ON THE SYSTEM OF LEVEL-ELEMENTS INDUCED BY AN L-SUBSET

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Abstract. This paper focuses on the relationship between an L-subset and the system of level-elements induced by it, where the underlying lattice L is a complete residuated lattice and the domain set of L-subset is an L-partially ordered set (X, P). Firstly, we obtain the sufficient and necessary condition that an L-subset is represented by its system of level-elements. Then, a new representation theorem of intersection-preserving L-subsets is shown by using union-preserving system of elements. At last, another representation theorem of compatible intersection-preserving L-subsets is obtained by means of compatible union-preserving system of elements.

1. Introduction

In the research of fuzzy sets, the mathematical representation of fuzzy sets is an important theoretic aspect. In the literature, there are various types of representation theorems which establish the links between fuzzy sets and their level-structures. For example, in 1983, Luo(see [8]) first proposed the concept of nested systems, and established a representation theorem of L-subsets(fuzzy sets) with them. Then based on different lattice-theoretic situation and different forms of nested systems, literature (see [1, 2, 10, 14])further studied the representation theorems of L-subsets. Recently, a number of related work are constantly in progress, such as B.Šešelja and A.Tepavčević(see [11, 12, 13]), W.Y.Zeng(see [16]), L.X.Meng(see [9]) and others.

It is valueable to note that these representation theorems of L-subsets have a common characteristic that domain X is just a set, and don't have any other mathematic structure. But in many areas of fuzzy mathematics, the domain X is often endowed with L-partial order P, e.g. lattice-valued topology. Therefore, L-subsets show up as a map from L-partially ordered set (X, P) to the underlying lattice L, where L is complete residuated lattice. It should be pointed out that these L-subsets also determine some kinds of "level-elements", and they can be described with certain "level-elements" as well, e.g. Han(see [6]) discussed the case of L-family (special L-subsets which are maps from L^X to L). Hence, it makes sense to find the relations between an L-subset and its "level-elements".

For this purpose, we introduce the concept of the system of level-elements. Moreover, we obtain the sufficient and necessary condition that an L-subset is

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represented by its system of level-elements. Then, a new representation theorem of intersection-preserving L-subsets is shown by using union-preserving system of elements. At last, another representation theorem of compatible intersection-preserving L-subsets is obtained by means of compatible union-preserving system of elements.

2. **Preliminaries**

Throughout this paper, $(L, \leq, \otimes, 1)$, or simply L, denotes a complete residuated lattice, which means that $(L, \leq)$ is a complete lattice with the top element 1, and the bottom element 0, 1 is the unit element for $\otimes$ and $\otimes : L \times L \longrightarrow L$, called a tensor on L, is a commutative, associative binary operation such that for every element $b \in L$ and for every subset $\{a_j \mid j \in J\}$ of L,

$$b \otimes (\bigvee_{j\in J} a_j) = \bigvee_{j\in J} (a_j \otimes b).$$

In this case, there exists an implication operator $\rightarrow: L \times L \longrightarrow L$ as the right adjoint for the tensor operation $\otimes$ by

$$a \rightarrow b = \bigvee\{c \mid a \otimes c \leq b\}, \forall a, b \in L.$$

Thus the pair $(\otimes, \rightarrow)$ forms a Galois correspondence on L, that is,

$$\forall a, b, c \in L,\ a \otimes c \leq b \Leftrightarrow\ c \leq (a \rightarrow b).$$

For $B \subseteq L$, write $\bigvee B$ for the least upper bound of B and $\bigwedge B$ for the greatest lower bound of B. In particular, $\bigvee \varnothing = 0$ and $\bigwedge \varnothing = 1$.

Some basic properties of the tensor $\otimes$ and the implication operation $\rightarrow$ are collected in the following lemma; they can be found in many works, for instance(see [1, 3, 7]).

Lemma 2.1. *Suppose that $(L, \leq, \otimes, 1)$ is a complete residuated lattice and $\rightarrow$ is the implication operation corresponding to $\otimes$. Then for all $a, b, c \in L, \{a_i\}_{i\in I} \subseteq L$, the following conditions hold:*
(I1) $a \otimes 1 = a$;
(I2) If $b \leq c$, then $a \otimes b \leq a \otimes c$;
(I3) If $b \leq c$, then $a \rightarrow b \leq a \rightarrow c$;
(I4) $b \rightarrow \bigwedge_{i\in I} a_i = \bigwedge_{i\in I}(b \rightarrow a_i)$.

An L-subset on a set X is a mapping from X to L, and the family of all L-subsets on X will be denoted by L^X, called the L-power set of X. By 0_X and 1_X, we denote the constant L-subsets on X taking the value 0 and 1, respectively. We do not distinguish between an element $a \in L$ and the constant function $a : X \rightarrow L$ such that $a(x) = a$ for all $x \in X$. All algebraic operations on L can be extended to the L-power set L^X pointwise. That is, for all $U, V \in L^X, a \in L$, $x \in X$,
(1) $U \leq V$ iff $U(x) \leq V(x)$;
(2) $(U \wedge V)(x) = U(x) \wedge V(x)$;
(3) $(U \otimes V)(x) = U(x) \otimes V(x)$;
(4) $(a \rightarrow A)(x) = a \rightarrow A(x)$.

Let us recall that an L-partially ordered set (X, P) is a set X together with a binary mapping $P(-,-) : X \times X \longrightarrow L$, called an L-partial order, such that
(1) $P(x, x) = 1$ for every $x \in X$;
(2) $P(x, y) = P(y, x) = 1 \Rightarrow (x = y)$ for every $x, y \in X$;
(3) $P(x, y) \otimes P(y, z) \leq P(x, z)$ for all $x, y, z \in X$.

Remark 2.2. L-partial order P induces a crisp partial order "$\leq_P$" on X as follows:

$$\forall x, y \in X,\ x \leq_P y \Leftrightarrow P(x, y) = 1.$$

Example 2.3. For a given set X, define a binary mapping $S(-,-) : L^X \times L^X \longrightarrow L$ by $S(U, V) = \bigwedge_{x \in X}(U(x) \to V(x))$ for each pair $(U, V) \in L^X \times L^X$, where "$\to$" is the implication operation on L. Then $S(-,-)$ is an L-partial order on L^X. For $U, V \in L^X$, $S(U, V)$ can be interpreted as the degree to which U is a subset of V. This L-partial order has been known in the literature for some time. It was called fuzzy inclusion order(see [1, 17]) of L-subsets or the subsethood degree(see [5]).

Definition 2.4. [15] Let (X, P) be an L-partially ordered set, $x_0 \in X, A \in L^X$. The element x_0 is called the supremum (resp., infimum) of A, in symbols $x_0 = \sup A$ (resp., $x_0 = \inf A$), if
(1) $\forall x \in X, A(x) \leq P(x, x_0)$ (resp., $A(x) \leq P(x_0, x)$);
(2) $\forall y \in X, \bigwedge_{x \in X} A(x) \to P(x, y) \leq P(x_0, y)$ (resp., $\bigwedge_{x \in X} A(x) \to P(y, x) \leq P(y, x_0)$).

Definition 2.5. [15] Let (X, P) be an L-partially ordered set. If for all $A \in L^X$, $\sup A$ and $\inf A$ exist, (X, P) is called a fuzzy complete lattice.

Example 2.6. For the L-partially ordered set $(L^X, S(-,-))$, it follows former([1]) and latter([16] that both $\inf \mathscr{R}$ and $\sup \mathscr{R}$ exist for each L-family $\mathscr{R} : L^X \longrightarrow L$, and they can be determined by the following formulas:

$$\sup \mathscr{R} = \bigvee_{B \in L^X} (\mathscr{R}(B) \otimes B),$$

$$\inf \mathscr{R} = \bigwedge_{B \in L^X} (\mathscr{R}(B) \to B).$$

Hence, $(L^X, S(-,-))$ is a fuzzy complete lattice in the sense of Definition 2.4.

The following proposition is useful to the subsequent section; and its proof is routine and is omitted.

Proposition 2.7. *If (X, P) is a fuzzy complete lattice, $(X, \leq_P)$ is a complete lattice.*

In the complete lattice $(X, \leq_P)$, the infimum and supremum of $A \subseteq X$, are respectively computed by $\bigwedge A$ and $\bigvee A$. The greatest element and the least element of $(X, \leq_P)$ are denoted by $\top_X$ and $\bot_X$, respectively.

Definition 2.8. Let (X, P) be an L-partially ordered set, $A \in L^X$. If A satisfies

$$\forall x, y \in X,\ A(x) \otimes P(x, y) \leq A(y).$$

Then A is called a compatible L-subset w.r.t. P on X. The family of all compatible L-subsets on X will be denoted by $L^{(X,P)}$.
Note that, a compatible L-subset A w.r.t. P on X has another name, namely a fuzzy upper set in the literature.

Example 2.9. [3, 4] Let X be a non-void set. An L-family $\mathscr{F} : L^X \longrightarrow L$ is called a stratified L-filter on X if it satisfies the following conditions:
(F1) $\mathscr{F}(1_X) = 1, \mathscr{F}(0_X) = 0$,
(F2) $A \leq B \Rightarrow \mathscr{F}(A) \leq \mathscr{F}(B), \forall A, B \in L^X$,
(F3) $\mathscr{F}(A) \wedge \mathscr{F}(B) \leq \mathscr{F}(A \wedge B), \forall A, B \in L^X$,
(F4) $a \otimes \mathscr{F}(A) \leq \mathscr{F}(a \otimes A), \forall A \in L^X, a \in L$.

In [4], Fang observes that a stratified L-filter $\mathscr{F} : L^X \longrightarrow L$ satisfies

$$\forall A, B \in L^X, \mathscr{F}(A) \otimes S(A, B) \leq \mathscr{F}(B).$$

Thus stratified L-filters could be considered as examples of being compatible with the fuzzy inclusion order $S(-,-)$ of L-subsets.

Proposition 2.10. [16] *Let (X, P) be an L-partially ordered set. The following statements are equivalent:*
(1) (X, P) is complete;
(2) For any $A \in L^X, \sup A$ exist; (3) For any $A \in L^X, \inf A$ exist.

3. Union-preserving System of Elements and Representation Theorem

In this section, (X, P) is always assumed to be a fuzzy complete lattice, which means that $(X, \leq_P)$ is a complete lattice. When establishing a representation theorem, there are different tools to fit different objectives. In order to establish the representation theorem of this paper, we need to propose a new tool, called the system of level-elements.

Definition 3.1. Let $A : (X, \leq_P) \longrightarrow L$ be an L-subset. For each $a \in L$, an element $x_A(a)$ of X defined by

$$x_A(a) = \bigwedge \{x \mid A(x) \geq a\}$$

is called $a-$level element of A. $\{x_A(a)\}_{a \in L}$ is called the system of level-elements of A.

Following Definition 3.1, we observe that there is the system of level-elements $\{x_A(a)\}_{a \in L}$ determined by an L-subset A. Naturally, we could ask whether an L-subset A could be represented by its system of level-elements. In order to answer this problem, we need to introduce the following definition, and then explore the condition that an L-subset A is represented by its system of level-elements.

Definition 3.2. Let $A : (X, \leq_P) \longrightarrow L$ be an L-subset, $\{x_A(a)\}_{a \in L}$ be the system of level-elements. Then we say that A can be represented by $\{x_A(a)\}_{a \in L}$ if

$$A(x) = \bigvee \{a \in L \mid x_A(a) \leq_P x\}$$

holds for all $x \in X$.

Now, let us explore by the lemma below, the sufficient and necessary condition that an $L-$subset is represented by its system of level-elements.

Lemma 3.3. *Let $A : (X, \leq_P) \longrightarrow L$ be an L-subset, $\{x_A(a)\}_{a\in L}$ be the system of level elements of A. Then A can be represented by $\{x_A(a)\}_{a\in L}$ iff A is an intersection-preserving map in the sense that $A(\bigwedge_{i\in I} x_i) = \bigwedge_{i\in I} A(x_i)$ holds for all $\{x_i\}_{i\in I} \subseteq X$.*

Proof. Necessity. For $\{x_i\}_{i\in I} \subseteq X$, putting $b = \bigwedge_{i\in I} A(x_i)$, then $x_A(b) \leq_P x_i$ holds for all $i \in I$. This shows $x_A(b) \leq_P \bigwedge_{i\in I} x_i$. Further, since $A : (X, \leq_P) \longrightarrow L$ can be represented by $\{x_A(a)\}_{a\in L}$, it follows that

$$A(\bigwedge_{i\in I} x_i) = \bigvee\{a \,|\, x_A(a) \leq_P \bigwedge_{i\in I} x_i\} \geq b = \bigwedge_{i\in I} A(x_i).$$

On the other hand, for each $i \in I$, it follows from $\bigwedge_{i\in I} x_i \leq_P x_i$ that

$$\begin{aligned} A(\bigwedge_{i\in I} x_i) &= \bigvee\{a \,|\, x_A(a) \leq_P \bigwedge_{i\in I} x_i\} \\ &\leq \bigvee\{a \,|\, x_A(a) \leq_P x_i\} \\ &= A(x_i), \end{aligned}$$

which shows $A(\bigwedge_{i\in I} x_i) \leq \bigwedge_{i\in I} A(x_i)$. Hence, $\bigwedge_{i\in I} A(x_i) = A(\bigwedge_{i\in I} x_i)$ holds.

Sufficiency. On one hand, for all $a \in L$, we have

$$A(x_A(a)) = A(\bigwedge_{A(x)\geq a} x) = \bigwedge_{A(x)\geq a} A(x) \geq a.$$

Further, for all $x \in X$ with $x_A(a) \leq_P x$, $A(x) \geq A\big(x_A(a)\big) \geq a$ always holds. So

$$A(x) \geq \bigvee\{a \in L \,|\, x_A(a) \leq_P x\}.$$

On the other hand, for each $x \in X$, putting $A(x) = c$, then $x_A(c) \leq_P x$. Thus,

$$A(x) = c \leq \bigvee\{a|\; x_A(a) \leq_P x\}.$$

Hence, we obtain $A(x) = \bigvee\{a \,|\, x_A(a) \leq_P x\}$ for every $x \in X$, which means A can be represented by $\{x_A(a)\}_{a\in L}$. □

In the following, we will explore the fact that every intersection-preserving L-subset could be represented by a kind of union-preserving system of elements. For this, a concept of union-preserving system of elements is proposed in the following definition.

Among them, $\top_X$ and $\perp_X$ respectively denote the greatest element and the least element of $(X, \leq_P)$.

Definition 3.4. Let $H : L \longrightarrow (X, \leq_P)$ be a map. If H satisfies the following conditions:
(H1) If non-void set $M \subseteq L$, then $H(\bigvee M) = \bigvee_{a\in M} H(a)$,
(H2) $H(0) = \perp_X$,
then H is called a union-preserving system of elements. The family of all union-preserving system of elements will be denoted by $H_L(X, \leq_P)$.

Let $H, G \in H_L(X, \leq_P)$. We define a partial order "$\leq_H$" on $H_L(X, \leq_P)$ as follows:

$$H \leq_H G \Leftrightarrow \forall a \in L,\, G(a) \leq_P H(a).$$

Then $H_L(X, \leq_P)$ has the greatest element $H^1 : L \longrightarrow (X, \leq_P)$ defined by $H^1(a) = \perp_X$ for all $a \in L$ and the least element $H^0 : L \longrightarrow (X, \leq_P)$ defined by

$$H^0(a) = \begin{cases} \perp_X,\ a = 0, \\ \top_X,\ a \neq 0. \end{cases}$$

Thus both H^0 and H^1 are considered as example of union-preserving system of elements. Moreover,the following example also shows that the system of level elements of an intersection-preserving L-subset is a union-preserving system of elements, that is to say a union-preserving system of elements defined above exists besides H^0 and H^1.

Example 3.5. Let $A : (X, \leq_P) \longrightarrow L$ be an L-subset which preserves arbitrary intersections. For a mapping $H_A : L \longrightarrow (X, \leq_P)$ defined by $H_A(a) = x_A(a)$, where $a \in L$, then $H_A : L \longrightarrow (X, \leq_P)$ is a canonical example of union-preserving system of elements.

Proof. Obviously, by Definition 3.1, H_A satisfies (H2). We verify that H_A fulfills (H1). Let $M \subseteq L$ be a non-void set. Then $H_A(\bigvee M) = \bigvee_{a\in M} H_A(a)$ follows from the following two steps:

Step 1: For any $a \in M$, we have

$$\begin{aligned} H_A(a) = x_A(a) &= \bigwedge\{x \mid A(x) \geq a\} \\ &\leq_P \bigwedge\{x \mid A(x) \geq \bigvee M\} \\ &= x_A(\bigvee M) \\ &= H_A(\bigvee M), \end{aligned}$$

which means $\bigvee_{a\in M} H_A(a) \leq H_A(\bigvee M)$.

Step 2: Under the condition of A being an intersection-preserving map, we observe that for each $b \in M$,

$$\begin{aligned} A(\bigvee_{a\in M} H_A(a)) &\geq A(H_A(b)) = A(x_A(b)) \\ &= A(\bigwedge_{A(x)\geq b} x) = \bigwedge_{A(x)\geq b} A(x) \\ &\geq b. \end{aligned}$$

Hence, $A(\bigvee_{a\in M} H_A(a)) \geq \bigvee M$, which means $H_A(\bigvee M) = x_A(\bigvee M) \leq_P \bigvee_{a\in M} H_A(a)$.

To sum up, by Step 1,2, we have proved $H_A(\bigvee M) = \bigvee_{a\in M} H_A(a)$, as desired. □

Proposition 3.6. *If $(X, \leq_P)$ is a complete lattice, then $(H_L(X, \leq_P), \leq_H)$ is a complete lattice.*

Proof. To prove that $(H_L(X, \leq_P), \leq_H)$ is a complete lattice, our strategy is to show that $(H_L(X, \leq_P), \leq_H)$ has the greatest element and each non-void set $\{H_t\}_{t\in T} \subseteq H_L(X, \leq_P)$ has an infimum $\bigwedge_{t\in T} H_t$ w.r.t. $\leq_H$.

Let $\{H_t\}_{t\in T} \in H_L(X, \leq_P)$. A map, denoted by $H : L \longrightarrow (X, \leq_P)$, is determined by

$$H(a) = \bigvee_{t\in T} H_t(a),\ \forall a \in L.$$

Firstly, to show $H \in H_L(X, \leq_P)$, we have to show that H satisfies (H1)and (H2) in Definition 3.4. Obviously, it follows from the definition of H that H satisfies the axiom (H2). Now, We demonstrate H satisfies the axiom (H1) as follows. In fact, we observe that

$$\begin{aligned} H(\bigvee M) &= \bigvee_{t\in T} H_t(\bigvee M) = \bigvee_{t\in T}\bigvee_{a\in M} H_t(a) \\ &= \bigvee_{a\in M}\bigvee_{t\in T} H_t(a) = \bigvee_{a\in M} H(a). \end{aligned}$$

From all above, we conclude $H \in H_L(X, \leq_P)$.

Next we show that H is the infimum of that $\{H_t | t \in T\}$.

i) For every $a \in L$ and $t \in T$, $H_t(a) \leq_P \bigvee_{t\in T} H_t(a) = H(a)$ holds, which means $\bigwedge_{t\in T} H_t \leq_H H_t$. Therefore, H is a lower bound of $\{H_t | t \in T\}$.

ii) Let G is a lower bound of $\{H_t | t \in T\}$. Thus $G \leq_H H_t$ holds for all $t \in T$, which means that $\forall a \in L, H(a) = \bigvee H_t(a) \leq_P G(a)$ is true, and equivalently, $G \leq_H H$. To sum up, H is the greatest lower bound of $\{H_t | t \in T\}$. That is to say H is the infimum of $\{H_t | t \in T\}$ in symbol $H = \bigvee_{t\in T} H_t$.

To sum up, $(H_L(X, \leq_P), \leq_H)$ is a complete lattice. □

In the following, the set of all L-subsets which preserve arbitrary intersections is denoted by $H(L^{(X,\leq_P)})$. Now, we list our main result in the following theorem.

Theorem 3.7. *Let (X, P) be a fuzzy complete lattice and $f : H_L(X, \leq_P) \longrightarrow H(L^{(X,\leq_P)})$ be a map defined by*

$$\forall H \in H_L(X, \leq_P), \forall x \in X,\ f(H)(x) = \bigvee\{a \in L \,|\, H(a) \leq_P x\}.$$

Then f is surjective and order-embedding, i.e. $(H_L(X, \leq_P), \wedge, \vee)$ is order-isomorphic to $(H(L^{(X,\leq_P)}), \wedge, \vee)$.

Proof. Firstly, we confirm that f is surjective. Since $A \in H(L^{(X,\leq_P)}), H_A \in H_L(X, \leq_P)$ holds (see Example 3.5). By Lemma 3.3, it follows that

$$\begin{aligned} f(H_A)(x) &= \bigvee\{a \in L \,|\, H_A(a) \leq_P x\} \\ &= \bigvee\{a \in L \,|\, x_A(a) \leq_P x\} \\ &= A(x). \end{aligned}$$

Next, we show that f is an order-embedding, in the sense that f fulfills "$H \leq_H G$ iff $f(H) \leq f(G)$" for all $H, G \in H_L(X, \leq_P)$.

(1) Let $H, G \in H_L(X, \leq_P)$ satisfy $H \leq_H G$,. Then $G(a) \leq_P H(a)$ for all $a \in L$. Further, for any $x \in X$,

$$\begin{aligned} f(H)(x) &= \bigvee\{a \in L \mid H(a) \leq_P x\} \\ &\leq \bigvee\{a \in L \mid G(a) \leq_P x\} \\ &= f(G)(x) \end{aligned}$$

holds. Hence, we obtain $f(H) \leq f(G)$.
(2) If $H, G \in H_L(X, \leq_P)$ satisfy $f(H) \leq f(G)$, then $f(H)(x) \leq f(G)(x)$ for all $x \in X$, i.e. $\bigvee\{a \in L \mid H(a) \leq_P x\} \leq \bigvee\{a \in L \mid G(a) \leq_P x\}$. Note that, for each $c \in L$ with $H(c) \leq_P x$, we have

$$c \leq \bigvee\{a \in L \mid H(a) \leq_P x\} \leq \bigvee\{a \in L \mid G(a) \leq_P x\}.$$

Put $M = \{a \in L \mid G(a) \leq_P x\}$. As G is a union-preserving system of elements, we have

$$G(c) \leq_P G(\bigvee M) = \bigvee_{b \in M} G(b) \leq_P x.$$

Therefore, $G(c) \leq_P H(c)$, i.e. $H \leq_H G$.

To sum up, f is surjective and order-embedding, i.e. f is a order-isomophism between $(H_L(X, \leq_P), \wedge, \vee)$ and $(H(L^{(X, \leq_P)}), \wedge, \vee)$. □

Note that Theorem 3.7 establishes the one-to-one correspondence between the set of all intersection-preserving L-subsets and that of all union-preserving systems of elements. That is so called representation theorem of intersection-preserving L-subsets.

4. **The Compatible Union-preserving System of Elements and Representation Theorem**

In this section, we will show that every compatible intersection-preserving L-subset could be represented by a kind of compatible union-preserving system of elements. First of all, the concept of compatible union-preserving system of elements is proposed in the following definition.

Definition 4.1. Let (X, P) be a fuzzy complete lattice, $H : L \longrightarrow (X, \leq_P)$ be the union-preserving system of elements. If in addition, H satisfies:

$$\forall x, y \in X, a \in L, H(a) \leq_P x \Longrightarrow H(a \otimes P(x, y)) \leq_P y.$$

Then H is called compatible w.r.t. P. The family of all compatible union-preserving systems of elements will be denoted by $G_L(X, P)$.

Remark 4.2. Let $A \in L^{(X,P)}$. If A satisfies

$$\forall \{x_i\}_{i \in I} \subseteq X, A(\bigwedge_{i \in I} x_i) = \bigwedge_{i \in I} A(x_i),$$

then A is called compatible intersection-preserving L-subset.

In the following, the set of all the compatible intersection-preserving L-subsets is denoted by $G(L^{(X,P)})$. The example below shows that the compatible union-preserving system of elements exists.

Example 4.3. Let $A \in G(L^{(X,P)})$. For each $a \in L$, we define a map $H_A : L \longrightarrow (X, \leq_P)$ by $H_A(a) = x_A(a)$. Then $H_A : L \longrightarrow (X, \leq_P)$ is a canonical example of compatible union-preserving system of elements.

Proof. By Example 3.5, we know that H_A is a union-preserving system of elements.

Hence we have to demonstrate that H_A is compatible w.r.t. P. For this purpose, let x and y be any two element in X and $a \in L$ such that $H_A(a) \leq_P x$. Our object is to show $H_A(a \otimes P(x, y)) \leq_P y$.

It follows from $H_A(a) = x_A(a) \leq_P x$, or equivalently $P(H_A(a), x) = 1$ and $A \in G(L^{(X,P)})$ that

$$A(x_A(a)) = A(H_A(a)) = A(H_A(a) \otimes P(H_A(a), x)) \leq A(x),$$

i.e. $A(x_A(a)) \leq A(x)$.

Hence, by using the property of A being intersection-preserving L-subset, we have

$$A(x) \geq A(x_A(a)) = A(\bigwedge_{A(x)\geq a} x) = \bigwedge_{A(x)\geq a} A(x) \geq a.$$

Finally, by using the property of A being compatible L-subset, we observe that

$$A(y) \geq A(x) \otimes P(x, y) \geq a \otimes P(x, y),$$

which already means that

$$H_A(a \otimes P(x, y)) = x_A(a \otimes P(x, y)) \leq_P y,$$

i.e. $H_A(a \otimes P(x, y)) \leq_P y$ as desired. □

Note that $(G_L(X, P), \leq_H)$ has the greatest element H^1, and each non-void family $\{H_t\}_{t\in T} \subseteq G_L(X, P)$ has the infimum $\bigvee_{t\in T} H_t$ w.r.t. $\leq_H$ in the sense of

$$\forall a \in L, (\bigvee_{t\in T} H_t)(a) = \bigvee_{t\in T} H_t(a).$$

From these facts, we obtain the following proposition.

Proposition 4.4. *Let (X, P) be a fuzzy complete lattice, then $(G_L(X, P), \leq_H)$ is complete lattice.*

In order to obtain the new representation theorem of compatible intersection-preserving L-subsets, we need the following lemma for preparation.

Lemma 4.5. *Let (X, P) be a fuzzy complete lattice and $H \in H_L(X, \leq_P)$. If H is compatible w.r.t. P, then $f(H)$ (defined in Theorem 3.7) is compatible w.r.t. P.*

Proof. Let H be compatible w.r.t. P. We observe that for all $x, y \in X$,

$$\begin{aligned} f(H)(x) \otimes P(x,y) &= \bigvee\{a \in L \mid H(a) \leq_P x\} \otimes P(x,y) \\ &= \bigvee\{a \otimes P(x,y) \mid H(a) \leq_P x\} \\ &\leq \bigvee\{a \otimes P(x,y) \mid H(a \otimes P(x,y)) \leq_P y\} \\ &\leq \bigvee\{b \in L \mid H(b) \leq_P y\} \\ &= f(H)(y). \end{aligned}$$

From this, it follows from Definition 2.8 that $f(H)$ is compatible L-subset. □

At the end of this section, according to the proof of Example 3.7 and combining Example 4.3, Lemma 4.5, we obtain the representation theorem of compatible intersection-preserving L-subsets based on the compatible union-preserving system of elements.

Theorem 4.6. *Let (X, P) be a fuzzy complete lattice. Then there exists an isomorphism between $(G_L(X,P), \vee, \wedge)$ and $(G(L^{(X,P)}), \vee, \wedge)$, i.e.*

$$(G_L(X,P), \vee, \wedge) \cong (G(L^{(X,P)}), \vee, \wedge).$$

Proposition 4.7. *Let (X, P) be a fuzzy lattice. $(G_L(X,P), \vee, \wedge)$ is a fuzzy complete lattice.*

Proof. Since $(G_L(X,P), \vee, \wedge) \cong (G(L^{(X,P)}), \vee, \wedge)$, our strategy is to show that $(G(L^{(X,P)}), S^*)$ is a fuzzy complete lattice, here $S^* : G(L^{(X,P)}) \times G(L^{(X,P)}) \longrightarrow L$ given by $S^*(A, B) = S(A, B)$, for all $A, B \in G(L^{(X,P)})$.

For each L-family $\Re : G(L^{(X,P)}), \longrightarrow L$, putting $A = \bigwedge_{B \in G(L^{(X,P)})} \Re(B) \to B$. Firstly, to show $A \in G(L^{(X,P)})$, it follows that

Step 1: For $\{x_i\}_{i \in I} \subseteq X$,

$$\begin{aligned} A(\bigwedge_{i\in I} x_i) &= \bigwedge_{B \in G(L^{(X,P)})} (\Re(B) \to B)(\bigwedge_{i \in I} x_i) \\ &= \bigwedge_{B \in G(L^{(X,P)})} \Re(B) \to B(\bigwedge_{i \in I} x_i) \\ &= \bigwedge_{B \in G(L^{(X,P)})} \Re(B) \to \bigwedge_{i \in I} B(x_i) \\ &= \bigwedge_{B \in G(L^{(X,P)})} \bigwedge_{i \in I} \Re(B) \to B(x_i) \\ &= \bigwedge_{i \in I} \bigwedge_{B \in G(L^{(X,P)})} \Re(B) \to B(x_i) \\ &= \bigwedge_{i \in I} A(x_i). \end{aligned}$$

Step 2: For any $x, y \in X$,

$$\begin{aligned} A(x) \otimes P(x,y) &= \Big(\bigwedge_{B \in G(L^{(X,P)})} (\Re(B) \to B)\Big) \otimes P(x,y) \\ &= \bigwedge_{B \in G(L^{(X,P)})} \Re(B) \to B \otimes P(x,y) \\ &\leq \bigwedge_{B \in G(L^{(X,P)})} \Re(B) \to B(y) \\ &= A(y). \end{aligned}$$

Hence, by step 1 and step 2, we know that $A \in G(L^{(X,P)})$.

Next to show that A satisfies (1) and (2) in Definition 2.8. We verify (1)(2) as follows:

(1)For $B \in G(L^{(X,P)})$,

$$\begin{aligned} S(A,B) &= \bigwedge_{x \in X} \Big(\bigwedge_{C \in G(L^{(X,P)})} \Re(C) \to C(x)\Big) \to B(x) \\ &\geq \bigwedge_{x \in X} (\Re(B) \to B(x)) \to B(x) \\ &\geq \Re(B). \end{aligned}$$

(2)For $\forall C \in G(L^{(X,P)})$,

$$\begin{aligned} \bigwedge_{B \in G(L^{(X,P)})} \Re(B) \to S(C,B) &= \bigwedge_{B \in G(L^{(X,P)})} \Re(B) \to \bigwedge_{x \in X} C(x) \to B(x) \\ &= \bigwedge_{x \in X} \bigwedge_{B \in G(L^{(X,P)})} \Re(B) \to (C(x) \to B(x)) \\ &= \bigwedge_{x \in X} \bigwedge_{B \in G(L^{(X,P)})} C(x) \to (\Re(B) \to B(x)) \\ &= \bigwedge_{x \in X} C(x) \to \bigwedge_{B \in G(L^{(X,P)})} (\Re(B) \to B(x)) \\ &= \bigwedge_{x \in X} C(x) \to A(x) \\ &= S(C,A). \end{aligned}$$

To sum up, by Proposition 2.10, we have proved $(G(L^{(X,P)}), S^*)$ is a fuzzy complete lattice i.e. $(G_L(X,P)$ is a fuzzy complete lattice. □

5. **Conclusion**

The research of representation theorem of L-subsets is an important part of the theory of fuzzy set. In previous literature, it is not required that domain X is endowed with L-partial order P. But in some areas of fuzzy mathematics, the domain X is often endowed with L-partial order P. Therefore, L-subsets considered in the present paper show up as a map from L-partially ordered set (X,P) to L and their "level structure" show up as a map from L to $(X, \leq_P)$. In this paper, a new representation theorem of intersection-preserving L-subsets is given by using union-preserving system of elements. At the same time, another representation theorem of compatible intersection-preserving L-subsets is obtained by means of compatible union-preserving system of elements. Hence, we predict that these representation theorems will be main methods to define intersection-preserving L-subsets and compatible intersection-preserving L-subsets by using union-preserving system of elements and compatible union-preserving system of elements, respectively.

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