FUZZY TRAIN ENERGY CONSUMPTION MINIMIZATION MODEL AND ALGORITHM

X. LI, D. RALESCU AND T. TANG

Abstract. Train energy saving problem investigates how to control train's velocity such that the quantity of energy consumption is minimized and some system constraints are satisfied. On the assumption that the train's weights on different links are estimated by fuzzy variables when making the train scheduling strategy, we study the fuzzy train energy saving problem. First, we propose a fuzzy energy consumption minimization model, which minimizes the average value and entropy of the fuzzy energy consumption under the maximal allowable velocity constraint and traversing time constraint. Furthermore, we analyze the properties of the optimal solution, and then design an iterative algorithm based on the Karush-Kuhn-Tucker conditions. Finally, we illustrate a numerical example to show the effectiveness of the proposed model and algorithm.

1. Introduction

The train energy saving problem studies how to control the train's velocity profile such that the energy consumption is minimized, which is usually treated as a part of the train scheduling problem. For example, Kraay et al. [10] studied a train pacing problem in which the energy consumption and the delay time of train are minimized. The authors designed a branch-and-bound algorithm by linearizing the objective function and ignoring the train interactions. In 2004, Ghoseiri et al. [5] proposed a two-objective train scheduling model minimizing the energy consumption and the total passenger-time, and employed the ideal point method to transform the two-objective optimization model to a single-objective optimization model. In 2009, Castillo et al. [2] proposed a three-stage scheduling model, in which the relative travel time is considered as the objective of the first stage, the prompt allocation of trains is considered as the objective of the second stage, and the energy consumption is selected as the objective of the third stage.

As the energy saving problem receives more attention from both political scientists and economists, some researchers studied the train energy saving problem separately from the train scheduling problem. In 1997, Chang and Sim [3] investigated a coasting control optimization model for saving energy consumption. In their work, a genetic algorithm is designed for solving the optimal coasting control strategy. In 2000, Khmelnitsky [9] investigated the operation of a train on a

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variable grade profile, which minimizes the energy consumption on a given route subject to arbitrary velocity restrictions. By using the maximum principle, the author analyzed the properties of optimal solution, and designed an efficient numerical algorithm. In the same year, Howlett [6] investigated an energy consumption minimization model with a fixed traversing time constraint, and designed an algorithm based on the Karush-Kuhn-Tucker conditions for the optimal solution. In 2003, Liu and Golovitcher [14] presented an analytical solution that gives the sequence of optimal equations to find the control change points, and then designed an algorithm for finding the energy efficient train control strategy. In 2006, Effati and Roohparvar [4] presented a linear programming approach and a dynamic programming approach to solve the energy saving problem. In 2009, Bai et al. [1] investigated the energy-efficient driving strategy for freight trains, which minimizes the energy consumption for overcoming the resistance and the energy loss caused by braking. In the same year, Howlett et al. [7] studied the optimal control strategy for the local minimization of energy consumption. In 2010, Miyatake and Ko [15] presented a good survey on the train energy saving models and algorithms.

In train scheduling problem, the uncertainty for some parameters have been recognized and studied. For example, Zhou and Zhong [19] proposed a two-objective scheduling model by minimizing the waiting time of high-speed trains and the running time of all trains in the railway system, in which the passenger demand and waiting time are treated as random variables. Yang et al. [16] proposed a fuzzy train scheduling model by minimizing the total passenger-time and train delay, in which the numbers of passengers boarding and leaving the train at stations are treated as fuzzy variables. Khan and Zhou [8] studied a two-stage random resource model by minimizing the trip time and train delay, which incorporates random segment travel time disturbances into a medium-term planning timetable. Yang et al. [17] studied the passenger train scheduling problem on a single-track with stochastic information. Yang et al. [18] analyzed the railway freight transportation planning problem with mixed uncertainty of randomness and fuzziness. Although train energy saving problem has been widely studied on both models and algorithms, according to our knowledge, there is currently no study on the uncertainty of parameters. The purpose of this paper is to study a train energy saving model and algorithm with fuzzy parameters.

The rest of this paper is organized as follows. Section 2 recalls some basic concepts and properties about fuzzy variables. In Section 3, we propose a fuzzy train energy consumption minimization model, which minimizes the average value and entropy of the fuzzy energy consumption. In Section 4, we first analyze the theoretical properties of the optimal solution, and then design an iterative algorithm based on the Karush-Kuhn-Tucker conditions. For illustrating the effectiveness of our model, we present a numerical example in Section 5, which shows that our model may reduce the quantity of energy consumption significantly.

2. Preliminaries

Intending to investigate the train energy saving problem with fuzzy parameters, we shall briefly introduce some basic knowledge on fuzzy variables.

Suppose that ξ is a fuzzy variable with membership function μ. For any $x \in \Re$, membership degree $\mu(x)$ expresses the possibility of ξ taking value x. For any $B \subseteq \Re$, the possibility measure of ξ taking values in B is defined as:

$$\mathrm{Pos}\{\xi \in B\} = \sup_{x \in B} \mu(x).$$

It is proved that possibility measure satisfies the properties of normality, nonnegativity and monotonicity. However, it is not self-dual. Since the self-duality is intuitive and important in real problems, Liu and Liu [13] defined a credibility measure as:

$$\mathrm{Cr}\{\xi \in B\} = \frac{1}{2}\left(\sup_{x \in B} \mu(x) + 1 - \sup_{x \in B^c} \mu(x)\right).$$

In 2006, Li and Liu [11] proved a sufficient and necessary condition for credibility measure, which tells us that a set function is a credibility measure if and only if it satisfies normality, monotonicity, self-duality and partial maximality.

Let ξ_1 and ξ_2 be two fuzzy variables with membership functions μ_1 and μ_2, respectively. If fuzzy vector (ξ_1, ξ_2) has joint membership function μ, then ξ_1 and ξ_2 are said to be independent if and only if for any $x_1, x_2 \in \Re$, we have

$$\mu(x_1, x_2) = \min\{\mu_1(x_1), \mu_2(x_2)\}.$$

In order to measure the average values of fuzzy variables, Liu and Liu [13] defined a fuzzy expected value operator as:

$$E[\xi] = \int_0^{+\infty} \mathrm{Cr}\{\xi \geq r\}\mathrm{d}r - \int_{-\infty}^0 \mathrm{Cr}\{\xi \leq r\}\mathrm{d}r,$$

provided that at least one of the two integrals is finite. For fuzzy variables ξ and η being independent, it has been proved that $E[a\xi + b\eta] = aE[\xi] + bE[\eta]$, for any real numbers a and b.

Let ξ be a continuous fuzzy variable with membership function μ. Its entropy was defined by Li and Liu [12] as:

$$H[\xi] = -\int_{-\infty}^{+\infty} \frac{\mu(x)}{2} \ln \frac{\mu(x)}{2} + \left(1 - \frac{\mu(x)}{2}\right) \ln \left(1 - \frac{\mu(x)}{2}\right) \mathrm{d}x.$$

The entropy is used to characterize the uncertainty of a fuzzy variable resulting from information deficiency.

Example 2.1. *A triangular fuzzy variable ξ is defined by a piecewise linear membership function (see Figure 1),*

$$\mu(x) = \begin{cases} (x-a)/(b-a), & \text{if } a \le x < b \\ (c-x)/(c-b), & \text{if } b \le x < c \\ 0, & \text{otherwise,} \end{cases}$$

where $a \le b \le c$. Since the membership function of a triangular fuzzy variable ξ is fully determined by triplet (a, b, c), we write $\xi = (a, b, c)$ in what follows. It can be proved that

$$E[\xi] = (a + 2b + c)/4, \;\; H[\xi] = (c - a)/2.$$

If ξ has a symmetric membership function, that is, $b - a = c - b$, then we have $E[\xi] = b$ and $H[\xi] = b - a$.

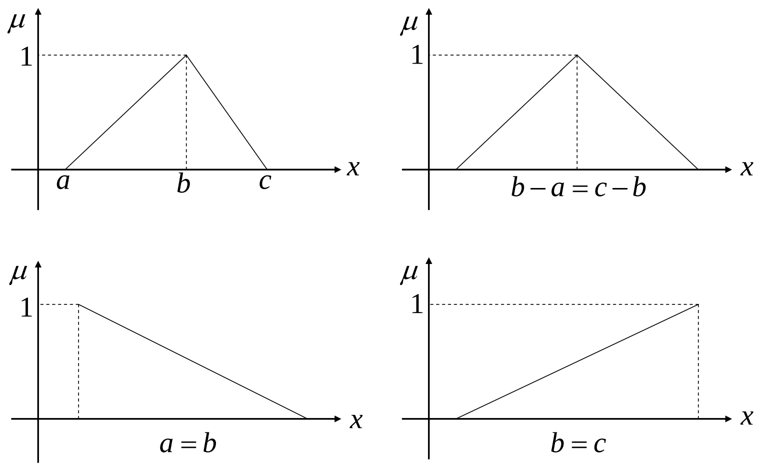

Figure 1. Membership Function of Triangular Fuzzy Variable $\xi = (a, b, c)$

3. Mathematical Model

Suppose that there is a train running on a rail line which consists of $n+1$ stations (see Figure 2). For simplicity, the link between the ith station and the $(i+1)$th station is labeled with its starting station. It is clear that there are n links along the rail line. If we use m_i and v_i to denote the weight and velocity of the train when it runs on the ith link, then according to the Davis formula, the resistance imposed on the train is

$$R_i = m_i(r_0 + r_1 v_i + r_2 v_i^2),$$

where r_0 is the resistance coefficient due to grade, r_1 is the resistance coefficient due to rail friction, and r_2 is the resistance coefficient due to air friction. In order to keep the train moving with constant velocity v_i, the required traction to the train is R_i. If the length of the ith link is d_i, then the required power for overcoming the resistance on this link is $R_i \cdot d_i$, and the required energy consumption is

$$C_i = m_i \left(r_0 + r_1 v_i + r_2 v_i^2\right) d_i s,$$

where s is the quantity of energy consumption for providing per unit of power. Furthermore, the total energy consumption on the rail line is calculated as:

$$C = \sum_{i=1}^{n} m_i \left(r_0 + r_1 v_i + r_2 v_i^2\right) d_i s.$$

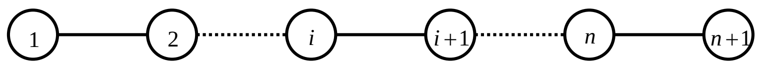

FIGURE 2. A Rail Line Which Consists of $n+1$ Stations

The train's weight is the sum of the weights of engine and coaches and the weights of freights. Since the weights of freights are in general unknown when making scheduling strategy, it is more reasonable to estimate the train's weights $m_1, m_2, \cdots, m_n$ by fuzzy variables $\xi_1, \xi_2, \cdots, \xi_n$. In this case, for each velocity vector $(v_1, v_2, \cdots, v_n)$, the total energy consumption is also a fuzzy variable,

$$\mathcal{C}(v_1, v_2, \cdots, v_n) = \sum_{i=1}^{n} \xi_i \left(r_0 + r_1 v_i + r_2 v_i^2\right) d_i s.$$

Since it is in general not meaningful to minimize a fuzzy objective function, i.e., it is in general not meaningful to rank two fuzzy variables, we should take some measure to transform the fuzzy objective value to a crisp one. Expected value principle has been widely employed to rank fuzzy variables, which tells us that a fuzzy variable is said to be smaller than another one if and only if it has a smaller expected value. Here, we will employ the expected value principle to minimize the expected value (or average value) of the fuzzy energy consumption. On the other hand, we should minimize the uncertainty that the possible value of energy consumption is larger than the average value. If we use entropy to measure the uncertainty, then we get the following expectation entropy minimization (EEM) model,

$$\begin{cases} \min & \alpha E\left[\mathcal{C}(v_1, v_2, \cdots, v_n)\right] + (1-\alpha) H[\mathcal{C}(v_1, v_2, \cdots, v_n)] \\ \text{s.t.} & \displaystyle\sum_{i=1}^{n} d_i / v_i \le T \\ & v_i > 0, \ i = 1, 2, \cdots, n, \end{cases}$$

where $\alpha \in (0,1)$ denotes the preference coefficient of the decision maker on the average energy consumption, the first constraint ensures that the traversing time is less than a predetermined real number T, and the second constraint ensures that the train's velocity is positive.

In a real railway system, the train's running velocity is always confined to a given range due to the different physical conditions of tracks. If the maximal allowable velocity on the ith link is $v_i^{\max}$, then we get the following expectation entropy minimization model with maximal allowable velocity constraint,

$$\begin{cases} \min & \alpha E[\mathcal{C}(v_1, v_2, \cdots, v_n)] + (1-\alpha) H[\mathcal{C}(v_1, v_2, \cdots, v_n)] \\ \text{s.t.} & \displaystyle\sum_{i=1}^{n} d_i / v_i \le T \\ & 0 < v_i \le v_i^{\max}, \ i = 1, 2, \cdots, n, \end{cases}$$

where the second constraint ensures that the train's velocity is less than a predetermined maximal allowable value. For simplicity, this model will be denoted by the EEMV model.

Suppose that fuzzy variables $\xi_1, \xi_2, \cdots, \xi_n$ are independent and have unimodal membership functions. Then, the expected value operator and entropy operator are proved to be linear. That is, for any $x_1, x_2, \cdots, x_n \in \Re$, we have

$$E[x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n] = \sum_{i=1}^{n} x_i E[\xi_i],$$

$$H[x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n] = \sum_{i=1}^{n} |x_i| H[\xi_i].$$

According to these linear properties, it is easy to prove that the EEM model is equivalent to the following nonlinear optimization model,

$$\begin{cases} \min & \displaystyle\sum_{i=1}^{n} m_i(\alpha) \left(r_0 + r_1 v_i + r_2 v_i^2 \right) d_i s \\ \text{s.t.} & \displaystyle\sum_{i=1}^{n} d_i / v_i \le T \\ & v_i > 0, \ i = 1, 2, \cdots, n, \end{cases}$$

and the EEMV model is equivalent to the following nonlinear optimization model,

$$\begin{cases} \min & \displaystyle\sum_{i=1}^{n} m_i(\alpha) \left(r_0 + r_1 v_i + r_2 v_i^2 \right) d_i s \\ \text{s.t.} & \displaystyle\sum_{i=1}^{n} d_i / v_i \le T \\ & 0 < v_i \le v_i^{\max}, \ i = 1, 2, \cdots, n, \end{cases}$$

where $m_i(\alpha) = \alpha E[\xi_i] + (1-\alpha) H[\xi_i]$, for all $1 \le i \le n$. For example, if ξ_i is a triangular fuzzy variable (a_i, b_i, c_i), then we have $E[\xi_i] = (a_i + 2b_i + c_i)/4$, $H[\xi_i] = (c_i - b_i)/2$, and $m_i(\alpha) = (3\alpha - 2)a_i + 2\alpha b_i + (2-\alpha)c_i$.

4. **Algorithms**

In this section, we first analyze some theoretical properties of the optimal solutions of the EEM and EEMV models. Furthermore, we design an iterative algorithm for solving the EEM model by using the Karush-Kuhn-Tucker conditions. Finally, for solving the EEMV model, we design a numerical algorithm by modifying the optimal solution of the EEM model gradually.

Theorem 4.1. *Suppose that fuzzy variables* $\xi_1, \xi_2, \cdots, \xi_n$ *are independent and have unimodal membership functions. Let* $(v_1, v_2, \cdots, v_n)$ *be the optimal solution of the EEM model. For all* $1 \leq i, j \leq n$ *and* $0 < \alpha < 1$*, if* $m_i(\alpha) < m_j(\alpha)$*, then we have* $v_i \geq v_j$*.*

Proof. Assume that $(v_1, v_2, \cdots, v_n)$ is the optimal solution of the EEM model. Let i and j be two indices satisfying $m_i(\alpha) < m_j(\alpha)$. If $v_i < v_j$, then it is easy to prove that

$$m_i(\alpha)(r_1 + 2r_2 v_i)v_i^2 < m_j(\alpha)(r_1 + 2r_2 v_j)v_j^2,$$

and that there is a pair of small positive numbers ε_i and ε_j satisfying

$$v_i + \varepsilon_i > 0, \ v_j - \varepsilon_j > 0, \tag{1}$$

$$d_i/v_i + d_j/v_j = d_i/(v_i + \varepsilon_i) + d_j/(v_j - \varepsilon_j), \tag{2}$$

$$m_i(\alpha)v_i(v_i + \varepsilon_i)(r_1 + 2r_2 v_i + r_2\varepsilon_i) < m_j(\alpha)v_j(v_j - \varepsilon_j)(r_1 + 2r_2 v_j - r_2\varepsilon_j). \tag{3}$$

First, according to conditions (1)-(2), it is clear that $(\cdots, v_i + \varepsilon_i, \cdots, v_j - \varepsilon_j, \cdots)$ is a feasible solution of the EEM model. In addition, according to condition (2), we have $d_i\varepsilon_i/v_i(v_i + \varepsilon_i) = d_j\varepsilon_j/v_j(v_j - \varepsilon_j)$. Then, it follows from inequality (3) that

$$\begin{aligned} &r_1 m_i(\alpha)d_i\varepsilon_i + 2r_2 m_i(\alpha)v_i d_i\varepsilon_i + r_2 m_i(\alpha)d_i\varepsilon_i^2 \\ &< r_1 m_j(\alpha)d_j\varepsilon_j + 2r_2 m_j(\alpha)v_j d_j\varepsilon_j - r_2 m_j(\alpha)d_j\varepsilon_j^2, \end{aligned}$$

which is equivalent to the following inequality:

$$\begin{aligned} &m_i(\alpha)\left(r_0 + r_1(v_i + \varepsilon_i) + r_2(v_i + \varepsilon_i)^2\right)d_i s \\ &\quad + m_j(\alpha)\left(r_0 + r_1(v_j - \varepsilon_j) + r_2(v_j - \varepsilon_j)^2\right)d_j s \\ &< m_i(\alpha)\left(r_0 + r_1 v_i + r_2 v_i^2\right)d_i s + m_j(\alpha)\left(r_0 + r_1 v_j + r_2 v_j^2\right)d_j s. \end{aligned}$$

The inequality tells us that $(\cdots, v_i + \varepsilon_i, \cdots, v_j - \varepsilon_j, \cdots)$ has a smaller energy consumption than $(v_1, v_2, \cdots, v_n)$, which contradicts the assumption that it is the optimal solution of the EEM model. Hence, we have $v_i \geq v_j$, and the proof is complete. □

Theorem 4.2. *Suppose that fuzzy variables* $\xi_1, \xi_2, \cdots, \xi_n$ *are independent and have unimodal membership functions. If* $(v_1, v_2, \cdots, v_n)$ *is the optimal solution of the EEMV model (or the EEM model), then we have*

$$\sum_{i=1}^{n} d_i/v_i = T. \tag{4}$$

Proof. Assume that $(v_1, v_2, \cdots, v_n)$ is the optimal solution of the EEMV model. If equation (4) does not hold, then there is a sequence of positive real numbers $\{\varepsilon_i\}$ such that $0 < v_i - \varepsilon_i < v_i^{\max}$, for all i, and

$$\sum_{i=1}^{n} d_i/v_i \le T,$$

which implies that vector $(v_1 - \varepsilon_1, v_2 - \varepsilon_2, \cdots, v_n - \varepsilon_n)$ satisfies the maximal allowable velocity constraint and the traversing time constraint, i.e., it is a feasible solution of the EEMV model. Since $v_i > \varepsilon_i$, for all i, it is easy to prove that

$$\begin{aligned}
&\sum_{i=1}^{n} m_i(\alpha)(r_0 + r_1 v_i + r_2 v_i^2) d_i s - \sum_{i=1}^{n} m_i(\alpha)(r_0 + r_1(v_i - \varepsilon_i) + r_2(v_i - \varepsilon_i)^2) d_i s \\
&= \sum_{i=1}^{n} m_i(\alpha)(r_1 + 2r_2 v_i - r_2\varepsilon_i) d_i \varepsilon_i s \\
&> \sum_{i=1}^{n} m_i(\alpha)(r_1 + r_2\varepsilon_i) d_i \varepsilon_i s \\
&> 0,
\end{aligned}$$

which contradicts the assumption that $(v_1, v_2, \cdots, v_n)$ is the optimal solution, since $(v_1 - \varepsilon_1, v_2 - \varepsilon_2, \cdots, v_n - \varepsilon_n)$ has a smaller energy consumption than $(v_1, v_2, \cdots, v_n)$. Hence, condition (4) holds. For each optimal solution of the EEM model, a similar procedure can prove that condition (4) holds. The proof is now complete. □

Remark 4.3. *According to Theorem 4.2, the traversing time constraint in the EEM and EMMV models may be simplified to the following equality constraint:*

$$\sum_{i=1}^{n} d_i/v_i = T.$$

Now, let us design the algorithm for solving the EEM model. First, it is easy to prove that the Lagrange operator is

$$L(v_1, v_2, \cdots, v_n, \lambda) = \sum_{i=1}^{n} m_i(\alpha)\left(r_0 + r_1 v_i + r_2 v_i^2\right) d_i s + \lambda \left(\sum_{i=1}^{n} d_i/v_i - T\right).$$

According to the Karush-Kuhn-Tucker conditions, the optimal solution should satisfy the following equations:

$$\begin{cases}
\dfrac{\mathrm{d}L}{\mathrm{d}v_i} = m_i(\alpha)\left(r_1 + 2r_2 d_i v_i\right) s - \lambda d_i/v_i^2 = 0, \ i = 1, 2, \cdots, n \\
\dfrac{\mathrm{d}L}{\mathrm{d}\lambda} = \displaystyle\sum_{i=1}^{n} d_i/v_i - T = 0.
\end{cases} \tag{5}$$

For simplicity, define the following parameters,

$$q = r_1/6r_2, \ p_i = \lambda/4r_2 m_i(\alpha)s, \ i = 1, 2, \cdots, n. \tag{6}$$

For each $1 \le i \le n$, applying Kaerdannuo formula to the first equation of (5), the optimal velocity v_i may be determined by parameter λ as:

$$v_i = \sqrt[3]{p_i - q^3 + \sqrt{(p_i - 2q^3)p_i}} + \sqrt[3]{p_i - q^3 - \sqrt{(p_i - 2q^3)p_i}} - q. \tag{7}$$

On the other hand, according to the second equation of (5), it is easy to prove that parameter λ may be determined by $(v_1, v_2, \cdots, v_n)$ via the following formulation,

$$\lambda = \frac{1}{T}\sum_{i=1}^{n} m_i(\alpha)(r_1 v_i + 2r_2 v_i^2)d_i s. \tag{8}$$

Since the EEM model is a convex programming problem, the Karush-Kuhn-Tucker conditions are also sufficient conditions for the optimal solution. Then, we may solve the EEM model by finding a point which satisfies conditions (6)-(8).

Algorithm 4.4. *An iterative algorithm for solving the EEM model.*

Step 1: *Set ε to be a small positive real number.*

Step 2: *Initialize a feasible solution $v_0 = (v_1^0, v_2^0, \cdots, v_n^0)$, and calculate the corresponding parameter λ_0 according to formula (8).*

Step 3: *Calculate parameters $p_1, p_2, \cdots, p_n$ and q according to (6).*

Step 4: *Calculate a new feasible solution $v = (v_1, v_2, \cdots, v_n)$ according to formula (7), and calculate the corresponding parameter λ according to formula (8).*

Step 5: *If $|\lambda_0 - \lambda| > \varepsilon$, then set $v_0 = v, \lambda_0 = \lambda$ and go to Step 3.*

Step 6: *Return v as the optimal solution.*

Theorem 4.5. *Suppose that fuzzy variables $\xi_1, \xi_2, \cdots, \xi_n$ are independent and have unimodal membership functions. Let $(v_1, v_2, \cdots, v_n)$ and $(u_1, u_2, \cdots, u_n)$ be the optimal solutions of the EEM and EEMV models, respectively. For any $1 \le i \le n$, if $v_i > v_i^{\max}$, then we have $u_i = v_i^{\max}$.*

Proof. Without loss of generality, assume that $v_1 > v_1^{\max}$ and there is at least one index k such that $v_k < v_k^{\max}$. If $u_1 < v_1^{\max}$, then we prove that $(u_1, u_2, \cdots, u_n)$ is not the optimal solution of the EEMV model. Let i be an index satisfying $v_i < v_i^{\max}$ and $u_i > v_i$. First, since $(v_1, v_2, \cdots, v_n)$ is the optimal solution of the EEM model, for small positive numbers ε_1 and ε_i satisfying $d_1/v_1 + d_i/v_i = d_1/(v_1 - \varepsilon_1) + d_i/(v_i + \varepsilon_i)$, we have

$$\begin{aligned} & m_1(\alpha)(r_0 + r_1 v_1 + r_2 v_1^2)d_1 s + m_i(\alpha)(r_0 + r_1 v_i + r_2 v_i^2)d_i s \\ < & \; m_1(\alpha)(r_0 + r_1(v_1 - \varepsilon_1) + r_2(v_1 - \varepsilon_1)^2)d_1 s \\ & + m_i(\alpha)(r_0 + r_1(v_i + \varepsilon_i) + r_2(v_i + \varepsilon_i)^2)d_i s, \end{aligned}$$

which is equivalent to:

$$m_1(\alpha)v_1(v_1 - \varepsilon_1)(r_1 + 2r_2 v_1 - r_2\varepsilon_1) < m_i(\alpha)v_i(v_i + \varepsilon_i)(r_1 + 2r_2 v_i + r_2\varepsilon_i).$$

ε_1 and ε_i being arbitrary, let $\varepsilon_1 \to 0$ and $\varepsilon_i \to 0$. Then, we have

$$m_1(\alpha)v_1^2(r_1 + 2r_2v_1) \le m_i(\alpha)v_i^2(r_1 + 2r_2v_i).$$

On the other hand, since it is known that $u_1 < v_1$ and $u_i > v_i$, it is easy to prove that

$$m_1(\alpha)u_1^2(r_1 + 2r_2u_1) < m_i(\alpha)u_i^2(r_1 + 2r_2u_i).$$

Then, there is a pair of positive numbers θ_1 and θ_i satisfying

$$u_1 + \theta_1 \le v_1^{\max},\ u_i - \theta_i > 0,$$
$$d_1/u_1 + d_i/u_i = d_1/(u_1 + \theta_1) + d_i/(u_i - \theta_i),$$
$$m_1(\alpha)u_1(u_1 + \theta_1)(r_1 + 2r_2u_1 + r_2\theta_1) < m_i(\alpha)u_i(u_i - \theta_i)(r_1 + 2r_2u_i - r_2\theta_i),$$

which implies that

$$m_1(r_0 + r_1(u_1 + \theta_1) + r_2(u_1 + \theta_1)^2)d_1s + m_i(r_0 + r_1(u_i - \theta_i) + r_2(u_i - \theta_i)^2)d_is$$
$$< m_1(r_0 + r_1u_1 + r_2u_1^2)d_1s + m_i(r_0 + r_1u_i + r_2u_i^2)d_is.$$

Hence, $(u_1, u_2, \cdots, u_n)$ is not the optimal solution of the EEMV model, because it has a larger energy consumption than the feasible solution $(u_1+\theta_1, \cdots, u_i-\theta_i, \cdots)$. The contradiction implies that $u_1 = v_1^{\max}$. The proof is now complete. □

For each $S > 0$ and subset $I \subseteq \{1, 2, \cdots, n\}$, we use the notation EEM(S, I) to denote the following optimization model:

$$\begin{cases} \min & \displaystyle\sum_{i\in I} m_i(\alpha)\left(r_0 + r_1v_i + r_2v_i^2\right)d_is \\ \text{s.t.} & \displaystyle\sum_{i\in I} d_i/v_i = S \\ & v_i > 0,\ \forall i \in I. \end{cases}$$

According to Theorem 4.3, we may solve the EEMV model by modifying the optimal solution of the EEM(S, I) gradually. First, let S_1 be the maximal allowable traversing time T, and let I_1 be the universal set $\{1, 2, \cdots, n\}$. By Algorithm 4.4, we solve the optimal solution $(v_1, v_2, \cdots, v_n)$ of the EEM(S_1, I_1). If it satisfies the maximal allowable velocity constraint, then it is also the optimal solution of the EEMV model. Otherwise, we denote

$$I' = \{i \in I_1 \mid v_i > v_i^{\max}\},$$

and modify $v_i = v_i^{\max}$, for all $i \in I'$. After the modification, at least one component of the optimal solution is fixed. Now, let us consider the second modification. Set a new index set $I_2 = I_1 \setminus I'$, and

$$S_2 = S_1 - \sum_{i\in I'} d_i/v_i^{\max}.$$

Then, we get a new optimization model EEM(I_2, S_2). We solve the optimal solution $\{v_i, i \in I_2\}$ using Algorithm 4.4. If it satisfies the maximal allowable velocity constraint, then $(v_1, v_2, \cdots, v_n)$ is the optimal solution of the EMMV model. Otherwise, we modify the optimal solution again according to Theorem 4.3, and proceed with the next modification. Since the optimal solution consists of n components, the EEMV model may be solved by at most n loops.

Algorithm 4.6. *Algorithm for finding the optimal solution $(v_1, v_2, \cdots, v_n)$ of the EEMV model.*

Step 1: *Set $I = \{1, 2, \cdots, n\}$ and $S = T$.*

Step 2: *By Algorithm 4.4, calculate the optimal solution $(v_1, v_2, \cdots, v_n)$ of the EEM(S, I) model.*

Step 3: *Define $I' = \{i \in I \mid v_i > v_i^{\max}\}$. If $I' = \emptyset$, then return $(v_1, v_2, \cdots, v_n)$ as the optimal solution. Otherwise, set $v_i = v_i^{\max}$, for all $i \in I'$.*

Step 4: *Set $I \leftarrow I \setminus I'$ and $S \leftarrow S - \sum_{i \in I'} d_i / v_i^{\max}$.*

Step 5: *By Algorithm 4.4, calculate the optimal solution $\{v_i,\ i \in I\}$ of the EEM(S, I) model, and go to Step 3.*

5. **Numerical Example**

Here, we provide a numerical example to show the effectiveness of the proposed models and algorithms. Table 1 shows the scheduling strategies for four freight trains, where the second column shows the train velocity, and the third column shows the length of link. The first train is assumed to traverse eight links on its journey, the second train is assumed to traverse seven links, and the last two trains are assumed to traverse four links. Taking the first train as an example, the lengths of the eight links are $d_1 = 154$, $d_2 = 101$, $d_3 = 113$, $d_4 = 191$, $d_5 = 144$, $d_6 = 34$, $d_7 = 68$, and $d_8 = 348$. Since the running velocities of the train on the links are respectively known to be $v_1 = 144.38$, $v_2 = 147.80$, $v_3 = 141.25$, $v_4 = 138.07$, $v_5 = 123.43$, $v_6 = 102.00$, $v_7 = 94.88$, and $v_8 = 139.20$, the traversing time for the first train is

$$T = d_1/v_1 + d_2/v_2 + \cdots + d_{10}/v_{10} = 8.65.$$

In the same way, we can calculate the traversing time for other three trains, which are shown in the last column of Table 1. Table 2 shows the weights of trains on different links, which are all assumed to be triangular fuzzy variables. For example, the weights of the first train on the eight links are assumed to be $\xi_1 = (7, 9, 11)$, $\xi_2 = (10, 11, 12)$, $\xi_3 = (7, 10, 13)$, $\xi_4 = (12, 13, 14)$, $\xi_5 = (8, 12, 16)$, $\xi_6 = (9, 10, 11)$, $\xi_7 = (10, 12, 13)$, and $\xi_8 = (10, 12, 13)$. If we take $\alpha = 0.5$, then it is easy to calculate that $m_1(\alpha) = 550$, $m_2(\alpha) = 600$, $m_3(\alpha) = 650$, $m_4(\alpha) = 700$, $m_5(\alpha) = 800$, $m_6(\alpha) = 550$, $m_7(\alpha) = 662.5$, and $m_8(\alpha) = 662.5$. The other parameters used in this example are listed as follows: resistance coefficient due to grade $r_0 = 16.6$, resistance coefficient due to rail friction $r_1 = 0.366$, resistance coefficient due to air friction $r_2 = 0.0261$, and quantity of energy consumption for providing per unit of power $s = 1$.

	Train velocity (km/h)	Length of link (km)	Traversing time (h)
T_1	144.38 147.80 141.25 138.07	154 101 113 191 144	8.65
	123.43 102.00 94.88 139.20	34 68 348	
T_2	146.67 151.88 118.08 142.04	154 405 246 348 136	10.60
	143.16 106.36 121.93	39 126	
T_3	94.65 108.76 118.00 100.80	814 484 99 126	15.14
T_4	124.52 115.68 127.84 99.77	689 536 362 715	20.16

Table 1. Database for the Numerical Example

	T_1	T_2	T_3	T_4
Link 1	(7,9,11)	(9,10,12)	(6,7,9)	(6,9,10)
Link 2	(10,11,12)	(8,10,12)	(8,10,11)	(8,10,12)
Link 3	(7,10,13)	(10,11,13)	(9,10,13)	(12,14,15)
Link 4	(12,13,14)	(10,11,13)	(7,9,12)	(11,13,14)
Link 5	(8,12,16)	(11,12,14)	-	-
Link 6	(9,10,11)	(10,12,13)	-	-
Link 7	(10,12,13)	(10,12,15)	-	-
Link 8	(10,12,13)	-	-	-

Table 2. The Weights of Trains on Different Links ($\times 100$ ton)

For each train, we find the optimal solution of the EEM model by Algorithm 4.4. The maximal allowable traversing time in the EEM model is selected as 8.65, 10.60, 15.14, and 20.16, respectively. The computational results are summarized in Table 3. It can be readily proved that the smaller the value $m_i(\alpha)$ is, the larger is the corresponding optimal velocity v_i. In particular, since $m_7(\alpha) = m_8(\alpha)$ for the first train, the optimal solution takes the same value on the last two links. In order to illustrate the effectiveness of the EEM model on saving energy, we perform comparisons between the quantity of energy consumption with the given velocity and the quantity of energy consumption with the optimal velocity. The computational results are shown in Table 4, which tells us that, by using the same traversing time, the EEM model saves the energy consumption by 2.87% for the first train, 2.24% for the second train, 3.75% for the third train, and 2.16% for the fourth train.

Now, let us consider the maximal allowable velocity constraint. In a real railway system, the running velocity of a train is always confined to a given range due to the different physical conditions of tracks. Take the first train for example, and assume that the maximal allowable velocities on all links are $v_1^{\max} = 150$, $v_2^{\max} = 150$,

$v_3^{\max} = 150$, $v_4^{\max} = 140$, $v_5^{\max} = 130$, $v_6^{\max} = 120$, $v_7^{\max} = 125$, and $v_8^{\max} = 140$. Then, it is clear that the optimal solution of the EEM model does not satisfy this constraint, because $v_6 > 120$ and $v_7 > 125$. According to Algorithm 4.6, we modify $v_6 = 120$ and $v_7 = 125$. Define a new index set $I = \{1, 2, 3, 4, 5, 8\}$, and update the maximal allowable traversing time $S = 8.65-0.28-0.54 = 7.83$. A run of Algorithm 4.4 shows that the optimal solution of EEM$(7.83, I)$ is $v_1 = 143.04$, $v_2 = 138.88$, $v_3 = 135.17$, $v_4 = 131.81$, $v_5 = 125.98$, and $v_8 = 134.30$. Since this satisfies the maximal allowable velocity constraint, we get the optimal solution of the EEMV model to be $(143.04, 138.88, 135.17, 131.81, 125.98, 120.00, 125.00, 134.30)$.

	Link 1	Link 2	Link 3	Link 4	Link 5	Link 6	Link 7	Link 8
T_1	141.66	137.54	133.86	130.54	124.76	141.66	133.00	133.00
T_2	139.06	140.04	137.19	137.19	133.73	135.42	130.58	-
T_3	121.74	116.34	107.23	109.74	-	-	-	-
T_4	116.69	111.51	102.77	105.18	-	-	-	-

Table 3. The Optimal Solutions of the EEM Models with $\alpha = 0.5$

	T_1	T_2	T_3	T_4
Energy consumption with the given velocity ($\mathcal{C}_0$)	4.136×10^8	5.292×10^8	2.507×10^8	5.948×10^8
Energy consumption with the optimal velocity ($\mathcal{C}$)	4.018×10^8	5.174×10^8	2.413×10^8	5.820×10^8
Relative reduction $((\mathcal{C}_0 - \mathcal{C})/\mathcal{C}_0 \times 100\%)$	2.87%	2.24%	3.75%	2.16%

Table 4. The Energy Consumption Corresponding to the Optimal Solution

6. Conclusions and Future Research

We studied the fuzzy train energy saving problem on the assumption that the train's weights were estimated by fuzzy variables when making the train scheduling strategy. We proposed an energy consumption minimization model, which minimized the average value and entropy of the fuzzy energy consumption. For solving the model, we first analyzed the properties of the optimal solution, and then designed an iterative algorithm based on the Karush-Kuhn-Tucker conditions. Finally, we illustrated a numerical example to show the effectiveness of the proposed model and algorithm.

Although the proposed model and algorithm are proved to be effective on a numerical example, we need to conduct empirical studies for testing the effectiveness on real railway systems. New algorithms are needed to consider safety constraints. In addition, further research is needed to study the energy saving problem by using coasting control strategy since it is known to be effective on reducing the energy consumption.

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References

[1] Y. Bai, B. Mao, F. Zhou, Y. Ding and C. Dong, *Energy-efficient driving strategy for freight trains based on power consumption analysis*, Journal of Transportation Systems Engineering and Information Technology, **9(3)** (2009), 43-50.

[2] E. Castillo, I. Gallego, J. M. Urena and J. M. Coronado, *Timetabling optimization of a single railway track line with sensitivity analysis*, Top, **17(2)** (2009), 256-287.

[3] C. S. Chang and S. Sim, *Optimising train movements through coast control using genetic algorithms*, IEE Proceedings-Electric Power Applications, **144(1)** (1997), 65-73.

[4] S. Effati and H. Roohparvar, *The minimization of the fuel costs in the train transportation*, Applied Mathematics and Computation, **175** (2006), 1415-1431.

[5] K. Ghoseiri, F. Szidarovszky and M. J. Asgharpour, *A multi-objective train scheduling model and solution*, Transportation Research Part B, **38** (2004), 927-952.

[6] P. Howlett, *The optimal control of a train*, Annals of Operations Research, **98** (2000), 65-87.

[7] P. Howlett, P. Pudney and X. Vu, *Local energy minimization in optimal train control*, Automatica, **45(11)** (2009), 2692-2698.

[8] K. B. Khan and X. S. Zhou, *Stochastic optimization model and solution algorithm for robust double-track train-timetabling problem*, IEEE Transactions on Intelligent Transportation Systems, **11(1)** (2010), 81-89.

[9] E. Khmelnitsky, *On an optimal control problem of train operation*, IEEE Transactions on Automatic Control, **45(7)** (2000), 1257-1266.

[10] D. R. Kraay, P. T. Harker and B. Chen, *Optimal pacing of trains in freight railroads: model formulation and solution*, Operations Research, **39** (1991), 82-99.

[11] X. Li and B. Liu, *A sufficient and necessary condition for credibility measures*, International Journal of Uncertainty, Fuzziness & Knowledge-Based Systems, **14(5)** (2006), 527-535.

[12] P. Li and B. Liu, *Entropy of credibility distributions for fuzzy variables*, IEEE Transactions on Fuzzy Systems, **16(1)** (2008), 123-129.

[13] B. Liu and Y. Liu, *Expected value of fuzzy variable and fuzzy expected value models*, IEEE Transactions on Fuzzy Systems, **10(4)** (2002), 445-450.

[14] R. Liu and I. M. Golovitcher, *Energy-efficient operation of rail vehicles*, Transportation Research Part A, **37** (2003), 917-932.

[15] M. Miyatake and H. Ko, *Optimization of train speed profile for minimum energy consumption*, IEEJ Transactions on Electrical and Electronic Engineering, **5** (2010), 263-269.

[16] L. Yang, K. P. Li and Z. Y. Gao, *Train timetable problem on a single-line railway with fuzzy passenger demand*, IEEE Transactions on Fuzzy Systems, **17(3)** (2009), 617-629.

[17] L. Yang, Z. Y. Gao and K. P. Li, *Passenger train scheduling on a single-track or partially double-track railway with stochastic information*, Engineering Optimization, **42(11)** (2010), 1003-1022.

[18] L. Yang, Z. Y. Gao and K. P. Li, *Railway freight transportation planning with mixed uncertainty of randomness and fuzziness*, Applied Soft Computing, **11(1)** (2011), 778-792.
[19] X. S. Zhou and M. Zhong, *Bicriteria train scheduling for high-speed passenger railroad planning applications*, European Journal of Operational Research, **167** (2005), 752-771.

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