

## ESTIMATING THE PARAMETERS OF A FUZZY LINEAR REGRESSION MODEL

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ABSTRACT. Fuzzy linear regression models are used to obtain an appropriate linear relation between a dependent variable and several independent variables in a fuzzy environment. Several methods for evaluating fuzzy coefficients in linear regression models have been proposed. The first attempts at estimating the parameters of a fuzzy regression model used mathematical programming methods. In this thesis, we generalize the metric defined by Diamond and use it as a criterion to estimate these parameters. Our method, is not only computationally easy to handle, but, when compared with earlier methods, has a smaller the sum of errors of estimation.

### 1. Introduction

Classical regression analysis is helpful in ascertaining the probable form of the relationship between variables, and usually the ultimate objective is to predict, or estimate, the value of one variable corresponding to a given value of another variable. The method usually employed for obtaining the “regression surface” is known as the method of least squares and the parameters are estimated by minimizing the sum of squares of the difference between observed and predicted values.

A fuzzy linear regression model was first introduced by Tanaka et al. [8]. They formulated a linear regression model with fuzzy response data, crisp predictor data and fuzzy parameters as a mathematical programming problem. Their approach was later improved by Tanaka and others [9, 10, 11, 12]. Diamond [1] proposed the fuzzy least squares approach to determine fuzzy parameters by defining a metric between two fuzzy numbers. However most of the articles on fuzzy regression analysis until now use linear programming to estimate the parameters. When using this approach, each additional observation results in several additional constraints, and the linear programming problem becomes unwieldy very quickly, especially if the fuzzy triangular numbers involved are not symmetric. Moreover, trapezoidal fuzzy numbers are not even considered because of the complexity of computation.

We use the metric defined by Diamond to estimate the parameters of a fuzzy linear regression model in both simple and multivariate cases. The estimators obtained by our method are in explicit form, and computations are very simple. In fact, by generalizing Diamond’s metric to trapezoidal fuzzy numbers, we also obtain estimators for the case when either the predictor variables or the parameters are trapezoidal fuzzy numbers.

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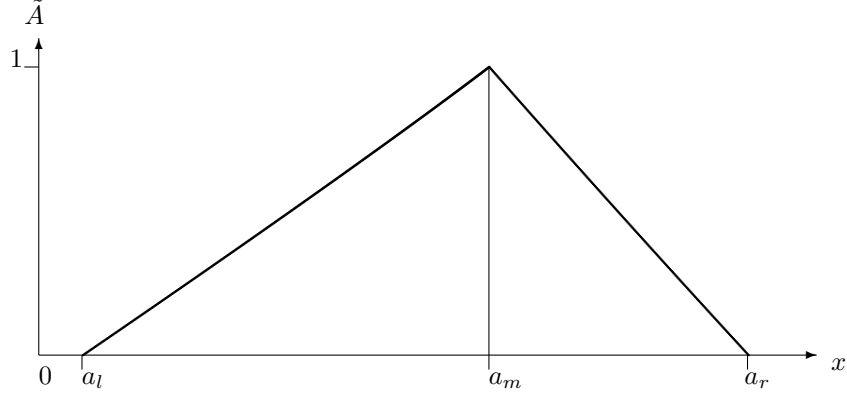


FIGURE 1. Triangular fuzzy number.

This paper is organized as follows: in Section 2 we recall the definition of distance between two triangular fuzzy numbers as given by Diamond [1] and generalize it to trapezoidal fuzzy numbers. In Section 3 we estimate the parameters of several fuzzy regression models and compare our estimations with those previously in Section 4. Finally, Section 5 provides a conclusion.

## 2. A Fuzzy Topology

**2.1. Preliminaries and Definitions.** Let  $\mathcal{F}(\mathbb{R})$  denote the set of normalized fuzzy numbers: that is, the set of upper semicontinuous convex functions  $\tilde{X} : \mathbb{R} \rightarrow [0, 1]$  such that  $\{x \in \mathbb{R} : \tilde{X}(x) = 1\}$  is nonempty. Denote the set of fuzzy numbers with compact  $\alpha$ -level sets  $\tilde{X}_\alpha$ , for  $\alpha > 0$ , by  $\mathcal{F}_c(\mathbb{R})$ . Now define a metric  $d^*$  on  $\mathcal{F}_c(\mathbb{R})$  by  $d^*(\tilde{X}, \tilde{Y}) = \sup_{\alpha > 0} d_H(X_\alpha, Y_\alpha)$ , where the Hausdorff metric  $d_H$  is given by

$$d_H(A, B) = \max\left(\sup_{a \in A} \inf_{b \in B} \|a - b\|, \sup_{b \in B} \inf_{a \in A} \|a - b\|\right). \quad (1)$$

The metric space  $(\mathcal{F}_c(\mathbb{R}), d^*)$  is complete [6].

In the usual notation if  $\tilde{A}$  is triangular, we write  $\tilde{A} = (a_l, a_m, a_r)$ , where  $a_m$  is the center, and  $a_l$  and  $a_r$  are respectively the left and right points of the triangular fuzzy number  $\tilde{A}$  (Figure 1). Also if  $\tilde{A}$  is trapezoidal, we write  $\tilde{A} = (a_l, a_m, a_u, a_r)$ , where  $a_l$  and  $a_r$  are the left and right “end” points of the corresponding trapezoid, and  $a_m$  and  $a_u$  are the left and right “middle” points (Figure 2). If  $a_m = a_u$ ,  $\tilde{A}$  reduces to the triangular fuzzy number  $(a_l, a_m, a_r)$ .

Let  $\mathcal{T}(\mathbb{R})$  denote the set of trapezoidal fuzzy numbers, and  $\mathcal{P}(\mathbb{R})$  be the subspace of  $\mathcal{T}(\mathbb{R})$  all of whose elements have nonnegative support: that is, for each  $\tilde{A} = (a_l, a_m, a_u, a_r)$ ,  $a_l > 0$ . Then  $\mathcal{P}(\mathbb{R})$  is a cone in  $\mathcal{T}(\mathbb{R})$  and is a closed convex subset of  $\mathcal{T}(\mathbb{R})$ .

It may be shown [1] that for two triangular fuzzy numbers  $\tilde{A}_1 = (a_{1l}, a_{1m}, a_{1r})$  and  $\tilde{A}_2 = (a_{2l}, a_{2m}, a_{2r})$ , using a suitable norm, (1) reduces to  $d(\tilde{A}_1, \tilde{A}_2)$ , where

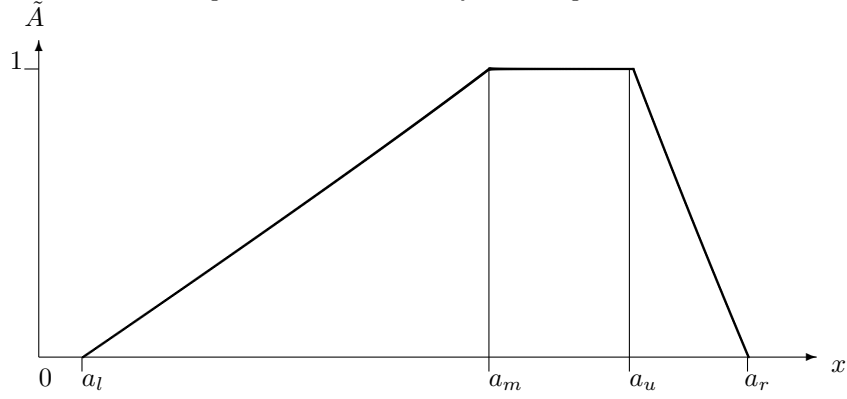


FIGURE 2. Trapezoidal fuzzy number.

$$d^2(\tilde{A}_1, \tilde{A}_2) = (a_{1l} - a_{2l})^2 + (a_{1m} - a_{2m})^2 + (a_{1r} - a_{2r})^2. \quad (2)$$

It is obvious that this distance satisfies the usual distance properties i.e.:

- $d(\tilde{A}_1, \tilde{A}_2) \geq 0$ ,
- $d(\tilde{A}_1, \tilde{A}_2) = 0 \iff \tilde{A}_1 = \tilde{A}_2$ , (i.e.,  $a_{1l} = a_{2l}$ ,  $a_{1m} = a_{2m}$  and  $a_{1r} = a_{2r}$ ),
- $d(\tilde{A}_1, \tilde{A}_2) \leq d(\tilde{A}_1, \tilde{A}_3) + d(\tilde{A}_3, \tilde{A}_2)$ .

We may generalize this definition of distance between two trapezoidal fuzzy numbers  $\tilde{A}_1 = (a_{1l}, a_{1m}, a_{1u}, a_{1r})$  and  $\tilde{A}_2 = (a_{2l}, a_{2m}, a_{2u}, a_{2r})$ , as follows:

$$d^2(\tilde{A}_1, \tilde{A}_2) = (a_{1l} - a_{2l})^2 + 0.5(a_{1m} - a_{2m})^2 + 0.5(a_{1u} - a_{2u})^2 + (a_{1r} - a_{2r})^2. \quad (3)$$

Note that if  $a_{1m} = a_{1u}$  and  $a_{2m} = a_{2u}$ , i.e. if both numbers are triangular, this distance reduces to (2.1).

**2.2. Existence of Minimizing Fuzzy Numbers.** Let  $\mathcal{V}$  be a cone in  $\mathcal{P}(\mathbb{R})$ , and let  $\tilde{X} = (x_l, x_m, x_u, x_r) \in \mathcal{P}(\mathbb{R})$ . Write the elements of  $\mathcal{V}$  as  $\tilde{V} = (v_l, v_m, v_u, v_r)$ . Then the supports of  $\tilde{X}$  and  $\tilde{V}$  are closed intervals  $[x_l, x_r]$  and  $[v_l, v_r]$ , respectively. If there exists a trapezoidal fuzzy numbers  $\tilde{V}_0 \in \mathcal{V}$  such that for every  $\tilde{V} \in \mathcal{V}$ ,

$$\begin{aligned} & (v_{0l} - v_l)(x_l - v_{0l}) + (v_{0r} - v_r)(x_r - v_{0r}) \\ & + 0.5(v_{0m} - v_m)(x_m - v_{0m}) + 0.5(v_{0u} - v_u)(x_u - v_{0u}) \geq 0, \end{aligned} \quad (4)$$

then  $\tilde{X}$  is said to be  $\tilde{V}_0$ -orthogonal to  $\mathcal{V}$ .

**Theorem 2.1.** *Let  $\mathcal{V}$  be a closed cone in  $\mathcal{P}(\mathbb{R})$ . For any  $\tilde{X}$  in  $\mathcal{P}(\mathbb{R})$ , there is a unique trapezoidal fuzzy number  $\tilde{V}_0$  in  $\mathcal{V}$ , such that  $d(\tilde{X}, \tilde{V}_0) \leq d(\tilde{X}, \tilde{V})$  for all  $\tilde{V}$  in  $\mathcal{V}$ . A necessary and sufficient condition for  $\tilde{V}_0$  to be the unique minimizing fuzzy number in  $\mathcal{V}$  is that  $\tilde{X}$  is  $\tilde{V}_0$ -orthogonal to  $\mathcal{V}$ .*

The following two lemmas facilitate the proof of Theorem 2.1.

**Lemma 2.2.** For  $\tilde{A}, \tilde{B}$  in  $\mathcal{T}(\mathbb{R})$  and the metric defined in (3) we have

$$d^2(\tilde{A}, \tilde{B}) = 2d^2(\tilde{A}, \tilde{X}) + 2d^2(\tilde{X}, \tilde{B}) - 4d^2(\tilde{X}, \frac{\tilde{A} + \tilde{B}}{2})$$

*Proof.* The proof follows immediately from the identity:

$$(a - b)^2 = 2(a - x)^2 + 2(b - x)^2 - 4(x - \frac{a + b}{2})^2, \quad a, b, x \in \mathbb{R}.$$

□

**Lemma 2.3.** Let  $\mathcal{V}$  be a closed cone in  $\mathcal{P}(\mathbb{R})$ , and  $\tilde{X}$  an arbitrary element of  $\mathcal{P}(\mathbb{R})$ . If there is an element  $\tilde{V}_0$  in  $\mathcal{V}$  such that  $d(\tilde{X}, \tilde{V}_0) \leq d(\tilde{X}, \tilde{V})$  for all  $\tilde{V}$  in  $\mathcal{V}$ , then  $\tilde{V}_0$  is unique. A necessary and sufficient condition for  $\tilde{V}_0$  to be a unique minimizing fuzzy number in  $\mathcal{V}$  is that  $\tilde{X}$  is  $\tilde{V}_0$ -orthogonal to  $\mathcal{V}$ .

*Proof.* prove sufficiency:

If (4) is satisfied for all  $\tilde{V}$  in  $\mathcal{V}$ , and  $\tilde{V}$  is any element in  $\mathcal{V}$  distinct from  $\tilde{V}_0$ , then

$$\begin{aligned} d^2(\tilde{X}, \tilde{V}) &= (x_l - v_l)^2 + 0.5(x_m - v_m)^2 + 0.5(x_u - v_u)^2 + (x_r - v_r)^2 \\ &= (x_l - v_{0l} + v_{0l} - v_l)^2 + 0.5(x_m - v_{0m} + v_{0m} - v_m)^2 \\ &\quad + 0.5(x_u - v_{0u} + v_{0u} - v_u)^2 + (x_r - v_{0r} + v_{0r} - v_r)^2 \\ &= (x_l - v_{0l})^2 + 0.5(x_m - v_{0m})^2 + 0.5(x_u - v_{0u})^2 + (x_r - v_{0r})^2 \\ &\quad + (v_l - v_{0l})^2 + 0.5(v_m - v_{0m})^2 + 0.5(v_u - v_{0u})^2 + (v_r - v_{0r})^2 \\ &\quad + (v_{0l} - v_l)(x_l - v_{0l}) + (v_{0r} - v_r)(x_r - v_{0r}) \\ &\quad + 0.5(v_{0m} - v_m)(x_m - v_{0m}) + 0.5(v_{0u} - v_u)(x_u - v_{0u}), \end{aligned}$$

therefore

$$\begin{aligned} d^2(\tilde{X}, \tilde{V}) &= d^2(\tilde{X}, \tilde{V}_0) + d^2(\tilde{V}, \tilde{V}_0) \\ &\quad + 2\{(v_{0l} - v_l)(x_l - v_{0l}) + (v_{0r} - v_r)(x_r - v_{0r}) \\ &\quad + 0.5(v_{0m} - v_m)(x_m - v_{0m}) + 0.5(v_{0u} - v_u)(x_u - v_{0u})\}, \end{aligned}$$

and by (4), since  $d^2(\tilde{V}, \tilde{V}_0) > 0$ , we see that for  $\tilde{V} \neq \tilde{V}_0$ ,  $d^2(\tilde{X}, \tilde{V}_0) < d^2(\tilde{X}, \tilde{V})$ .

To prove necessity, suppose that for some  $\tilde{V}$  in  $\mathcal{V}$

$$\begin{aligned} &(v_{0l} - v_l)(x_l - v_{0l}) + (v_{0r} - v_r)(x_r - v_{0r}) \\ &+ 0.5(v_{0m} - v_m)(x_m - v_{0m}) + 0.5(v_{0u} - v_u)(x_u - v_{0u}) = -\lambda, \quad 0 < \lambda < 1. \end{aligned}$$

Without loss of generality, suppose that  $d(\tilde{V}, \tilde{V}_0) = 1$ . Consider  $\tilde{V}_1 = (1 - \lambda)\tilde{V}_0 + \lambda\tilde{V}$ , which is in  $\mathcal{V}$  by convexity. We now have

$$\begin{aligned} d^2(\tilde{X}, \tilde{V}_1) &= d^2(\tilde{X}, \tilde{V}_0) + \lambda^2 d^2(\tilde{V}, \tilde{V}_0) \\ &\quad + 2\lambda\{(v_{0l} - v_l)(x_l - v_{0l}) + (v_{0r} - v_r)(x_r - v_{0r}) \\ &\quad + 0.5(v_{0m} - v_m)(x_m - v_{0m}) + 0.5(v_{0u} - v_u)(x_u - v_{0u})\}, \end{aligned}$$

therefore

$$d^2(\tilde{X}, \tilde{V}_1) = d^2(\tilde{X}, \tilde{V}_0) - \lambda^2,$$

and  $\tilde{V}_0$  is not a minimizing element of  $\mathcal{V}$ . □

*Proof.* We prove the Theorem 2.1 by using Lemma 2.2 and 2.3. Uniqueness and the sufficiency of  $\tilde{V}_0$ -orthogonality follows from Lemma 2.3. It remains to show the existence of  $\tilde{V}_0$ . If  $\tilde{X}$  is in  $\mathcal{V}$  there is nothing to prove. Now assume  $\tilde{X}$  is not in  $\mathcal{V}$ , and let  $\delta = \inf\{d(\tilde{X}, \tilde{V}) : \tilde{V} \in \mathcal{V}\}$ . Let  $\tilde{V}_i$  be a sequence of trapezoidal fuzzy numbers in  $\mathcal{V}$  such that  $d(\tilde{X}, \tilde{V}_i) \rightarrow \delta$ . By Lemma 2.2,

$$d^2(\tilde{V}_i, \tilde{V}_j) = 2d^2(\tilde{V}_i, \tilde{X}) + 2d^2(\tilde{X}, \tilde{V}_j) - 4d^2\left(\tilde{X}, \frac{\tilde{V}_i + \tilde{V}_j}{2}\right).$$

For all  $i, j$ ,  $\frac{\tilde{V}_i + \tilde{V}_j}{2}$  is in  $\mathcal{V}$  because the cone  $\mathcal{V}$  is convex. Thus  $d(\tilde{X}, \frac{\tilde{V}_i + \tilde{V}_j}{2}) \geq \delta$  and

$$d^2(\tilde{V}_i, \tilde{V}_j) \leq 2d^2(\tilde{V}_i, \tilde{X}) + 2d^2(\tilde{X}, \tilde{V}_j) - 4\delta^2,$$

so  $d^2(\tilde{V}_i, \tilde{V}_j) \rightarrow 0$  as  $i, j \rightarrow \infty$ . Therefore  $\{\tilde{V}_i\}$  is a Cauchy sequence, and since  $\mathcal{T}(\mathbb{R})$  is complete and  $\mathcal{V}$  is closed,  $\tilde{V}_0 = \lim \tilde{V}_i$  is in  $\mathcal{V}$ .  $\square$

**Corollary 2.4.** *Let  $n$  be a positive integer, and suppose that  $\mathcal{V}$  is a closed cone in  $\mathcal{P}(\mathbb{R})^n$ . Denote by  $d_n$  the metric on  $\mathcal{P}(\mathbb{R})^n$  defined by*

$$d_n^2(\tilde{\mathbf{V}}, \tilde{\mathbf{W}}) = \sum_{i=1}^n d^2(\tilde{V}_i, \tilde{W}_i), \quad \tilde{\mathbf{V}}, \tilde{\mathbf{W}} \in \mathcal{P}(\mathbb{R})^n,$$

where  $\tilde{V}_i, \tilde{W}_i \in \mathcal{P}(\mathbb{R}), i = 1, 2, \dots, n$ , are the components of  $\tilde{\mathbf{V}}, \tilde{\mathbf{W}}$ . Then for any  $\tilde{\mathbf{X}}$  in  $\mathcal{P}(\mathbb{R})^n$  there is a unique  $n$ -vector  $\tilde{\mathbf{V}}_0$  in  $\mathcal{V}$  such that  $d_n(\tilde{\mathbf{X}}, \tilde{\mathbf{V}}_0) \leq d_n(\tilde{\mathbf{X}}, \tilde{\mathbf{V}})$  for all  $\tilde{\mathbf{V}}$  in  $\mathcal{V}$ .

*Proof.* The parallelogram-like law of Lemma 2.2 extends to  $d_n$ , so the existence of  $\tilde{\mathbf{V}}_0$  follows from Theorem 2.1. Uniqueness comes from a similar argument to that of Lemma 2.3.  $\square$

### 3. Fuzzy Regression Models

The functional form of the classical simple linear regression model is as follows:

$$Y = \beta_0 + \beta_1 x + \varepsilon, \quad (5)$$

where  $Y$  is the observed response variable,  $x$  is the predictor variable,  $\beta_0$  and  $\beta_1$  are unknown parameters and  $\varepsilon$  is the error term. Hence for  $n$  pairs of observations  $(y_i, x_i)$  we have:

$$y_i = \beta_0 + \beta_1 x_i + \varepsilon_i, \quad i = 1, 2, \dots, n.$$

The least squares estimators of  $\beta_0$  and  $\beta_1$  denoted by  $\hat{\beta}_0$  and  $\hat{\beta}_1$  respectively, minimize the sum of the squared errors  $S(\beta_0, \beta_1) = \sum_{i=1}^n (y_i - \beta_0 - \beta_1 x_i)^2$ , and are as follows:

$$\hat{\beta}_1 = \frac{\sum_{i=1}^n x_i y_i - n\bar{x}\bar{y}}{\sum_{i=1}^n x_i^2 - n\bar{x}^2}, \quad \text{and} \quad \hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x} \quad (6)$$

In some cases we may need to consider that the relationship expressed in (5) may be fuzzy. We shall first consider the case where the predictor variable is fuzzy, but the parameters are crisp, then the case of a crisp predictor and fuzzy parameters and finally the case of a fuzzy predictor and fuzzy parameters. The observed responses

in all cases are naturally fuzzy, and satisfy the fuzzy equations below:

Case 1:

$$\tilde{y}_i = \beta_0 + \beta_1 \tilde{x}_i + \tilde{\varepsilon}_i, \quad i = 1, 2, \dots, n, \quad (7)$$

Case 2:

$$\tilde{y}_i = \tilde{\beta}_0 + \tilde{\beta}_1 x_i + \tilde{\varepsilon}_i, \quad i = 1, 2, \dots, n, \quad (8)$$

Case 3:

$$\tilde{y}_i = \tilde{\beta}_0 + \tilde{\beta}_1 \tilde{x}_i + \tilde{\varepsilon}_i, \quad i = 1, 2, \dots, n, \quad (9)$$

where  $\tilde{y}_i$ 's are the fuzzy responses. In (7), the parameters  $\beta_0$  and  $\beta_1$  are crisp and the  $\tilde{x}_i$ 's are fuzzy, whereas in (8), the parameters  $\tilde{\beta}_0$  and  $\tilde{\beta}_1$  are fuzzy but the  $x_i$ 's are crisp. In (9), the predictors and parameters are all fuzzy.

For a linear regression model, if  $\hat{\beta}_0$  and  $\hat{\beta}_1$  are respectively the estimates of the parameters  $\beta_0$  and  $\beta_1$ , then the predicted value of  $y$  when  $x = x_0$  is:

$$\hat{y}_0 = \hat{\beta}_0 + \hat{\beta}_1 x_0.$$

In all cases we shall obtain  $\beta_0$  and  $\beta_1$  which minimize the sum of squared distances  $\sum_{i=1}^n d^2(y_i, \hat{y}_i)$ . In general, we write:

$$S(\beta_0, \beta_1) = \sum_{i=1}^n d^2(y_i, \beta_0 + \beta_1 x_i),$$

where  $x_i$ ,  $\beta_0$  and  $\beta_1$  may be fuzzy or crisp, depending on the model.

In what follows, we use (2) and (3) to estimate the unknown parameters  $\beta_0$  and  $\beta_1$  of the fuzzy regression models.

**3.1. The Triangular Case.** In this section we estimate the parameters of models (7)-(9) when the predictor or unknown parameters are triangular fuzzy numbers.

**3.1.1. Crisp Parameters and Fuzzy Predictor.** To estimate  $\beta_0$  and  $\beta_1$  in model (7), we write  $\tilde{y}_i = (y_{il}, y_{im}, y_{ir})$  and  $\tilde{x}_i = (x_{il}, x_{im}, x_{ir})$ , where for  $i = 1, 2, \dots, n$ ,  $x_{im}$ ,  $x_{il}$  and  $x_{ir}$  are known. Two cases arise according as  $\beta_1 \geq 0$  or  $\beta_1 < 0$ . We estimate  $\beta_0$  and  $\beta_1$  in the first case as follows:

By the arithmetic of triangular fuzzy numbers,  $\beta_0 + \beta_1 \tilde{x}_i$  is the triangular fuzzy number  $(\beta_0 + \beta_1 x_{il}, \beta_0 + \beta_1 x_{im}, \beta_0 + \beta_1 x_{ir})$ . Hence we have:

$$\begin{aligned} S^+(\beta_0, \beta_1) &= \sum_{i=1}^n d^2(\tilde{y}_i, \beta_0 + \beta_1 \tilde{x}_i) \\ &= \sum_{i=1}^n [(y_{im} - \beta_0 - \beta_1 x_{im})^2 + (y_{il} - \beta_0 - \beta_1 x_{il})^2 \\ &\quad + (y_{ir} - \beta_0 - \beta_1 x_{ir})^2]. \end{aligned} \quad (10)$$

Differentiating with respect to  $\beta_0$  and  $\beta_1$  we obtain the estimates of  $\beta_0$  and  $\beta_1$  denoted by  $\hat{\beta}_0^+$  and  $\hat{\beta}_1^+$  respectively, as follows:

$$\hat{\beta}_1^+ = \frac{\sum_{i=1}^n (x_{il}y_{il} + x_{im}y_{im} + x_{ir}y_{ir}) - 3n\bar{x}\bar{y}}{\sum_{i=1}^n (x_{il}^2 + x_{im}^2 + x_{ir}^2) - 3n\bar{x}^2}, \quad (11)$$

and

$$\hat{\beta}_0^+ = \bar{y} - \hat{\beta}_1^+ \bar{x}. \quad (12)$$

where  $\bar{y} = \frac{\sum_{i=1}^n (y_{il} + y_{im} + y_{ir})}{3n}$  and  $\bar{x} = \frac{\sum_{i=1}^n (x_{il} + x_{im} + x_{ir})}{3n}$ .

Similarly for the case when  $\beta_1 < 0$  we have:

$$\begin{aligned} S^-(\beta_0, \beta_1) &= \sum_{i=1}^n d^2(\tilde{y}_i, \beta_0 + \beta_1 \tilde{x}_i) \\ &= \sum_{i=1}^n [(y_{im} - \beta_0 - \beta_1 x_{im})^2 + (y_{il} - \beta_0 - \beta_1 x_{ir})^2 \\ &\quad + (y_{ir} - \beta_0 - \beta_1 x_{il})^2]. \end{aligned} \quad (13)$$

Again, differentiating with respect to  $\beta_0$  and  $\beta_1$  we obtain:

$$\hat{\beta}_1^- = \frac{\sum_{i=1}^n (x_{ir}y_{il} + x_{im}y_{im} + x_{il}y_{ir}) - 3n\bar{x}\bar{y}}{\sum_{i=1}^n (x_{il}^2 + x_{im}^2 + x_{ir}^2) - 3n\bar{x}^2}, \quad (14)$$

and

$$\hat{\beta}_0^- = \bar{y} - \hat{\beta}_1^- \bar{x}. \quad (15)$$

**Definition 3.1.** The fuzzy data set  $\tilde{y}_i = (y_{il}, y_{im}, y_{ir})$  and  $\tilde{x}_i = (x_{il}, x_{im}, x_{ir})$ , for  $i = 1, 2, \dots, n$ , is said to be tight if either  $\hat{\beta}_1^- \geq 0$  or  $\hat{\beta}_1^+ \leq 0$ . If  $\hat{\beta}_1^- \geq 0$  the data set is said to be tight positive; if  $\hat{\beta}_1^+ \leq 0$ , tight negative [1].

Diamond [1] shows that  $\hat{\beta}_1^+ \geq \hat{\beta}_1^-$  and the estimators of the parameters in model (7) are unique if the nondegenerate fuzzy data set (not all observations in a data set are made at the same datum) is tight. If the data set is tight positive, the least-squares estimators of parameters  $\beta_0$  and  $\beta_1$  in model (7) are given by (11) and (12), and if tight negative, by (14) and (15).

In what follows we assume tightness. If a data set is not tight, either  $\beta_1$  is close to zero or there is no linear trend, and neither of these cases are of interest to us.

**Remark 3.2.** We note that in the case of classical regression  $x_{il} = x_{im} = x_{ir}$  and  $y_{il} = y_{im} = y_{ir}$  the estimates (3.1.1)-(3.1.1) or (3.1.1)-(3.1.1) reduce to (3). This is not true of estimates obtained by using linear programming [4, 3, 5, 7, 8, 10].

**3.1.2. Fuzzy Parameters and Crisp Predictor.** In model (8) suppose  $\tilde{\beta}_0$  and  $\tilde{\beta}_1$  are triangular fuzzy numbers as follows:

$$\tilde{\beta}_0 = (\beta_{0l}, \beta_{0m}, \beta_{0r}) \quad \text{and} \quad \tilde{\beta}_1 = (\beta_{1l}, \beta_{1m}, \beta_{1r}).$$

Therefore, if  $x_i \geq 0$  then  $\tilde{\beta}_0 + \tilde{\beta}_1 x_i$  is the triangular fuzzy number

$$(\beta_{0l} + \beta_{1l}x_i, \beta_{0m} + \beta_{1m}x_i, \beta_{0r} + \beta_{1r}x_i)$$

If  $x_i < 0$  then  $\tilde{\beta}_0 + \tilde{\beta}_1 x_i$  is the triangular fuzzy number

$$(\beta_{0l} + \beta_{1r}x_i, \beta_{0m} + \beta_{1m}x_i, \beta_{0r} + \beta_{1l}x_i).$$

We first suppose  $x_i \geq 0$ , for  $i = 1, 2, \dots, n$ . To find an estimate for  $\tilde{\beta}_0$  and  $\tilde{\beta}_1$  for model (8) we again minimize the following sum of squared distances:

$$\begin{aligned} S(\beta_0, \beta_1) &= \sum_{i=1}^n d^2(\tilde{y}_i, \tilde{\beta}_0 + \tilde{\beta}_1 x_i) \\ &= \sum_{i=1}^n [(y_{im} - \beta_{0m} - \beta_{1m}x_i)^2 + (y_{il} - \beta_{0l} - \beta_{1l}x_i)^2 \\ &\quad + (y_{ir} - \beta_{0r} - \beta_{1r}x_i)^2] \end{aligned} \quad (16)$$

Differentiating with respect to  $\beta_{0l}$ ,  $\beta_{0m}$ ,  $\beta_{0r}$ ,  $\beta_{1l}$ ,  $\beta_{1m}$  and  $\beta_{1r}$ , we obtain the estimates of  $\tilde{\beta}_0$  and  $\tilde{\beta}_1$  as follows:

$$\hat{\beta}_{1l} = \frac{\sum_{i=1}^n x_i y_{il} - n\bar{x}\bar{y}_l}{\sum_{i=1}^n x_i^2 - n\bar{x}^2}, \quad \text{and} \quad \hat{\beta}_{0l} = \bar{y}_l - \hat{\beta}_{1l}\bar{x}, \quad (17)$$

$$\hat{\beta}_{1m} = \frac{\sum_{i=1}^n x_i y_{im} - n\bar{x}\bar{y}_m}{\sum_{i=1}^n x_i^2 - n\bar{x}^2}, \quad \text{and} \quad \hat{\beta}_{0m} = \bar{y}_m - \hat{\beta}_{1m}\bar{x}, \quad (18)$$

$$\hat{\beta}_{1r} = \frac{\sum_{i=1}^n x_i y_{ir} - n\bar{x}\bar{y}_r}{\sum_{i=1}^n x_i^2 - n\bar{x}^2}, \quad \text{and} \quad \hat{\beta}_{0r} = \bar{y}_r - \hat{\beta}_{1r}\bar{x}, \quad (19)$$

where  $\hat{\beta}_1 = (\hat{\beta}_{1l}, \hat{\beta}_{1m}, \hat{\beta}_{1r})$ ,  $\hat{\beta}_0 = (\hat{\beta}_{0l}, \hat{\beta}_{0m}, \hat{\beta}_{0r})$ ,  $\bar{y}_l = \frac{\sum_{i=1}^n y_{il}}{n}$ ,  $\bar{y}_m = \frac{\sum_{i=1}^n y_{im}}{n}$ ,  $\bar{y}_r = \frac{\sum_{i=1}^n y_{ir}}{n}$  and  $\bar{x} = \frac{\sum_{i=1}^n x_i}{n}$ .

If  $x_i < 0$ , for  $i = 1, 2, \dots, n$ , then the estimates of parameters are obtained as follows:

$$\hat{\beta}_{1r} = \frac{\sum_{i=1}^n x_i y_{il} - n\bar{x}\bar{y}_l}{\sum_{i=1}^n x_i^2 - n\bar{x}^2}, \quad \text{and} \quad \hat{\beta}_{0l} = \bar{y}_l - \hat{\beta}_{1r}\bar{x}, \quad (20)$$

$$\hat{\beta}_{1m} = \frac{\sum_{i=1}^n x_i y_{im} - n\bar{x}\bar{y}_m}{\sum_{i=1}^n x_i^2 - n\bar{x}^2}, \quad \text{and} \quad \hat{\beta}_{0m} = \bar{y}_m - \hat{\beta}_{1m}\bar{x}, \quad (21)$$

$$\hat{\beta}_{1l} = \frac{\sum_{i=1}^n x_i y_{ir} - n\bar{x}\bar{y}_r}{\sum_{i=1}^n x_i^2 - n\bar{x}^2}, \quad \text{and} \quad \hat{\beta}_{0r} = \bar{y}_r - \hat{\beta}_{1l}\bar{x}. \quad (22)$$



The case when some of the  $x_i$ 's are positive and others negative is also easily handled.

**Remark 3.3.** We note that if  $\beta_{0l} = \beta_{0m} = \beta_{0r}$  and  $\beta_{1l} = \beta_{1m} = \beta_{1r}$  the estimates (17)-(19) reduce to (6).

**3.1.3. Fuzzy Parameters and Fuzzy Predictor.** Suppose in model (9)  $\tilde{x}_i$ 's and  $\tilde{\beta}_0$  and  $\tilde{\beta}_1$  are all triangular fuzzy numbers. An approximation for product of two positive triangular fuzzy numbers  $\tilde{A}_1 = (a_{1l}, a_{1m}, a_{1r})$  and  $\tilde{A}_2 = (a_{2l}, a_{2m}, a_{2r})$  is as follows [13]:

$$\tilde{A}_1 \tilde{A}_2 \simeq (a_{1l}a_{2l}, a_{1m}a_{2m}, a_{1r}a_{2r}).$$

Therefore,  $\tilde{\beta}_0 + \tilde{\beta}_1 \tilde{x}_i$  is the approximately triangular fuzzy number

$$(\beta_{0l} + \beta_{1l}x_{il}, \beta_{0m} + \beta_{1m}x_{im}, \beta_{0r} + \beta_{1r}x_{ir}).$$

To find an estimate for  $\tilde{\beta}_0$  and  $\tilde{\beta}_1$  we minimize the following sum of squared distances:

$$\begin{aligned} S(\beta_0, \beta_1) &= \sum_{i=1}^n d^2(\tilde{y}_i, \tilde{\beta}_0 + \tilde{\beta}_1 \tilde{x}_i) \\ &= \sum_{i=1}^n [(y_{im} - \beta_{0m} - \beta_{1m}x_{im})^2 + (y_{il} - \beta_{0l} - \beta_{1l}x_{il})^2 \\ &\quad + (y_{ir} - \beta_{0r} - \beta_{1r}x_{ir})^2]. \end{aligned} \quad (23)$$

Differentiating with respect to  $\beta_{0l}$ ,  $\beta_{0m}$ ,  $\beta_{0r}$ ,  $\beta_{1l}$ ,  $\beta_{1m}$  and  $\beta_{1r}$ , we obtain the following the estimates of  $\tilde{\beta}_1$  and  $\tilde{\beta}_0$ :

$$\hat{\beta}_1 = (\hat{\beta}_{1l}, \hat{\beta}_{1m}, \hat{\beta}_{1r}) \quad \text{and} \quad \hat{\beta}_0 = (\hat{\beta}_{0l}, \hat{\beta}_{0m}, \hat{\beta}_{0r})$$

where  $\bar{x}_l = \frac{\sum_{i=1}^n x_{il}}{n}$ ,  $\bar{x}_m = \frac{\sum_{i=1}^n x_{im}}{n}$ ,  $\bar{x}_r = \frac{\sum_{i=1}^n x_{ir}}{n}$ ,  $\bar{y}_l$ ,  $\bar{y}_m$  and  $\bar{y}_r$  are as defined in the previous section and

$$\hat{\beta}_{1l} = \frac{\sum_{i=1}^n x_{il}y_{il} - n\bar{x}_l\bar{y}_l}{\sum_{i=1}^n x_{il}^2 - n\bar{x}_l^2}, \quad \text{and} \quad \hat{\beta}_{0l} = \bar{y}_l - \hat{\beta}_{1l}\bar{x}_l, \quad (24)$$

$$\hat{\beta}_{1m} = \frac{\sum_{i=1}^n x_{im}y_{im} - n\bar{x}_m\bar{y}_m}{\sum_{i=1}^n x_{im}^2 - n\bar{x}_m^2}, \quad \text{and} \quad \hat{\beta}_{0m} = \bar{y}_m - \hat{\beta}_{1m}\bar{x}_m, \quad (25)$$

$$\hat{\beta}_{1r} = \frac{\sum_{i=1}^n x_{ir}y_{ir} - n\bar{x}_r\bar{y}_r}{\sum_{i=1}^n x_{ir}^2 - n\bar{x}_r^2}, \quad \text{and} \quad \hat{\beta}_{0r} = \bar{y}_r - \hat{\beta}_{1r}\bar{x}_r. \quad (26)$$

The case when the fuzzy numbers  $\tilde{x}_i$ 's and  $\tilde{\beta}_1$  are not all positive is similarly handled.

**Remark 3.4.** Again, this is a true generalization of the previous models. In other words, if either the predictor variable or the parameters are crisp, the model reduces to a corresponding earlier model.

**3.2. The Trapezoidal Case.** In this section we estimate the parameters  $\beta_0$  and  $\beta_1$  of the models (7) and (8), when the predictors variables in (7) and the parameters in (8) are trapezoidal fuzzy numbers.

**3.2.1. Fuzzy Predictor and Crisp Parameters.** Let  $\tilde{x}_i = (x_{il}, x_{im}, x_{iu}, x_{ir})$  and  $\tilde{y}_i = (y_{il}, y_{im}, y_{iu}, y_{ir})$ , for  $i = 1, 2, \dots, n$ , be trapezoidal fuzzy numbers. Again in model (7) two cases arise according as  $\beta_1 \geq 0$  or  $\beta_1 < 0$ .

For the first case  $\beta_0 + \beta_1 \tilde{x}_i$  is the trapezoidal fuzzy number  $(\beta_0 + \beta_1 x_{il}, \beta_0 + \beta_1 x_{im}, \beta_0 + \beta_1 x_{iu}, \beta_0 + \beta_1 x_{ir})$ . Hence by definition of distance between two trapezoidal fuzzy numbers in (3) we have:

$$S^+(\beta_0, \beta_1) = \sum_{i=1}^n [(y_{il} - \beta_0 - \beta_1 x_{il})^2 + 0.5(y_{im} - \beta_0 - \beta_1 x_{im})^2 + 0.5(y_{iu} - \beta_0 - \beta_1 x_{iu})^2 + (y_{ir} - \beta_0 - \beta_1 x_{ir})^2] \quad (27)$$

Differentiating with respect to  $\beta_0$  and  $\beta_1$ , we obtain the estimates of  $\beta_0$  and  $\beta_1$  as follows:

$$\hat{\beta}_1^+ = \frac{\sum_{i=1}^n (x_{il} y_{il} + 0.5 x_{im} y_{im} + 0.5 x_{iu} y_{iu} + x_{ir} y_{ir}) - 3n \bar{x} \bar{y}}{\sum_{i=1}^n (x_{il}^2 + 0.5 x_{im}^2 + 0.5 x_{iu}^2 + x_{ir}^2) - 3n \bar{x}^2}, \quad (28)$$

and

$$\hat{\beta}_0^+ = \bar{y} - \hat{\beta}_1^+ \bar{x}, \quad (29)$$

where  $\bar{y} = \frac{\sum_{i=1}^n (y_{il} + 0.5 y_{im} + 0.5 y_{iu} + y_{ir})}{3n}$  and  $\bar{x} = \frac{\sum_{i=1}^n (x_{il} + 0.5 x_{im} + 0.5 x_{iu} + x_{ir})}{3n}$ .

Also if  $\beta_1 < 0$ , then we have:

$$S^-(\beta_0, \beta_1) = \sum_{i=1}^n [(y_{il} - \beta_0 - \beta_1 x_{ir})^2 + 0.5(y_{im} - \beta_0 - \beta_1 x_{iu})^2 + 0.5(y_{iu} - \beta_0 - \beta_1 x_{im})^2 + (y_{ir} - \beta_0 - \beta_1 x_{il})^2] \quad (30)$$

We again minimize (30) to estimates  $\beta_0$  and  $\beta_1$ . The estimators obtain as follows:

$$\hat{\beta}_1^- = \frac{\sum_{i=1}^n (x_{ir} y_{il} + 0.5 x_{iu} y_{im} + 0.5 x_{im} y_{iu} + x_{il} y_{ir}) - 3n \bar{x} \bar{y}}{\sum_{i=1}^n (x_{il}^2 + 0.5 x_{im}^2 + 0.5 x_{iu}^2 + x_{ir}^2) - 3n \bar{x}^2} \quad (31)$$

and

$$\hat{\beta}_0^- = \bar{y} - \hat{\beta}_1^- \bar{x} \quad (32)$$

**Lemma 3.5.** For every nondegenerate data set

$$\hat{\beta}_1^+ \geq \hat{\beta}_1^-.$$

*Proof.* Let  $T_{xx} = \sum_{i=1}^n (x_{il}^2 + 0.5 x_{im}^2 + 0.5 x_{iu}^2 + x_{ir}^2) - 3n \bar{x}^2$ , we can write  $T_{xx}$  as  $T_{xx} = \sum_{i=1}^n (x_{il} - \bar{x})^2 + 0.5 \sum_{i=1}^n (x_{im} - \bar{x})^2 + 0.5 \sum_{i=1}^n (x_{iu} - \bar{x})^2 + \sum_{i=1}^n (x_{ir} - \bar{x})^2$ , therefore  $T_{xx} \geq 0$ .

By (28) and (31) we have:

$$\begin{aligned}
 T_{xx}(\hat{\beta}_1^+ - \hat{\beta}_1^-) &= \left[ \sum_{i=1}^n (x_{il}y_{il} + 0.5x_{im}y_{im} + 0.5x_{iu}y_{iu} + x_{ir}y_{ir}) - 3n\bar{x}\bar{y} \right] \\
 &\quad - \left[ \sum_{i=1}^n (x_{ir}y_{il} + 0.5x_{iu}y_{im} + 0.5x_{im}y_{iu} + x_{il}y_{ir}) - 3n\bar{x}\bar{y} \right] \\
 &= \sum_{i=1}^n [(x_{il} - x_{ir})y_{il} + 0.5(x_{im} - x_{iu})y_{im} \\
 &\quad + 0.5(x_{iu} - x_{im})y_{iu} + (x_{ir} - x_{il})y_{ir}] \\
 &= \sum_{i=1}^n [(x_{il} - x_{ir})(y_{il} - y_{ir}) + 0.5(x_{iu} - x_{im})(y_{iu} - y_{im})],
 \end{aligned}$$

since  $x_{ir} - x_{il}$ ,  $x_{iu} - x_{im}$ ,  $y_{ir} - y_{il}$ ,  $y_{iu} - y_{im}$  and  $T_{xx}$  are all positive, therefore  $T_{xx}(\hat{\beta}_1^+ - \hat{\beta}_1^-) \geq 0$ , proving the lemma.  $\square$

**Theorem 3.6.** *The estimators (28) and (31) are unique if the nondegenerate data set is tight. If the data set is tight positive, the fuzzy least squares regression line is  $\tilde{y}_i = \hat{\beta}_0 + \hat{\beta}_1^+ \tilde{x}_i$ , and if tight negative it is  $\tilde{y}_i = \hat{\beta}_0 + \hat{\beta}_1^- \tilde{x}_i$ .*

*Proof.* Suppose that the data set is tight positive. By Lemma 3.5,  $\hat{\beta}_1^+ \geq \hat{\beta}_1^- \geq 0$ . However, it is easily seen that if  $\hat{\beta}_1^- \geq 0$ , then  $S^-(\beta_0, \beta_1) = \sum_{i=1}^n d^2(\tilde{y}_i, \beta_0 + \beta_1 \tilde{x}_i)$  in (30) simply cannot arise as a possible solution to the minimization of the sum of squared distances between  $\tilde{y}_i$  and  $\beta_0 + \beta_1 \tilde{x}_i$ .

Let  $\mathcal{V}$  be the cone in  $\mathcal{P}(\mathbb{R})^n$  generated by the  $n$ -vectors of trapezoidal fuzzy numbers  $\tilde{\mathbf{1}}' = (\tilde{1}, \tilde{1}, \dots, \tilde{1})$  and  $\tilde{\mathbf{X}}' = (\tilde{X}_1, \tilde{X}_2, \dots, \tilde{X}_n)$ , where  $\tilde{1}$  denotes trapezoidal fuzzy number  $\tilde{1} = (1, 1, 1, 1)$  and  $\tilde{X}_i = (x_{il}, x_{im}, x_{iu}, x_{ir})$  for  $i = 1, 2, \dots, n$ . By Corollary 2.4, if a pair  $\beta_0^*$  and  $\beta_1^*$  exists that minimizes the distance between  $\tilde{\mathbf{Y}}' = (\tilde{Y}_1, \tilde{Y}_2, \dots, \tilde{Y}_n)$  and the cone  $\{\beta_0 \tilde{\mathbf{1}} + \beta_1 \tilde{\mathbf{X}} : \beta_0, \beta_1 \geq 0\}$ , it is unique. A similar argument holds for the tight negative case.  $\square$

**Remark 3.7.** We note that if  $x_{im} = x_{iu}$  the trapezoidal fuzzy numbers reduce to triangular fuzzy numbers and then the estimates (28) and (29) ((31) and (32)) reduce to (11) and (12) ((14) and (15)), respectively.

**3.2.2. Crisp Predictor and Fuzzy Parameters.** In model (8) if  $\tilde{\beta}_0$  and  $\tilde{\beta}_1$  are trapezoidal fuzzy numbers, where  $\tilde{\beta}_0 = (\beta_{0l}, \beta_{0m}, \beta_{0u}, \beta_{0r})$  and  $\tilde{\beta}_1 = (\beta_{1l}, \beta_{1m}, \beta_{1u}, \beta_{1r})$ , then for  $x_i > 0$ ,  $\tilde{\beta}_0 + \tilde{\beta}_1 x_i$  is the trapezoidal fuzzy number

$$(\beta_{0l} + \beta_{1l}x_i, \beta_{0m} + \beta_{1m}x_i, \beta_{0u} + \beta_{1u}x_i, \beta_{0r} + \beta_{1r}x_i).$$

To find an estimate for  $\tilde{\beta}_0$  and  $\tilde{\beta}_1$  we minimize the sum of squared distances

$$\begin{aligned} S(\beta_0, \beta_1) &= \sum_{i=1}^n d^2(\tilde{y}_i, \tilde{\beta}_0 + \tilde{\beta}_1 x_i) \\ &= \sum_{i=1}^n [(y_{il} - \beta_{0l} - \beta_{1l} x_i)^2 + 0.5(y_{im} - \beta_{0m} - \beta_{1m} x_i)^2 \\ &\quad + 0.5(y_{iu} - \beta_{0u} - \beta_{1u} x_i)^2 + (y_{ir} - \beta_{0r} - \beta_{1r} x_i)^2] \end{aligned} \quad (33)$$

Differentiating with respect to  $\beta_{0l}, \beta_{0m}, \beta_{0u}, \beta_{0r}$  and  $\beta_{1l}, \beta_{1m}, \beta_{1u}, \beta_{1r}$ , we obtain the estimates of  $\hat{\beta}_1$  and  $\hat{\beta}_0$  as follows:

$$\hat{\beta}_1 = (\hat{\beta}_{1l}, \hat{\beta}_{1m}, \hat{\beta}_{1u}, \hat{\beta}_{1r}) \quad \text{and} \quad \hat{\beta}_0 = (\hat{\beta}_{0l}, \hat{\beta}_{0m}, \hat{\beta}_{0u}, \hat{\beta}_{0r}),$$

where  $\bar{x} = \frac{\sum_{i=1}^n x_i}{n}$ ,  $\bar{y}_l = \frac{\sum_{i=1}^n y_{il}}{n}$ ,  $\bar{y}_m = \frac{\sum_{i=1}^n y_{im}}{n}$ ,  $\bar{y}_u = \frac{\sum_{i=1}^n y_{iu}}{n}$ ,  $\bar{y}_r = \frac{\sum_{i=1}^n y_{ir}}{n}$ , and

$$\hat{\beta}_{1l} = \frac{\sum_{i=1}^n x_i y_{il} - n \bar{x} \bar{y}_l}{\sum_{i=1}^n x_i^2 - n \bar{x}^2}, \quad \text{and} \quad \hat{\beta}_{0l} = \bar{y}_l - \hat{\beta}_{1l} \bar{x}, \quad (34)$$

$$\hat{\beta}_{1m} = \frac{\sum_{i=1}^n x_i y_{im} - n \bar{x} \bar{y}_m}{\sum_{i=1}^n x_i^2 - n \bar{x}^2}, \quad \text{and} \quad \hat{\beta}_{0m} = \bar{y}_m - \hat{\beta}_{1m} \bar{x}, \quad (35)$$

$$\hat{\beta}_{1u} = \frac{\sum_{i=1}^n x_i y_{iu} - n \bar{x} \bar{y}_u}{\sum_{i=1}^n x_i^2 - n \bar{x}^2}, \quad \text{and} \quad \hat{\beta}_{0u} = \bar{y}_u - \hat{\beta}_{1u} \bar{x}, \quad (36)$$

$$\hat{\beta}_{1r} = \frac{\sum_{i=1}^n x_i y_{ir} - n \bar{x} \bar{y}_r}{\sum_{i=1}^n x_i^2 - n \bar{x}^2}, \quad \text{and} \quad \hat{\beta}_{0r} = \bar{y}_r - \hat{\beta}_{1r} \bar{x}. \quad (37)$$

**Remark 3.8.** Again, if  $\beta_{0m} = \beta_{0u}$  and  $\beta_{1m} = \beta_{1u}$  and  $y_{im} = y_{iu}$  then the estimates (34)-(37) reduce to (17)-(19).

**3.3. A Generalization to Multivariate Linear Regression.** For the multivariate fuzzy regression model with  $n$  observations  $(\tilde{y}_i, \tilde{x}_{i1}, \dots, \tilde{x}_{ik})$  we may write

$$y_i = \beta_1(x_{i1} - \bar{x}_1) + \beta_2(x_{i2} - \bar{x}_2) + \dots + \beta_k(x_{ik} - \bar{x}_k) + \varepsilon_i, \quad i = 1, 2, \dots, n,$$

where,  $\bar{x}_j = \frac{\sum_{i=1}^n x_{ij}}{n}$ , for  $j = 1, 2, \dots, k$ . The  $x_{ij}$ 's or the positive parameters  $\beta_1, \beta_2, \dots, \beta_k$  may be trapezoidal fuzzy numbers. In the case of fuzzy predictors, the positive crisp parameters using distance (3) as a criterion, the estimate of the vector of parameters  $\beta' = (\beta_1, \dots, \beta_k)$  satisfies the following "normal" equation:

$$(\mathbf{X}'_l \mathbf{X}_l + 0.5 \mathbf{X}'_m \mathbf{X}_m + 0.5 \mathbf{X}'_u \mathbf{X}_u + \mathbf{X}'_r \mathbf{X}_r) \hat{\beta} = (\mathbf{X}'_l \mathbf{y}_l + 0.5 \mathbf{X}'_m \mathbf{y}_m + 0.5 \mathbf{X}'_u \mathbf{y}_u + \mathbf{X}'_r \mathbf{y}_r),$$

where

$$\begin{aligned} \mathbf{y}'_l &= (y_{1l}, y_{2l}, \dots, y_{nl}), & \mathbf{y}'_m &= (y_{1m}, y_{2m}, \dots, y_{nm}), \\ \mathbf{y}'_u &= (y_{1u}, y_{2u}, \dots, y_{nu}), & \mathbf{y}'_r &= (y_{1r}, y_{2r}, \dots, y_{nr}), \end{aligned} \quad (38)$$

and the  $n \times k$  matrices  $\mathbf{X}_l, \mathbf{X}_m, \mathbf{X}_u$  and  $\mathbf{X}_r$  are defined as follows:

$$\mathbf{X}_l = [x_{ij.l} - \bar{x}_j], \quad \mathbf{X}_m = [x_{ij.m} - \bar{x}_j], \quad \mathbf{X}_u = [x_{ij.u} - \bar{x}_j], \quad \mathbf{X}_r = [x_{ij.r} - \bar{x}_j].$$

The case of positive crisp predictors and fuzzy parameters is treated similarly by minimize the sum of squared distance as follows:

$$\begin{aligned}
 S(\tilde{\beta}_0, \tilde{\beta}_1, \dots, \tilde{\beta}_k) &= \sum_{i=1}^n d^2(\tilde{y}_i, \tilde{\beta}_0 + \tilde{\beta}_1 x_{i1} + \dots + \tilde{\beta}_k x_{ik}) \\
 &= \sum_{i=1}^n [(y_{il} - \beta_{0l} - \beta_{1l} x_{i1} - \dots - \beta_{kl} x_{ik})^2 \\
 &\quad + 0.5(y_{im} - \beta_{0m} - \beta_{1m} x_{i1} - \dots - \beta_{km} x_{ik})^2 \\
 &\quad + 0.5(y_{iu} - \beta_{0u} - \beta_{1u} x_{i1} - \dots - \beta_{ku} x_{ik})^2 \\
 &\quad + (y_{ir} - \beta_{0r} - \beta_{1r} x_{i1} - \dots - \beta_{kr} x_{ik})^2] \quad (39)
 \end{aligned}$$

Differentiating with respect to parameters we can obtain the estimators by the following “normal” equation:

$$\begin{aligned}
 (\mathbf{X}'\mathbf{X})\hat{\beta}_l &= \mathbf{X}'\mathbf{y}_l, \\
 (\mathbf{X}'\mathbf{X})\hat{\beta}_m &= \mathbf{X}'\mathbf{y}_m, \\
 (\mathbf{X}'\mathbf{X})\hat{\beta}_u &= \mathbf{X}'\mathbf{y}_u, \\
 (\mathbf{X}'\mathbf{X})\hat{\beta}_r &= \mathbf{X}'\mathbf{y}_r, \quad (40)
 \end{aligned}$$

where the  $n \times k$  matrix  $\mathbf{X}$  is the response observation and

$$\begin{aligned}
 \hat{\beta}'_l &= (\hat{\beta}_{1l}, \hat{\beta}_{2l}, \dots, \hat{\beta}_{kl}), & \hat{\beta}'_m &= (\hat{\beta}_{1m}, \hat{\beta}_{2m}, \dots, \hat{\beta}_{km}), \\
 \hat{\beta}'_u &= (\hat{\beta}_{1u}, \hat{\beta}_{2u}, \dots, \hat{\beta}_{ku}), & \hat{\beta}'_r &= (\hat{\beta}_{1r}, \hat{\beta}_{2r}, \dots, \hat{\beta}_{kr}). \quad (41)
 \end{aligned}$$

#### 4. Simulation Studies and Numerical Examples

In this section we use simulation studies and numerical examples to illustrate that when the observations are triangular or trapezoidal fuzzy numbers, our estimators are better than the previous ones even with respect to the earlier criteria. All computing has been done using SAS software.

To evaluate the performance of a fuzzy regression model, Kim and Bishu [5], Kao and Chyu [4, 3], Yang and Lin [13] and the others use the absolute difference between the membership value of the observed and estimated fuzzy response, as a measure of error ( $a+b$  area in Figure 3). In these papers this measure is also adopted as the criterion for computing the performance of difference methods; smaller values of this measure indicate a better fit.

**Example 4.1.** To show the effectiveness of our method we performed a simulation study as follows:

100 samples, each containing 100 “ $x_i$ -values”, were drawn from the uniform distribution on  $(0, 10)$ , and  $x_{il} = x_i - \alpha_i$ ,  $x_{ir} = x_i + \gamma_i$  corresponding to each  $x_i$ , where  $\alpha_i$  and  $\gamma_i$  are a random point in the unit square (This may lead to asymmetrical

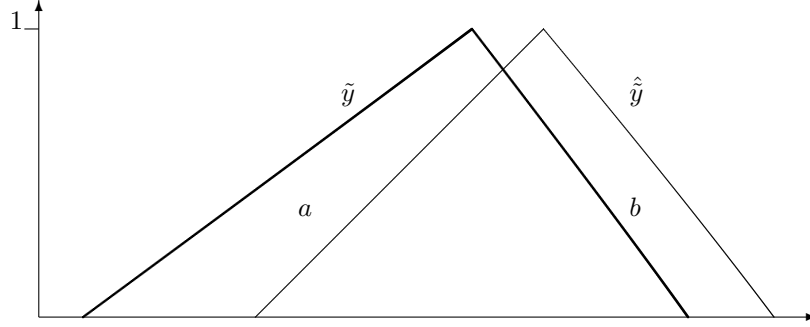


FIGURE 3. The prediction error for an observation.

triangular fuzzy numbers). Also, for each  $x_i$ , the “observed” response was chosen as:

$$\tilde{y}_i = \beta_0 + \beta_1 \tilde{x}_i + \tilde{\varepsilon}_i,$$

Where  $\varepsilon_i \stackrel{iid}{\sim} N(0, 1)$ ,  $\beta_0 = 1$ , and  $\beta_1$  varies over the constants 0.6, 0.8, 1.0, 1.2, 1.4.

By applying the fuzzy least squares method described in Section 3.1 we estimate the parameters  $\beta_0$  and  $\beta_1$  for these simulated fuzzy observations. In order to compare our method with that of Kao-Chyu [3], we suppose  $\tilde{\beta}_0$  is triangular fuzzy number and  $\beta_1$  is positive crisp. Figure 4 shows a plot of our errors vs the errors of the Kao-Chyu method as computed in [3]. Only 15 of the 100 points lie below the bisector and we conclude that our method is better about (85%) of the times.

**Example 4.2.** We now apply the proposed method to fit the fuzzy linear regression model (8) to the data taken from Hong and Hwang [2]. In that example there are nine pairs of observations  $(\tilde{y}_i, x_i)$  as shown in Table 1.

By applying (17)-(19) the fuzzy linear regression model obtained by our method is as follows:

$$\tilde{y}_i = (-5.249, -4.895, -4.541) + (1.138, 1.195, 1.252)x_i$$

It is interesting that the spreads for  $\hat{\tilde{\beta}}_0$  are both zero and the fuzzy model obtained by the Hong and Hwang [2] approach is:

$$\tilde{Y}_{HH} = -5.451 + (0.709, 1.217, 1.725)x$$

Applying the Kao-Chyu method [3] to this data, we have the following fuzzy regression model:

$$\tilde{Y}_{KC} = -4.89487 + 1.19542 x + (-2.99, 0, 1.80)$$

We note that if one adds the “constant” terms, since one of them is fuzzy,  $\hat{\tilde{\beta}}_0$  is in fact fuzzy. The errors of the observations are shown in Table 1. We see that the sum of errors in our method (10.3944) is considerably less than that of the Hong-Hwang and Kao-Chyu method.

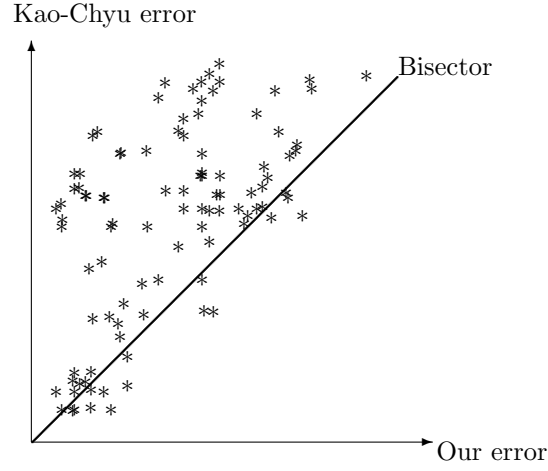


FIGURE 4. Error of simulation data for model (7).

Obs.	Response Variable	Predictor Variable	The data Error terms		
			Hong-Hwang Approach	Kao-Chyu Approach	Our method
1	(-2.1, -1.6, -1.1)	1.0	2.0160	2.8875	0.9113
2	(-2.3, -1.8, -1.3)	3.0	2.0480	1.9324	0.7473
3	(-1.5, -1.0, -0.5)	4.0	3.1537	1.9943	1.0471
4	(0.7, 1.2, 1.7)	5.6	4.6965	1.9458	0.8928
5	(1.2, 2.2, 3.2)	7.8	3.4238	2.6143	1.7993
6	( 5.8, 6.8, 7.8)	10.2	4.1836	1.4666	0.8685
7	( 9.0, 10.0, 11.0)	11.0	4.8699	3.0077	1.9537
8	( 9.0, 10.0, 11.0)	11.5	4.9699	2.4298	1.6401
9	( 9.0, 10.0, 11.0)	12.7	5.4516	1.4254	0.5342
Total error			34.8130	19.7039	10.3944

Table 1. The data and error in estimation for Example 4.2

**Example 4.3.** To show the effectiveness of our method for model (8), we performed a simulation study as follows:

100 samples, each containing 100 “ $x_i$ -values”, were drawn from the uniform distribution on (0, 10). Also, for each  $x_i$ , the “observed” response was chosen as

$$\tilde{y}_i = \beta_0 + \beta_1 x_i + \tilde{\varepsilon}_i,$$

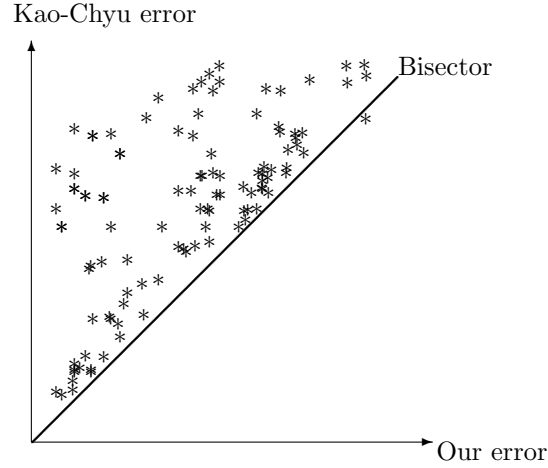


FIGURE 5. Error of simulation data for model (8).

where  $\varepsilon_i \stackrel{iid}{\sim} N(0, 1)$ ,  $\beta_0 = 1$  and  $\beta_1$  varies over the constants 0.6, 0.8, 1.0, 1.2, 1.4. And  $y_{il} = y_i - \xi_i, y_{ir} = y_i + \eta_i$  corresponding to each  $y_i$ , where  $\xi_i$  and  $\eta_i$  are a random point in the unit square.

By applying (17)-(19) we obtained the fuzzy regression line for this simulated fuzzy observation. Figure 5 shows a plot of our errors vs the errors of the Kao-Chyu method as computed in [3]. It is clear that almost all of the 100 points lie above the bisector and hence in most (99.9%) cases our errors were smaller than the errors estimating by the Kao-Chyu method.

**Example 4.4.** In order to compare the proposed method with Yang and Lin for the fuzzy linear regression model (9), we apply this method to the data as show in Table 2. In this example  $\tilde{\beta}_0$  and  $\tilde{\beta}_1$  and  $\tilde{x}_i$  are triangular fuzzy numbers. We estimate the fuzzy parameters  $\tilde{\beta}_0$  and  $\tilde{\beta}_1$  by (24)-(26).

The fuzzy regression model estimated by our method is as follows:

$$\tilde{y}_i = (3.27580, 3.57243, 3.83865) + (0.518804, 0.519348, 0.523524)\tilde{x}_i$$

To evaluate our model we compare it with the fuzzy model obtained by Yang and Lin [13] which is as follows:

$$\tilde{Y}_{YL} = (3.2052, 3.4967, 3.7882) + (0.5251, 0.5293, 0.5335)\tilde{x}$$

Table 2 lists both our errors and those obtained by the Yang and Lin approach [13]. Again the sum of errors in our method (7.69) is less than that obtained by the Yang and Lin method. This implies that our fit is slightly better than that of Yang and Lin.



Obs.	The data		Error terms	
	Response Variable( $\tilde{y}_i$ )	Predictor Variable( $\tilde{x}_i$ )	Yang-Lin Approach	Our method
1	(3.5, 4.0, 4.5)	(1.5, 2.0, 2.5)	0.82247	0.85856
2	(5.0, 5.5, 6.0)	(3.0, 3.5, 4.0)	0.27822	0.21580
3	(6.5, 7.5, 8.5)	(4.5, 5.5, 6.5)	1.53468	1.51221
4	(6.0, 6.5, 7.0)	(6.5, 7.0, 7.5)	0.95111	0.94164
5	(8.0, 8.5, 9.0)	(8.0, 8.5, 9.0)	0.77397	0.77724
6	(7.0, 8.0, 9.0)	(9.5, 10.5, 11.5)	1.51546	1.47983
7	(10.0, 10.5, 11.0)	(10.5, 11.0, 11.5)	1.10235	1.06797
8	(9.0, 9.5, 10.0)	(12.0, 12.5, 13.0)	0.88847	0.83765
	Total error		7.86673	7.69090

Table 2. The error in estimation for Example 4.4

**Example 4.5.** In this example we use the trapezoidal fuzzy observations as shown in Table 3, and apply the equations (28) and (29) to these data to estimate  $\beta_0$  and  $\beta_1$ , as explained in Section 3.2.1. In fact Table 3 is a modification of Table 2; the unique central number has been replaced by an interval. The fuzzy least squares line obtained is as  $\tilde{y}_i = 3.4773 + 0.5319\tilde{x}_i$ .

The errors of the observations are shown in Table 3. Since trapezoidal fuzzy numbers are less precise than the corresponding triangular fuzzy numbers, it is not surprising that these errors are large. Of course the method of Kao-Chyu or Yang-Lin cannot be applied here.

Obs.	Response Variable	Predictor Variable	Error Terms
1	(3.5, 3.75, 4.25, 4.5)	(1.5, 1.75, 2.25, 2.5)	1.6039
2	(5.0, 5.25, 5.75, 6.0)	(3.0, 3.25, 3.75, 4.0)	0.3627
3	(6.5, 7.00, 8.00, 8.5)	(4.5, 5.00, 6.00, 6.5)	2.0510
4	(6.0, 6.25, 6.75, 7.0)	(6.5, 6.75, 7.25, 7.5)	1.3080
5	(8.0, 8.25, 8.75, 9.0)	(8.0, 8.25, 8.75, 9.0)	0.9661
6	(7.0, 7.50, 8.50, 9.0)	(9.5, 10.00, 11.00, 11.5)	3.2359
7	(10.0, 10.25, 10.75, 11.0)	(10.5, 10.75, 11.25, 11.5)	1.1489
8	(9.0, 9.25, 9.75, 10.0)	(12.0, 12.25, 12.75, 13)	1.4626
	Total error		12.1391

Table 3. The trapezoidal fuzzy observations and errors for Example 4.5

**Example 4.6.** In this example we use the trapezoidal fuzzy observations for  $\tilde{y}_i$  and crisp observation for  $x_i$  as shown in Table 4, and then apply the equations (34)-(37) to estimate  $\tilde{\beta}_0$  and  $\tilde{\beta}_1$ . Based on our formula we have

$$\tilde{\beta}_0 = (3.29, 3.60, 4.22, 4.54), \tilde{\beta}_1 = 0.798.$$

Thus, the fuzzy regression model obtained is as follows:

$$\tilde{y}_i = (3.29, 3.60, 4.22, 4.54) + 0.798 x_i$$

The errors of the observations are shown in Table 4. Notice that the sum of errors in Table 4 is similar to the sum of errors in Table 3.

Obs.	Response Variable	Predictor Variable	Error Terms
1	(3.5, 3.75, 4.25, 4.5)	1	2.1196
2	(5.0, 5.25, 5.75, 6.0)	2	0.0075
3	(6.5, 7.00, 8.00, 8.5)	3	2.2114
4	(6.0, 6.25, 6.75, 7.0)	4	2.1309
5	(8.0, 8.25, 8.75, 9.0)	5	1.1997
6	(7.0, 7.50, 8.50, 9.0)	6	1.3875
7	(10.0, 10.25, 10.75, 11.0)	7	1.6623
8	(9.0, 9.25, 9.75, 10.0)	8	2.0826
		Total error	12.8020

Table 4. The trapezoidal fuzzy observations and errors for Example 4.6

## 5. Conclusion

In this paper we have used the method of least squares to estimate the parameters of various fuzzy regression models. This method is especially attractive since the estimators are very similar to those of the classical linear regression model and when the fuzzy data or the fuzzy parameters becomes crisp, the estimators and the fuzzy regression line are identical to those in the classical case. Moreover additional observations do not significantly affect the difficulty of computation, even when the fuzzy numbers involved are not symmetric.

Simulation studies show that the total error of this simple but powerful method is usually significantly smaller than that of the earlier methods.

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