim}\mathcal{F}(x) \leq \overline{\text{lim}}\mathcal{F}(x) \leq \overline{\text{lim}}\mathcal{G}(x)$$

and, consequently,  $\overline{\text{lim}}\mathcal{G}(x) \rightarrow \text{lim}\mathcal{G}(x) \leq \text{lim}\mathcal{F}(x) \rightarrow \text{lim}\mathcal{G}(x)$ , whenever  $\mathcal{F} \leq \mathcal{G}$ . Hence

$$\begin{aligned} \omega_{LP}(X, \text{lim}) &= \bigwedge_{\mathcal{G}, x} [\overline{\text{lim}}\mathcal{G}(x) \rightarrow \text{lim}\mathcal{G}(x)] \\ &\leq \bigwedge_{\mathcal{F}, \mathcal{G} \text{ s.t. } \mathcal{F} \leq \mathcal{G}, x} [\text{lim}\mathcal{F}(x) \rightarrow \text{lim}\mathcal{G}(x)] = \omega_{L2}(X, \text{lim}). \end{aligned}$$

This corresponds to the classical result that the axiom (Lp) entails the axiom (L2).

**Remark 5.4.** It is well-known that e.g. the axiom (L3) does not imply the axiom (L2). As an example one may consider a frame  $L$  which is not a complete Boolean algebra and an infinite set  $X$ . We define

$$\text{lim}\mathcal{F}(x) = \begin{cases} \top & \text{if } \mathcal{F} = \bigwedge_{x \in A} [x] \text{ with } A \subseteq X \text{ finite} \\ \perp & \text{else.} \end{cases}$$

Then  $(X, \text{lim})$  satisfies (L1) and (L3) but fails to satisfy (L2) as can be seen by taking a stratified  $L$ -ultrafilter  $\mathcal{U} \geq [x]$ . Note that  $[x]$  is not ultra if  $L$  is not a complete Boolean algebra.

## 6. Measuring the ‘‘Topologiness’’ of a Lattice-valued Convergence Space

It is known that a stratified  $L$ -pretopological space  $(X, \text{lim})$  can be identified with a stratified  $L$ -topological space if the axiom (Lt) is satisfied [8, 9]. Hence we can quantify how close a stratified  $L$ -pretopological space is to being  $L$ -topological with the following  $L$ -class (where  $\mathcal{U}^x = \mathcal{U}_{\text{lim}}^x$  is the stratified neighbourhood  $L$ -filter defined by  $\mathcal{U}^x(a) = \bigwedge_{\mathcal{F} \in \mathbb{F}_L^s(X)} (\text{lim}\mathcal{F}(x) \rightarrow \mathcal{F}(a))$ ):

$$\begin{aligned} \omega_{Lt}(X, \text{lim}) &= \bigwedge_{a \in L^X} \bigwedge_{x \in X} \left[ \mathcal{U}^x(a) \rightarrow \bigvee \{ \mathcal{U}^x(b) \mid b(y) \leq \mathcal{U}^y(a) \forall y \in X \} \right] \\ &= \bigwedge_{a \in L^X} \bigwedge_{x \in X} \left[ \mathcal{U}^x(a) \rightarrow \mathcal{U}^x(\mathcal{U}^{(\cdot)}(a)) \right], \end{aligned}$$

Clearly,  $(X, \text{lim}) \in |SL\text{-}PCS|$  satisfies the axiom (Lt) if and only if  $\omega_{Lt}(X, \text{lim}) = \top$ . In order to characterize this ‘‘measure of being  $L$ -topological’’ in a different way, we fuzzify the diagonal axiom (LK), see [10]. We define the following fuzzy predicate for a stratified  $L$ -preconvergence space:

$$\begin{aligned} & \omega_{LK}(X, \text{lim}) \\ &= \bigwedge_{\mathcal{G} \in \mathbb{F}_L^s(X)} \bigwedge_{\mathcal{F}_y \in \mathbb{F}_L^s(X)} \bigwedge_{(y \in X)} \bigwedge_{x \in X} \left[ \left( \lim \mathcal{G}(x) \wedge \bigwedge_{y \in X} \lim \mathcal{F}_y(y) \right) \rightarrow \lim \mathcal{G}(\mathcal{F}_{(\cdot)})(x) \right]. \end{aligned}$$

Then  $\omega_{LK}(X, \text{lim}) = \top$  if and only if  $(X, \text{lim})$  satisfies the axiom (LK).

**Lemma 6.1.** *Let  $(X, \text{lim})$  be a stratified  $L$ -pretopological space (i.e.  $\omega_{PCS}(X, \text{lim}) = \top$ ). Then  $\omega_{Lt}(X, \text{lim}) = \omega_{LK}(X, \text{lim})$ .*

*Proof.* Choosing  $\mathcal{G} = \mathcal{U}^x$  and  $\mathcal{F}_y = \mathcal{U}^y$  for all  $y \in X$ , we obtain

$$\begin{aligned} \omega_{LK}(X, \text{lim}) &\leq \bigwedge_{x \in X} \bigwedge_{y \in Y} \left[ \left( \underbrace{\lim \mathcal{U}^x(x)}_{=\top} \wedge \bigwedge_{y \in X} \underbrace{\lim \mathcal{U}^y(y)}_{=\top} \right) \rightarrow \lim \mathcal{U}^x(\mathcal{U}^{(\cdot)})(x) \right] \\ &= \bigwedge_{x \in X} \lim \mathcal{U}^x(\mathcal{U}^{(\cdot)})(x) \\ &= \bigwedge_{x \in X} \bigwedge_{a \in L^X} \left[ \mathcal{U}^x(a) \rightarrow \mathcal{U}^x(\mathcal{U}^{(\cdot)})(a) \right] = \omega_{Lt}(X, \text{lim}). \end{aligned}$$

On the other hand we have for  $\mathcal{G} \in \mathbb{F}_L^s(X)$ ,  $\mathcal{F}_y \in \mathbb{F}_L^s(X)$  ( $y \in Y$ ) and  $x \in X$  with  $\beta = \bigwedge_{y \in Y} \lim \mathcal{F}_y(y)$  that  $\mathcal{G}(\mathcal{F}_{(\cdot)})(a) \geq \beta \wedge \mathcal{G}(\mathcal{U}^{(\cdot)})(a)$  for all  $a \in L^X$  (see [10]). This implies

$$\begin{aligned} & (\lim \mathcal{G}(x) \wedge \beta) \rightarrow \mathcal{G}(\mathcal{F}_{(\cdot)})(x) \\ &= \left[ \beta \wedge \bigwedge_{b \in L^X} (\mathcal{U}^x(b) \rightarrow \mathcal{G}(b)) \right] \rightarrow \bigwedge_{a \in L^X} (\mathcal{U}^x(a) \rightarrow \mathcal{G}(\mathcal{F}_{(\cdot)})(a)) \\ &\geq \left[ \beta \wedge \bigwedge_{a \in L^X} (\mathcal{U}^x(\mathcal{U}^{(\cdot)}(a)) \rightarrow \mathcal{G}(\mathcal{U}^{(\cdot)}(a))) \right] \rightarrow \left( \bigwedge_{a \in L^X} (\mathcal{U}^x(a) \rightarrow (\beta \wedge \mathcal{G}(\mathcal{U}^{(\cdot)}(a)))) \right) \\ &\geq \left[ \beta \wedge \bigwedge_{a \in L^X} (\mathcal{U}^x(\mathcal{U}^{(\cdot)}(a)) \rightarrow \mathcal{G}(\mathcal{U}^{(\cdot)}(a))) \right] \rightarrow \left[ \beta \wedge \bigwedge_{a \in L^X} (\mathcal{U}^x(a) \rightarrow \mathcal{G}(\mathcal{U}^{(\cdot)}(a))) \right] \\ &\geq \left[ \bigwedge_{a \in L^X} (\mathcal{U}^x(\mathcal{U}^{(\cdot)}(a)) \rightarrow \mathcal{G}(\mathcal{U}^{(\cdot)}(a))) \right] \rightarrow \left[ \bigwedge_{a \in L^X} (\mathcal{U}^x(a) \rightarrow \mathcal{G}(\mathcal{U}^{(\cdot)}(a))) \right] \\ &\geq \bigwedge_{a \in L^X} \left[ (\mathcal{U}^x(\mathcal{U}^{(\cdot)}(a)) \rightarrow \mathcal{G}(\mathcal{U}^{(\cdot)}(a))) \rightarrow (\mathcal{U}^x(a) \rightarrow \mathcal{G}(\mathcal{U}^{(\cdot)}(a))) \right] \\ &\geq \bigwedge_{a \in L^X} (\mathcal{U}^x(a) \rightarrow \mathcal{U}^x(\mathcal{U}^{(\cdot)}(a))) = \lim \mathcal{U}^x(\mathcal{U}^{(\cdot)})(x). \end{aligned}$$

Taking the infimum over all  $\mathcal{G}, \mathcal{F}_y \in \mathbb{F}_L^s(X)$  and all  $x \in X$  we thus conclude

$$\begin{aligned} \omega_{Lt}(X, \text{lim}) &= \bigwedge_{x \in X} \lim \mathcal{U}^x(\mathcal{U}^{(\cdot)})(x) \\ &\leq \bigwedge_{\mathcal{G}, \mathcal{F}_y, x} \left[ \left( \lim \mathcal{G}(x) \wedge \bigwedge_{y \in Y} \lim \mathcal{F}_y(y) \right) \rightarrow \lim \mathcal{G}(\mathcal{F}_{(\cdot)})(x) \right] \\ &= \omega_{LK}(X, \text{lim}). \end{aligned}$$

This completes the proof.  $\square$

We note now that if  $(X, \text{lim})$  satisfies the axiom (L1), then  $\mathcal{U}^x$  is a stratified  $L$ -filter. In this case the  $L$ -class  $\omega_{Lt}$  is well-defined. We can thus define, for a stratified  $L$ -preconvergence space  $(X, \text{lim})$ , the following  $L$ -classes.

$$\begin{aligned} \omega_{TCS1}(X, \text{lim}) &= \begin{cases} \omega_{Lt}(X, \text{lim}) \wedge \omega_{Lp}(X, \text{lim}) & \text{if } \omega_{L1}(X, \text{lim}) = \top \\ \perp & \text{else;} \end{cases} \quad \text{and} \\ \omega_{TCS2}(X, \text{lim}) &= \omega_{LK}(X, \text{lim}) \wedge \omega_{Lp}(X, \text{lim}) \wedge \omega_{L1}(X, \text{lim}). \end{aligned}$$

and with these the  $L$ -subcategories

$$\begin{aligned} L\text{-}SL\text{-}TCS1 &= (|SL\text{-}PreCS|, \omega_{TCS1}, \text{Mor}(SL\text{-}PreCS, \mu, \circ)) \\ L\text{-}SL\text{-}TCS2 &= (|SL\text{-}PreCS|, \omega_{TCS2}, \text{Mor}(SL\text{-}PreCS, \mu, \circ)) \end{aligned}$$

of  $L\text{-}SL\text{-}PCS$ . As  $(X, \text{lim}) \in |SL\text{-}TCS|$  if  $\omega_{TCS1}(X, \text{lim}) = \omega_{TCS2}(X, \text{lim}) = \top$ , the fuzzy predicates  $\omega_{TCS1}$  and  $\omega_{TCS2}$  both measure the grade to which a space  $(X, \text{lim})$  is in fact a stratified  $L$ -topological (convergence) space. However, only in case  $\omega_{Lp}(X, \text{lim}) \wedge \omega_{L1}(X, \text{lim}) = \top$ , both these grades will be the same.

There is a further characterization of stratified  $L$ -topological spaces in terms of convergence of stratified  $L$ -filters. The axiom (LF) is used. It was shown in [11] that a stratified  $L$ -generalized convergence space which satisfies the axiom (Lpw1) (see below) is an  $L$ -topological space if and only if the axiom (LF) is satisfied. We will from now on assume that  $(X, \text{lim}) \in |SL\text{-}GCS|$ . We first note that the axiom (Lp) splits into two (independent) axioms [10]

$$\begin{aligned} \text{(Lpw1)} \quad & [\alpha \wedge \mathcal{U}^x] = \mathcal{U}_\alpha^x \text{ for all } \alpha \in L, x \in X; \\ \text{(Lpw2)} \quad & \bigwedge_{i \in J} \lim \mathcal{F}_i(x) \leq \lim(\bigwedge_{i \in J} \mathcal{F}_i)(x) \text{ for all } J, \mathcal{F}_i \in \mathbb{F}_L^s(X), i \in J. \end{aligned}$$

Here, it is defined

$$\begin{aligned} \mathcal{U}_\alpha^x &= \bigwedge \{ \mathcal{F} \in \mathbb{F}_L^s(X) \mid \lim \mathcal{F}(x) \geq \alpha \} \\ [\alpha \wedge \mathcal{U}^x] &= \bigwedge \{ \mathcal{F} \in \mathbb{F}_L^s(X) \mid \alpha \wedge \mathcal{U}^x(a) \leq \mathcal{F}(a) \forall a \in L^X \}. \end{aligned}$$

We have then (Lp)  $\iff$  (Lpw1) and (Lpw2) [10]. We “fuzzify” the axiom (Lpw2) with the following  $L$ -class.

$$\omega_{Lpw2}(X, \text{lim}) = \bigwedge_J \bigwedge_{\mathcal{F}_i \in \mathbb{F}_L^s(X), i \in J} \bigwedge_{x \in X} \left[ \left( \bigwedge_{i \in J} \lim \mathcal{F}_i(x) \right) \rightarrow \lim \left( \bigwedge_{i \in J} \mathcal{F}_i \right)(x) \right].$$

Then clearly  $\omega_{Lpw2}(X, \text{lim}) = \top$  if  $(X, \text{lim})$  satisfies the axiom (Lpw2).

**Lemma 6.2.** *Let  $(X, \lim) \in |SL-GCS|$ . Then  $\omega_{Lp}(X, \lim) \leq \omega_{Lpw2}(X, \lim)$  and if  $(X, \lim)$  satisfies the axiom (Lpw1) then we have equality.*

*Proof.* We note that with  $\overline{\lim}\mathcal{F}(x) = \bigwedge_{\alpha \in L^X} (\mathcal{U}^\alpha(a) \rightarrow \mathcal{F}(a))$  we have that  $(X, \overline{\lim})$  satisfies (Lpw2) and  $\lim \mathcal{F}(x) \leq \overline{\lim}\mathcal{F}(x)$  for any  $\mathcal{F} \in \mathbb{F}_L^s(X)$  and  $x \in X$ . Hence we have

$$\begin{aligned} \omega_{Lpw2}(X, \lim) &\geq \bigwedge_{\mathcal{F}_i, x} \left[ \left( \bigwedge_{i \in J} \overline{\lim}\mathcal{F}_i(x) \right) \rightarrow \lim \left( \bigwedge_{i \in J} \mathcal{F}_i \right) (x) \right] \\ &= \bigwedge_{\mathcal{F}_i, x} \left[ \overline{\lim} \left( \bigwedge_{i \in J} \mathcal{F}_i \right) (x) \rightarrow \lim \left( \bigwedge_{i \in J} \mathcal{F}_i \right) (x) \right] = \omega_{Lp}(X, \lim). \end{aligned}$$

Let us now assume that (Lpw1) is satisfied. We take  $\mathcal{F} \in \mathbb{F}_L^s(X)$  and denote  $\alpha_0 = \overline{\lim}\mathcal{F}(x)$ . Then  $\mathcal{U}_{\alpha_0}^x = [\alpha_0 \wedge \mathcal{U}^x] = \bigwedge \{ \mathcal{G} \mid \alpha_0 \leq \overline{\lim}\mathcal{G}(x) \} \leq \mathcal{F}$ . Hence

$$\bigwedge_{\alpha \in L} \bigwedge_{\mathcal{G} \geq \mathcal{U}_{\alpha}^x} (\alpha \rightarrow \lim \mathcal{G}(x)) \leq \bigwedge_{\mathcal{G} \geq \mathcal{U}_{\alpha_0}^x} (\alpha_0 \rightarrow \lim \mathcal{G}(x)) \leq \overline{\lim}\mathcal{F}(x) \rightarrow \lim \mathcal{F}(x)$$

and therefore  $\bigwedge_{\alpha \in L} \bigwedge_{\mathcal{G} \geq \mathcal{U}_{\alpha}^x} (\alpha \rightarrow \lim \mathcal{G}(x)) \leq \omega_{Lp}(X, \lim)$ . Let now  $\alpha \in L$  and  $\mathcal{F} \geq \mathcal{U}_{\alpha}^x$ .

Then, with (L2),  $\lim \mathcal{U}_{\alpha}^x(x) \leq \lim \mathcal{F}(x)$  and hence

$$\begin{aligned} &\bigwedge_{\alpha \in L} \bigwedge_{\mathcal{F} \geq \mathcal{U}_{\alpha}^x} (\alpha \rightarrow \lim \mathcal{F}(x)) \\ &\geq \bigwedge_{\alpha \in L} (\alpha \rightarrow \lim \mathcal{U}_{\alpha}^x(x)) \\ &= \bigwedge_{\alpha \in L} (\alpha \rightarrow \lim \left( \bigwedge \lim \mathcal{F}(x) \geq \alpha \mathcal{F} \right) (x)) \\ &\geq \bigwedge_{\alpha \in L} \left[ \left( \bigwedge_{\lim \mathcal{F}(x) \geq \alpha} \lim \mathcal{F}(x) \right) \rightarrow \lim \left( \bigwedge_{\lim(x) \geq \alpha} \mathcal{F} \right) (x) \right] \geq \omega_{Lpw2}(X, \lim). \end{aligned}$$

□

Motivated by the proof of Lemma 6.2 we define

$$\omega_{Lpw2*}(X, \lim) = \bigwedge_{\alpha \in L} \bigwedge_{x \in X} (\alpha \rightarrow \lim \mathcal{U}_{\alpha}^x(x)).$$

The above proof then shows  $\omega_{Lpw2*}(X, \lim) \leq \omega_{Lpw2}(X, \lim)$ . To show the converse, let  $\mathcal{F}_i \in \mathbb{F}_L^s(X)$  ( $i \in J$ ) and define  $\alpha_0 = \bigwedge_{i \in J} \lim \mathcal{F}_i(x)$ . Then  $\mathcal{U}_{\alpha_0}^x \leq \mathcal{F}_i$  for all  $i \in J$  and therefore  $\mathcal{U}_{\alpha_0}^x \leq \bigwedge_{i \in J} \mathcal{F}_i$ . With (L2) we obtain

$$\alpha_0 \rightarrow \lim \mathcal{U}_{\alpha_0}^x(x) \leq \left( \bigwedge_{i \in J} \lim \mathcal{F}_i(x) \right) \rightarrow \lim \left( \bigwedge_{i \in J} \mathcal{F}_i \right) (x)$$

and hence

$$\bigwedge_{\alpha \in L, x \in X} (\alpha \rightarrow \lim \mathcal{U}_\alpha^x(x)) \leq \bigwedge_{x \in X} \left( \bigwedge_{i \in J} \lim \mathcal{F}_i(x) \rightarrow \lim \left( \bigwedge_{i \in J} \mathcal{F}_i \right)(x) \right).$$

Taking the infimum over all families  $(\mathcal{F}_i)$  we thus see that  $\omega_{Lpw2^*}(X, \lim) \leq \omega_{Lpw2}(X, \lim)$ . We collect these results in the following lemma.

**Lemma 6.3.** *Let  $(X, \lim) \in |SL-GCS|$ . Then  $\omega_{Lpw2^*}(X, \lim) = \omega_{Lpw2}(X, \lim)$ .*

After these preparations we can go back to the axiom (LF). Let  $(X, \lim) \in |SL-GCS|$ . We define

$$\begin{aligned} & \omega_{LF}(X, \lim) \\ = & \bigwedge_{J, \psi: J \rightarrow X} \bigwedge_{\mathcal{G} \in \mathbb{F}_L^s(J)} \bigwedge_{\mathcal{F}_i \in \mathbb{F}_L^s(X)} \bigwedge_{(i \in J)} \bigwedge_{x \in X} \left[ \left( \lim \psi(\mathcal{G})(x) \wedge \bigwedge_{i \in J} \lim \mathcal{F}_i(\psi(i)) \right) \right. \\ & \left. \rightarrow \lim \mathcal{G}(\mathcal{F}_{(\cdot)})(x) \right]. \end{aligned}$$

We come to the main result of this section.

**Theorem 6.4.** *Let  $(X, \lim) \in |SL-GCS|$ . Then  $\omega_{LF}(X, \lim) = \omega_{LK}(X, \lim) \wedge \omega_{Lpw2}(X, \lim)$ .*

*Proof.* If we choose  $J = X$  and  $\psi = id_X$ , then we obtain

$$\begin{aligned} & \omega_{LF}(X, \lim) \\ \leq & \bigwedge_{\mathcal{G} \in \mathbb{F}_L^s(X)} \bigwedge_{\mathcal{F}_y \in \mathbb{F}_L^s(X)} \bigwedge_{(y \in X)} \bigwedge_{x \in X} \left[ \left( \lim \mathcal{G}(x) \wedge \bigwedge_{y \in X} \lim \mathcal{F}_y(y) \right) \rightarrow \lim \mathcal{G}(\mathcal{F}_{(\cdot)})(x) \right] \\ = & \omega_{LK}(X, \lim). \end{aligned}$$

Moreover, with (L1) and choosing  $\psi(i) = x$  for all  $i \in J$  and  $\mathcal{G} = [J]$  we have  $\psi(\mathcal{G}) = [x]$  and  $\mathcal{G}(\mathcal{F}_{(\cdot)}) = \bigwedge_{i \in J} \mathcal{F}_i$  (see [11]). Hence

$$\begin{aligned} \omega_{LF}(X, \lim) & \leq \bigwedge_J \bigwedge_{\mathcal{F}_i \in \mathbb{F}_L^s(X), i \in J} \left[ \left( \underbrace{\lim [x](x)}_{= \top} \wedge \bigwedge_{i \in J} \lim \mathcal{F}_i(x) \right) \rightarrow \lim \bigwedge_{i \in J} \mathcal{F}_i(x) \right] \\ & = \omega_{Lpw2}(X, \lim). \end{aligned}$$

Hence we have  $\omega_{LF}(X, \lim) \leq \omega_{LK}(X, \lim) \wedge \omega_{Lpw2}(X, \lim)$ . To show the converse inequality, we fix  $\alpha, \beta \in L$ . Then we have

$$\begin{aligned} \omega_{Lpw2^*}(X, \lim) \wedge \alpha & \leq \lim \mathcal{U}_\alpha^y(y) \quad \text{for all } y \in X; \\ \omega_{Lpw2^*}(X, \lim) \wedge \beta & \leq \lim \mathcal{U}_\beta^x(x). \end{aligned}$$

Hence, we have also  $\omega_{Lpw2^*}(X, \lim) \wedge \alpha \leq \bigwedge_{y \in X} \lim \mathcal{U}_\alpha^y(y)$ . We conclude from this

$$\begin{aligned} \omega_{LK}(X, \lim) & \leq \left( \lim \mathcal{U}_\beta^x(x) \wedge \bigwedge_{y \in X} \lim \mathcal{U}_\alpha^y(y) \right) \rightarrow \lim \mathcal{U}_\beta^x(\mathcal{U}_\alpha^{(\cdot)})(x) \\ & \leq (\omega_{Lpw2^*}(X, \lim) \wedge \alpha \wedge \beta) \rightarrow \lim \mathcal{U}_\beta^x(\mathcal{U}_\alpha^{(\cdot)})(x). \end{aligned}$$

Hence  $\omega_{LK}(X, \text{lim}) \wedge \omega_{Lpw2*}(X, \text{lim}) \leq (\alpha \wedge \beta) \rightarrow \lim \mathcal{U}_\beta^x(\mathcal{U}_\alpha^{(\cdot)})(x)$ . Taking the infimum over all  $\alpha, \beta \in L$  we thus have shown

$$\omega_{LK}(X, \text{lim}) \wedge \omega_{Lpw2*}(X, \text{lim}) \leq \bigwedge_{\alpha, \beta \in L} \left[ (\alpha \wedge \beta) \rightarrow \lim \mathcal{U}_\beta^x(\mathcal{U}_\alpha^{(\cdot)})(x) \right] =: \bar{\omega}(X, \text{lim}).$$

Let now  $J$  be a set,  $\psi : J \rightarrow X$  be a mapping,  $\mathcal{G} \in \mathbb{F}_L^s(J)$  and  $\mathcal{F}_i \in \mathbb{F}_L^s(X)$  ( $i \in J$ ) be stratified  $L$ -filters. We define  $\alpha = \bigwedge_{i \in J} \lim \mathcal{F}_i(\psi(i))$  and  $\beta = \lim \psi(\mathcal{G})(x)$ . Then  $\mathcal{U}_\alpha^{\psi(i)} \leq \mathcal{F}_i$  for all  $i \in J$  and  $\mathcal{U}_\beta^x \leq \psi(\mathcal{G})$ . As a consequence (see [11]) we obtain  $\mathcal{U}_\beta^x(\mathcal{U}_\alpha^{(\cdot)}) \leq \mathcal{G}(\mathcal{F}_{(\cdot)})$  and therefore

$$\begin{aligned} (\alpha \wedge \beta) \rightarrow \lim \mathcal{U}_\beta^x(\mathcal{U}_\alpha^{(\cdot)})(x) &\leq (\alpha \wedge \beta) \rightarrow \lim \mathcal{G}(\mathcal{F}_{(\cdot)})(x) \\ &= \left( \lim \psi(\mathcal{G})(x) \wedge \bigwedge_{i \in J} \lim \mathcal{F}_i(\psi(i)) \right) \rightarrow \lim \mathcal{G}(\mathcal{F}_{(\cdot)})(x). \end{aligned}$$

Taking the infimum over all  $\alpha, \beta \in L$  we therefore have

$$\bar{\omega}(X, \text{lim}) \leq \left( \lim \psi(\mathcal{G})(x) \wedge \bigwedge_{i \in J} \lim \mathcal{F}_i(\psi(i)) \right) \rightarrow \lim \mathcal{G}(\mathcal{F}_{(\cdot)})(x).$$

Taking the infimum over all  $J, \psi, \mathcal{G}, \mathcal{F}_i$ 's we finally obtain  $\bar{\omega}(X, \text{lim}) \leq \omega_{LF}(X, \text{lim})$  and the proof is complete.  $\square$

**Corollary 6.5.** *Let  $(X, \text{lim}) \in |SL-GCS|$  satisfy the axiom (Lpw1). If  $\omega_{LF}(X, \text{lim}) = \top$ , then  $(X, \text{lim})$  is a stratified  $L$ -topological convergence space.*

*Proof.* If (Lpw1) is true, then  $\omega_{Lpw2}(X, \text{lim}) = \omega_{Lp}(X, \text{lim})$  and from  $\omega_{LF}(X, \text{lim}) = \top$  we conclude therefore that the axioms (LK) and (Lp) are satisfied. According to [10] this means that  $(X, \text{lim})$  is a stratified  $L$ -topological convergence space.  $\square$

**Corollary 6.6.** *Let  $(X, \text{lim}) \in |SL-GCS|$ . If  $\omega_{Lp}(X, \text{lim}) = \top$ , then  $\omega_{LF}(X, \text{lim}) = \omega_{LK}(X, \text{lim}) = \omega_{Lt}(X, \text{lim})$ .*

There is thus a further possibility of obtaining an  $L$ -subcategory of  $SL-GCS$  where the grade of being  $L$ -topological is measured. We define

$$\omega_{TCS3}(X, \text{lim}) = \omega_{LF}(X, \text{lim}) \wedge \omega_{Lp}(X, \text{lim}) \wedge \omega_{L1}(X, \text{lim}),$$

and with this the  $L$ -subcategory

$$L\text{-}SL\text{-}TCS3 = (|SL\text{-}PreCS|, \omega_{TCS3}, \text{Mor}(SL\text{-}PreCS), \mu, \circ).$$

Again, only if  $\omega_{Lp}(X, \text{lim}) \wedge \omega_{L1}(X, \text{lim}) = \top$  we will have  $\omega_{TCS1}(X, \text{lim}) = \omega_{TCS2}(X, \text{lim}) = \omega_{TCS3}(X, \text{lim})$ .

## 7. Conclusions

Motivated by the fact that continuous convergence in general fails to satisfy the axiom (L1) of a lattice-valued convergence structure, we quantified the grade to which certain axioms of a lattice-valued convergence space are satisfied. Together with quantifying the grades of continuity of mappings this leads to lattice-valued categories of lattice-valued convergence spaces. We studied initial constructions and function spaces. Finally we showed that there are different ways of “measuring” the grade that a stratified  $L$ -convergence space is  $L$ -topological.

The theory developed in this paper mainly exploits the residual implication that is available in a frame,  $L$ . To accommodate further examples of  $L$ -categories of lattice-valued topological (convergence) spaces in a similar way, a more general lattice-theoretic situation is necessary. As Šostak [18, 19, 20, 21, 22] points out, complete quantales, i.e. complete lattices enriched by an additional operation  $*$  satisfying  $\alpha * \bigvee_{i \in J} \beta_i = \bigvee_{i \in J} (\alpha * \beta_i)$  are appropriate. A more general set-up would be the lattice-theoretic background used in [6], where a so-called  $cl$ -premonoid is enriched by a suitable quantale operation.

Apart from this general direction for further research, one might consider e.g. grades of compactness [13] or regularity [12] or the grade to which certain algebraic operations are continuous [2] for a lattice-valued convergence space (or a lattice-valued topological space). In all these cases,  $L$ -subcategories of  $L$ - $SL$ - $GCS$  would arise and their relationships could be of interest. This then also leads to studying  $L$ -categories more deeply, especially with regard to reflective  $L$ -subcategories or cartesian closedness of  $L$ -categories.

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