COALITIONAL GAME WITH FUZZY PAYOFFS AND CREDIBILISTIC SHAPLEY VALUE

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Abstract. Coalitional game deals with situations that involve cooperations among players, and there are different solution concepts such as the core, the Shapley value and the kernel. In many situations, there is no way to predict the payoff functions except for the expert experiences and subjective intuitions, which leads to the coalitional game with fuzzy payoffs. Within the framework of credibility theory, this paper employs two credibilistic approaches to define the behaviors of players under fuzzy situations. Correspondingly, two variations of Shapley value are proposed as the solutions of the coalitional game with fuzzy payoffs. Meanwhile, some characterizations of the credibilistic Shapley value are investigated. Finally, an example is provided for illustrating the usefulness of the theory developed in this paper.

1. Introduction

Since von Neumann and Morgenstern's seminal work [22], game theory has been used extensively to analyze conflict and cooperative situations in economics, sociology, etc. Theoretical game models may fall into three categories: the strategic games, the extensive games and the coalitional games. The first two are called noncooperative games that focus on the independent individual behavior of game players. While the coalitional game focuses on the behavior of the group of players as an entity (which is called a coalition). For instance, in business competition, strategic coalition has become one of the efficient ways to obtain more profits. In literature, the solutions of cooperative game have been investigated from different perspectives (e.g., Shapley [33], Aumann [1], Maschler [28], Schmeidler [34]), amongst which the core and the Shapley value are mostly used.

In many real-world games, as Harsanyi [14] pointed out, "the player may lack full information about the other players' (or even their own) payoffs, etc." Moreover, many situations of interest contain few historical records available for probabilistic reasoning because of the complicated decision environments. In such situations, fuzzy set theory offers an appropriate and powerful alternative to deal with the incomplete information. With it, we can make use of human experiences, subjective judgements and intuitions, and specify the incomplete information as fuzzy variables. Within the framework of possibility theory, the fuzzy cooperative game was

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introduced by Aubin [3] and Butnaria [4] in around 1980. Mareš [25] and Nishizaki and Sakawa [30][31] discussed the solutions of coalition game by fuzzification of payoffs and goals. Mareš [26] also introduced the fuzzy coalition structures based on the membership of coalitions in a coalitional game. The strategic game with fuzzy information were also investigated by many researchers (e.g., Campos et al [7], Maeda [23][24], Nishizaki and Sakawa [30])

As an important advance, credibility theory was found by Liu [18] in 2004, and refined by Liu [21] in 2007 as a branch of mathematics for studying the behavior of fuzzy phenomena. Now, we can see numerous applications of credibility theory in literature. Within the framework of credibility theory, a spectrum of credibilistic strategic game has been developed by Gao and his coworkers [9]-[12][15]. The credibilistic extensive game with fuzzy payoffs was proposed by Gao and Yu [13], and the solution of the game was defined as credibilistic equilibrium and credibilistic subgame perfect equilibrium. Shen and Gao [35] investigated the credibilistic coalitional game and presented two definitions of credibilistic cores as the solution of the game.

In this work, we reconsider the fuzzy coalitional game and discuss another solution concept—Shapley value. Section 2 calls on the concepts of Shapley value and briefly reviews some basic results of credibility theory including two credibilistic approaches to define the behaviors of players under fuzzy situations. In Section 3, two variations of Shapley value are proposed as the solutions of the coalitional game with fuzzy payoffs. One is the expected Shapley value, and the other is α-optimistic Shapley value. Meanwhile, the uniqueness of the credibilistic Shapley value is proved. Finally, an example is provided to illustrate the usefulness of the theory developed in this paper.

2. **Preliminaries**

In this section, we recall some basic results in coalitional game and credibility theory.

2.1. **Shapley Value.** In 1953, Shapley [32] introduced a special function that describes the value of players in a coalitional game, it was called the Shapley value later. Because this value reacts the intuitive consideration of players about the payoffs and is simple to be processed with mathematical methods, it is used widely in both economy and political science.

Firstly, we give the definition of coalitional game with transferable payoffs.

Definition 2.1. [32] A coalitional game with transferable payoff $\langle \boldsymbol{N}, \boldsymbol{v} \rangle$ consists of:

(1) a finite set $\boldsymbol{N}$ (the set of the players);
(2) a function $\boldsymbol{v}$ that associates a real number $\boldsymbol{v}(\boldsymbol{S})$ the worth of $\boldsymbol{S}$ with every nonempty subset $\boldsymbol{S}$ of $\boldsymbol{N}$ (a coalition).

And there is an assumption on this model that

$$\boldsymbol{v}(\boldsymbol{S} \cup \boldsymbol{T}) \geq \boldsymbol{v}(\boldsymbol{S}) + \boldsymbol{v}(\boldsymbol{T})$$

for all $\boldsymbol{S}$ and $\boldsymbol{T}$ with $\boldsymbol{S} \cap \boldsymbol{T} = \emptyset$. That is to say, the payoff of a coalition must be more than the sum of the payoff that each player could receive if she or he does not join the coalition.

Definition 2.2. [32] The Shapley value $\boldsymbol{\varphi}$ is defined by the condition

$$\boldsymbol{\varphi}_i(\boldsymbol{N}, \boldsymbol{v}) = \frac{1}{|\ \boldsymbol{N}\ |!} \sum_{\boldsymbol{S} \subseteq \boldsymbol{N}} (\boldsymbol{s}-1)!(|\ \boldsymbol{N}\ | -\boldsymbol{s})!(\boldsymbol{v}(\boldsymbol{S}) - \boldsymbol{v}(\boldsymbol{S} \setminus i)),$$

where $\boldsymbol{s}$ is the number of players in $\boldsymbol{S}$, $\forall\, i \in \boldsymbol{N}$.

The value means the payoff of the player i in a coalitional game $\langle \boldsymbol{N}, \boldsymbol{v} \rangle$. So, the payoff profile in this game is presented as a vector $\boldsymbol{\varphi}(\boldsymbol{N}, \boldsymbol{v}) = \boldsymbol{\varphi}_i(\boldsymbol{N}, \boldsymbol{v})_{i \in \boldsymbol{N}}$. Obviously, the Shapley value is distributed the overall payoff of game $\boldsymbol{v}(\boldsymbol{N})$.

To discuss Shapley value further, we turn to an axiomatic characterization of the Shapley value. Before giving the axioms, some definitions are given first.

Definition 2.3. [32] Player i is a dummy in $\boldsymbol{v}$ if $\boldsymbol{v}(\boldsymbol{S}) - \boldsymbol{v}(\boldsymbol{S} \setminus \{i\}) = \boldsymbol{v}(i)$ for each coalition $\boldsymbol{S}$ that includes i.

Definition 2.4. [32] Players i and j are interchangeable in $\langle \boldsymbol{N}, \boldsymbol{v} \rangle$, if $\boldsymbol{v}(\boldsymbol{S} \setminus \{i\}) = \boldsymbol{v}(\boldsymbol{S} \setminus \{j\})$, $\forall\, \boldsymbol{S} \subseteq \boldsymbol{N}$.

The three axioms are as follows:

(1) SYM(symmetry): if i and j are interchangeable in $\langle \boldsymbol{N}, \boldsymbol{v} \rangle$, then $\boldsymbol{\varphi}_i(\boldsymbol{N}, \boldsymbol{v}) = \boldsymbol{\varphi}_j(\boldsymbol{N}, \boldsymbol{v})$;
(2) DUM(dummy player): if i is a dummy in $\langle \boldsymbol{N}, \boldsymbol{v} \rangle$, then $\boldsymbol{\varphi}_i(\boldsymbol{N}, \boldsymbol{v}) = \boldsymbol{v}(i)$;
(3) ADD(additivity): for any two games $\boldsymbol{v}$ and $\boldsymbol{w}$, there is $\boldsymbol{\varphi}_i(\boldsymbol{v}+\boldsymbol{w}) = \boldsymbol{\varphi}_i(\boldsymbol{v}) + \boldsymbol{\varphi}_i(\boldsymbol{w})$ for all $i \in \boldsymbol{N}$, where $\boldsymbol{v}+\boldsymbol{w}$ is defined as $(\boldsymbol{v}+\boldsymbol{w})(\boldsymbol{S}) = \boldsymbol{v}(\boldsymbol{S}) + \boldsymbol{w}(\boldsymbol{S})$, $\forall\, \boldsymbol{S} \subseteq \boldsymbol{N}$.

Lemma 2.5. [32] *The Shapley value is the only value that satisfies SYM, DUM, and ADD.*

2.2. **Credibility Theory.** In 1965, Zadeh [36] initiated the concept of fuzzy set via membership function. Then possibility theory was proposed by Zadeh [37], and developed by many researchers such as Dubios and Prade [8]. In 2002, Liu and Liu [16] proposed the concept of credibility measure. Then credibility theory was founded [18] and refined by Liu [21] as a branch of mathematics for dealing with the behavior of fuzzy phenomena. Particularly, credibility theory has an axiomatic foundation like that of probability theory. Hence, this theory has been applied widely in the area of fuzzy optimization and decision-making [19]. Let ξ be a fuzzy variable with membership function μ, and B a set of real numbers. The credibility measure Cr is defined by

$$\mathrm{Cr}\{\xi \in B\} = \frac{1}{2}\left(\sup_{x \in B} \mu(x) + 1 - \sup_{x \in B^c} \mu(x)\right). \tag{1}$$

The formula above is also called the credibility inversion theorem.

Definition 2.6. [16] Let ξ be a fuzzy variable. The expected value of ξ is defined as

$$\mathrm{E}[\xi] = \int_0^{\infty} \mathrm{Cr}\{\xi \ge r\}\mathrm{d}r - \int_{-\infty}^{0} \mathrm{Cr}\{\xi \le r\}\mathrm{d}r, \tag{2}$$

provided that at least one of the two integrals is finite.

Definition 2.7. [20] The fuzzy variables $\xi_1, \xi_2, \cdots, \xi_m$ are said to be independent if

$$\mathrm{Cr}\left\{\bigcap_{i=1}^{m} \{\xi_i \in B_i\}\right\} = \min_{1\le i\le m} \mathrm{Cr}\left\{\xi_i \in B_i\right\}. \tag{3}$$

for any sets $B_1, B_2, \cdots, B_m$ of real numbers.

Definition 2.8. [17] Let ξ be a fuzzy variable, and $\alpha \in (0, 1]$. Then

$$\xi_{\sup}(\alpha) = \sup\{r \mid \mathrm{Cr}\{\xi \ge r\} \ge \alpha\} \tag{4}$$

is called the critical value at confidence level α.

Let ξ and η be independent fuzzy variables. Then for any real numbers a, b and $\alpha \in (0, 1]$, we have $\mathrm{E}[a\xi + b\eta] = a\mathrm{E}[\xi] + b\mathrm{E}[\eta]$ and $(a\xi + b\eta)_{\sup}(\alpha) = a\xi_{\sup}(\alpha) + b\eta_{\sup}(\alpha)$. That is, both the expected value operator and the critical value operator have linear property under the assumption of independence. Let r be a real number. Based on the credibility theory, we may have the following fuzzy ranking criteria:

(1) Expected value criterion: $\xi < \eta$ iff $E[\xi] < E[\eta]$;
(2) Critical value criterion: $\xi < \eta$ iff $\xi_{\sup}(\alpha) < \eta_{\sup}(\alpha)$ for some predetermined confidence level $\alpha \in (0, 1]$.

3. Credibilistic Shapley Value of Fuzzy Coalitional Game

In this paper, we assume the payoffs of coalition for players are fuzzy variables. Under the fuzzy situation, we discuss the definitions and characteristics of Shapley value. First, the definition of fuzzy coalitional game model is presented.

Definition 3.1. A coalitional game with fuzzy transferable payoff $\langle \boldsymbol{N}, \tilde{\boldsymbol{v}}\rangle$ consists of:

(1) a finite set $\boldsymbol{N}$ (the set of the players);
(2) a fuzzy variable $\tilde{\boldsymbol{v}}(\boldsymbol{S})$ is the worth of $\boldsymbol{S}$(a coalition), where $\boldsymbol{S}$ is any nonempty subseteq of $\boldsymbol{N}$.

As the traditional definition of transferable payoffs, the fuzzy payoff $\tilde{\boldsymbol{v}}(\boldsymbol{S})$ is the total payoff that is available for the division among the members of $\boldsymbol{S}$, for the coalition $\boldsymbol{S}$. Meanwhile, $\langle \boldsymbol{N}, \tilde{\boldsymbol{v}}\rangle$ is cohesive if

$$\tilde{\boldsymbol{v}}(\boldsymbol{N})_{\sup}(\alpha) \ge \sum_{k=1}^{K} \tilde{\boldsymbol{v}}(\boldsymbol{S}_k)_{\sup}(\alpha), \quad \left(\Longrightarrow \mathrm{E}[\tilde{\boldsymbol{v}}(\boldsymbol{N})] \ge \sum_{k=1}^{K} \mathrm{E}[\tilde{\boldsymbol{v}}(\boldsymbol{S}_k)]\right)$$

for all $\alpha \in (0, 1)$ and each partition $\{\boldsymbol{S}_1, S_2, \cdots, \boldsymbol{S}_K\}$ of $\boldsymbol{N}$.

3.1. **The Expected Shapley Value.** When the players' goals are to maximize the expected value of their fuzzy objectives in the decision process, we will use the expected value criterion.

Definition 3.2. The expected Shapley value of coalitional game with fuzzy transferable $\boldsymbol{\varphi}$ is defined by the condition

$$\boldsymbol{\varphi}_i(\boldsymbol{N}, \tilde{\boldsymbol{v}}) = \frac{1}{\mid \boldsymbol{N} \mid !} \sum_{\boldsymbol{S} \subseteq \boldsymbol{N}} (\boldsymbol{s}-1)!(\mid \boldsymbol{N} \mid - \boldsymbol{s})! \{\mathrm{E}[\tilde{\boldsymbol{v}}(\boldsymbol{S})] - \mathrm{E}[\tilde{\boldsymbol{v}}(\boldsymbol{S} \setminus \{i\})]\},$$

where $\boldsymbol{s}$ is the number of players in $\boldsymbol{S}$, $\forall\, i \in \boldsymbol{N}$.

Definition 3.3. Player i is a dummy in $\tilde{\boldsymbol{v}}$ if $\mathrm{E}[\tilde{\boldsymbol{v}}(\boldsymbol{S})] - \mathrm{E}[\tilde{\boldsymbol{v}}(\boldsymbol{S} \setminus i)] = \mathrm{E}[\tilde{\boldsymbol{v}}(i)]$ for each coalition $\boldsymbol{S}$ that includes i.

Definition 3.4. Players i and j are interchangeable in $\langle \boldsymbol{N}, \tilde{\boldsymbol{v}} \rangle$, if $\mathrm{E}[\tilde{\boldsymbol{v}}(\boldsymbol{S} \setminus \{i\})] = \mathrm{E}[\tilde{\boldsymbol{v}}(\boldsymbol{S} \setminus \{j\})]$, $\forall\, \boldsymbol{S} \subseteq \boldsymbol{N}$.

To be similar with the crisp situation, we also introduce the axiomatic characterization of the expected Shapley value and prove that this is the unique value that is fit for all these axoims.

(1) SYM(symmetry): if i and j are interchangeable in $\langle \boldsymbol{N}, \tilde{\boldsymbol{v}} \rangle$, then $\boldsymbol{\varphi}_i(\boldsymbol{N}, \tilde{\boldsymbol{v}}) = \boldsymbol{\varphi}_j(\boldsymbol{N}, \tilde{\boldsymbol{v}})$;
(2) DUM(dummy player): if i is a dummy in $\langle \boldsymbol{N}, \tilde{\boldsymbol{v}} \rangle$, then $\boldsymbol{\varphi}_i(\boldsymbol{N}, \tilde{\boldsymbol{v}}) = \mathrm{E}[\tilde{\boldsymbol{v}}(i)]$;
(3) ADD(additivity): for any two games $\tilde{\boldsymbol{v}}$ and $\tilde{\boldsymbol{w}}$, there is $\boldsymbol{\varphi}_i(\tilde{\boldsymbol{v}} + \tilde{\boldsymbol{w}}) = \boldsymbol{\varphi}_i(\tilde{\boldsymbol{v}}) + \boldsymbol{\varphi}_i(\tilde{\boldsymbol{w}})$ for all $i \in \boldsymbol{N}$, where $\tilde{\boldsymbol{v}} + \tilde{\boldsymbol{w}}$ is defined as $\mathrm{E}[(\tilde{\boldsymbol{v}} + \tilde{\boldsymbol{w}})(\boldsymbol{S})] = \mathrm{E}[\tilde{\boldsymbol{v}}(\boldsymbol{S})] + \mathrm{E}[\tilde{\boldsymbol{v}}(\boldsymbol{S})]$, $\forall\, \boldsymbol{S} \subseteq \boldsymbol{N}$.

Theorem 3.5. *The Expected Shapley value is the only value that satisfies SYM, DUM, and ADD.*

Proof. First, we verify that the Shapley value satisfies the three axioms.
SYM: If i and j are interchangeable in $\langle \boldsymbol{N}, \tilde{\boldsymbol{v}} \rangle$, then $\mathrm{E}[\tilde{\boldsymbol{v}}(\boldsymbol{S} \setminus \{i\})] = \mathrm{E}[\tilde{\boldsymbol{v}}(\boldsymbol{S} \setminus \{j\})]$, $\forall\, \boldsymbol{S} \subseteq \boldsymbol{N}$. Based on the definition of the expected Shapley value, it is obviously $\boldsymbol{\varphi}_i(\boldsymbol{N}, \tilde{\boldsymbol{v}}) = \boldsymbol{\varphi}_j(\boldsymbol{N}, \tilde{\boldsymbol{v}})$.
DUM: If i is a dummy in $\langle \boldsymbol{N}, \tilde{\boldsymbol{v}} \rangle$, then $\mathrm{E}[\tilde{\boldsymbol{v}}(\boldsymbol{S})] - \mathrm{E}[\tilde{\boldsymbol{v}}(\boldsymbol{S} \setminus \{i\}] = \mathrm{E}[\tilde{\boldsymbol{v}}(i)]$ for each coalition $\boldsymbol{S}$ that includes i. Therefore,

$$\begin{aligned} &\boldsymbol{\varphi}_i(\boldsymbol{N}, \tilde{\boldsymbol{v}}) \\ &= \tfrac{1}{|\boldsymbol{N}|!} \textstyle\sum_{\boldsymbol{S} \subseteq \boldsymbol{N}} (\boldsymbol{s}-1)!(\mid \boldsymbol{N} \mid - \boldsymbol{s})! \\ &\qquad \times \{\mathrm{E}[\tilde{\boldsymbol{v}}(\boldsymbol{S})] - \mathrm{E}[\tilde{\boldsymbol{v}}(\boldsymbol{S} \setminus \{i\})]\} \\ &= \tfrac{1}{|\boldsymbol{N}|!} \textstyle\sum_{\boldsymbol{S} \subseteq \boldsymbol{N}} (\boldsymbol{s}-1)!(\mid \boldsymbol{N} \mid - \boldsymbol{s})! \mathrm{E}[\tilde{\boldsymbol{v}}(\{i\})] \end{aligned} \tag{5}$$

then

$$\boldsymbol{\varphi}_i(\boldsymbol{N}, \tilde{\boldsymbol{v}}) = \mathrm{E}[\tilde{\boldsymbol{v}}(i)].$$

ADD: It indirectly follows that $\mathrm{E}[(\tilde{\boldsymbol{v}}+\tilde{\boldsymbol{w}})(\boldsymbol{S})]-\mathrm{E}[(\tilde{\boldsymbol{v}}+\tilde{\boldsymbol{w}})(\boldsymbol{S}\setminus\{i\})]=\mathrm{E}[\tilde{\boldsymbol{v}}(\boldsymbol{S})]-\mathrm{E}[\tilde{\boldsymbol{v}}(\boldsymbol{S}\setminus\{i\})]+\mathrm{E}[\tilde{\boldsymbol{w}}(\boldsymbol{S})]-\mathrm{E}[\tilde{\boldsymbol{w}}(\boldsymbol{S}\setminus\{i\})]$.

Secondly, we discuss that the Shapley value is the only value to satisfy the axioms. Let $\boldsymbol{\varphi}$ be the value which is fit for the three axioms. There is a coalition $\tilde{\boldsymbol{v}}_T$,which is defined by

$$\tilde{\boldsymbol{v}}_T(\boldsymbol{S})=\begin{cases}1, & if \;\; T\subseteq \boldsymbol{S},\\ 0, & otherwise.\end{cases} \tag{6}$$

So we treat a coalition game $\tilde{\boldsymbol{v}}$ as a collection of $2^{\boldsymbol{N}}-1$ numbers $\tilde{\boldsymbol{v}}(\boldsymbol{S})\}_{\boldsymbol{S}\subseteq\boldsymbol{N}}$. And then we prove that there is a unique collection $(\beta_T)_{T\subseteq\boldsymbol{N}}$ of real numbers such that $\mathrm{E}[\tilde{\boldsymbol{v}}(\boldsymbol{S})]=\sum_{T\subseteq\boldsymbol{N}}\beta_T\{\tilde{\boldsymbol{v}}_T(\boldsymbol{S})$, for any $\tilde{\boldsymbol{v}}$ and any $\boldsymbol{S}\subseteq\boldsymbol{N}$.

As a result, this collection $\mathrm{E}[\tilde{\boldsymbol{v}}(T)_{T\in\boldsymbol{N}}]$ includes $2^{\boldsymbol{N}}-1$ members. It suffices to show that these games are linear independent. To prove the proposition above, we suppose that these games are not linear independent, which means that there exists some coalition T with $\gamma_T\neq 0$ when $\sum_{\boldsymbol{S}\in\boldsymbol{N}}\gamma_{\boldsymbol{S}}\tilde{\boldsymbol{v}}_{\boldsymbol{S}}(R)=0$, for any $R\subseteq\boldsymbol{N}$. Then we get the contrary that $\sum_{\boldsymbol{S}\in\boldsymbol{N}}\gamma_{\boldsymbol{S}}\tilde{\boldsymbol{v}}_{\boldsymbol{S}}(T)=\gamma_T\neq 0$ by choosing a coalition T, there is $\gamma_{\boldsymbol{S}}=0$, for any $\boldsymbol{S}\subseteq T$.

By SYM and DUM, the value of any game $\beta\tilde{\boldsymbol{v}}_T,\beta\geq 0$ is defined uniquely by:

$$\boldsymbol{\varphi}_i(\beta\tilde{\boldsymbol{v}}_T)=\begin{cases}\beta/|T|, & if \;\; i\in T\\ 0, & otherwise\end{cases} \tag{7}$$

By ADD, if $\mathrm{E}[\tilde{\boldsymbol{v}}\boldsymbol{S}]=\sum_{T\subseteq\boldsymbol{N}}\beta_T\tilde{\boldsymbol{v}}_T(\boldsymbol{S})$, then $\mathrm{E}[\tilde{\boldsymbol{v}}\boldsymbol{S}]=\sum_{\{T\subseteq\boldsymbol{N};\beta_T>0\}}\beta_T\tilde{\boldsymbol{v}}_T(\boldsymbol{S})-\sum_{\{T\subseteq\boldsymbol{N};\beta_T<0\}}\beta_T\tilde{\boldsymbol{v}}_T(\boldsymbol{S})$. So the value of coalition game $\tilde{\boldsymbol{v}}$ is determined uniquely.

So, we proved that the Shapley value is the only value that satisfies SYM, DUM, and ADD. □

3.2. **The Optimistic Shapley Value.** When the players are risk averse and want to avoid potential considerable losses, they could adopt the optimistic value criterion. Under a predetermined confidence level α, they maximize their optimistic profits. Under the α-optimistic criterion, we present the the α-optimistic Shapley value as follows.

Definition 3.6. The α-optimistic Shapley value of the coalitional game with fuzzy transferable payoff $\boldsymbol{\varphi}$ is defined by the condition

$$\boldsymbol{\varphi}_i(\boldsymbol{N},\tilde{\boldsymbol{v}})=\frac{1}{|\,\boldsymbol{N}\,|!}\sum_{\boldsymbol{S}\subseteq\boldsymbol{N}}(\boldsymbol{s}-1)!(\boldsymbol{N}-\boldsymbol{s})!\left\{[\tilde{\boldsymbol{v}}(\boldsymbol{S}))]_{sup}(\alpha)-[\tilde{\boldsymbol{v}}(\boldsymbol{S}\setminus\{i\})]_{sup}(\alpha)\right\},$$

where $\boldsymbol{s}$ is the number of players in $\boldsymbol{S}$, $\forall\, i\in\boldsymbol{N}$.

Definition 3.7. Player i is a dummy in $\tilde{\boldsymbol{v}}$ if $[\tilde{\boldsymbol{v}}(\boldsymbol{S})]_{\sup}(\alpha)-[\tilde{\boldsymbol{v}}(\boldsymbol{S}\setminus i)]_{\sup}(\alpha)=[\tilde{\boldsymbol{v}}(i)]_{\sup}(\alpha)$ for each coalition $\boldsymbol{S}$ that includes i.

Definition 3.8. Players i and j are interchangeable in $\langle\boldsymbol{N},\tilde{\boldsymbol{v}}\rangle$, if $[\tilde{\boldsymbol{v}}(\boldsymbol{S}\setminus\{i\})]_{\sup}(\alpha)=[\tilde{\boldsymbol{v}}(\boldsymbol{S}\setminus\{j\})]_{\sup}(\alpha)$, $\forall\,\boldsymbol{S}\subseteq\boldsymbol{N}$.

Now, we introduce the axiomatic characterization of the α-optimistic Shapley value and show that this is the unique value that is fit for all these axioms.

(1) SYM(symmetry): if i and j are interchangeable in $\langle \boldsymbol{N}, \tilde{\boldsymbol{v}} \rangle$, then $\boldsymbol{\varphi}_i(\boldsymbol{N}, \tilde{\boldsymbol{v}}) = \boldsymbol{\varphi}_j(\boldsymbol{N}, \tilde{\boldsymbol{v}})$;

(2) DUM(dummy player): if i is a dummy in $\langle \boldsymbol{N}, \tilde{\boldsymbol{v}} \rangle$, then $\boldsymbol{\varphi}_i(\boldsymbol{N}, \tilde{\boldsymbol{v}}) = [\tilde{\boldsymbol{v}}(i)]_{sup}(\alpha)$

(3) ADD(additivity): for any two games $\tilde{\boldsymbol{v}}$ and $\tilde{\boldsymbol{w}}$, there is $\boldsymbol{\varphi}_i(\tilde{\boldsymbol{v}}+\tilde{\boldsymbol{w}}) = \boldsymbol{\varphi}_i(\tilde{\boldsymbol{v}})+\boldsymbol{\varphi}_i(\tilde{\boldsymbol{w}})$ for all $i \in \boldsymbol{N}$, where $\tilde{\boldsymbol{v}} + \tilde{\boldsymbol{w}}$ is defined as $[(\tilde{\boldsymbol{v}} + \tilde{\boldsymbol{w}})(\boldsymbol{S})]_{sup}(\alpha) = [\tilde{\boldsymbol{v}}(\boldsymbol{S})]_{sup}(\alpha) + [\tilde{\boldsymbol{v}}(\boldsymbol{S})]_{sup}(\alpha)$, $\forall \boldsymbol{S} \subseteq \boldsymbol{N}$.

Theorem 3.9. *The α-optimistic Shapley value is the only value that satisfies SYM, DUM, and ADD.*

The proof is similar to that of the excepted Shapley value. We omit it here.

4. An Example

Suppose that there are 4 petroleum production countries in a game, that is, $\boldsymbol{N} = \{1,2,3,4\}$. Denote by $\boldsymbol{S}_4^0 = \{1,2,3,4\}$ the coalition that includes all four players. Denote by $\boldsymbol{S}_3^1 = \{1,2,3\}$ the coalition that includes three players but not player 4; similarly denote $\boldsymbol{S}_3^2 = \{1,2,4\}$, $\boldsymbol{S}_3^1 = \{1,3,4\}$ and $\boldsymbol{S}_3^1 = \{2,3,4\}$. Denote by $\boldsymbol{S}_2^1 = \{1,2\}$ the coalition that includes players 1 and 2, the action of which is independent of players 3 and 4, where players 3 and 4 act independently; similarly denote $\boldsymbol{S}_2^2 = \{1,3\}$, $\boldsymbol{S}_2^3 = \{1,4\}$, $\boldsymbol{S}_2^4 = \{2,3\}$, $\boldsymbol{S}_2^5 = \{2,4\}$ and $\boldsymbol{S}_2^6 = \{3,4\}$. Finally, denote by
S_1 the game in which the four players act independently, i.e., there does not exist any coalition.

Then the payoff of $\boldsymbol{S}_4^0$ is a triangular fuzzy number, denoted by $(a_4^0, b_4^0, c_4^0) = (300, 500, 700)$. The payoff of the coalition that includes three players is a triangular fuzzy number, denoted by $(a_3^1, b_3^1, c_3^1) = (200, 400, 500)$, $(a_3^2, b_3^2, c_3^2) = (200, 350, 600)$, $(a_3^3, b_3^3, c_3^3) = (200, 375, 550)$ and $(a_3^4, b_3^4, c_3^4) = (150, 400, 550)$, respectively. The payoff of any coalition that includes two players is also a triangular fuzzy number, denoted by $(a_2^1, b_2^1, c_2^1) = (100, 200, 300)$, $(a_2^2, b_2^2, c_2^2) = (100, 150, 400)$, $(a_2^3, b_2^3, c_2^3) = (100, 210, 280)$, $(a_2^4, b_2^4, c_2^4) = (50, 100, 550)$, $(a_2^5, b_2^5, c_2^5) = (150, 200, 250)$ and $(a_2^6, b_2^6, c_2^6) = (60, 220, 300)$, respectively. Finally for $\boldsymbol{S}_1$, the payoff of which is again a triangular fuzzy number, denoted by $(a_1^0, b_1^0, c_1^0) = (50, 100, 150)$.

This is a four-player coalitional game with fuzzy payoffs, in which the membership functions of fuzzy payoffs are defined as follows:

$$\mu(\tilde{\tilde{v}}(k)) = \begin{cases} 0, & \text{if } \tilde{\tilde{v}}(k) < a_k^j, \\ \frac{\tilde{\tilde{v}}(k) - a_k^j}{b_k^j - a_k^j}, & \text{if } a_k^j \leq \tilde{\tilde{v}}(k) \leq b_k^j, \\ \frac{\tilde{\tilde{v}}(k) - c_k^j}{b_k^j - c_k^j}, & \text{if } b_k^j \leq \tilde{\tilde{v}}(k) \leq c_k^j, \\ 0, & \text{if } \tilde{\tilde{v}}(k) > c_k^j, \end{cases} \tag{8}$$

where $k = 1,2,3,4,\ j = 1,\ldots,6$.

We assume that the players all adopt the expected criterion. According to the conclusion that the triangular fuzzy variable $\boldsymbol{\xi} = (a, b, c)$ has an expected value $\mathrm{E}[\xi] = (a + 2b + c)/4$, we can get the expected values as follows:

$$\mathrm{E}[\tilde{\tilde{v}}(4)] = 500, \quad \mathrm{E}[\tilde{\tilde{v}}(3)] = 375,$$
$$\mathrm{E}[\tilde{\tilde{v}}(2)] = 200, \quad \mathrm{E}[\tilde{\tilde{v}}(1)] = 100.$$

By

$$\boldsymbol{\varphi}_i(N, \tilde{\tilde{v}}) = \frac{1}{|\boldsymbol{N}|!} \sum_{\boldsymbol{S} \subseteq \boldsymbol{N}} (\boldsymbol{s} - 1)!(|\boldsymbol{N}| - \boldsymbol{s})! \left\{ \mathrm{E}[\tilde{\tilde{v}}(\boldsymbol{S})] - \mathrm{E}[\tilde{\tilde{v}}(\boldsymbol{S} \setminus \{i\})] \right\},$$

we can easily get

$$\boldsymbol{\varphi}_1(\boldsymbol{N}, \tilde{\tilde{v}}) = \boldsymbol{\varphi}_2(\boldsymbol{N}, \tilde{\tilde{v}}) = \boldsymbol{\varphi}_3(\boldsymbol{N}, \tilde{\tilde{v}}) = \boldsymbol{\varphi}_4(\boldsymbol{N}, \tilde{\tilde{v}}) = 125.$$

From which we can get the Shapley value of the coalitional game as $(125, 125, 125, 125)$.

In what follows, we present the α-optimistic values under different confidence levels. Firstly, we have the following conclusion: letting $\boldsymbol{\xi} = (a, b, c)$ be a fuzzy variable with $a < b < c$, then the α-optimistic value of ξ is

$$\boldsymbol{\xi}_{\sup}(\alpha) = \begin{cases} 2\alpha b + (1 - 2\alpha)c, & \text{if } \alpha \le 0.5, \\ (2\alpha - 1)a + (2 - 2\alpha)b, & \text{if } \alpha > 0.5. \end{cases}$$

We do the calculation for $\alpha = 0.6$ as an example as follows:

$$\begin{aligned}
&\boldsymbol{\xi}^{\tilde{\tilde{v}}(4)}_{\sup}(0.6) = 460, \quad \xi^{\tilde{\tilde{v}}(3)}_{\sup 1}(0.6) = 360, \quad \boldsymbol{\xi}^{\tilde{\tilde{v}}(3)}_{\sup 2}(0.6) = 320, \\
&\xi^{\tilde{\tilde{v}}(3)}_{\sup 3}(0.6) = 340, \quad \boldsymbol{\xi}^{\tilde{\tilde{v}}(3)}_{\sup 4}(0.6) = 350, \quad \boldsymbol{\xi}^{\tilde{\tilde{v}}(2)}_{\sup 1}(0.6) = 350, \\
&\boldsymbol{\xi}^{\tilde{\tilde{v}}(2)}_{\sup 1}(0.6) = 180, \quad \boldsymbol{\xi}^{\tilde{\tilde{v}}(2)}_{\sup 2}(0.6) = 140, \quad \boldsymbol{\xi}^{\tilde{\tilde{v}}(2)}_{\sup 3}(0.6) = 188, \\
&\boldsymbol{\xi}^{\tilde{\tilde{v}}(2)}_{\sup 4}(0.6) = 90, \quad \boldsymbol{\xi}^{\tilde{\tilde{v}}(2)}_{\sup 5}(0.6) = 190, \quad \boldsymbol{\xi}^{\tilde{\tilde{v}}(2)}_{\sup 6}(0.6) = 188, \\
&\boldsymbol{\xi}^{\tilde{\tilde{v}}(1)}_{\sup}(0.6) = 90.
\end{aligned}$$

Then we can get that

$$\boldsymbol{\varphi}_1^{0.6} = 124.17, \quad \boldsymbol{\varphi}_2^{0.6} = 103.33,$$
$$\boldsymbol{\varphi}_3^{0.6} = 127.17, \quad \boldsymbol{\varphi}_4^{0.6} = 97.83.$$

By the same way as above, it is obvious that the 0.6-optimistic core is nonempty and the payoff profile $(124.17, 103.33, 127.17, 97.83)$ is in the core. If we change the confidence level from 0.3 to 0.9, we get that the core is still nonempty, but the payoffs are different. The table followed is the profiles at different confidence levels. It can be seen from Table 1 that as the confidence level α increases, the Shapley value decreases correspondingly, from which we can get the conclusions as follows. If one player has more confidence in the coalition than other players, i.e., this player is more risk-affordable, he would choose a lower confidence level to get more payoff; conversely, if a player is risk-averse, then he would prefer not to afford the risk of coalition and hopes to keep lower but stable payoff under high confidence level.

α	profiles
0.3	(134.25, 151.58, 144.58, 159.58)
0.4	(132.33, 137.67, 143.33, 126.67)
0.5	(130.42, 123.75, 142.08, 103.75)
0.6	(124.17, 110.83, 127.17, 97.83)
0.7	(118.42, 97.92, 112.25, 89.42)
0.8	(111.67, 85.00, 97.33, 86.00)
0.9	(105.42, 72.08, 82.42, 80.08)

Table 1. Shapley Values at Different Levels

Therefore, both the α-optimistic Shapley value and the expected Shapley value can be used to describe the coalitional game with fuzzy payoffs, which provide new methods to solve fuzzy coalitional game, meanwhile more strategies are presented to satisfy variable preferences of players.

5. Conclusion

In this paper, we presented a new coalitional game model — coalitional game with fuzzy transferable payoff. For this model, we gave two definitions of credibilistic Shapley value — the solution of coalitional game, which enable us to analyze the fuzzy coalitional games. Finally, we provided an example to illustrate the usefulness of the discussion in this paper.

References

[1] R. Aumann and M. Maschler, *The bargainning set for cooperative games*, Advances in Game Theory, Annals of Mathematics Studies, **52** (1964), 443-476.

[2] R. Aumann and L. S. Shapley, *Values of non-atomic games*, Princeton University, Princeton, 1974.

[3] J. P. Aubin, *Cooperative fuzzy games*, Mathematical Operation Research, **6** (1981), 1-13.

[4] D. Butnariu, *Fuzzy games: a description of the concept*, Fuzzy Sets and Systems, **1** (1978), 181-192.

[5] D. Butnariu, *Stability and shapley value for an n-persons fuzzy games*, Fuzzy Sets and Systems, **4** (1980), 63-72.

[6] D. Butnariu and E. P. Klement, *Core, value and equilibria for market games: on a problem of Aumann and Shapley*, International Journal of Game Theory, **18** (1996), 149-160.

[7] L. Campos, A. Gonzalez and M. A. Vila, *On the use of the ranking function approach to solve fuzzy matrix games in a direct way*, Fuzzy Sets and Systems, **49** (1992), 193-203.

[8] D. Dubois and H. Prade, *Possibility theory*, Plenum, New York, 1988.

[9] J. Gao, and B. Liu, *Fuzzy multilevel programming with a hybrid intelligent algorithm*, Computer & Mathmatics with applications, **49** (2005), 1539-1548.

[10] J. Gao, *Credibilistic game with fuzzy information*, Journal of Uncertain Systems, **1(1)** (2007), 72-78.

[11] J. Gao, Z. Q. Liu and P. Shen, *On characterization of fuzzy-payoff two-player zero-sum game*, Soft Computing, **13(2)** (2009), 127-132.

[12] J. Gao, *Fuzzy dependent-chance multilevel programming with applications*, Journal of Universal Computer Science, to be Published.

[13] J. Gao and Y. Yu, *Credibilistic extensive game with fuzzy information*, Soft Computing, to be Published.
[14] J. C. Harsanyi, *Games with incomplete information*, The American Economic Review, **85(3)** (1995), 291-303.
[15] R. Liang, Y. Yu, J. Gao and Z. Q. Liu, *N-person credibilistic strategic game*, Frontiers of Computer Science in China, **4(2)** (2010), 212-219.
[16] B. Liu and Y. Liu, *Expected value of fuzzy variable and fuzzy expected value models*, IEEE Transactions on Fuzzy Systems, **10** (2002), 445-450.
[17] Y. Liu and B. Liu, *Expected value operator of random fuzzy variable and random fuzzy expected value models*, International Journal of Uncertainty, Fuzziness & Knowledge-Based Systems, **11(2)** (2003), 195-215.
[18] B. Liu, *Uncertainty theory: an introduction to its axiomatic foundations*, Springer-Verlag, Berlin, 2004.
[19] B. Liu, *A survey of credibility theory*, Fuzzy Optimization and Decision Making, **5(4)** (2006), 387-408.
[20] Y. Liu and J. Gao, *The dependence of fuzzy variables with applications to fuzzy random optimization*, International Journal of Uncertainty, Fuzziness & Knowledge-Based Systems, **15** (2007), 1-20.
[21] B. Liu, *Uncertainty theory*, 2nd Edition, Springer-Verlag, Berlin, 2007.
[22] J. Von Neumann and D. Morgenstern, *The theory of games in economic bahavior*, New York: Wiley, 1944.
[23] T. Maeda, *Characterization of the equilibrium strategy of the bimatrix game with fuzzy payoff*, Journal of Mathematical Analysis and Applications, **251** (2000), 885-896.
[24] T. Maeda, *Characterization of the equilibrium strategy of the two-person zero-sum game with fuzzy payoff*, Fuzzy Sets and Systems, **139** (2003), 283-296.
[25] M. Mareš, *Computation over fuzzy quantities*, CRC-Press, 1994.
[26] M. Mareš, *Fuzzy coalitions structures*, Fuzzy Sets and Systems, **114(1)** (2000), 23-33.
[27] M. Mareš, *Fuzzy cooperative games*, Physica-Verlag, Heidelberg, 2001.
[28] M. Mashler, B. Peleg and L. S. Shapley, *Geometric properties of the kernel, nucleolus and related solution concepts*, Mathematics Operational Research, **4** (1979), 303-337.
[29] I. Nishizaki and M. Sakawa, *Equilibrium solutions for multiobjective bimatrix games with fuzzy payoffs and fuzzy goals*, Fuzzy Sets and Systems, **111(1)** (2000), 99-116.
[30] I. Nishiizaki and M. Sakawa, *Fuzzy cooperative games arising from linear production programming problems with fuzzy parameters*, Fuzzy sets and Systems, **114(1)** (2000), 11-21.
[31] I. Nishiizaki and M. Sakawa, *Solutions based on fuzzy goals in fuzzy linear programming games*, Fuzzy Sets and Systems, **115(1)** (2000), 105-109.
[32] L. S. Shapley, *A value for n-persons games*, Annals of Mathematics Studies, **28** (1953), 307-318.
[33] L. S. Shapley and M. Shubik, *A method for evaluating the distribution of power in committee systerm*, American Politics Science Review, **48** (1954), 787-792.
[34] D. Schmeidler, *The nucleolus of a characteristic function games*, Journal of Applied Mathematics **17** (1969), 1163-1170.
[35] P. Shen and J. Gao, *Coalitional game with fuzzy information and credibilistic core*, Soft Computing, DOI: 10.1007/s00500-010-0632-9, 2010.
[36] L. A. Zadeh, *Fuzzy sets*, Information and Control, **8**(1965), 338-353.
[37] L. A. Zadeh, *Fuzzy sets as a basis for a theory of possibility*, Fuzzy Sets and Systems, **1**(1978), 3-28.

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