TAUBERIAN THEOREMS FOR THE EULER-NÖRLUND MEAN-CONVERGENT SEQUENCES OF FUZZY NUMBERS

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Abstract. Fuzzy set theory has entered into a large variety of disciplines of sciences, technology and humanities having established itself as an extremely versatile interdisciplinary research area. Accordingly different notions of fuzzy structure have been developed such as fuzzy normed linear space, fuzzy topological vector space, fuzzy sequence space etc. While reviewing the literature in fuzzy sequence space, we have seen that the notion of Tauberian theorems for the Euler-Nörlund mean-convergent sequences of fuzzy numbers has not been developed. In the present paper, we introduce some new concepts about statistical convergence of sequences of fuzzy numbers. The main purpose of this paper is to study Tauberian theorems for the Euler-Nörlund mean-convergent sequences of fuzzy numbers and investigate some other kind of convergences named Euler-Nörlund mean-level convergence so as to fill up the existing gaps in the literature. The results which we obtained in this study are much more general than those obtained by others.

1. **Introduction**

The concepts of fuzzy sets and fuzzy set operations were first introduced by Zadeh [17] and subsequently several authors have discussed various aspects of the theory and applications of fuzzy sets such as fuzzy topological spaces, similarity relations and fuzzy orderings, fuzzy measures of fuzzy events, fuzzy mathematical programming. Matloka [11] introduced bounded and convergent sequences of fuzzy numbers, studied some of their properties and showed that every convergent sequence of fuzzy numbers is bounded. In addition, sequences of fuzzy numbers have been discussed by Altin et al. [1], Aytar et al. [2], Et et al. [5], Gokhan et al. [8], Kwon [9], Nanda [13] and many others.

The idea of statistical convergence was given by Zygmund [18] in the first edition of his monograph published in Warsaw in 1935. Later on the concept of statistical convergence was introduced by Steinhaus [15] and Fast [6] and later reintroduced by Schoenberg [14], independently. Over the years and under different names statistical convergence has been discussed different areas of mathematics such as Fourier analysis, Ergodic theory, Number theory, Measure theory, Trigonometric series, Turnpike theory and Banach spaces. Later on it was further investigated from the sequence space point of view and linked with summability theory by Moricz [12],

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Kwon [9], Altin et al. [1], Et et al. [5], Gokhan et al. [8] and many others. In recent years, generalizations of statistical convergence have appeared in the study of strong integral summability and the structure of ideals of bounded continuous functions on locally compact spaces. Statistical convergence and its generalizations are also connected with subsets of the Stone-Čech compactification of the natural numbers. Moreover, statistical convergence is closely related to the concept of convergence in probability.

In the present paper we will prove Tauberian theorems for fuzzy sequence spaces. In this section we give a brief overview about statistical convergence, fuzzy numbers and sequences of fuzzy numbers. In section 2 we prove main results of this paper. In section 3 we give some level convergence of fuzzy numbers which is the generalization of that given by [16].

The idea of statistical convergence depends upon the density of subsets of the set $\mathbb{N}$ of natural numbers. We shall denote by $\mathbb{N}$ the set of all natural numbers. Let $K \in \mathbb{N}$ and $K_n = \{k \leq n : k \in K\}$. Then the natural density of K is defined by $d(K) = \lim_{n\to\infty} \frac{|K_n|}{n}$ if the limit exists, where the vertical bars indicate the number of elements in the enclosed set. The sequence $x = (x_k)$ is said to be statistically convergent to L if for every $\epsilon > 0$, the set $K_\epsilon = \{k \in \mathbb{N} : |x_k - L| \geq \epsilon\}$ has natural density zero, i.e. for each $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : |x_k - L| \geq \epsilon\}| = 0.$$

In this case, we write $st - \lim x = L$. Note that every convergent sequence is statistically convergent but not conversely.

Let us define the $(EC)^1_n-$ summability method as follows:

$$(EC)^1_n = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} C^1_k,$$

where C^1_k denotes the Cesàro summability method. The summability method $(EC)^1_n$ is a regular.

We say that the series $\sum_{n=1}^{\infty} x_n$ is $(EC)^1_n-$ summable to L if

$$\lim_n \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} \frac{1}{k+1} \sum_{v=0}^{k} x_v = L.$$

Definition 1.1. A sequence (x_n) is weighted $(EC)^1_n-$statistically convergent to L if for every $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{2^n} \left| \left\{ k \leq 2^n : \left| \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} \frac{1}{k+1} \sum_{v=0}^{k} x_v - L \right| \geq \epsilon \right\} \right| = 0.$$

Denote by

$$L(\mathbb{R}^n) = \{u : \mathbb{R}^n \to [0,1] : u \quad \text{satisfies} \quad (1) - (4) \quad \text{bellow}\}$$

where

(1) u is normal, there exists an $x_0 \in \mathbb{R}^n$ such that $u(x_0) = 1$,

(2) u is fuzzy convex, for any $x, y \in \mathbb{R}^n$ and $0 \leq \lambda \leq 1$, $u(\lambda x + (1-\lambda)y) \geq \min[u(x), u(y)]$,

(3) u is upper semicontinuous,

(4) the closure of $\{x \in \mathbb{R}^n : u(x) > 0\}$, denoted by $[u]_0$, is compact.

If $u \in L(\mathbb{R}^n)$, then u is called fuzzy number, and $L(\mathbb{R}^n)$ is said to be fuzzy number space. For $0 < \alpha \leq 1$, the $\alpha-$ level set $[u]_\alpha$ of u is defined by $[u]_\alpha = \{x \in \mathbb{R}^n : u(x) \geq \alpha\}$. Then from (1)-(4), it follows that the $\alpha-$level sets $[u]_\alpha \in C(\mathbb{R}^n)$, where $C(\mathbb{R}^n) = \{A \subset \mathbb{R}^n : A \text{ is compact and convex}\}$.

We will denote by E the set of all fuzzy numbers on $\mathbb{R}$. The set of real numbers can be embedded in E, since each $r \in \mathbb{R}$ can be regarded as a fuzzy number $\overline{r}$ defined by

$$\overline{r}(x) = \begin{cases} 1; & \text{if} \quad x = r, \\ 0; & \text{if} \quad x \neq r. \end{cases}$$

Let $u, v, w \in E$ and $k \in \mathbb{R}$. Then the operations addition and scalar multiplications are defined on E as follows:

$$u + v = w \Leftrightarrow [w]_\alpha = [u]_\alpha + [v]_\alpha \quad \text{for all} \quad \alpha \in [0,1],$$

$$\Leftrightarrow w^-(\alpha) = u^-(\alpha) + v^-(\alpha) \quad \text{and} \quad w^+(\alpha) = u^+(\alpha) + v^+(\alpha) \quad \text{for all} \quad \alpha \in [0,1],$$

$$[ku]_\alpha = k[u]_\alpha \quad \text{for all} \quad \alpha \in [0,1],$$

where

$$[u]_\alpha = [u^-(\alpha), u^+(\alpha)], \qquad [v]_\alpha = [v^-(\alpha), v^+(\alpha)] \quad \textit{and} \quad [w]_\alpha = [w^-(\alpha), w^+(\alpha)].$$

Further details related to the structural properties of the fuzzy numbers, are given in [3]. Let us denote by W the set of all closed bounded intervals A of real numbers with endpoints $\underline{A}$ and $\overline{A}$, i.e., $A = [\underline{A}, \overline{A}]$. Define the relation d on W by

$$d(A, B) = \max\{|\underline{A} - \underline{B}|, |\overline{A} - \overline{B}|\}.$$

Then it can be easily observed that d is a metric on W and (W, d) is a complete metric space, ([13]). Now, we may define the metric D on E by means of the Hausdorff metric d as follows

$$D(u, v) = \sup_{\alpha \in [0,1]} d([u]_\alpha, [v]_\alpha) = \sup_{\alpha \in [0;1]} \max\{|u^-(\alpha) - v^-(\alpha)|, |u^+(\alpha) - v^+(\alpha)|\},$$

and

$$D(u, 0) = \sup_{\alpha \in [0;1]} \max\{|u^-(\alpha)|, |u^+(\alpha)|\} = \max\{|u^-(\alpha)|, |u^+(\alpha)|\}.$$

A sequence $u = (u_k)$ of fuzzy numbers is a function u from the set $\mathbb{N}$, into the set E. The fuzzy number u_k denotes the value of the function at $k \in \mathbb{N}$ and is called as the $k-$th term of the sequence. By $w(F)$, we denote the set of all sequences of fuzzy numbers. A sequence $(u_n) \in w(F)$ is said to be convergent to $u \in E$, if for every $\epsilon > 0$ there exists an $n_0 = n_0(\epsilon) \in \mathbb{N}$ such that

$$D(u_n, u) < \epsilon \quad \text{for all} \quad n > n_0.$$

Definition 1.2. Let $X = (X_k)$ be a sequence of fuzzy numbers. The sequence X is said to converge weighted statistically to a fuzzy number X_0, if for every $\epsilon > 0$

$$\lim_{n\to\infty} \frac{1}{2^n} \left|\{k \leq 2^n : D(X_k, X_0) \geq \epsilon\}\right| = 0.$$

The above type of convergence will be denoted as

$$st_F - \lim_n X_n = X_0.$$

Definition 1.3. Let $X = (X_k)$ be a sequence of fuzzy numbers. The sequence X is said to be statistically Euler-Cesaro summable to a fuzzy number X_0 if the sequence

$$(EC)^1_n(X) = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} \frac{1}{k+1} \sum_{v=0}^{k} X_v,$$

is statistically convergent to X_0, where the sum in $(EC)^1_n(X)$ is usual addition of fuzzy real numbers through $\alpha-$ level sets. That is (X_k) is statistically Euler-Nörlund summable to the fuzzy number X_0, if for every $\epsilon > 0$

$$\lim_{n\to\infty} \frac{1}{2^n} \left|\{k \leq 2^n : D((EC)^1_k, X_0) \geq \epsilon\}\right| = 0.$$

The above type of convergence will be denoted as

$$st_F - \lim_n (EC)^1_n = X.$$

Theorem 1.4. *Let us suppose that (X_k) is a bounded sequence of fuzzy numbers such that $st_F - \lim_k X_k = L$. Then there exists $st_F - \lim_k (EC)^1_k = L$, but the converse is not true.*

Proof. Let us suppose that $st_F - \lim_k X_k = L$. Let $\epsilon > 0$ be any given number. Then

$$\begin{aligned} D((EC)^1_k, L) &= D\left(\frac{1}{2^k}\sum_{i=0}^{k}\binom{k}{i}\frac{1}{i+1}\sum_{v=0}^{i}X_v, L\right) = D\left(\frac{1}{2^k}\sum_{i=0}^{k}\binom{k}{i}\frac{1}{i+1}\sum_{v=0}^{i}X_v, \frac{2^k}{2^k}L\right) \\ &= \left|\frac{1}{2^k}\right| D\left(\sum_{i=0}^{k}\binom{k}{i}\frac{1}{i+1}\sum_{v=0}^{i}X_v, 2^k L\right) \leq \frac{1}{2^k} D\left(\sum_{i=0}^{k}\binom{k}{i}\frac{1}{i+1}\sum_{v=0}^{i}X_v, L\right) \\ &\leq \frac{1}{2^n} \sup_{j\in\{0,1,2,\cdots,n\}} \{D(X_j, L)\} \to 0, \quad \text{as} \quad n \to \infty, \end{aligned}$$

because

$$\sup_{k\leq 2^n} \{D(X_k, L)\} \subseteq \sup\{k \leq 2^n : D(X_k, L) \leq \epsilon\} \cup \sup\{k \leq 2^n : D(X_k, L) \geq \epsilon\},$$

and from last relation we have:

$$\lim_n \frac{|\{k \le 2^n : D(X_k, L)\}|}{2^n} \le \lim_n \frac{|\{k \le 2^n : D(X_k, L) \le \epsilon\}|}{2^n} + \lim_n \frac{|\{k \le 2^n : D(X_k, L) \ge \epsilon\}|}{2^n}.$$

To prove that converse is not true, we construct the following example

Example 1.5. Let us consider the following sequence (X_k), which is defined as follows:

$$X_k(t) = \begin{cases} \left.\begin{matrix} t+1, & -1 \le t \le 0 \\ -t+1, & 0 \le t \le 1 \\ \bar{0}, & \text{otherwise} \end{matrix}\right\}, & \text{for } k = m^2 - m, ..., m^2 - 1 \\ \left.\begin{matrix} 1 - \frac{t}{m}, & -m \le t \le 0 \\ 1 + \frac{t}{m}, & 0 \le t \le m \\ \bar{0}, & \text{otherwise} \end{matrix}\right\}, & \text{for } k = m^2, m = 2, ... \\ \bar{0}, \ \text{otherwise} \ \}, & \text{otherwise} \end{cases}$$

Then, we calculate the $\alpha-$level sets of sequences (X_k) as follows

$$[X_k]^\alpha = \begin{cases} [\alpha - 1, 1 - \alpha] & \text{for} \quad k = m^2 - m, ..., m^2 - 1 \\ [-m(\alpha - 1), -m(1 - \alpha)] & k = m^2; m = 2, 3, \cdots \\ \bar{0} & \text{otherwise} \end{cases}$$

The sequence of fuzzy numbers (X_k) is $(EC)^1_n$ summable to $\bar{0}$, and hence statistically $(EC)^1_n-$ summable. On the other hand, the sequence $(m^2; m = 2, 3 \cdots ,)$ has natural density zero and it is clear that $st-\liminf_n X_n = \bar{0}$ and $st-\limsup_n X_n = \bar{1}$. Thus, $X = (X_n)$ is not statistically convergent, nor weighted statistically convergent. □

In this paper our aim is to find conditions (so-called Tauberian) under which the converse implication holds, for defined convergence. Exactly, we will prove under which conditions weighted statistically convergence follows from statistically Euler-Cesaro summable $(EC)^1_n$.

The theory of Tauberian theorems are intensively investigated by several authors, see ([4], [10], [12]).

2. Main Results

Theorem 2.1. *Let us consider that*

$$st_F - \liminf_n \frac{2^{t_n}}{2^n} > 1, \quad t > 1 \tag{1}$$

where t_n, denotes the integral parts of the $[t_n]$ for every $n \in \mathbb{N}$, and let (X_k) be a sequence of fuzzy real numbers such that $st_F - \lim_k (EC)^1_k = L$. Then (X_k) is weighted statistically convergent to the same fuzzy number L if and only if the following conditions hold:

$$\inf_{t>1} \limsup_n \frac{1}{2^n} \left| \left\{ k \le 2^n : D\left(\frac{1}{2^{t_k} - 2^k} \sum_{j=k+1}^{t_k} \binom{t_k}{j} \frac{1}{j+1} \sum_{v=0}^{j} X_v, X_k \right) \ge \epsilon \right\} \right| = 0, \tag{2}$$

and

$$\inf_{0<t<1} \limsup_n \frac{1}{2^n} \left| \left\{ k \le 2^n : D\left(\frac{1}{2^k - 2^{t_k}} \sum_{j=t_k+1}^{k} \binom{k}{j} \frac{1}{j+1} \sum_{v=0}^{j} X_v, X_k \right) \ge \epsilon \right\} \right| = 0. \quad (3)$$

Remark 2.2. Let us suppose that $st_F - \lim_k X_k = L$; $st_F - \lim_k (EC)^1_k = L$ and relation (1) satisfies, then for every $t > 1$, is valid the following relations:

$$st_F - \lim_k \frac{1}{2^{t_k} - 2^k} \sum_{j=k+1}^{t_k} \binom{t_k}{j} \frac{1}{j+1} \sum_{v=0}^{j} X_v = 0, \quad (4)$$

and in case where $0 < t < 1$,

$$st_F - \lim_k \frac{1}{2^k - 2^{t_k}} \sum_{j=t_k+1}^{k} \binom{k}{j} \frac{1}{j+1} \sum_{v=0}^{j} X_v = 0. \quad (5)$$

In what follows we will show some auxiliary lemmas which are needful in the sequel.

Lemma 2.3. *The condition given by relation (1) is equivalent to this one:*

$$st_F - \liminf_n \frac{2^n}{2^{t_n}} > 1, \quad 0 < t < 1. \quad (6)$$

Proof. Let us suppose that relation (1) is valid, $0 < t < 1$ and $m = t_n = [t \cdot n]$, $n \in \mathbb{N}$. Then it follows that

$$\frac{1}{t} > 1 \Rightarrow \frac{m}{t} = \frac{[t \cdot n]}{t} \le n,$$

we have:

$$\frac{2^n}{2^{t_n}} \ge \frac{2^{\left[\frac{m}{t}\right]}}{2^m} \Rightarrow st_F - \liminf_n \frac{2^n}{2^{t_n}} \ge st_F - \liminf_n \frac{2^{\left[\frac{m}{t}\right]}}{2^{t_n}} > 1.$$

Conversely, let us suppose that relation (6) is valid. Let $t > 1$ and be given and let t_1 be chosen such that $1 < t_1 < t$. Set $m = t_n = [t \cdot n]$. From $0 < \frac{1}{t} < \frac{1}{t_1} < 1$, it follows that:

$$n \le \frac{tn - 1}{t_1} < \frac{[tn]}{t_1} = \frac{m}{t_1},$$

provided $t_1 \le t - \frac{1}{n}$, which is a case where if n is large enough.Under this conditions we have:

$$\frac{2^{t_n}}{2^n} \ge \frac{2^{t_n}}{2^{\left[\frac{m}{t_1}\right]}} \Rightarrow st_F - \liminf_n \frac{2^{t_n}}{2^n} \ge st_F - \liminf_n \frac{2^{t_n}}{2^{\left[\frac{m}{t_1}\right]}} > 1.$$

□

Lemma 2.4. *Let us suppose that $X = (X_k)$ is a sequence of fuzzy numbers which is $(EC)^1_n$−statistically convergent to a fuzzy number X_0. Then for every $t > 0$, we have*

$$st_F - \lim_n (EC)^1_{t_n} = X_0,$$

where by $t_n = [t \cdot n]$, is denoted the integral part of the product $t \cdot n$.

Proof. From given facts we have

$$\lim_n \frac{1}{2^{t_n}} \sum_{k=0}^{t_n} \binom{t_n}{k} \frac{1}{k+1} \sum_{v=0}^{k} X_v = \lim_n \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} \frac{1}{k+1} \sum_{v=0}^{k} X_v, \tag{7}$$

and

$$D((EC)^1_{t_n}, X_0) \leq D((EC)^1_{t_n}, (EC)^1_n) + D((EC)^1_n, X_0).$$

Now we obtain

$$\frac{1}{2^n} \left|\{k \leq 2^n : D((EC)^1_{t_n}, X_0) \geq \epsilon\}\right| \leq \frac{1}{2^n} \left|\{k \leq 2^n : D((EC)^1_n, X_0) \geq \epsilon\}\right| +$$

$$\frac{1}{2^n} \left|\{k \leq 2^n : D((EC)^1_n, (EC)^1_{t_n}) \geq \epsilon\}\right| = I_1 + I_2.$$

$I_1 \to 0$, as $n \to \infty$, from $st_F - \lim_n (EC)^1_n = X_0$, and $I_2 \to 0$, as $n \to \infty$ from relation (7). Hence, $st_F - \lim_n (EC)^1_{t_n} = X_0$. □

Lemma 2.5. *Let us suppose that $X = (X_k)$ is a sequence of fuzzy numbers which is $(EC)^1_n-$statistically convergent to fuzzy number L. Then for every $t > 1$,*

$$st_F - \lim_n (2^{t_n} - 2^n)^{-1} \sum_{j=n+1}^{t_n} \binom{t_n}{j} \frac{1}{j+1} \sum_{v=0}^{j} X_v = L; \tag{8}$$

and for every $0 < t < 1$,

$$st_F - \lim_n (2^n - 2^{t_n})^{-1} \sum_{j=t_n+1}^{n} \binom{n}{j} \frac{1}{j+1} \sum_{v=0}^{j} X_v = L. \tag{9}$$

Proof. Let us suppose that $t > 1$, then $2^{t_n} > 2^n$, and

$$D\left(\frac{1}{2^{t_n} - 2^n} \sum_{k=n+1}^{t_n} \binom{t_n}{k} \frac{1}{k+1} \sum_{v=0}^{k} X_v, L\right) \leq$$

$$D\left(\frac{1}{2^{t_n} - 2^n} \sum_{k=n+1}^{t_n} \binom{t_n}{k} \frac{1}{k+1} \sum_{v=0}^{k} X_v, (EC)^1_n\right) + D\left((EC)^1_n, L\right). \tag{10}$$

It can be easily shown that for any three fuzzy numbers X, Y and Z, the following relation is valid:

$$D(X + Y, Z) \leq c \cdot D(X, Z),$$

for some positive constant $c > 1$.

(I) Let us suppose that $t > 1$. After some calculations we obtain and knowing that $D(x, y) = D(x + u, y + u)$(see [3]), we have

$$D\left(\frac{1}{2^{t_n} - 2^n} \sum_{k=n+1}^{t_n} \binom{t_n}{k} \frac{1}{k+1} \sum_{v=0}^{k} X_v, (EC)^1_n\right) =$$

$$D\left(\frac{1}{2^{t_n} - 2^n} \sum_{k=1}^{t_n} \binom{t_n}{k} \frac{1}{k+1} \sum_{v=0}^{k} X_v, (EC)^1_n + \frac{1}{2^{t_n} - 2^n} \sum_{k=1}^{n} \binom{t_n}{k} \frac{1}{k+1} \sum_{v=0}^{k} X_v\right) \leq$$

$$c_1 \cdot D\left((EC)^1_n, \frac{2^{t_n}}{2^{t_n}-2^n}(EC)^1_{t_n}\right) \quad \text{for some} \quad c_1 > 1,$$

$$= c_1 \frac{2^{t_n}}{2^{t_n}-2^n} D\left(\frac{2^{t_n}-2^n}{2^{t_n}}(EC)^1_n, (EC)^1_{t_n}\right) \leq c_1 \frac{2^{t_n}}{2^{t_n}-2^n} D\left((EC)^1_n, (EC)^1_{t_n}\right). \tag{11}$$

From definition of the sequence (t_n), we get

$$st_F - \limsup_n \frac{2^{t_n}}{2^{t_n}-2^n} < \infty. \tag{12}$$

Now relation (8) follows from relations (11), (12) and Lemma 2.4.

(II) The case $0 < t < 1$. In this case we have

$$D\left(\frac{1}{2^n - 2^{t_n}} \sum_{k=t_n+1}^{n} \binom{n}{k} \frac{1}{k+1} \sum_{v=0}^{k} X_v, L\right) \leq$$

$$D\left(\frac{1}{2^n - 2^{t_n}} \sum_{k=t_n+1}^{n} \binom{n}{k} \frac{1}{k+1} \sum_{v=0}^{k} X_v, (EC)^1_{t_n}\right) + D\left((EC)^1_{t_n}, L\right).$$

As in the previous case we obtain:

$$D\left(\frac{1}{2^n - 2^{t_n}} \sum_{k=t_n+1}^{n} \binom{n}{k} \frac{1}{k+1} \sum_{v=0}^{k} X_v, (EC)^1_{t_n}\right) =$$

$$D\left(\frac{1}{2^n - 2^{t_n}} \sum_{k=1}^{n} \binom{n}{k} \frac{1}{k+1} \sum_{v=0}^{k} X_v, (EC)^1_{t_n} + \frac{1}{2^n - 2^{t_n}} \sum_{k=1}^{t_n} \binom{n}{k} \frac{1}{k+1} \sum_{v=0}^{k} X_v\right) \leq$$

$$c_2 \cdot D\left((EC)^1_{t_n}, \frac{2^n}{2^n - 2^{t_n}}(EC)^1_n\right) \quad \text{for some} \quad c_2 > 1,$$

$$= c_2 \frac{2^n}{2^n - 2^{t_n}} D\left(\frac{2^n - 2^{t_n}}{2^n}(EC)^1_{t_n}, (EC)^1_n\right) \leq c_2 \frac{2^n}{2^n - 2^{t_n}} D\left((EC)^1_n, (EC)^1_{t_n}\right). \tag{13}$$

From definition of the sequence (t_n), we get

$$st_F - \limsup_n \frac{2^n}{2^n - 2^{t_n}} < \infty. \tag{14}$$

Now relation (9) follows from relations (13), (14) and Lemma 2.4. □

In what follows we will prove the Theorem 2.1.

Proof. Necessity. Let us suppose that $st_F - \lim_k X_k = X_0$, and $st_F - \lim_k (EC)^1_k = X_0$. For every t following Lemma 2.5 we get relation (2) and relation (3), respectively.

Sufficiency: Assume that $st_F - \lim_n (EC)^1_n = X_0$, and conditions (1), (2) and (3) are satisfied. In what follows we will prove that $st_F - \lim_n X_n = X_0$, or equivalently, $st_F - \lim_n D((EC)^1_n, X_n) = 0$.

First we consider the case where $t > 1$. We will start from this estimation:

$$D((EC)^1_n, X_n) \le D\left((EC)^1_n, \frac{1}{2^{t_n} - 2^n} \sum_{k=n+1}^{t_n} \binom{t_n}{k} \frac{1}{k+1} \sum_{v=0}^{k} X_v\right) + D\left(\frac{1}{2^{t_n} - 2^n} \sum_{k=n+1}^{t_n} \binom{t_n}{k} \frac{1}{k+1} \sum_{v=0}^{k} X_v, X_n\right).$$

As in the proof of Lemma 2.5, we obtain

$$D\left((EC)^1_n, \frac{1}{2^{t_n} - 2^n} \sum_{k=n+1}^{t_n} \binom{t_n}{k} \frac{1}{k+1} \sum_{v=0}^{k} X_v\right) \le c_1 \frac{2^{t_n}}{2^{t_n} - 2^n} \cdot D\left((EC)^1_n; (EC)^1_{t_n}\right), \tag{15}$$

for some $c_1 > 1$.

For any $\epsilon > 0$, we get:

$$\{k \le 2^n : D(X_k, (EC)^1_k) \ge \epsilon\} \subset \left\{k \le 2^n : D\left(\frac{1}{2^{t_k} - 2^k} \sum_{j=k+1}^{t_k} \binom{t_k}{j} \frac{1}{j+1} \sum_{v=0}^{j} X_v, X_n\right) \ge \epsilon\right\} \cup \left\{k \le 2^n : c_1 \frac{2^{t_n}}{2^{t_n} - 2^n} D((EC)^1_n; (EC)^1_{t_n}) \ge \frac{\epsilon}{2}\right\}. \tag{16}$$

From relation (2), it follows that for every $\delta > 0$, exists a $t > 1$ such that

$$\limsup_n \frac{1}{2_n} \left|\left\{k \le 2^n : D\left(\frac{1}{2^{t_k} - 2^k} \sum_{j=k+1}^{t_k} \binom{t_k}{j} \frac{1}{j+1} \sum_{v=0}^{j} X_v, X_k\right) \ge \epsilon\right\}\right| \le \delta. \tag{17}$$

On the other side, from Lemma 2.5, it follows that

$$\lim_{n\to\infty} \left|\left\{k \le 2^n : c_1 \frac{2^{t_k}}{2^{t_k} - 2^k} D((EC)^1_k; (EC)^1_{t_k}) \ge \frac{\epsilon}{2}\right\}\right| = 0. \tag{18}$$

From relations (16), (17), (18) and (1), it yields

$$\lim_{n\to\infty} \left|\left\{k \le 2^n : D((EC)^1_k; X_k) \ge \epsilon\right\}\right| \le \delta.$$

Since $\delta > 0$ is arbitrary , we can conclude that for every $\epsilon > 0$,

$$\lim_{n\to\infty} \left|\left\{k \le 2^n : D((EC)^1_k; X_k) \ge \epsilon\right\}\right| = 0.$$

Now, in similar way as above we can prove the second case. □

3. Tauberian Theorems for Euler-Nörlund Mean-level Convergence

In this section we will introduce some level convergence of fuzzy numbers which is the generalization of that given by [16] and also we will give some results related to the statistical limit inferior and superior for sequence of fuzzy numbers related to the new definition.

Let $X = (X_k)$ and $Y = (Y_k)$ be two real sequences of fuzzy numbers. We say that (X_k) is statistically bounded from above if there exists a fuzzy number X_0, such that

$$\lim_n \frac{1}{2^n} |\{k \leq 2^n : X_k > X_0\} \cup \{k \leq 2^n : X_k \nsim X_0\}| = 0,$$

where $v \nsim w$, denotes that fuzzy numbers v and w are not comparable, which means that if neither $v \leq w$ nor $w \leq v$.

Similarly, (X_k) is statistically bounded from bellow if there exists a fuzzy number Y_0, such that

$$\lim_n \frac{1}{2^n} |\{k \leq 2^n : X_k < Y_0\} \cup \{k \leq 2^n : X_k \nsim Y_0\}| = 0.$$

If the sequence $X = (X_k)$ is both statistically bounded from above and bellow we will say that it is statistically bounded.

Definition 3.1. Let $X = (X_n)$ be a statistically bounded sequence of fuzzy numbers. Then the statistical limit inferior of the $X = (X_n)$ is given by

$$st_F - \liminf_n X_n = \inf \left\{ X_0 \in E : \lim_n \frac{1}{2^n} |k \leq 2^n : X_k < X_0| \neq 0 \right\},$$

and, the statistical limit superior of the $X = (X_n)$ is given by

$$st_F - \limsup_n X_n = \inf \left\{ X_0 \in E : \lim_n \frac{1}{2^n} |k \leq 2^n : X_k > X_0| \neq 0 \right\}.$$

Theorem 3.2. *Let $X = (X_n)$ be a statistically bounded sequence of fuzzy numbers. If $st_F - \lim_n \inf X_n = X_0$, then*

$$\lim_n \frac{1}{2^n} |\{k \leq 2^n : X_k < X_0 - \bar{\epsilon}\}| = 0 \tag{19}$$

and

$$\lim_n \frac{1}{2^n} |\{k \leq 2^n : X_k < X_0 + \bar{\epsilon}\} \cup \{k \leq 2^n : X_k \nsim X_0 + \bar{\epsilon}\}| \neq 0, \tag{20}$$

for every $\epsilon > 0$.

Proof. Proof of the theorem is similar to Theorem 2, given in [2]. For this reason we omit it. □

The same results are valid for the statistically limit superior, as is shown in the following

Theorem 3.3. *Let $X = (X_n)$ be a statistically bounded sequences of fuzzy numbers. If $st_F - \lim_n \sup X_n = X_0$, then*

$$\lim_n \frac{1}{2^n} |\{k \leq 2^n : X_k > X_0 + \bar{\epsilon}\}| = 0 \tag{21}$$

and

$$\lim_n \frac{1}{2^n} \left|\{k \le 2^n : X_k > X_0 - \bar{\epsilon}\} \cup \{k \le 2^n : X_k \nsim X_0 - \bar{\epsilon}\}\right| \neq 0, \tag{22}$$

for every $\epsilon > 0$.

The level convergence of sequence of fuzzy numbers was given by [7], as follows. Let $X = (X_k)$ be a sequence of fuzzy numbers, it is level-convergent to fuzzy number X_0, if

$$\lim_{n\to\infty} X_n^-(\alpha) = X_0^-(\alpha), \lim_{n\to\infty} X_n^+(\alpha) = X_0^+(\alpha), \tag{23}$$

for any $\alpha \in [0,1]$.

Remark 3.4. The level convergence differs from the standard convergence and it is shown in the following example.

Example 3.5. Let

$$\underline{u_n}(r) = \left\{ \begin{array}{ll} (r-\frac{1}{4})^n; & \frac{1}{4} < r \le 1 \\ 0; & 0 \le r \le \frac{1}{4} \end{array} \right. , \quad \overline{u_n}(r) = 1,$$

and

$$\underline{u_0}(r) = \left\{ \begin{array}{ll} 1; & \frac{1}{4} < r \le 1 \\ 0; & 0 \le r \le \frac{1}{4} \end{array} \right. , \quad \overline{u_0}(r) = 1.$$

From representation theorem in [16], there exists a unique fuzzy number u_n and a unique fuzzy number u_0 such that $[u_n]^r = [\underline{u_n}(r), \overline{u_n}(r)], [u_0]^r = [\underline{u_0}(r), \overline{u_0}(r)]$. Obviously, the sequences $\underline{u_n}(r), \overline{u_n}(r)$ converges to $\underline{u_0}(r), \overline{u_0}(r)$, respectively at any $r \in [0,1]$, as $n \to \infty$. And $D(u_n, u_0) = \sup_{r\in(\frac{1}{4},1]} \left\{1 - (r-\frac{1}{4})^{\frac{1}{n}}\right\} = 1$, for any natural number n. But u_n does not converges to u_0.

Following it and our definition related to the Euler-Nörlund mean-convergent, we give this

Definition 3.6. The fuzzy sequence $X = (X_k)$ is mean-level convergent by Euler-Nörlund, to a fuzzy number X_0, if

$$\lim_{n\to\infty} \frac{1}{2^n} \sum_{k=1}^{n} \binom{n}{k} \frac{1}{k+1} \sum_{v=0}^{k} X_k^-(\alpha) = X_0^-(\alpha),$$

$$\lim_{n\to\infty} \frac{1}{2^n} \sum_{k=1}^{n} \binom{n}{k} \frac{1}{k+1} \sum_{v=0}^{k} X_k^+(\alpha) = X_0^+(\alpha),$$

for all $\alpha \in [0,1]$.

In what follows we prove a kind of Tauberian theorem for the new defined concept of Euler-Nörlund mean-level convergent.

Theorem 3.7. *Let $X = (X_k)$ be a bounded sequence of fuzzy numbers. Assume that (X_k) is Euler-Nörlund mean-level convergent to fuzzy number X_0. Also we* *suppose that* $st_F - \lim_n \sup X_k = X_0$ *and there is a number* $\epsilon_0 > 0$ *such that for every* $\epsilon \in (0, \epsilon_0)$,

$$\lim_n \frac{1}{2^n} \left| \left\{ k \le 2^n : \binom{n}{k} \frac{1}{k+1} \sum_{v=0}^{k} X_k \nsim X_0 - \overline{\epsilon} \right\} \right| = 0,$$

$$, \lim_n \frac{1}{2^n} \left| \left\{ k \le 2^n : \binom{n}{k} \frac{1}{k+1} \sum_{v=0}^{k} X_k \nsim X_0 + \overline{\epsilon} \right\} \right| = 0.$$

Then $st_F - lim_n X_n = X_0$.

Proof. From $st_F - \lim_n \sup X_k = X_0$ and Theorem 3.3, we have

$$\lim_n \frac{1}{2^n} \left| \{ k \le 2^n : X_k > X_0 + \overline{\epsilon} \} \right| = 0,$$

for every $\overline{\epsilon} > 0$. Let us suppose for the moment that

$$\lim_n \frac{1}{2^n} \left| \{ k \le 2^n : D(X_k, X_0) > \epsilon \} \right| \ne 0.$$

Then there exists a $\overline{\epsilon}_1 \in (0, \overline{\epsilon}_0)$ such that

$$\lim_n \frac{1}{2^n} \left| \{ k \le 2^n : X_k < X_0 - \overline{\epsilon}_1 \} \right| \ne 0.$$

We define the following sets of fuzzy numbers:

$$A_1 = \left\{ k \le 2^n : \binom{n}{k} \frac{1}{k+1} \sum_{v=0}^{k} X_k < X_0 - \overline{\epsilon} \right\},$$

$$A_2 = \left\{ k \le 2^n : X_0 - \overline{\epsilon} < \binom{n}{k} \frac{1}{k+1} \sum_{v=0}^{k} X_k < X_0 + \overline{\epsilon} \right\},$$

$$A_3 = \left\{ k \le 2^n : \binom{n}{k} \frac{1}{k+1} \sum_{v=0}^{k} X_k > X_0 + \overline{\epsilon} \right\} \cup \left\{ k \le 2^n : \binom{n}{k} \frac{1}{k+1} \sum_{v=0}^{k} X_k \nsim X_0 - \overline{\epsilon} \right\} \cup$$

$$\left\{ k \le 2^n : \binom{n}{k} \frac{1}{k+1} \sum_{v=0}^{k} X_k \nsim X_0 + \overline{\epsilon} \right\}.$$

From definitions of the sets $A_i, i \in \{1, 2, 3\}$, we get

$$\lim_n \frac{1}{2^n} |A_3| = 0, \lim_n \frac{1}{2^n} |A_1| \ne 0, \lim_n \frac{1}{2^n} |A_2| = 1 - \lim_n \frac{1}{2^n} |A_1|.$$

On the other side, from second relation in the expression we have that

$$\frac{1}{\lambda_n} |A_1| \ge a > 0,$$

for infinitely n and after some calculations we obtain

$$(EC)_n^1 = \frac{1}{2^n}\sum_{k=1}^{n}\binom{n}{k}\frac{1}{k+1}\sum_{v=0}^{k}X_v = \frac{1}{2^n}\sum_{k\in A_1}\binom{n}{k}\frac{1}{k+1}\sum_{v=0}^{k}X_v +$$
$$\frac{1}{2^n}\sum_{k\in A_2}\binom{n}{k}\frac{1}{k+1}\sum_{v=0}^{k}X_v + \frac{1}{2^n}\sum_{k\in A_3}\binom{n}{k}\frac{1}{k+1}\sum_{v=0}^{k}X_v <$$
$$\frac{X_0-\overline{\epsilon}}{2^n}|A_1| + \frac{X_0+\overline{\epsilon}}{2^n}|A_2| + \frac{A}{2^n}|A_3|,$$

for some fuzzy number A. There exists an $\alpha \in [0,1]$ such that

$$(EC)_n^{1^-}(\alpha) < X_0^-(\alpha) + \overline{\epsilon}(1-2a) + 0(1).$$

since $\overline{\epsilon} \in (0, \overline{\epsilon}_0)$ is arbitrary, then we get

$$\liminf\,(EC)_n^{1^-}(\alpha) \le X_0^-(\alpha).$$

Hence $X = (X_n)$ is not Euler-Nörlund mean-level convergent to fuzzy number X_0, which prove Theorem. □

Remark 3.8. The above theorem is valid also for the $st_F - \liminf_n X_k = X_0$.

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