GENERALIZED REGULAR FUZZY MATRICES

A. R. MEENAKSHI AND P. JENITA

Abstract. In this paper, the concept of k-regular fuzzy matrix as a generalization of regular matrix is introduced and some basic properties of a k-regular fuzzy matrix are derived. This leads to the characterization of a matrix for which the regularity index and the index are identical. Further the relation between regular, k-regular and regularity of powers of fuzzy matrices are discussed.

1. **Introduction**

Throughout this paper we deal with fuzzy matrices, that is matrices over the fuzzy algebra $\mathcal{F}$=[0,1] under max-min operations(+,.) defined as a + b = max{a, b} and a . b = min {a, b} for all a, b $\in \mathcal{F}$ and the standard order '$\leq$'of real numbers. Let $\mathcal{F}_{mn}$ be the set of all $m \times n$ matrices over $\mathcal{F}$. In short $\mathcal{F}_n$ denotes $\mathcal{F}_{nn}$. For A $\in \mathcal{F}_n$, A^T, $R(A)$, $C(A)$, $\rho_r(A)$, $\rho_c(A)$ and $\rho(A)$ denotes the transpose, row space, column space, row rank, column rank and rank of A respectively. The algebraic operations on matrices are max-min operations, which are different from that of the standard operations on real matrices. In practice, fuzzy matrices have been proposed to represent fuzzy relations in a system based on fuzzy sets theory[3], the behavior of the dynamic fuzzy systems depends heavily on the products of fuzzy matrices in the matrix representations of the system. A $\in \mathcal{F}_{mn}$ is regular if there exists X such that $AXA = A$; X is called a generalized(g^-) inverse of A and is denoted by A^-. $A\{1\}$ denotes the set of all g-inverses of a regular matrix A. A regular matrix as one that has a generalized inverse lays the foundation in the study on fuzzy relational equations. Regular fuzzy matrices play an important role in estimation and inverse problem in fuzzy relational equations [7, pp.1-14] and in fuzzy optimization problems [6, pp.533-539].

This motivates us to develop the study on generalized regular fuzzy matrices. The powers of a fuzzy matrix are either convergent to a fuzzy matrix (or) oscillating with finite period. For a fuzzy matrix A, A^{k+d} =A^k for some integers k, $d > 0$. Therefore, all fuzzy matrices have an index and a period. On the other hand, most matrices over the non-negative real numbers will not have an index and a period [3]. Spectral inverses, such as group inverse and Drazin inverse are defined for fuzzy matrices, analogous to that for complex matrices [1]. For A $\in \mathcal{F}_n$, the Drazin inverse of A is a solution of the equations:A^kXA=A^k, XAX=X, AX=XA, for some positive integer k. Group inverse is the solution of the equations: AXA=A,

Received: June 2009; Revised: April 2010; Accepted: July 2010
Key words and phrases: Fuzzy matrices, K-regular fuzzy matrices, Index, Period.

$XAX{=}X$, $AX{=}XA$. Hence Drazin inverse and group inverse are identical when k=1.

In this paper in section 2, we introduce the concept of k-regular fuzzy matrices as a generalization of regular matrices. The row and column ranks of a k-regular fuzzy matrix are determined. Conditions for products of k-regular fuzzy matrix to be k-regular are obtained. The relation between regular, k-regular and k^{th} power of a fuzzy matrix are discussed.

In section 3, we define the regularity index of a matrix A $\in \mathcal{F}_n$ as a generalization of the index of A. A characterization of a matrix whose regularity index coincides with the index of A is established. It is shown that for a matrix, regularity index is less than (or) equal to the index of the matrix. Numerical examples are provided to illustrate the relation between the regularity index and index of a fuzzy matrix.

2. K-regular Matrices

Definition 2.1. A matrix A $\in \mathcal{F}_n$, is said to be right k-regular if there exists a matrix X $\in \mathcal{F}_n$ such that $A^kXA{=}A^k$, for some positive integer k. X is called a right k-g-inverse of A.
Let $A_r\{1^k\}$={X/$A^kXA = A^k$}.

Definition 2.2. A matrix A $\in \mathcal{F}_n$, is said to be left k-regular if there exists a matrix Y $\in \mathcal{F}_n$ such that $AYA^k{=}A^k$, for some positive integer k. Y is called a left k-g-inverse of A.
Let $A_\ell\{1^k\}$={Y/$AYA^k = A^k$}.

In general, right k-regular is different from left k-regular. Hence a right k-g-inverse need not be a left k-g-inverse (refer Example (2.22)).

Throughout this paper, we have proved the results for right k-regular and the results for left k-regular can be derived in the same manner. Hence forth we call a right k-regular (or) left k-regular matrix as a k-regular matrix. Let $A\{1^k\}{=}A_r\{1^k\}\cup A_\ell\{1^k\}$.

Remark 2.3. Each element of the set $A\{1^k\}$ is called a k-g-inverse of A. If A is k-regular then A is q-regular for all integers $q \geq k$. For k=1, $A\{1^k\}$ reduces to the set of all g-inverses of a regular matrix A.

In the sequel, we shall make use of the following results.

Lemma 2.4. [3] *For A, B* $\in \mathcal{F}_n$, $R(B) \subseteq R(A) \Leftrightarrow B{=}XA$ *for some* $X \in \mathcal{F}_n$, $C(B) \subseteq C(A) \Leftrightarrow B{=}AY$ *for some* $Y \in \mathcal{F}_n$.

Lemma 2.5. [3] *For* $A \in \mathcal{F}_{mn}$ *and* $B \in \mathcal{F}_{np}$,$R(AB) \subseteq R(B)$,$C(AB) \subseteq C(A)$.

Lemma 2.6. *For A, B* $\in \mathcal{F}_n$, *and a positive integer k, the following hold.*

(i) *If A is right k-regular and* $R(B) \subseteq R(A^k)$ *then* $B{=}BXA$ *for each right k-g-inverse X of A.*
(ii) *If A is left k-regular and* $C(B) \subseteq C(A^k)$ *then* $B{=}AYB$ *for each left k-g-inverse Y of A.*

Proof. (i) Since $R(B) \subseteq R(A^k)$ by Lemma(2.4), there exists Y such that $B=YA^k$. By Definition (2.1), $A^kXA = A^k$.
Hence, $B=YA^k= YA^kXA=BXA$. Thus (i) holds.

(ii) Can be proved in the same manner. □

Remark 2.7. From Lemma(2.5), the converse relation implies $R(B) \subseteq R(A)$ and $C(B) \subseteq C(A)$ but need not imply $R(B) \subseteq R(A^k)$ and $C(B) \subseteq C(A^k)$ (refer Example (2.20)).

However, in particular for k=1, it reduces to the following.

Lemma 2.8. [4] *For A, $B \in \mathcal{F}_{mn}$, if A is regular then*

(i) *$R(B) \subseteq R(A) \Leftrightarrow B=BA^-A$ for each A^- of A.*
(ii) *$C(B) \subseteq C(A) \Leftrightarrow B=AA^-B$ for each A^- of A.*

Theorem 2.9. *Let $A \in \mathcal{F}_n$ and k be a positive integer, then $X \in A_r\{1^k\} \Leftrightarrow X^T \in A_\ell^T\{1^k\}$.*

Proof.

$$\begin{aligned} X \in A_r\{1^k\} &\Leftrightarrow A^kXA = A^k \\ &\Leftrightarrow (A^kXA)^T = (A^k)^T \\ &\Leftrightarrow A^TX^T(A^T)^k = (A^T)^k \\ &\Leftrightarrow X^T \in A_\ell^T\{1^k\} \end{aligned}$$

□

Theorem 2.10. *Let $A \in \mathcal{F}_n$ and k be a positive integer,*

(i) *if $X \in A_r\{1^k\}$ then $\rho_c(A^k)=\rho_c(A^kX)$ and $\rho_r(A^k) \leq \rho_r(XA) \leq \rho_r(A)$.*
(ii) *if $X \in A_\ell\{1^k\}$ then $\rho_r(A^k)=\rho_r(XA^k)$ and $\rho_c(A^k) \leq \rho_c(AX) \leq \rho_c(A)$.*

Proof. (i) Since X $\in A_r\{1^k\}$, $A^kXA=A^k$, by Lemma(2.5), $C(A^k) = C(A^kXA) \subseteq C(A^kX) \subseteq C(A^k)$. Therefore, $C(A^k)=C(A^kX)$ and $\rho_c(A^k)=\rho_c(A^kX)$. Since, $A^kXA=A^k$, we have $A^k=A^kXA = A^k(XA)^2=......=A^k(XA)^k$. Therefore, $A^k=A^k(XA)^k$.
Hence, by Lemma(2.5), $R(A^k)= R(A^k(XA)^k) \subseteq R((XA)^k) \subseteq R(XA) \subseteq R(A)$. Therefore, $R(A^k) \subseteq R(XA) \subseteq R(A)$ and $\rho_r(A^k) \leq \rho_r(XA) \leq \rho_r(A)$.

(ii) Proof is similar to (i) and hence omitted. □

Remark 2.11. In particular for k=1, A is regular and by Theorem (2.3) of [2], $\rho(A)=\rho_r(A)= \rho_c(A)$. Hence it reduces to $\rho(A)=\rho(XA)=\rho(AX)$. This rank property always holds for a complex matrix([1], pp.11).

Theorem 2.12. *Let $A \in \mathcal{F}_n$ and k be a positive integer. The following statements are equivalent:*

(i) *A is k-regular.*
(ii) *λA is k-regular for $\lambda \neq 0 \in \mathcal{F}$.*

(iii) PAP^T *is k-regular for some permutation matrix P.*

Proof.

$$\begin{aligned} A \;\; is \;\; k-regular &\Rightarrow A^kXA = A^k \\ &\Rightarrow (\lambda A)^k X(\lambda A) = (\lambda A)^k for \;\; \lambda \neq 0 \in \mathcal{F} \\ &\Rightarrow \lambda A \;\; is \;\; k-regular \end{aligned}$$

If λA is k-regular, then for λ=1, A is k-regular. Thus, (i)⇔(ii) hold.

$$\begin{aligned} A \text{ is k-regular} &\Leftrightarrow A^kXA = A^k \text{for some } X \in \mathcal{F}_n \\ &\Leftrightarrow (PA^kP^T)(PXP^T)(PAP^T) = PA^kP^T \\ &\quad\;\; \text{for some permutation matrix P and some } X \in \mathcal{F}_n \\ &\Leftrightarrow (PAP^T)^k(PXP^T)(PAP^T) = (PAP^T)^k \\ &\Leftrightarrow PAP^T \text{ is k-regular} \end{aligned}$$

Thus (i) ⇔ (iii) hold. Hence the proof. □

Theorem 2.13. *If Y, Z* $\in A_r\{1^k\}$ *then YAZ* $\in A_r\{1^k\}$.

Proof. Since Y, Z $\in A_r\{1^k\}$, $A^kYA = A^k$ and $A^kZA = A^k$.
Now, take $X = YAZ$, then we have $A^kXA = A^k(YAZ)A = (A^kYA)ZA = A^kZA = A^k$.
Hence, X=YAZ $\in A_r\{1^k\}$. □

Theorem 2.14. *For A, B* $\in \mathcal{F}_n$, *with* $R(A) = R(B)$ *and* $R(A^k) = R(B^k)$ *then, A is right k-regular ⇔ B is right k-regular .*

Proof. Let A be a right k-regular matrix, satisfying $R(B^k) \subseteq R(A^k)$ and $R(A) \subseteq R(B)$.
Since $R(B^k) \subseteq R(A^k)$, by Lemma (2.6), $B^k = B^kXA$ for each k-g-inverse X of A.
Since $R(A) \subseteq R(B)$, by Lemma (2.4), $A = YB$ for some Y $\in \mathcal{F}_n$. Substituting for A in $B^k = B^kXA$, we get $B^k = B^kXA = B^kXYB = B^kZB$ where $XY = Z$.
Hence B is right k-regular.
Conversely, if B is a right k-regular matrix satisfying $R(A^k) \subseteq R(B^k)$ and $R(B) \subseteq R(A)$, then A is right k-regular can be proved in the same manner.
Hence the theorem. □

Theorem 2.15. ′ *For A, B* $\in \mathcal{F}_n$, *with* $C(A) = C(B)$ *and* $C(A^k) = C(B^k)$ *then, A is left k-regular ⇔ B is left k-regular.*

Proof. This is similar to Theorem (2.14) and hence omitted. □

Theorem 2.16. *For A, B* $\in \mathcal{F}_n$, *if A is right k-regular,* $R(B) \subseteq R(A^k)$ *and B is idempotent then AB is right k-regular and* $A_r\{1^k\} \subseteq (AB)_r\{1^k\}$.

Proof. Since $R(B) \subseteq R(A^k)$, by Lemma (2.6), B=BXA, for each right k-g-inverse X of A. Since B is idempotent, $B^2 = B$.

$$\begin{aligned} (AB)^k &= (AB)^{k-1}(AB) \\ &= (AB)^{k-1}AB^2 \\ &= (AB)^{k-1}ABB \\ &= (AB)^{k-1}A(BXA)B \\ &= (AB)^k X(AB) \end{aligned}$$

Thus AB is right k-regular. X is a right k-g-inverse of AB. Hence $A_r\{1^k\} \subseteq (AB)_r\{1^k\}$. □

Theorem 2.17. *For A, $B \in \mathcal{F}_n$, if A is left k-regular, $C(B) \subseteq C(A^k)$ and B is idempotent then BA is left k-regular and $A_\ell\{1^k\} \subseteq (BA)_\ell\{1^k\}$.*

Proof. This is similar to Theorem (2.16) and hence omitted. □

Remark 2.18. In particular for k=1 and A commutes with B, it reduces to the following result found in [5]:

Theorem 2.19. *For a regular fuzzy matrix A if $R(B) \subseteq R(A)$ and B is idempotent then AB is a regular fuzzy matrix and $A\{1^k\} \subseteq (AB)\{1^k\}$.*

Now, we shall discuss the relation between regular, k-regular and k^{th} power of a matrix A in $\mathcal{F}_n$, We assume $A^0 = I$. Throughout k is a positive integer.

If A is regular, then $AXA = A$ for some X in $\mathcal{F}_n$, for $k \geq 1$, pre(post) multiplying by A^{k-1} on both sides, we get $A^kXA = A^k (AXA^k = A^k)$. Thus A is k-regular for all $k \geq 1$. If A^k is regular then $A^kYA^k = A^k$ for some Y in $\mathcal{F}_n$. This can written as $A^kZA = A^k$ where $Z = YA^{k-1}$, hence A is k-regular. On the other hand if A is k-regular for some positive integer k, need not imply A^k is regular (or) A is regular. These are illustrated in the following examples.

Example 2.20. Let us consider the non-regular matrix given in [3],
$A = \begin{bmatrix} 1 & 0.5 & 0 \\ 0.5 & 1 & 0.5 \\ 0.5 & 0.5 & 1 \end{bmatrix}$. For this A, $A^2 = \begin{bmatrix} 1 & 0.5 & 0.5 \\ 0.5 & 1 & 0.5 \\ 0.5 & 0.5 & 1 \end{bmatrix}$.

For $X = \begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0.5 \\ 0 & 0 & 1 \end{bmatrix}$, $A^2XA = A^2$ holds. A is 2-regular, but A is not regular.

Example 2.21. Let us consider $A = \begin{bmatrix} 1 & 0.5 & 0 \\ 0 & 0 & 0.5 \\ 0.5 & 0 & 0 \end{bmatrix}$.

For this A, $A^2 = \begin{bmatrix} 1 & 0.5 & 0.5 \\ 0.5 & 0 & 0 \\ 0.5 & 0.5 & 0 \end{bmatrix}$, $A^3 = \begin{bmatrix} 1 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0 \\ 0.5 & 0.5 & 0.5 \end{bmatrix}$.

For $X = \begin{bmatrix} 1 & 0 & 0.5 \\ 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0.5 \end{bmatrix}$, $A^3XA = A^3 \neq AXA^3$ holds.

Therefore A is 3-regular. $A^5 = A^4$.
Put $A^3 = B$, now we prove that B is not regular. If B is regular then $B = BYB$ for some Y $\in \mathcal{F}_3$. The $(1,1)^{th}$ entry of B is b_{11}=1.
Therefore, $b_{11} = \sum_{j,k=1}^{3} b_{1j}y_{jk}b_{k1} \Rightarrow 1=b_{11}y_{1k}b_{k1}+b_{12}y_{2k}b_{k1}+b_{13}y_{3k}b_{k1} \Rightarrow b_{11}y_{1k}b_{k1}=1$ (or)$b_{12}y_{2k}b_{k1}=1$ (or) $b_{13}y_{3k}b_{k1}=1$. Since $b_{12} = b_{13}=0.5$ and $b_{11}=1$, the only possibility is $y_{1k}b_{k1}=1$ for all k. Now for k=1, $y_{11}b_{11}=1 \Rightarrow y_{11}=1$. For k=2, $y_{12}b_{21}=1$, for k=3, $y_{13}b_{31}=1$ are not possible, since $b_{21} = b_{31}=0.5$. Hence $y_{11}=1$.
The $(2,3)^{th}$ entry of B is $b_{23}=0$.
Therefore, $b_{23} = \sum_{j,k=1}^{3} b_{2j}y_{jk}b_{k3} \Rightarrow 0=b_{21}y_{1k}b_{k3}+b_{22}y_{2k}b_{k3}+b_{23}y_{3k}b_{k3} \Rightarrow b_{21}y_{1k}b_{k3}=0$, $b_{22}y_{2k}b_{k3}=0$, $b_{23}y_{3k}b_{k3}=0$. Suppose $b_{21}y_{1k}b_{k3}=0 \Rightarrow b_{21}y_{11}b_{13}=0$, $b_{21}y_{12}b_{23}=0$, $b_{21}y_{13}b_{33}=0$ but $b_{21}y_{11}b_{13} \neq 0$. Since $y_{11}=1$ and $b_{21} = b_{13}=0.5$, therefore $b_{21}y_{1k}b_{k3} \neq 0$.
Hence $\sum_{j,k=1}^{3} b_{2j}y_{jk}b_{k3} \neq 0 \Rightarrow b_{23} \neq 0$. Therefore, $BYB \neq B$ for any Y $\in \mathcal{F}_3 \Rightarrow$ B=A^3 is not regular.

Example 2.22. The following example shows that, for A $\in \mathcal{F}_n$ and k is a positive integer, $A_r\{1^k\} \neq A_\ell\{1^k\}$.

Let us consider $A = \begin{bmatrix} 0 & 0.5 & 0 \\ 0 & 1 & 0.5 \\ 0.5 & 0 & 0 \end{bmatrix}$.

For this A, $A^2 = \begin{bmatrix} 0 & 0.5 & 0.5 \\ 0.5 & 1 & 0.5 \\ 0 & 0.5 & 0 \end{bmatrix}$, $A^3 = \begin{bmatrix} 0.5 & 0.5 & 0.5 \\ 0.5 & 1 & 0.5 \\ 0 & 0.5 & 0.5 \end{bmatrix}$.

For $X = \begin{bmatrix} 0.5 & 0 & 0.5 \\ 0.5 & 1 & 0 \\ 0 & 0.5 & 0 \end{bmatrix}$, $A^3XA = A^3$. HenceA is 3-regular. For k=3, $A^3XA = A^3$ but $AXA^3 \neq A^3$.
Hence $A_r\{1^k\} \neq A_\ell\{1^k\}$.

3. Regularity Index

It is well known that [3] every matrix A $\in \mathcal{F}_n$ has index 'k' and period 'd' that is, least positive integers k and d exist such that $A^{k+d} = A^k$. Let i(A) and p(A), denote the index and period of A respectively. It is clear that, if $i(A)$=1 then A is regular but the converse is not true (refer Example(3.5)).

In this section first we shall prove that every matrix of index k is k-regular.

Theorem 3.1. *For $A \in \mathcal{F}_n$ with $i(A)$=k, we have the following:*

(i) *The smallest exponent for which $R(A^k) = R(A^{k+1})$ and $C(A^k) = C(A^{k+1})$ hold is k.*
(ii) *The smallest positive integer k for which $\rho_r(A^k) = \rho_r(A^{k+1})$ and $\rho_c(A^k) = \rho_c(A^{k+1})$ hold is k.*
(iii) *The smallest exponent for which $YA^{k+1} = A^k$ and $A^{k+1}X = A^k$ have solutions is k.*
(iv) *A is right k-regular and left k-regular.*
(v) *$A_r\{1^k\} = A_\ell\{1^k\} = A\{1^k\} \neq \phi$.*

(vi) *A is q-regular for all integers $q \geq k$.*

Proof. By definition of index of A, k is the smallest positive integer and $d > 0$ such that $A^{k+d} = A^k$. For $d \geq 1$ this can be written as,
$A^{d-1}A^{k+1} = A^k = A^{k+1}A^{d-1}$ .... (3.1)
Therefore by Lemma (2.4), we have $R(A^k) \subseteq R(A^{k+1})$ and $C(A^k) \subseteq C(A^{k+1})$. By Lemma (2.5), $R(A^{k+1}) \subseteq R(A^k)$ and $C(A^{k+1}) \subseteq C(A^k)$ always hold, hence k is the smallest exponent for which, $R(A^k) = R(A^{k+1})$ and $C(A^k) = C(A^{k+1})$. Thus (i) holds. (ii) Automatically follows from (i). Again by Lemma (2.4), there exists Y,X $\in \mathcal{F}_n$ satisfying $YA^{k+1} = A^k$ and $A^{k+1}X = A^k$. Thus (iii) holds. In (3.1), put $X = A^{d-1}$, then $XA^{k+1} = A^k = A^{k+1}X$, premultiplying by A, $AXA^{k+1} = A^{k+1}$ and post multiplying by A, $A^{k+1}XA = A^{k+1}$. By multiplying these equations suitably by A^{d-1}, yield $AXA^{k+d} = A^{k+d}$ and $A^{k+d}XA = A^{k+d}$. Again by using $A^{k+d} = A^k$, we get $AXA^k = A^k = A^kXA$. Thus A is both right k-regular and left k-regular and X $\in A_r\{1^k\} = A_\ell\{1^k\} \neq \phi$. Thus (iv) and(v) hold. (vi) Follows from Remark(2.3). Hence the theorem. □

Remark 3.2. In particular for $i(A)$=1, A is regular and the group inverse exists. From Theorem (3.1), we observe that every fuzzy matrix of index k is k-regular, but k is not the least positive integer for which A is k-regular.

Hence $A_r\{1^k\} = A_\ell\{1^k\} = A\{1^k\} \neq \phi$, but k is not the smallest positive integer satisfying $A^kXA = A^k(\text{or})AYA^k = A^k$.
Further $R(A^{h-1})$ is not contained in $R(A^h)$ and $C(A^{h-1})$ is not contained in $C(A^h)$ for all $h \leq k$.

Definition 3.3. The smallest positive integer k for which A is k-regular, is called the regularity index of A and is denoted as reg.$i(A)$.

Remark 3.4. It is clear that from the Definition (3.3) and Remark (2.3), reg.$i(A) = k_0 \Leftrightarrow k_0$ is the smallest positive integer for which A has a k_0-g-inverse$\Leftrightarrow A\{1^{k_0}\} \neq \phi$ and $A\{1^h\}=\phi$ for all $h < k_0$. Further A is regular $\Leftrightarrow$ reg.$i(A)$=1. From Theorem(3.1), it follows that $A\{1^{k_0}\} \subseteq A\{1^k\}$. Therefore reg.$i(A) \leq i(A)$.
This is illustrated in the following.

Example 3.5. $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ is regular and hence k-regular for all $k \geq 1$; reg.$i(A)$=1.

By using Algorithm (1) in [3], $A\{1\}$=\{ $X = \begin{bmatrix} 0 & 1 \\ 1 & \alpha \end{bmatrix}$, $\alpha \in \mathcal{F}$\} is the set of all g-inverses of A. It can be verified that $AX \neq XA$ for all $X \in A\{1\}$. Therefore the group inverse of A and Drazin inverse of A does not exist. $R(A)$ is not contained in $R(A^2)$, $C(A)$ is not contained in $C(A^2)$. Here, $A^2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = A^3$, therefore $i(A)$=2 and $A^2XA = A^2 = AXA^2$ hold, but 2 is not the smallest positive integer. Hence reg.$i(A) < i(A)$.

Corollary 3.6. *For a non-regular matrix $A \in \mathcal{F}_n$ if $i(A)$=2, then reg. $i(A)$=2.*

Proof. Since $i(A)$=2, by Theorem (3.1), A is 2-regular and $A^2XA = A^2 = AXA^2$, for X $\in \mathcal{F}_n$.

Since A is not regular, reg.$i(A) \neq 1$. Therefore k=2 is the smallest integer satisfying $A^kXA = A^k = AXA^k$. Hence reg.$i(A) = i(A) = 2$.
Relation between regularity index and index of a matrix are illustrated in the following examples. □

Example 3.7. Let us consider $A = \begin{bmatrix} 1 & 0.8 & 0 \\ 0.8 & 0.7 & 0 \\ 0.7 & 0.6 & 0 \end{bmatrix}$. Here, $\rho_r(A) = 3$, $\rho_c(A) = 2$.
Therefore by Theorem (2.3) of [2] A is not regular. Hence, reg.$i(A) \neq 1$.
Since $A^2 = \begin{bmatrix} 1 & 0.8 & 0 \\ 0.8 & 0.8 & 0 \\ 0.7 & 0.7 & 0 \end{bmatrix} = A^3$, $i(A)=2$, $1 < reg.i(A) \leq i(A)=2$.
Thus reg.$i(A) = i(A) = 2$.

Example 3.8. Let us consider the matrix A in Example (2.22). For
$X = \begin{bmatrix} 0.5 & 0 & 0.5 \\ 0.5 & 1 & 0 \\ 0 & 0.5 & 0 \end{bmatrix}$, $A^3XA = A^3$ holds but $A^2XA \neq A^2$, $AXA \neq A$ and 3 is the smallest positive integer for which X is a 3-g-inverse of A. $X \in A\{1^3\}$, $X \notin A\{1^2\}$, $X \notin A\{1\}$. Hence $A\{1^3\} \neq \phi$. Therefore reg.$i(A)$=3. For this X, $AX \neq XA$. Hence X is not a Drazin inverse of A. Further $A^5 = A^4$. Hence $i(A)$=4. Thus reg.$i(A) = 3 < 4 = i(A)$.

Theorem 3.9. *For $A \in \mathcal{F}_n$ with $i(A) = k > 1$ and $p(A) = d$ divides $(k-1)$ then A^k is regular.*

Proof. Since $i(A)=k$, by Theorem (3.1), A is k-regular. Therefore, $A^kXA = A^k$ ...... (3.2) .
Post multiplying Equation (3.2) by A^{k-1} on both sides, we get $A^kXA^k = A^{k+(k-1)}$.
Since d divides $(k-1)$, $A^kXA^k = A^{k+(k-1)} = A^k$. Hence A^k is regular. □

Theorem 3.10. *For $A \in \mathcal{F}_n$ and k be a positive integer, the following statements are equivalent:*

(i) *$reg.i(A) = i(A) = k$.*
(ii) *The smallest exponent for which $A^kXA = A^k$ holds is k.*
(iii) *The smallest exponent for which $AXA^k = A^k$ holds is k.*
(iv) *k is the smallest positive integer such that A is k-regular.*
(v) *k is the smallest positive integer such that A^T is k-regular.*
(vi) *The Drazin inverse A_D exists and unique.*

Proof. This follows from Theorem (3.1) and the definition of regularity index. □

References

[1] A. Ben-Israel and T. N. E. Greville, *Generalized inverses*, Theory and Applications, Wiley, Newyork, 1974.
[2] H. H. Cho, *Regular fuzzy matrices and fuzzy equations*, Fuzzy sets and systems, **105** (1999), 445-451.

[3] K. H. KIM and F. W. Roush, *On generalised fuzzy matrices*, Fuzzy Sets and Systems, **4** (1980), 293-375.
[4] A. R. Meenakshi, *On regularity of block triangular fuzzy matrices*, Applied Math & Computing, **15** (2004), 207-220.
[5] A. R. Meenakshi and S. Sriram, *On regularity of sums of fuzzy matrices*, Bulletin of Pure and Applied sciences., **22E(2)** (2003), 395-403.
[6] T. J. Ross, *Fuzzy logic with engineering applications*, McGraw Hill Inc., 1995.
[7] W. Pedrycz, *Fuzzy relational calculus*, Hand Book of Fuzzy Computations, IOP.4d, 1998.

A. R. Meenakshi, Department of Mathematics, Karpagam college of Engineering, Coimbatore 641 032, India
E-mail address: arm_meenakshi@yahoo.co.in

P. Jenita*, Department of Mathematics, Karpagam college of Engineering, Coimbatore 641 032, India
E-mail address: sureshjenita@yahoo.co.in

*Corresponding author