THE CATEGORY OF ⊤-CONVERGENCE SPACES AND ITS CARTESIAN-CLOSEDNESS

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Abstract. In this paper, we define a kind of lattice-valued convergence spaces based on the notion of ⊤-filters, namely ⊤-convergence spaces, and show the category of ⊤-convergence spaces is Cartesian-closed. Further, in the lattice valued context of a complete MV-algebra, a close relation between the category of ⊤-convergence spaces and that of strong L-topological spaces is established. In details, we show that the category of strong L-topological spaces is concretely isomorphic to that of strong L-topological ⊤-convergence spaces categorically and bireflectively embedded in that of ⊤-convergence spaces.

1. Introduction

As pointed out by E. Lowen and R. Lowen in [14], the category of stratified $[0,1]$-topological spaces (or fuzzy topological spaces in the original terminology of [13]) is not completely satisfactory for certain application in Algebra topology or Functional analysis, here $[0,1]$ is the unital interval. The main reason is the fact that it is not Cartesian-closed and hence there is no natural function space for the sets of morphisms. In order to overcome this deficiency, by starting from convergence theory in stratified $[0,1]$-topological spaces developed by R. Lowen in [13], E. Lowen et al. [14, 15] considered fuzzy convergence spaces as a generalization of Choquet's convergence spaces [1] and obtained the resulting Cartesian-closed category containing the category of stratified $[0,1]$-topological spaces as a fully embedded subcategory.

For more general lattice L instead of the unital interval $[0,1]$, stemming from stratified L-topological spaces, Jäger [10] developed a theory of convergence based on the notion of stratified L-filters, where L is a complete Heyting Algebra. The resulting category, namely the category of stratified L-generalized convergence spaces, has the desired structural property of Cartesian-closedness and contains the category of stratified L-topological spaces as an embedded reflective subcategory. The convergence theory was developed to a significant extent in recent years [3, 18, 19, 11, 12, 16, 20, 21]. On this basis, in the same lattice valued context, Fang [2] defined a subcategory of the category of stratified L-generalized convergence spaces, namely the category of stratified L-ordered convergence spaces, which also is Cartesian-closed and contains the category of stratified L-topological spaces as an embedded reflective subcategory.

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Thus in case of the underlying lattice L being a complete Heyting Algebra, we see that Jäger and Fang all got nice conclusions. Regretfully, the idempotency about the meet operation $\wedge$ of L is indispensable for Cartesian-closedness of the resulting categories in [2, 10]. The requirement of idempotency is not very convenient because the semigroup operation in most of underlying lattices (e.g., complete MV-algebras) is not idempotent, but commutative.

Apart from the Cartesian-closedness depending on the idempotency of the meet operation $\wedge$, all convergence spaces mentioned above, such as fuzzy convergence space, stratified L-generalized convergence spaces and stratified L-ordered convergence spaces, start from a kind of stratified L-topological spaces. In fact, besides stratified L-topological spaces, there exists another kind of lattice-valued topological spaces, namely strong L-topological spaces introduced by Zhang [22], which actually are probabilistic topological spaces [8, 9] in a complete MV-algebra.

Hence when the underlying lattice is possessed of a semigroup operation with non-idempotency, it is necessary to find a kind of lattice-valued convergence spaces starting from strong L-topological spaces such that the resulting category is Cartesian-closed and contains the category of strong L-topological spaces as an embedded reflective subcategory.

By this paper, we try to propose a kind of lattice-valued convergence spaces based on the notion of $\top$-filters which was introduced by Höhle [9], namely $\top$-convergence spaces, and show that when the lattice theoretical setting is a complete MV-algebra, the category of $\top$-convergence spaces is Cartesian-closed and the idempotency of the semigroup operation is not required here. Further, we also want to establish a close relation between the category of $\top$-convergence spaces and that of strong L-topological spaces in case the lattice L is a complete MV-algebra. In fact, we will show that the category of strong L-topological $\top$-convergence spaces is concretely isomorphic to that of strong L-topological spaces categorically, and the category of $\top$-convergence spaces contains that of strong L-topological spaces as an embedded bireflective subcategory.

The paper is organized as follows: In Section 2, we provide the lattice theoretical context and recall some notions used in this paper. In Section 3, a concept of $\top$-convergence spaces is proposed and the category of $\top$-convergence spaces is introduced. Then after showing the category of $\top$-convergence spaces is topological, the Cartesian-closedness of the category of $\top$-convergence spaces is obtained. In Section 4, it is presented the relation between the category of $\top$-convergence spaces and that of strong L-topological spaces.

2. **Preliminaries**

A triple $(L, \leq, *)$ is called a *complete residuated lattice*, if $(L, \leq)$ is a complete lattice with $\top$ and $\perp$ respectively being the top and the bottom element of L, and $* : L \times L \to L$, called a tensor on L, is a commutative, associative binary operation such that

(1) $*$ is monotone on each variable,

(2) For each $\alpha \in L$, the monotone mapping $\alpha * (-) : L \to L$ has a right adjoint $\alpha \to (-) : L \to L$ in the sense that $\alpha * \beta \leqslant \gamma \Longleftrightarrow \beta \leqslant \alpha \to \gamma$ for all $\beta, \gamma \in L$,
(3) The top element $\top$ is a unit element for $*$, i.e. $\top * \alpha = \alpha$ for all $\alpha \in L$.

For a given complete residuated lattice, the binary operation $\to$ on L can be computed by $\alpha \to \beta = \bigvee\{\gamma \in L \mid \alpha * \gamma \leq \beta\}$ for all $\alpha,\ \beta \in L$. The binary operation $\to$ is called the *implication* operation with respect to $*$. Some basic properties of the tensor $*$ and the implication operation $\to$ are collected in the following lemma; they can be found in many works, for instance [4, 19, 22].

Lemma 2.1. *Let $(L, \leq, *)$ be a complete residuated lattice. Then for all $\alpha, \beta, \gamma, \delta \in L$, $\{\beta_i\}_{i\in I} \subseteq L$, the following conditions hold:*

(a) $\top \to \alpha = \alpha$,
(b) $\alpha * (\alpha \to \beta) \leq \beta$,
(c) $\alpha \leq \beta$ *if and only if* $\alpha \to \beta = \top$,
(d) $(\alpha \to \beta) * (\beta \to \gamma) \leq \alpha \to \gamma$,
(e) $(\alpha \to \beta) * (\gamma \to \delta) \leq (\alpha * \gamma) \to (\beta * \delta)$ *and* $(\alpha \to \beta) \wedge (\gamma \to \delta) \leq (\alpha \wedge \gamma) \to (\beta \wedge \delta)$,
(f) $\alpha * \bigvee_{i\in I} \beta_i = \bigvee_{i\in I}(\alpha * \beta_i)$,
(g) $\alpha \to \bigwedge_{i\in I} \beta_i = \bigwedge_{i\in I}(\alpha \to \beta_i)$, *hence* $(\alpha \to \beta) \leq (\alpha \to \gamma)$ *whenever* $\beta \leq \gamma$,
(h) $\left(\bigvee_{i\in I} \beta_i\right) \to \beta = \bigwedge_{i\in I}(\beta_i \to \beta)$, *hence* $(\alpha \to \beta) \geq (\gamma \to \beta)$ *whenever* $\alpha \leq \gamma$.

A complete residuated lattice $(L, \leq, *)$, denoted by L simply, is called a *complete MV-algebra*, if L satisfies the condition:

(MV) $$\alpha \vee \beta = (\alpha \to \beta) \to \beta,\ \forall \alpha,\ \beta \in L.$$

A canonical example is the unital interval $[0, 1]$ with the tensor $\alpha * \beta = \max\{\alpha + \beta - 1, 0\}$, which means that $([0, 1], \leq, *)$ is a complete MV-algebra such that 1 is the unital element and the implication $\to$ with respect to $*$ is given by $\alpha \to \beta = \min\{1 - \alpha + \beta, 1\}$ for all $\alpha, \beta \in [0, 1]$.

Throughout this paper, we will assume L to be a complete MV-algebra although most of results are valid for more general lattice-valued cases.

In this paper, we will often use, without explicitly mentioning, the following properties of a complete MV-algebra.

Lemma 2.2. [7] *For all $\alpha \in L$ and $\{\beta_j\}_{j\in J} \subseteq L$, then the following properties are valid:*

(M1) $\alpha \wedge \left(\bigvee_{j\in J} \beta_j\right) = \bigvee_{j\in J}(\alpha \wedge \beta_j)$,
(M2) $\alpha \vee \left(\bigwedge_{j\in J} \beta_j\right) = \bigwedge_{j\in J}(\alpha \vee \beta_j)$,
(M3) $\alpha * \left(\bigwedge_{j\in J} \beta_j\right) = \bigwedge_{j\in J}(\alpha * \beta_j)$,
(M4) $\alpha \to \left(\bigvee_{j\in J} \beta_j\right) = \bigvee_{j\in J}(\alpha \to \beta_j)$.

An L-subset on a set X is a map from X to L, and the family of all L-subsets on X will be denoted by L^X, called the L-power set of X. For any $x \in X$, $A(x)$ is interpreted as the degree to which x is in A. By 1_X and 0_X, we denote the constant L-subsets on X taking the value $\top$ and $\bot$, respectively. We don't distinguish an element $\alpha \in L$ and the constant function $\alpha : X \to L$ such that $\alpha(x) = \alpha$ for all $x \in X$. As usual, for a universal set X, the set of all subsets of X is denoted by $\mathcal{P}(X)$, called the *power set* of X.

All algebraic operations on L can be extended to the L-power set L^X pointwisely. That is, for all $A,\ B \in L^X$ and $x \in X$,

(1) $(A \wedge B)(x) = A(x) \wedge B(x)$,
(2) $(A \vee B)(x) = A(x) \vee B(x)$,
(3) $(A * B)(x) = A(x) * B(x)$,
(4) $(A \to B)(x) = A(x) \to B(x)$.

Let $\varphi : X \to Y$ be a map. Define $\varphi^{\to} : L^X \to L^Y$ and $\varphi^{\leftarrow} : L^Y \to L^X$ respectively by $\varphi^{\to}(A)(y) = \bigvee_{\varphi(x)=y} A(x)$ for all $A \in L^X$ and $y \in Y$, $\varphi^{\leftarrow}(B) = B \circ \varphi$ for all $B \in L^Y$.

For a set X, there exists a binary map $\mathcal{S}_X(-,-) : L^X \times L^X \to L$ defined by $\mathcal{S}_X(A,B) = \bigwedge_{x \in X} \big(A(x) \to B(x)\big)$ for each pair $(A,B) \in L^X \times L^X$, where $\to$ is the implication operation corresponding to $*$. For all $A,\ B \in L^X$, $\mathcal{S}_X(A,B)$ can be interpreted as the degree to which A is a subset of B. It was called *fuzzy inclusion order* [22] or *subsethood degree* [4] of L-subsets.

Lemma 2.3. [2]*Let X and Y be nonempty sets. For any $A, B, C \in L^X$ and $E, F \in L^Y$, then the following statements hold:*

(1) $A \leq B$ if and only if $\top = \mathcal{S}_X(A,B)$.
(2) $\mathcal{S}_X(A,B) \leq \mathcal{S}_X(B,C) \to \mathcal{S}_X(A,C)$.
(3) $\mathcal{S}_X(A, B \wedge C) = \mathcal{S}_X(A,B) \wedge \mathcal{S}_X(A,C)$ and $\mathcal{S}_X(B \vee C, A) = \mathcal{S}_X(B,A) \wedge \mathcal{S}_X(C,A)$, hence $\mathcal{S}_X(C,A) \leq \mathcal{S}_X(B,A)$ when $B \leq C$.
(4) If $\varphi : X \to Y$ is a map, then

$$\mathcal{S}_X(A,B) \leq \mathcal{S}_Y(\varphi^{\to}(A), \varphi^{\to}(B)) \text{ and } \mathcal{S}_Y(E,F) \leq \mathcal{S}_X(\varphi^{\leftarrow}(E), \varphi^{\leftarrow}(F)).$$

J. Gutiérrez García and M.A. De Prada Vicente [5, 6] introduced the notion of characteristic value of a family of L-subsets extending that of characteristic value of a prefilter in [13] and provided the equivalent form of κ-condition [9]. Thereby they obtained the equivalent definitions of $\top$-filter and $\top$-filter base as follows.

Definition 2.4. [5, 6] Let X be a nonempty set. A $\top$-*filter* is a nonempty subset $\mathbb{F}$ of L^X with the following properties:

($\mathbb{F}$1) If $A \in L^X$ with $\bigvee_{C \in \mathbb{F}} \mathcal{S}_X(C,A) = \top$, then $A \in \mathbb{F}$,
($\mathbb{F}$2) $A_1 \wedge A_2 \in \mathbb{F}$ for all $A_1,\ A_2 \in \mathbb{F}$,
($\mathbb{F}$3) $\bigvee_{x \in X} A(x) = \top$ for all $A \in \mathbb{F}$.

The set of all $\top$-filters on X is denoted by $\mathcal{F}_L^{\top}(X)$.

Example 2.5. Let $[x]_\top = \{A \in L^X \mid A(x) = \top\}$ for given a point $x \in X$. Then $[x]_\top$ is a $\top$-filter, and called the *point* $\top$-*filter* of x. In case $X = \{x\}$, a single point set, $[x]_\top$ is the unique $\top$-filter on the X.

Definition 2.6. [6] A nonempty subset $\mathbb{B}$ of L^X is a ⊤-filter base, if it satisfies the following conditions:

($\mathbb{B}$1) If $B_1,\ B_2 \in \mathbb{B}$, then $\bigvee_{B\in\mathbb{B}} \mathcal{S}_X(B, B_1 \wedge B_2) = \top$,

($\mathbb{B}$2) $\bigvee_{x\in X} B(x) = \top$ for all $B \in \mathbb{B}$.

Remark 2.7. [6]Every ⊤-filter base $\mathbb{B}$ generates a ⊤-filter $\mathbb{F}_{\mathbb{B}}$ defined by

$$\mathbb{F}_{\mathbb{B}} = \{A \in L^X \mid \bigvee_{B\in\mathbb{B}} \mathcal{S}_X(B, A) = \top\}.$$

In this case, $\mathbb{B} \subseteq \mathbb{F}_{\mathbb{B}}$ holds and $\mathbb{B}$ is called a base of $\mathbb{F}_{\mathbb{B}}$. We could know every ⊤-filter $\mathbb{F}$ is a base of itself. And let $\mathbb{B}$ be any base of a ⊤-filter $\mathbb{F}$, then $\mathbb{F} = \{A \in L^X \mid \bigvee_{B\in\mathbb{B}} \mathcal{S}_X(B, A) = \top\}$ is true.

Definition 2.8. [6] Let $\varphi : X \to Y$ be a map, $\mathbb{F} \in \mathcal{F}_L^\top(X)$, $\mathbb{G} \in \mathcal{F}_L^\top(Y)$.

(1) A ⊤-filter $\varphi^{\Rightarrow}(\mathbb{F})$ on Y generated by the ⊤-filter base

$$\{\varphi^{\rightarrow}(B) \in L^Y \mid B \in \mathbb{F}\}$$

is called the *image* of $\mathbb{F}$ under φ.

(2) If the class $\{\varphi^{\leftarrow}(C) \in L^X \mid C \in \mathbb{G}\}$ satisfies $\bigvee_{y\in\varphi(X)} C(y) = \top$ for all $C \in \mathbb{G}$, then $\{\varphi^{\leftarrow}(C) \in L^X \mid C \in \mathbb{G}\}$ is a ⊤-filter base on X and a ⊤-filter on X generated by it is called the *inverse image* of $\mathbb{G}$ under φ, denoted by $\varphi^{\Leftarrow}(\mathbb{G})$. In this case, we say the inverse image $\varphi^{\Leftarrow}(\mathbb{G})$ exists sometimes.

Remark 2.9. [6] Let $\varphi : X \to Y$ be a map, $\mathbb{F} \in \mathcal{F}_L^\top(X)$ and $\mathbb{G} \in \mathcal{F}_L^\top(Y)$.

(1) By Remark 2.7, we have

$$\varphi^{\Rightarrow}(\mathbb{F}) = \{C \in L^Y \mid \bigvee_{B\in\mathbb{F}} \mathcal{S}_Y(\varphi^{\rightarrow}(B), C) = \top\}.$$

In fact, $\varphi^{\Rightarrow}(\mathbb{F})$ has another expression $\{C \in L^Y \mid \varphi^{\leftarrow}(C) \in \mathbb{F}\}$.

(2) From Definition 2.8(2), we know that the inverse image $\varphi^{\Leftarrow}(\mathbb{G})$ doesn't always exist, but if $\varphi^{\Leftarrow}(\mathbb{G})$ exists, we have the following expression

$$\varphi^{\Leftarrow}(\mathbb{G}) = \{A \in L^X \mid \bigvee_{B\in\mathbb{G}} \mathcal{S}_X(\varphi^{\leftarrow}(B), A) = \top\}.$$

(3) The following are satisfied:

(i) if $\mathbb{F} \subseteq \mathbb{H}$ for $\mathbb{H} \in \mathcal{F}_L^\top(X)$, then $\varphi^{\Rightarrow}(\mathbb{F}) \subseteq \varphi^{\Rightarrow}(\mathbb{H})$,

(ii) $\varphi^{\Leftarrow} \circ \varphi^{\Rightarrow}(\mathbb{F}) \subseteq \mathbb{F}$.

For more information on the categorical terminology we refer the reader to [17]. By a category we mean a construct $\mathcal{C}$ whose objects are structured sets, i.e. pairs (X, ξ) where X is a set and ξ a $\mathcal{C}$-structure on X, whose morphisms $\varphi : (X, \xi) \to (Y, \eta)$ are suitable maps from X to Y and whose composition is the usual composition of maps. The forgetful functors will not be mentioned explicitly. We simply write $\boldsymbol{X}$ for a categorical object (X, ξ) sometimes.

Definition 2.10. [17] A category $\mathcal{C}$ is said to be *topological* if the following conditions are satisfied:

(1) *Existence of initial structures*: For any set X, any family $\big((X_i, \xi_i)\big)_{i\in I}$ of $\mathcal{C}$-objects indexed by a class I and any family $(f_i : X \to X_i)_{i\in I}$ of maps indexed by I there exists a unique $\mathcal{C}$-structure ξ on X which is *initial* with respect to $(f_i : X \to X_i)_{i\in I}$ in the sense that for any $\mathcal{C}$-object (Y, η), a map $g : (Y, \eta) \to (X, \xi)$ is a $\mathcal{C}$-morphism iff for every $i \in I$ the composite map $f_i \circ g : (Y, \eta) \to (X_i, \xi_i)$ is a $\mathcal{C}$-morphism.

(2) *Fibre-smallness*: For any set X, the class $\{\xi \mid (X, \xi) \text{ is a } \mathcal{C}-\text{object}\}$ of all $\mathcal{C}$-structures with the underlying set X, called $\mathcal{C}$-$fibre$ of X, is a set.

(3) *Terminal separator property*: For any set X with cardinality at most one, there exists exactly one $\mathcal{C}$-structure on X.

Definition 2.11. [17] A category $\mathcal{C}$ is said to be *Cartesian-closed* provided that the following conditions are satisfied:

(1) For each pair $(\boldsymbol{X}, \boldsymbol{Y})$ of $\mathcal{C}$-objects there exists a product $\boldsymbol{X} \times \boldsymbol{Y}$ in $\mathcal{C}$.

(2) For any $\mathcal{C}$-objects $\boldsymbol{X}$ and $\boldsymbol{Y}$, there exists some $\mathcal{C}$-object $\boldsymbol{Y}^{\boldsymbol{X}}$ (called *power object*) and some $\mathcal{C}$-morphism $ev_{\boldsymbol{X},\boldsymbol{Y}} : \boldsymbol{Y}^{\boldsymbol{X}} \times \boldsymbol{X} \to \boldsymbol{Y}$ (called *evaluation morphism*) such that for each $\mathcal{C}$-object $\boldsymbol{Z}$ and each $\mathcal{C}$-morphism $\varphi : \boldsymbol{Z} \times \boldsymbol{X} \to \boldsymbol{Y}$, there exists a unique $\mathcal{C}$-morphism $\varphi^* : \boldsymbol{Z} \to \boldsymbol{Y}^{\boldsymbol{X}}$ such that $ev_{\boldsymbol{X},\boldsymbol{Y}} \circ (\varphi^* \times id_{\boldsymbol{X}}) = \varphi$.

Definition 2.12. [17] Let $\mathcal{A}$ be a subcategory of a category $\mathcal{C}$. $\mathcal{A}$ is said to be *reflective* in $\mathcal{C}$ provided that for each $\boldsymbol{X} \in |\mathcal{C}|$ there exists an $\mathcal{A}$-object $\boldsymbol{X}_{\mathcal{A}}$ and a $\mathcal{C}$-morphism $\gamma_{\boldsymbol{X}} : \boldsymbol{X} \to \boldsymbol{X}_{\mathcal{A}}$ such that for each $\mathcal{A}$-object $\boldsymbol{Y}$ and each $\mathcal{C}$-morphism $\varphi : \boldsymbol{X} \to \boldsymbol{Y}$ there is a unique $\mathcal{A}$-morphism $\overline{\varphi} : \boldsymbol{X}_{\mathcal{A}} \to \boldsymbol{Y}$ such that $\varphi = \overline{\varphi} \circ \gamma_{\boldsymbol{X}}$. If the $\mathcal{C}$-morphism $\gamma_{\boldsymbol{X}} : \boldsymbol{X} \to \boldsymbol{X}_{\mathcal{A}}$ is bimorphism, then $\mathcal{A}$ is said to be $bireflective$ in $\mathcal{C}$, and $\gamma_{\boldsymbol{X}}$ is called $bireflection$.

3. The Cartesian-closedness of $\top$-Conv

In this section, we define a kind of lattice-valued convergence spaces based on the notion of $\top$-filter, namely $\top$-convergence spaces. The class of all $\top$-convergence spaces and continuous maps forms a category. We prove the category is topological and Cartesian-closed which are very nice structural properties.

Definition 3.1. Let X be a nonempty set. A map $\lim : \mathcal{F}_L^{\top}(X) \to \mathcal{P}(X)$ satisfying the following conditions:

(TC1) $\forall x \in X$, $x \in \lim[x]_{\top}$,

(TC2) $\forall \mathbb{F}$, $\mathbb{G} \in \mathcal{F}_L^{\top}(X)$, $\mathbb{F} \subseteq \mathbb{G} \Rightarrow \lim \mathbb{F} \subseteq \lim \mathbb{G}$,

is called a $\top$-*convergence* on X, and the pair $(X, \lim)$ is called a $\top$-*convergence space*. The set of all $\top$-convergences on X is denoted by $C^{\top}(X)$. We say $\mathbb{F}$ converges to x instead of $x \in \lim \mathbb{F}$.

A map $\varphi : (X, \lim^X) \to (Y, \lim^Y)$ between $\top$-convergence spaces is said to be *continuous* provided that $x \in \lim^X \mathbb{F}$ means $\varphi(x) \in \lim^Y \varphi^{\Rightarrow}(\mathbb{F})$ for all $x \in X$ and $\mathbb{F} \in \mathcal{F}_L^{\top}(X)$. The class of all $\top$-convergence spaces and continuous maps forms a category, which is denoted $\top$-**Conv**.

Theorem 3.2. *The category* $\top$-**Conv** *is topological.*

Proof. Firstly, the existence of initial structures can be proved as follows. Let X be a nonempty set, $\{(X_j, \lim_j)\}_{j\in J}$ a family of $\top$-convergence spaces and $\{\varphi_j : X \to (X_j, \lim_j)\}_{j\in J}$ a family of maps. A structure map $\lim^X : \mathcal{F}_L^\top(X) \to \mathcal{P}(X)$ on X is defined by $\lim^X \mathbb{F} = \{x \in X \mid \varphi_j(x) \in \lim_j \varphi_j^{\Rightarrow}(\mathbb{F}),\ \forall j \in J\}$ for all $\mathbb{F} \in \mathcal{F}_L^\top(X)$. Then it is easy to verify that $\lim^X$ is a $\top$-convergence on X, and we only check its property of being initial below.

Let $(Y, \lim^Y)$ be a $\top$-convergence space and $\psi : Y \to X$ be a map. For any $y \in \lim^Y \mathbb{G}$, here $y \in Y$ and $\mathbb{G} \in \mathcal{F}_L^\top(Y)$, we can get

$$\begin{aligned}
&\varphi_j \circ \psi : (Y, \lim^Y) \to (X_j, \lim_j) \text{ is continuous for every } j \in J\\
\Longleftrightarrow\ &\varphi_j\big(\psi(y)\big) = \varphi_j \circ \psi(y) \in \lim_j(\varphi_j \circ \psi)^{\Rightarrow}(\mathbb{G}) = \lim_j \varphi_j^{\Rightarrow}\big(\psi^{\Rightarrow}(\mathbb{G})\big),\ \forall j \in J\\
\Longleftrightarrow\ &\psi(y) \in \lim^X \psi^{\Rightarrow}(\mathbb{G}) \text{ (by the definition of } \lim^X)\\
\Longleftrightarrow\ &\psi : (Y, \lim^Y) \to (X, \lim^X) \text{ is continuous.}
\end{aligned}$$

Next, since the class of all $\top$-convergences on X belongs to the set $\mathbf{2}^{(\mathcal{P}(X)^{\mathcal{F}_L^\top(X)})}$, here $\mathbf{2} = \{0,\ 1\}$, the $\top$-**Conv**-fibre of X is a set.

Finally, let $X = \{x\}$ be a singleton. $\mathcal{F}_L^\top(X) = \{[x]_\top\}$ holds by Example 2.5. Then the structure map $\lim : \mathcal{F}_L^\top(X) \to \mathcal{P}(X)$ is only determined by $\lim[x]_\top = \{x\}$. Thus lim is the unique $\top$-convergence on X.

From all above, we get that the category $\top$-**Conv** is topological. □

Every topological category has products [17], so the condition (1) in Definition 2.11 is automatically fulfilled by the category of $\top$-convergence spaces.

Let $(X, \lim^X)$ and $(Y, \lim^Y)$ be $\top$-convergence spaces and $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ be the projection maps. The product of $(X, \lim^X)$ and $(Y, \lim^Y)$ is denoted by $(X \times Y, \lim^X \times \lim^Y)$ explicitly. Of course, for any $\mathbb{F} \in \mathcal{F}_L^\top(X \times Y)$, $(x, y) \in (\lim^X \times \lim^Y)\mathbb{F}$ if and only if $x \in \lim^X p_X^{\Rightarrow}(\mathbb{F})$ and $y \in \lim^Y p_Y^{\Rightarrow}(\mathbb{F})$.

The set of all continuous mappings from $(X, \lim^X)$ to $(Y, \lim^Y)$ is denoted by $C_\top(X, Y)$. In the category **Set** of sets and maps, there exists the evaluation map $ev_{X,Y} : C_\top(X, Y) \times X \to Y$ defined by $ev_{X,Y}(\varphi, x) = \varphi(x)$ for all $(\varphi, x) \in C_\top(X, Y) \times X$. In order to explore the Cartesian-closedness of $\top$-**Conv**, we need some lemmas and propositions in preparation for it.

Lemma 3.3. *Let* $\mathbb{F}_i \in \mathcal{F}_L^\top(X_i)$ *and* $\mathbb{B}_i$ *be a base of* $\mathbb{F}_i$*, here* $i = 1, 2$*. Then*

$$\mathbb{B} = \{B_1 \times B_2 \mid B_1 \in \mathbb{B}_1 \text{ and } B_2 \in \mathbb{B}_2\}$$

is a $\top$*-filter base, where for any* $B_i \in \mathbb{B}_i$ *(i=1,2),*

$$B_1 \times B_2((x_1, x_2)) = B_1(x_1) \wedge B_2(x_2),\ \forall (x_1, x_2) \in X_1 \times X_2.$$

Proof. For any $A, C \in \mathbb{B}$, there exist $A_1, C_1 \in \mathbb{B}_1$ and $A_2, C_2 \in \mathbb{B}_2$ such that $A = A_1 \times A_2$ and $C = C_1 \times C_2$. $\bigvee_{B_i \in \mathbb{B}_i} \mathcal{S}_{X_i}(B_i, A_i) = \top$ $(i = 1, 2)$ follows immediately from Remark 2.7. From this, we observe that $\bigvee_{\substack{B_i \in \mathbb{B}_i\\ i=1,2}} \mathcal{S}_{X_1}(B_1, A_1) \wedge \mathcal{S}_{X_2}(B_2, A_2) = \top$. Since for each $B_i \in \mathbb{B}_i$ $(i = 1, 2)$, $\mathcal{S}_{X_1\times X_2}(B_1 \times B_2, A_1 \times A_2) \geqslant \mathcal{S}_{X_1}(B_1, A_1) \wedge \mathcal{S}_{X_2}(B_2, A_2)$ holds owing to Lemma 2.1 (e), we conclude that

$$\bigvee_{\substack{B_i\in\mathbb{B}_i\\ i=1,2}} \mathcal{S}_{X_1\times X_2}(B_1 \times B_2, A_1 \times A_2) = \top.$$

Certainly, $\bigvee_{\substack{B_i\in\mathbb{B}_i\\ i=1,2}} \mathcal{S}_{X_1\times X_2}(B_1 \times B_2, C_1 \times C_2) = \top$. From all above, we obtain that

$$\begin{aligned}
&\bigvee_{\substack{B_i\in\mathbb{B}_i\\ i=1,2}} \mathcal{S}_{X_1\times X_2}\big(B_1 \times B_2, A \wedge C\big)\\
=&\bigvee_{\substack{B_i\in\mathbb{B}_i\\ i=1,2}} \mathcal{S}_{X_1\times X_2}\big(B_1 \times B_2, (A_1 \times A_2) \wedge (C_1 \times C_2)\big)\\
=&\bigvee_{\substack{B_i\in\mathbb{B}_i\\ i=1,2}} \mathcal{S}_{X_1\times X_2}(B_1 \times B_2, A_1 \times A_2) \wedge \mathcal{S}_{X_1\times X_2}(B_1 \times B_2, C_1 \times C_2)\\
&\qquad\text{(by using Lemma 2.3 (3))}\\
=&\top,
\end{aligned}$$

i.e. $\mathbb{B}$ satisfies ($\mathbb{B}$1).

For any $B_1 \in \mathbb{B}_1$ and $B_2 \in \mathbb{B}_2$, we have

$$\bigvee_{\substack{x_i\in X_i\\ i=1,2}} B_1 \times B_2((x_1, x_2)) = \bigvee_{\substack{x_i\in X_i\\ i=1,2}} B_1(x_1) \wedge B_2(x_2) = \top$$

since $\bigvee_{x_i\in X_i} B_i(x_i) = \top$ for $i = 1, 2$ holds by $\mathbb{B}_i$ satisfying ($\mathbb{B}$2). Then the condition ($\mathbb{B}$2) is satisfied by $\mathbb{B}$. □

Let $\mathbb{F}_i \in \mathcal{F}_L^{\top}(X_i)$ for $i = 1, 2$. Because every $\top$-filter is a base of itself, from Lemma 3.3 above, $\{B_1 \times B_2 \,|\, B_i \in \mathbb{F}_i, i = 1, 2\}$ is a $\top$-filter base, which generates a $\top$-filter, denoted by $\mathbb{F}_1 \times \mathbb{F}_2$. And from Remark 2.7, $\mathbb{F}_1 \times \mathbb{F}_2$ can be determined by

$$\mathbb{F}_1 \times \mathbb{F}_2 = \{A \in L^{X_1\times X_2} \mid \bigvee_{\substack{B_i\in\mathbb{F}_i\\ i=1,2}} \mathcal{S}_{X_1\times X_2}\big(B_1 \times B_2, A\big) = \top\}.$$

Let $\varphi_i : X_i \to Y_i$ (i=1,2) be a map. By Remark 2.7 and Definition 2.8, we know $\{\varphi_i^{\to}(B_i) \in L^{Y_i} \mid B_i \in \mathbb{F}_i\}$ is a base of $\varphi_i^{\Rightarrow}(\mathbb{F}_i)$. So

$$\{\varphi_1^{\to}(B_1) \times \varphi_2^{\to}(B_2) \mid B_i \in \mathbb{F}_i, i = 1, 2\}$$

is a $\top$-filter base from Lemma 3.3 and generates a $\top$-filter $\varphi_1^{\Rightarrow}(\mathbb{F}_1) \times \varphi_2^{\Rightarrow}(\mathbb{F}_2)$.

In addition, we could check that $(\varphi_1 \times \varphi_2)^{\to}(B_1 \times B_2) = \varphi_1^{\to}(B_1) \times \varphi_2^{\to}(B_2)$ is true. Then $\{(\varphi_1 \times \varphi_2)^{\to}(B_1 \times B_2) \mid B_i \in \mathbb{F}_i, i = 1, 2\}$ equals

$$\{\varphi_1^{\to}(B_1) \times \varphi_2^{\to}(B_2) \mid B_i \in \mathbb{F}_i, i = 1, 2\}$$

and is also a $\top$-filter base. And it generates a $\top$-filter $(\varphi_1 \times \varphi_2)^{\Rightarrow}(\mathbb{F}_1 \times \mathbb{F}_2)$ by Definition 2.8. So we can get the following lemma.

Lemma 3.4. *Let $\varphi : X_1 \to Y_1$ and $\psi : X_2 \to Y_2$ be two maps, $\mathbb{F}_1 \in \mathcal{F}_L^{\top}(X_1)$, $\mathbb{F}_2 \in \mathcal{F}_L^{\top}(X_2)$. Then $(\varphi \times \psi)^{\Rightarrow}(\mathbb{F}_1 \times \mathbb{F}_2) = \varphi^{\Rightarrow}(\mathbb{F}_1) \times \psi^{\Rightarrow}(\mathbb{F}_2)$.*

Lemma 3.5. *Let $(X \times Y, \lim^X \times \lim^Y)$ be the product of two ⊤-**Conv**-objects $(X, \lim^X)$ and $(Y, \lim^Y)$. If $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ be the projection maps, then for any $\mathbb{F} \in \mathcal{F}_L^{\top}(X)$ and $\mathbb{G} \in \mathcal{F}_L^{\top}(Y)$ the following are valid:*

$$(1)\ p_X^{\Rightarrow}(\mathbb{F} \times \mathbb{G}) = \mathbb{F},\ \ (2)\ p_Y^{\Rightarrow}(\mathbb{F} \times \mathbb{G}) = \mathbb{G}.$$

Proof. In the proof, we only show the conclusion (1) for example. In general, $p_X^{\rightarrow}(A \times B) = A$ for each $A \in \mathbb{F}$, $B \in \mathbb{G}$ since for all $x \in X$,

$$p_X^{\rightarrow}(A \times B)(x) = \bigvee_{p_X(x,y)=x} A \times B((x,y)) = A(x) \wedge \bigvee_{y \in Y} B(y) = A(x) \wedge \top = A(x),$$

where $\bigvee_{y \in Y} B(y) = \top$ owing to $B \in \mathbb{G}$. Thus for each $C \in L^X$, $C \in p_X^{\Rightarrow}(\mathbb{F} \times \mathbb{G})$ if and only if $\bigvee_{A \in \mathbb{F}, B \in \mathbb{G}} \mathcal{S}_X\big(p_X^{\rightarrow}(A \times B), C\big) = \top$, i.e. $\bigvee_{A \in \mathbb{F}} \mathcal{S}_X(A, C) = \top$, which is equivalent to $C \in \mathbb{F}$. Consequently, $p_X^{\Rightarrow}(\mathbb{F} \times \mathbb{G}) = \mathbb{F}$. □

Lemma 3.6 (Jäger [10])**.** *If $\varphi \in C_{\top}(X, Y)$, then $\varphi^{\rightarrow}(A) = ev_{X,Y}^{\rightarrow}(1_{\varphi} \times A)$ holds for any $A \in L^X$, here $1_{\varphi} \in L^{C_{\top}(X,Y)}$ such that $1_{\varphi}(\varphi) = \top$ and $1_{\varphi}(\psi) = \bot$ for $\psi \neq \varphi$.*

Now, we begin to confirm the existence of power object and the continuity of the evaluation map by the following propositions.

Proposition 3.7. *Let $(X, \lim^X)$ and $(Y, \lim^Y)$ be two ⊤-**Conv**-objects. A map*

$$\lim_C : \mathcal{F}_L^{\top}\big(C_{\top}(X, Y)\big) \to \mathcal{P}\big(C_{\top}(X, Y)\big)$$

is defined for all $\mathbb{H} \in \mathcal{F}_L^{\top}\big(C_{\top}(X, Y)\big)$, by

$$\lim_C \mathbb{H} = \{\varphi \in C_{\top}(X, Y) \mid \forall x \in X,\ \forall \mathbb{F} \in \mathcal{F}_L^{\top}(X),$$
$$x \in \lim^X \mathbb{F} \Rightarrow \varphi(x) \in \lim^Y ev_{X,Y}^{\Rightarrow}(\mathbb{H} \times \mathbb{F})\}.$$

Then $\lim_C$ is a ⊤-convergence on $C_{\top}(X, Y)$.

Proof. We have to check the map $\lim_C$ satisfies the axioms (TC1) and (TC2). The axiom (TC2) follows immediately from the definition of $\lim_C$.

To check the map $\lim_C$ satisfies the axioms (TC1), we have to show $\varphi \in \lim_C [\varphi]_{\top}$ holds for each $\varphi \in C_{\top}(X, Y)$, where $[\varphi]_{\top} = \{D \in L^{C_{\top}(X,Y)} \mid D(\varphi) = \top\}$ is the point ⊤-filter of φ on $C_{\top}(X, Y)$. And it suffices to check $x \in \lim^X \mathbb{F}$ implies $\varphi(x) \in \lim^Y ev_{X,Y}^{\Rightarrow}([\varphi]_{\top} \times \mathbb{F})$ for all $x \in X$ and $\mathbb{F} \in \mathcal{F}_L^{\top}(X)$ by the definition of $\lim_C$. In fact, we firstly observe that for each $C \in \varphi^{\Rightarrow}(\mathbb{F})$,

$$\begin{aligned} \top &= \bigvee_{A \in \mathbb{F}} \mathcal{S}_Y\big(\varphi^{\rightarrow}(A), C\big) \\ &= \bigvee_{A \in \mathbb{F}} \mathcal{S}_Y\big(ev_{X,Y}^{\rightarrow}(1_{\varphi} \times A), C\big) \ \ \text{(by Lemma 3.6)} \\ &\leq \bigvee_{E \in [\varphi]_{\top} \times \mathbb{F}} \mathcal{S}_Y\big(ev_{X,Y}^{\rightarrow}(E), C\big) \ \ \text{(here, } 1_{\varphi} \times A \in ([\varphi]_{\top} \times \mathbb{F}) \text{ for } A \in \mathbb{F}), \end{aligned}$$

which deduces $C \in ev^{\Rightarrow}_{X,Y}([\varphi]_\top \times \mathbb{F})$ by Remark 2.9 (1). Thus from the arbitrariness of C, $\varphi^{\Rightarrow}(\mathbb{F}) \subseteq ev^{\Rightarrow}_{X,Y}([\varphi]_\top \times \mathbb{F})$ is obtained.

Finally, since $x \in \lim^X \mathbb{F}$ means $\varphi(x) \in \lim^Y \varphi^{\Rightarrow}(\mathbb{F})$ by the continuity of φ for each $x \in X$, we have $\varphi(x) \in \lim^Y \varphi^{\Rightarrow}(\mathbb{F}) \subseteq \lim^Y ev^{\Rightarrow}_{X,Y}([\varphi]_\top \times \mathbb{F})$ by using the property of $\lim^Y$ satisfying the axiom (TC2). □

Proposition 3.8. *Let $(X, \lim^X)$ and $(Y, \lim^Y)$ be two $\top$-**Conv**-objects. Then the evaluation map $ev_{X,Y} : \big(C_\top(X,Y) \times X, \lim_C \times \lim^X\big) \to (Y, \lim^Y)$ is continuous.*

Proof. Firstly, we point out the fact that $ev^{\Rightarrow}_{X,Y}\big(p^{\Rightarrow}_{C_\top(X,Y)}(\mathbb{K}) \times p^{\Rightarrow}_X(\mathbb{K})\big) \subseteq ev^{\Rightarrow}_{X,Y}(\mathbb{K})$ holds for each $\mathbb{K} \in \mathcal{F}^\top_L\big(C_\top(X,Y) \times X\big)$. To confirm the fact, it suffices to show $p^{\Rightarrow}_{C_\top(X,Y)}(\mathbb{K}) \times p^{\Rightarrow}_X(\mathbb{K}) \subseteq \mathbb{K}$. Since $p^{\rightarrow}_{C_\top(X,Y)}(A) \times p^{\rightarrow}_X(B) \geqslant A \wedge B$ holds for all $A,\ B \in \mathbb{K}$, we have

$$\begin{aligned}\bigvee_{C\in\mathbb{K}} \mathcal{S}_{C_\top(X,Y)\times X}(C,D) &\geq \bigvee_{A,B\in\mathbb{K}} \mathcal{S}_{C_\top(X,Y)\times X}(A\wedge B, D)\\ &\geq \bigvee_{A,B\in\mathbb{K}} \mathcal{S}_{C_\top(X,Y)\times X}\big(p^{\rightarrow}_{C_\top(X,Y)}(A) \times p^{\rightarrow}_X(B), D\big)\\ &= \top\end{aligned}$$

for every $D \in p^{\Rightarrow}_{C_\top(X,Y)}(\mathbb{K}) \times p^{\Rightarrow}_X(\mathbb{K})$. Thus whenever $D \in p^{\Rightarrow}_{C_\top(X,Y)}(\mathbb{K}) \times p^{\Rightarrow}_X(\mathbb{K})$, $D \in \mathbb{K}$ follows from ($\mathbb{F}$1), which is to say $p^{\Rightarrow}_{C_\top(X,Y)}(\mathbb{K}) \times p^{\Rightarrow}_X(\mathbb{K}) \subseteq \mathbb{K}$ holds.

Now, we show the continuity of $ev_{X,Y}$ as follows:

Take any $(\varphi, x) \in C_\top(X,Y) \times X$ and any $\mathbb{K} \in \mathcal{F}^\top_L\big(C_\top(X,Y) \times X\big)$ such that $(\varphi, x) \in (\lim_C \times \lim^X)\mathbb{K}$. Then $\varphi \in \lim_C p^{\Rightarrow}_{C_\top(X,Y)}(\mathbb{K})$ and $x \in \lim^X p^{\Rightarrow}_X(\mathbb{K})$ hold. We have $\varphi(x) \in \lim^Y ev^{\Rightarrow}_{X,Y}\big(p^{\Rightarrow}_{C_\top(X,Y)}(\mathbb{K}) \times p^{\Rightarrow}_X(\mathbb{K})\big)$ from this and the definition of $\lim_C$. Finally, by using the fact above and the axiom (TC2), we conclude

$$ev_{X,Y}(\varphi, x) = \varphi(x) \in \lim^Y ev^{\Rightarrow}_{X,Y}(\mathbb{K}).$$

Consequently, $ev_{X,Y}$ is continuous. □

Lemma 3.9. *Let $(X, \lim^X)$, $(Y, \lim^Y)$ and $(Z, \lim^Z)$ be $\top$-convergence spaces. If the map $\psi : (Z \times X, \lim^Z \times \lim^X) \to (Y, \lim^Y)$ is continuous, then for each $z \in Z$, the map $\psi(z, -) : (X, \lim^X) \to (Y, \lim^Y)$ is also continuous.*

Proof. To show the continuity of $\psi(z,-)$ for each $z \in Z$, take any $x \in X$ and $\mathbb{F} \in \mathcal{F}^\top_L(X)$ such that $x \in \lim^X \mathbb{F}$. $(z,x) \in (\lim^Z \times \lim^X)([z]_\top \times \mathbb{F})$ follows from $z \in \lim^Z [z]_\top$ and Lemma 3.5. Further, by the continuity of ψ, we observe that

$$\psi(z,-)(x) = \psi(z,x) \in \lim^Y \psi^{\Rightarrow}([z]_\top \times \mathbb{F}).$$

Notice that for $A \in [z]_\top$ and $B \in \mathbb{F}$, $\psi^{\rightarrow}(A \times B) \geq \psi(z,-)^{\rightarrow}(B)$ holds because

$$\psi^{\rightarrow}(A\times B)(y) = \bigvee_{\psi(u,v)=y} A(u) \wedge B(v) \geq \bigvee_{\psi(z,v)=y} B(v) = \psi(z,-)^{\rightarrow}(B)(y)$$

for all $y \in Y$. From this, we observe that for $C \in \psi^{\Rightarrow}([z]_\top \times \mathbb{F})$,

$$\bigvee_{B\in\mathbb{F}} \mathcal{S}_Y\big(\psi(z,-)^{\rightarrow}(B), C\big) \geq \bigvee_{A\in[z]_\top, B\in\mathbb{F}} \mathcal{S}_Y\big(\psi^{\rightarrow}(A\times B), C\big) = \top,$$

which deduces $C \in \psi(z,-)^{\Rightarrow}(\mathbb{F})$. Thus, $\psi^{\Rightarrow}([z]_\top \times \mathbb{F}) \subseteq \psi(z,-)^{\Rightarrow}(\mathbb{F})$ follows from the arbitrariness of C. Finally, by the axiom (TC2) we obtain that for each $z \in Z$,

$$\psi(z,-)(x) \in \lim{}^Y \psi^{\Rightarrow}([z]_\top \times \mathbb{F}) \subseteq \lim{}^Y \psi(z,-)^{\Rightarrow}(\mathbb{F}).$$

In sum, the continuity of $\psi(z,-)$ is proved as desired. □

Let $(X, \lim^X)$, $(Y, \lim^Y)$ and $(Z, \lim^Z)$ be ⊤-convergence spaces and

$$\psi : (Z \times X, \lim{}^Z \times \lim{}^X) \to (Y, \lim{}^Y)$$

be a continuous map. We define a map $\psi^* : Z \to C_\top(X,Y)$ by $\psi^*(z) = \psi(z,-)$ for all $z \in Z$. Then by Lemma 3.9, ψ^* is *well-defined* and the following lemma confirm that $\psi^* : (Z, \lim^Z) \to \big(C_\top(X,Y), \lim_C\big)$ is continuous.

Lemma 3.10. *Let $(X, \lim^X)$, $(Y, \lim^Y)$ and $(Z, \lim^Z)$ be ⊤-convergence spaces. If $\psi : (Z \times X, \lim^Z \times \lim^X) \to (Y, \lim^Y)$ is a continuous mapping, then the mapping $\psi^* : (Z, \lim^Z) \to \big(C_\top(X,Y), \lim_C\big)$ is continuous.*

Proof. Take any $z \in Z$ and $\mathbb{G} \in \mathcal{F}_L^\top(Z)$ such that $z \in \lim^Z \mathbb{G}$. We have to show

$$\psi^*(z) = \psi(z,-) \in \lim{}_C \psi^{*\Rightarrow}(\mathbb{G}).$$

For this, assume that $x \in \lim^X \mathbb{F}$, here $x \in X$ and $\mathbb{F} \in \mathcal{F}_L^\top(X)$. Then we have

$$(z,x) \in (\lim{}^Z \times \lim{}^X)(\mathbb{G} \times \mathbb{F}).$$

Immediately, $\psi(z,-)(x) = \psi(z,x) \in \lim^Y \psi^{\Rightarrow}(\mathbb{G}\times\mathbb{F})$ follows from the continuity of ψ. Further by means of $ev_{X,Y} \circ (\psi^* \times id_X) = \psi$ and Lemma 3.4, we observe

$$\lim{}^Y \psi^{\Rightarrow}(\mathbb{G}\times\mathbb{F}) = \lim{}^Y \big(ev_{X,Y} \circ (\psi^* \times id_X)\big)^{\Rightarrow}(\mathbb{G}\times\mathbb{F}) = \lim{}^Y ev_{X,Y}^{\Rightarrow}\big(\psi^{*\Rightarrow}(\mathbb{G}) \times \mathbb{F}\big).$$

Hence $\psi^*(z) = \psi(z,-) \in \lim_C \psi^{*\Rightarrow}(\mathbb{G})$ is obtained from the definition of $\lim_C$. □

Let $(X, \lim^X)$ and $(Y, \lim^Y)$ be two ⊤-**Conv**-objects. From Propositions 3.7, 3.8 and Lemma 3.10, there are the ⊤-**Conv**-object $\big(C_\top(X,Y), \lim_C\big)$ and the continuous map $ev_{X,Y} : \big(C_\top(X,Y) \times X, \lim_C \times \lim^X\big) \to (Y, \lim^Y)$ such that for each ⊤-**Conv**-object $(Z, \lim^Z)$ and each continuous map

$$\psi : (Z \times X, \lim{}^Z \times \lim{}^X) \to (Y, \lim{}^Y),$$

there exists a unique continuous map $\psi^* : (Z, \lim^Z) \to \big(C_\top(X,Y), \lim_C\big)$ satisfying the equality $ev_{X,Y} \circ (\psi^* \times id_X) = \psi$. Thus by the definition of Cartesian-closedness, we obtain the following theorem.

Theorem 3.11. *The category ⊤-**Conv** of ⊤-convergence spaces is Cartesian-closed.*

4. Relation Between $\top$-convergences and Strong L-topologies

In this section, we show that there is a close relation between $\top$-convergences and strong L-topologies on a universal set X. In details, we will demonstrate that the category of strong L-topological spaces is concretely isomorphic to that of strong L-topological $\top$-convergence spaces categorically and embedded in the category $\top$-**Conv** of $\top$-convergence spaces as a bireflective subcategory.

Firstly, we recall the definition of strong L-topological spaces [22] as follows.

Definition 4.1. Let X be a nonempty set. A *strong L-topological space* is a pair (X, τ), where τ a subset of L^X such that the following conditions are satisfied:

(ST1) $0_X,\ 1_X \in \tau$,

(ST2) $U_1 \wedge U_2 \in \tau$ for all $U_1,\ U_2 \in \tau$,

(ST3) $\bigvee_{j\in J} U_j \in \tau$ for every family $\{U_j \mid j \in J\} \subseteq \tau$,

(ST4) $\alpha * U \in \tau$ for all $\alpha \in L$ and $U \in \tau$,

(ST5) $\alpha \to U \in \tau$ for all $\alpha \in L$ and $U \in \tau$.

If (X, τ) is a strong L-topological space, then τ is called a *strong L-topology* on the set X. The set of all strong L-topologies on X is denoted by $ST_L(X)$. A map $\varphi : (X, \tau) \to (Y, \delta)$ between strong L-topological spaces is called *continuous map* provided that $\varphi^{\leftarrow}(V) \in \tau$ for each $V \in \delta$. We denote the category of strong L-topological spaces and continuous maps by **STOP**(L).

Definition 4.2. Let X be a nonempty set. If $\mathcal{U} = \{\mathbb{U}^x\}_{x\in X}$ is a family of $\top$-filters satisfying the axiom

(N) $$\forall x \in X,\ \forall B \in \mathbb{U}^x,\ B(x) = \top,$$

we call $\mathcal{U}$ a *system of $\top$-neighborhoods* on X. And if a system of $\top$-neighborhoods $\mathcal{U}$ still satisfies the axiom

(TT) For any $x \in X$ and each $B \in \mathbb{U}^x$, there exists $B^* \in \mathbb{U}^x$ with $B^* \leq B$ such that for every $y \in X$, there exists $B_y \in \mathbb{U}^y$ satisfying $B^*(y) \leq \mathcal{S}_X(B_y, B)$,

$\mathcal{U}$ is called a *strong L-topological system of $\top$-neighborhoods* on X.

Example 4.3. (1) Let $(X, \lim)$ be a $\top$-convergence space. We denote $\mathcal{U}_{\lim} = \{\mathbb{U}^x_{\lim}\}_{x\in X}$, where $\mathbb{U}^x_{\lim} = \bigcap\{\mathbb{F} \in \mathcal{F}^{\top}_L(X) \mid x \in \lim \mathbb{F}\}$ for each $x \in X$. Then $\mathcal{U}_{\lim}$ is a system of $\top$-neighborhoods.
(2) Let (X, τ) be a strong L-topological space. For each $x \in X$, $\mathbb{U}^x_\tau$ is defined by $\mathbb{U}^x_\tau = \{B \in L^X \mid \bigvee_{U\in\tau} U(x) * \mathcal{S}_X(U, B) = \top\}$. Then $\{\mathbb{U}^x_\tau\}_{x\in X}$ is a strong L-topological system of $\top$-neighborhoods on X. The detailed contents of the proof see [9].

Definition 4.4. A $\top$-convergence space $(X, \lim)$ is said to be *strong L-topological* provided that $\{\mathbb{U}^x_{\lim}\}_{x\in X}$ satisfies the axioms (TT) and

(TP) $$x \in \lim \mathbb{U}^x_{\lim} \text{ for all } x \in X.$$

The set of all strong L-topological $\top$-convergences on X is denoted by $STC^{\top}(X)$. The category of strong L-topological $\top$-convergence spaces and continuous maps is denoted by $\top$-**STConv**.

Example 4.5. Let (X,τ) be a strong L-topological space. The structure map $\lim_\tau : \mathcal{F}_L^\top(X) \to \mathcal{P}(X)$ induced by τ is defined by

$$\lim_\tau \mathbb{F} = \{x \in X \mid \mathbb{U}_\tau^x \subseteq \mathbb{F}\}, for all \ \ \mathbb{F} \in \mathcal{F}_L^\top(X), \tag{1}$$

Then $(X, \lim_\tau)$ is a strong L-topological ⊤-convergence space.

Höhle [9] demonstrated that a strong L-topological space (X,τ) could induce a strong L-topological system of ⊤-neighborhoods $\{\mathbb{U}_\tau^x\}_{x\in X}$ (see Example 4.3 (2)). Besides, he also showed a strong L-topological system of ⊤-neighborhoods $\mathcal{U}$ can induce a strong L-topology given by

$$\tau_{\mathcal{U}} = \{U \in L^X \mid U(x) \le \bigvee_{B\in\mathbb{U}^x} \mathcal{S}_X(B,U), \forall x \in X\}.$$

Significantly, for the proofs of the above contents, it is dispensable that the condition Lemma 2.2 (M4) (i.e. $\to$ preserving arbitrary union) which holds in the context of a complete MV-algebra.

Notice that if $(X,\lim)$ is a ⊤-convergence space, $\{\mathbb{U}_{\lim}^x\}_{x\in X}$ is a system of ⊤-neighborhoods on X. Naturally, we can get a strong L-topology

$$\tau_{\lim} = \{U \in L^X \mid U(x) \le \bigvee_{B\in\mathbb{U}_{\lim}^x} \mathcal{S}_X(B,U), \forall x \in X\}. \tag{2}$$

In fact, there exists a bijection between the set $STC^\top(X)$ of all strong L-topological ⊤-convergence structures on a set X and the set $ST_L(X)$ of all strong L-topologies on the X. We need a lemma in preparation for it.

Lemma 4.6. *Let $(X,\lim)$ be a ⊤-convergence space. Then*

(1) for every $x \in X$, $\mathbb{U}_{\tau_{\lim}}^x \subseteq \mathbb{U}_{\lim}^x$ is valid.

(2) if $(X,\lim)$ is a strong L-topological, then for any $x \in X$, $\mathbb{U}_{\lim}^x = \mathbb{U}_{\tau_{\lim}}^x$ holds.

Proof. (1) Take any $x \in X$ and let $B \in \mathbb{U}_{\tau_{\lim}}^x$, i.e. $\bigvee_{U\in\tau_{\lim}} U(x) * \mathcal{S}_X(U,B) = \top$ by Example 4.3 (2). From the formula (2) and Lemma 2.3 (2), we observe that $U \in \tau_{\lim}$ means

$$\begin{aligned} U(x) &\le \bigvee_{C\in\mathbb{U}_{\lim}^x} \mathcal{S}_X(C,U) \\ &\le \bigvee_{C\in\mathbb{U}_{\lim}^x} \big(\mathcal{S}_X(U,B) \to \mathcal{S}_X(C,B)\big) \\ &= \mathcal{S}_X(U,B) \to \bigvee_{C\in\mathbb{U}_{\lim}^x} \mathcal{S}_X(C,B), \end{aligned}$$

which is to say $U(x) * \mathcal{S}_X(U,B) \le \bigvee_{C\in\mathbb{U}_{\lim}^x} \mathcal{S}_X(C,B)$. By the arbitrariness of U, we get $\top = \bigvee_{U\in\tau_{\lim}} U(x) * \mathcal{S}_X(U,B) \le \bigvee_{C\in\mathbb{U}_{\lim}^x} \mathcal{S}_X(C,B)$. From this and (𝔽1), $B \in \mathbb{U}_{\lim}^x$ holds, i.e. $\mathbb{U}_{\tau_{\lim}}^x \subseteq \mathbb{U}_{\lim}^x$ is obtained and the proof of the conclusion is completed.

(2) Let $(X,\lim)$ be a strong L-topological space. For any $x \in X$, we only need to show $\mathbb{U}_{\lim}^x \subseteq \mathbb{U}_{\tau_{\lim}}^x$ by using (1). Now take any $B \in \mathbb{U}_{\lim}^x$ and define $\overline{B} \in L^X$

by $\overline{B}(y) = \bigvee_{A\in\mathbb{U}^y_{\lim}} \mathcal{S}_X(A,B)$ for each $y \in X$. Then $\overline{B}(x) = \top$ and $\overline{B} \le B$ follows respectively from $\overline{B}(x) = \bigvee_{A\in\mathbb{U}^x_{\lim}} \mathcal{S}_X(A,B) \ge \mathcal{S}_X(B,B) = \top$ and for each $y \in X$

$$\overline{B}(y) = \bigvee_{A\in\mathbb{U}^y_{\lim}} \mathcal{S}_X(A,B) \le \bigvee_{A\in\mathbb{U}^y_{\lim}} (A(y)\to B(y)) = \bigvee_{A\in\mathbb{U}^y_{\lim}} (\top\to B(y)) = B(y).$$

In the following, we additionally confirm $\overline{B} \in \tau_{\lim}$ is also true by means of the axiom (TT). In fact, for each $y \in X$, the axiom (TT) tell us $A \in \mathbb{U}^y_{\lim}$ implies there is $A^* \in \mathbb{U}^y_{\lim}$ with $A^* \le A$ such that for all $z \in X$, there exists $A_z \in \mathbb{U}^z_{\lim}$ to satisfy $A^*(z) \le \mathcal{S}_X(A_z, A)$. Then we get $\mathcal{S}_X(A,B) \le \mathcal{S}_X(A^*,\overline{B})$ because for all $z \in X$

$$\begin{aligned}\mathcal{S}_X(A,B) &\le \mathcal{S}_X(A_z,A) \to \mathcal{S}_X(A_z,B)\\ &\le A^*(z) \to \mathcal{S}_X(A_z,B)\\ &\le A^*(z) \to \big(\bigvee_{C\in\mathbb{U}^z_{\lim}} \mathcal{S}_X(C,B)\big)\\ &= A^*(z) \to \overline{B}(z).\end{aligned}$$

From this, we conclude that for all $y \in X$,

$$\overline{B}(y) = \bigvee_{A\in\mathbb{U}^y_{\lim}} \mathcal{S}_X(A,B) \le \bigvee_{A^*\in\mathbb{U}^y_{\lim}} \mathcal{S}_X(A^*,\overline{B}) \le \bigvee_{C\in\mathbb{U}^y_{\lim}} \mathcal{S}_X(C,\overline{B}).$$

Namely, $\overline{B} \in \tau_{\lim}$ by the definition of $\tau_{\lim}$. From all above, we have

$$\bigvee_{U\in\tau_{\lim}} U(x) * \mathcal{S}_X(U,B) \ge \overline{B}(x) * \mathcal{S}_X(\overline{B},B) = \top,$$

i.e. $B \in \mathbb{U}^x_{\tau_{\lim}}$ holds. Finally, $\mathbb{U}^x_{\lim} \subseteq \mathbb{U}^x_{\tau_{\lim}}$ follows from the arbitrariness of B. □

Theorem 4.7. *Let X be a nonempty set.*

(1) Given a strong L-topology τ on X, then $\tau_{\lim_\tau} = \tau$.

(2) Given a strong L-topological $\top$-convergence $\lim$ on X, then $\lim_{\tau_{\lim}} = \lim$.

Proof. (1) By the formula (2), we know

$$\tau_{\lim_\tau} = \{U \in L^X \mid U(x) \le \bigvee_{A\in\mathbb{U}^x_{\lim_\tau}} \mathcal{S}_X(A,U),\ \forall x \in X\}.$$

From Example 4.3 and the formula (1), we observe that for all $x \in X$,

$$\mathbb{U}^x_{\lim_\tau} = \bigcap\{\mathbb{F} \in \mathcal{F}^\top_L(X) \mid x \in \lim_\tau\mathbb{F}\} = \bigcap\{\mathbb{F} \in \mathcal{F}^\top_L(X) \mid \mathbb{U}^x_\tau \subseteq \mathbb{F}\} = \mathbb{U}^x_\tau.$$

Above all, $\tau_{\lim_\tau} = \{U \in L^X \mid U(x) \le \bigvee_{A\in\mathbb{U}^x_\tau} \mathcal{S}_X(A,U),\ \forall x \in X\}$. In fact, $U \in \tau$ if and only if $U(x) \le \bigvee_{B\in\mathbb{U}^x_\tau} \mathcal{S}_X(B,U)$ holds for every $x \in X$ (see [9]). Then $\tau_{\lim_\tau} = \tau$ is true.

(2) In order to show $\lim_{\tau_{\lim}} = \lim$, take any $\mathbb{F} \in \mathcal{F}^\top_L(X)$.

$$\lim_{\tau_{\lim}}\mathbb{F} = \{x \in X \mid \mathbb{U}^x_{\tau_{\lim}} \subseteq \mathbb{F}\} = \{x \in X \mid \mathbb{U}^x_{\lim} \subseteq \mathbb{F}\}$$

follows from Lemma 4.6 (2). Further, for each $x \in X$ and $\mathbb{F} \in \mathcal{F}_L^\top(X)$, we have

$$x \in \lim_{\tau_{\lim}} \mathbb{F} \Longleftrightarrow \mathbb{U}^x_{\lim} \subseteq \mathbb{F} \Longleftrightarrow x \in \lim \mathbb{F}.$$

Consequently, $\lim_{\tau_{\lim}} = \lim$ is true. □

Corollary 4.8. *Let X be a nonempty set. Then there is a bijection between the set of all strong L-topological ⊤-convergences and the set of all strong L-topologies on X.*

In order to explore the relation between the category of strong L-topological spaces and that of ⊤-convergence spaces, the following theorem is indispensable.

Theorem 4.9. *Let (X, τ) and (Y, δ) be strong L-topological spaces, $(X, \lim^X)$ and $(Y, \lim^Y)$ be ⊤-convergence spaces. Then the following are valid:*

(1) If $\varphi : (X, \tau) \to (Y, \delta)$ is a continuous map, then $\varphi : (X, \lim_\tau) \to (Y, \lim_\delta)$ is a continuous map.

(2) If $\varphi : (X, \lim^X) \to (Y, \lim^Y)$ is a continuous map, then $\varphi : (X, \tau_{\lim^X}) \to (Y, \delta_{\lim^Y})$ is a continuous map.

Proof. (1) First, we point out the fact given in [9] that a map $\varphi : (X, \tau) \to (Y, \delta)$ is continuous if and only if φ is continuous at each $x \in X$ in the sense that $\mathbb{U}^{\varphi(x)}_\delta \subseteq \varphi^\Rightarrow(\mathbb{U}^x_\tau)$ holds. Let $x \in X$ and $\mathbb{F} \in \mathcal{F}_L^\top(X)$ such that $x \in \lim_\tau \mathbb{F}$, equivalently $\mathbb{U}^x_\tau \subseteq \mathbb{F}$. Then $\varphi^\Rightarrow(\mathbb{U}^x_\tau) \subseteq \varphi^\Rightarrow(\mathbb{F})$ follows from (i) in Remark 2.9 (3). Because the mapping $\varphi : (X, \tau) \to (Y, \delta)$ is continuous, which is to say $\mathbb{U}^{\varphi(x)}_\delta \subseteq \varphi^\Rightarrow(\mathbb{U}^x_\tau)$ holds from the fact above. Hence $\mathbb{U}^{\varphi(x)}_\delta \subseteq \varphi^\Rightarrow(\mathbb{F})$ is true, which already means that $\varphi(x) \in \lim_\delta \varphi^\Rightarrow(\mathbb{F})$. Consequently, the continuity of φ from $(X, \lim_\tau)$ to $(Y, \lim_\delta)$ is proved as desired.

(2) Firstly, we are going to show that $\varphi^\Leftarrow(\mathbb{U}^{\varphi(x)}_{\lim^Y}) \subseteq \mathbb{U}^x_{\lim^X}$ holds, here

$$\mathbb{U}^x_{\lim^X} = \bigcap\{\mathbb{F} \in \mathcal{F}_L^\top(X) \mid x \in \lim^X \mathbb{F}\}$$

defined in Example 4.3 (1). For all $C \in \mathbb{U}^{\varphi(x)}_{\lim^Y}$, $C(\varphi(x)) = \top$ holds by Example 4.3 (1). So $\varphi^\Leftarrow(\mathbb{U}^{\varphi(x)}_{\lim^Y})$ exists from Definition 2.8 (2). For any $\mathbb{F} \in \mathcal{F}_L^\top(X)$ with $x \in \lim^X \mathbb{F}$, by the continuity of φ from $(X, \lim^X)$ to $(Y, \lim^Y)$, we have $\varphi(x) \in \lim^Y \varphi^\Rightarrow(\mathbb{F})$. Then $\mathbb{U}^{\varphi(x)}_{\lim^Y} \subseteq \varphi^\Rightarrow(\mathbb{F})$ holds. From this and (ii) in Remark 2.9 (3),

$$\varphi^\Leftarrow(\mathbb{U}^{\varphi(x)}_{\lim^Y}) \subseteq \varphi^\Leftarrow \circ \varphi^\Rightarrow(\mathbb{F}) \subseteq \mathbb{F}$$

is valid. Hence we get

$$\varphi^\Leftarrow(\mathbb{U}^{\varphi(x)}_{\lim^Y}) \subseteq \bigcap\{\mathbb{F} \in F_L^\top(X) \mid x \in \lim^X \mathbb{F}\} = \mathbb{U}^x_{\lim^X}.$$

Now, we prove $\varphi : (X, \tau_{\lim^X}) \to (Y, \delta_{\lim^Y})$ is continuous. Take any $V \in \delta_{\lim^Y}$, and hence $V\big(\varphi(x)\big) \leq \bigvee_{B \in \mathbb{U}^{\varphi(x)}_{\lim^Y}} \mathcal{S}_Y(B, V)$ for $x \in X$. In this case, we observe that

$$\begin{aligned}\bigvee_{A\in\mathbb{U}^x_{\lim^X}} \mathcal{S}_X\big(A,\varphi^{\leftarrow}(V)\big) &\geq \bigvee_{A\in\varphi^{\Leftarrow}(\mathbb{U}^{\varphi(x)}_{\lim^Y})} \mathcal{S}_X\big(A,\varphi^{\leftarrow}(V)\big)\\ &\geq \bigvee_{B\in\mathbb{U}^{\varphi(x)}_{\lim^Y}} \mathcal{S}_X\big(\varphi^{\leftarrow}(B),\varphi^{\leftarrow}(V)\big)\\ &\geq \bigvee_{B\in\mathbb{U}^{\varphi(x)}_{\lim^Y}} \mathcal{S}_Y(B,V) \qquad \text{(by Lemma 2.3(4))}\\ &\geq V\big(\varphi(x)\big)=\varphi^{\leftarrow}(V)(x),\end{aligned}$$

which is to say $\varphi^{\leftarrow}(V)(x) \leq \bigvee_{A\in\mathbb{U}^x_{\lim^X}} \mathcal{S}_X\big(A,\varphi^{\leftarrow}(V)\big)$ holds for all $x\in X$. By definition of $\tau_{\lim^X}$, $\varphi^{\leftarrow}(V)\in\tau_{\lim^X}$ holds. Consequently, $\varphi:(X,\tau_{\lim^X})\to(Y,\delta_{\lim^Y})$ is continuous. □

From Corollary 4.8 and Theorem 4.9, the following theorem is obtained.

Theorem 4.10. *The category* **STOP**(L) *of strong L-topological spaces is concretely isomorphic to the category* $\top$-**STConv** *of strong L-topological $\top$-convergence spaces, categorically.*

The following theorem shows the category $\top$-**STConv** is bireflective in the category $\top$-**Conv** in some detail.

Theorem 4.11. *The category* $\top$-**STConv** *of strong L-topological $\top$-convergence spaces is bireflective in the category* $\top$-**Conv** *of $\top$-convergence spaces.*

Proof. Let $(X,\lim)$ be a $\top$-**Conv**-object. We claim that $\mathrm{id}_X:(X,\lim)\to(X,\widetilde{\lim})$ is the bireflection of $(X,\lim)$ w.r.t. $\top$-**STConv**, where $\widetilde{\lim}=\lim_{\tau_{\lim}}$, i.e. for each $\mathbb{F}\in\mathcal{F}^{\top}_L(X)$, $\widetilde{\lim}\mathbb{F}=\{x\in X\mid \mathbb{U}^x_{\tau_{\lim}}\subseteq\mathbb{F}\}$. By Example 4.5, we know $\widetilde{\lim}$ is a strong L-topological $\top$-convergence on X. By Definition 2.12, we need to confirm

(1) $\mathrm{id}_X:(X,\lim)\to(X,\widetilde{\lim})$ is continuous.
(2) For any continuous map $\varphi:(X,\lim)\to(Y,\lim^Y)$, where $(Y,\lim^Y)$ is a $\top$-**STConv**-object, $\varphi:(X,\widetilde{\lim})\to(Y,\lim^Y)$ is continuous.

In fact, for the conclusion (1), take any $\mathbb{F}\in\mathcal{F}^{\top}_L(X)$ and $x\in X$ such that $x\in\lim\mathbb{F}$. Then $\mathbb{U}^x_{\lim}\subseteq\mathbb{F}$ holds. And by Lemma 4.6 (1), $\mathbb{U}^x_{\tau_{\lim}}\subseteq\mathbb{U}^x_{\lim}$. Hence we further obtain $\mathbb{U}^x_{\tau_{\lim}}\subseteq\mathbb{F}$, i.e. $\mathrm{id}_X(x)=x\in\widetilde{\lim}\mathbb{F}=\widetilde{\lim}(\mathrm{id}_X)^{\Rightarrow}(\mathbb{F})$, which can deduces that id_X is continuous.

For the conclusion (2), assume a map $\varphi:(X,\lim)\to(Y,\lim^Y)$ is continuous, where $(Y,\lim^Y)$ is a $\top$-**STConv**-object and hence satisfies $\lim^Y=\lim_{\delta_{\lim^Y}}$ by Theorem 4.7. By Theorem 4.9 (1) and (2), $\varphi:(X,\tau_{\lim})\to(Y,\delta_{\lim^Y})$ is continuous first, and then the map $\varphi:(X,\lim_{\tau_{\lim}})\to(Y,\lim_{\delta_{\lim^Y}})$ is continuous also, which is to say that $\varphi:(X,\widetilde{\lim})\to(Y,\lim^Y)$ is continuous. □

By Theorems 4.10 and 4.11, we can get the following corollary.

Corollary 4.12. *The category* **STOP**(L) *of strong L-topological spaces is embedded in the category* $\top$-**Conv** *of $\top$-convergence spaces as a bireflective subcategory.*

5. Conclusions

In this paper, we proposed the concept of ⊤-convergence spaces based on ⊤-filters. Afterwards, in the lattice valued context of a complete MV-algebra without the idempotency of the semigroup operation, we proved the category of ⊤-convergence spaces is topological and further is Cartesian-closed. Additionally, the category of strong L-topological spaces can be bireflectively embedded in the category of ⊤-convergence spaces.

Interestingly, we could study the corresponding subcategories of ⊤-**Conv** by generalizing the well-known categories of Kent convergence spaces, of limit spaces, of pseudo-topological spaces to the many-valued setting as well as their categorical relations. From the paper [3], we have known the category **STOP**(L) is a reflective subcategory of the category L-**OCS** of L-ordered convergence spaces. Then, is there any link between the category ⊤-**Conv** and the category L-**OCS**? We will study these problems in the future.

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