

## SOME COMPUTATIONAL RESULTS FOR THE FUZZY RANDOM VALUE OF LIFE ACTUARIAL LIABILITIES

J. DE ANDRÉS-SÁNCHEZ AND L. GONZÁLEZ-VILA PUCHADES

ABSTRACT. The concept of fuzzy random variable has been applied in several papers to model the present value of life insurance liabilities. It allows the fuzzy uncertainty of the interest rate and the probabilistic behaviour of mortality to be used throughout the valuation process without any loss of information. Using this framework, and considering a triangular interest rate, this paper develops closed expressions for the expected present value and its defuzzified value, the variance and the distribution function of several well-known actuarial liabilities structures, namely life insurances, endowments and life annuities.

### 1. Introduction

One extended use of fuzzy set theory (FST) in insurance financial pricing consists of modelling uncertain parameters such as interest rates, salary growth rates or claim amounts with fuzzy numbers (see [18], [22] and [5]) in the life insurance context, and [10] and [11] in property liability insurance). Following the arguments depicted in these papers, we also assume that the fuzzy uncertainty only arises from the interest rate used for valuing and that it can be modelled with fuzzy numbers.

The common approach in insurance fuzzy pricing is based on introducing the fuzzy interest rate over the classic equivalence principle. It reduces the probabilities of insured events to deterministic rates of occurrence. Thus, insurance fuzzy pricing becomes non-stochastic and can be calculated using the financial mathematics with fuzzy parameters developed in [6] and [19]. However, with this model, the information that provides the complete statistical description of the insured events is lost.

To avoid this drawback, [2], [3] and [4] developed an approach in a life-insurance context that combines the stochastic approach to life-insurance (see [13] under deterministic interest rates) and the quantification of interest rates with fuzzy numbers. In those papers, fuzzy random variables and measures of solvency in portfolios of policies are used to develop a general formulation for determining the fair value and risk of individual contracts. However, no shape for fuzzy interest rates is assumed a priori and consequently no closed expression for these magnitudes is developed. This paper extends those results by regarding the average interest

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rate throughout the pricing horizon of the liabilities as a triangular fuzzy number (TFN). Notice that TFNs are extensively used in practical applications (see the aforementioned papers in the actuarial field).

Firstly, this paper develops closed expressions for the mathematical expectation of the present value of insured life contingencies and its defuzzified value by means of the  $\beta$ -average value of a fuzzy number introduced by [8]. Following [20] this defuzzifying method has interesting properties in fuzzy-random decision-making environments because fundamentals of fuzzy utility function can be established by means of an axiomatic development in the fuzzy expected utility approach.

We also develop the closed expression of the variance measure by [12] for the present value of those life-insurance structures. Likewise, we obtain a distribution function of the present value of payments that could be useful when simulating the liabilities of life-insurance portfolios.

This paper is organized as follows: in section 2 we describe the concepts of FST used to develop our paper. In section 3 we introduce the fuzzy discount factor induced by a triangular discount rate. Finally, sections 4, 5 and 6 demonstrate how our fuzzy random approach can be used to price life insurance, endowments and life annuities, respectively.

## 2. Basic Concepts of Fuzzy Set Theory

### 2.1. Fuzzy Numbers.

A *fuzzy set*  $\tilde{A}$  is a subset defined over a reference set  $X$  for which the level of membership of an element  $x \in X$  to  $\tilde{A}$  accepts values other than 0 or 1. So,  $\tilde{A}$  can therefore be defined as  $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \mid x \in X\}$ , where  $\mu_{\tilde{A}}(x)$  is the membership function and it is a mapping  $\mu_{\tilde{A}}: X \rightarrow [0, 1]$ . Alternatively, a fuzzy set  $\tilde{A}$  can be represented by its *level sets*  $\alpha$  or  $\alpha$ -cuts. For a fuzzy set  $\tilde{A}$ , we will name an  $\alpha$ -cut with  $A_\alpha$ , its mathematical expression being  $A_\alpha = \{x \in X \mid \mu_{\tilde{A}}(x) \geq \alpha\}$ ,  $0 \leq \alpha \leq 1$ .  $\tilde{A}$  is said to be *normal* if  $\sup_{x \in X} \mu_{\tilde{A}}(x) = 1$ .  $\tilde{A}$  is *convex* if its  $\alpha$ -cuts are closed and bounded intervals for  $0 < \alpha \leq 1$ . In our paper we will use *fuzzy numbers* that are a particular case of convex and normal fuzzy sets where the referential set  $X$  is the set of real numbers  $\mathfrak{R}$ .

A *fuzzy number* (FN)  $\tilde{A}$  is a normal and convex fuzzy set defined over real numbers  $\mathfrak{R}$ . It is the main instrument used in FST for quantifying uncertain quantities. The convexity of  $\tilde{A}$  implies that the  $\alpha$ -cuts of  $\tilde{A}$  are confidence intervals where the lower and upper extremes are increasing (decreasing) functions respect to  $\alpha$ . So,  $\forall \alpha \in (0, 1]$ :

$$A_\alpha = [\underline{A}_\alpha, \bar{A}_\alpha] = \left[ \inf_{x \in X} \{\mu_{\tilde{A}}(x) \geq \alpha\}, \sup_{x \in X} \{\mu_{\tilde{A}}(x) \geq \alpha\} \right] \quad (1)$$

with the convention that  $A_{\tilde{\alpha}=0}$  is the smallest closed interval containing the support of  $\tilde{A}$ , where the support of  $\tilde{A}$  comprises all  $x \in \mathfrak{R}$  that  $\mu_{\tilde{A}}(x) > 0$ . From an intuitive point of view, a FN can be interpreted as a fuzzy quantity approximately equal to the real number for which the membership function takes the value 1.

Triangular fuzzy numbers (TFNs) are widely used in practical applications and, of course, in actuarial problems, as we have shown in the aforementioned literature. We will symbolize a TFN  $\tilde{A}$  as  $\tilde{A} = (a_l, a_c, a_u)$ , where  $a_c$  is the core of  $\tilde{A}$ , i.e.  $\mu_{\tilde{A}}(a_c) = 1$ , and  $a_l, a_u$  are the lower and upper values of the support of  $\tilde{A}$ . Analytically a TFN is characterized by its  $\alpha$ -cuts,  $A_\alpha$ , as:

$$A_\alpha = [\underline{A}_\alpha, \overline{A}_\alpha] = [a_l + (a_c - a_l)\alpha, a_u - (a_u - a_c)\alpha], \forall \alpha \in [0, 1] \quad (2)$$

It is very common in real insurance situations to estimate magnitudes as approximate quantities, for example, by means of a sentence like “the claim provisions must be around 9,000 monetary units”. Clearly, FNs can be used to represent these magnitudes. However, these values often need to be quantified with crisp values or intervals as well. For example, in our context, this will occur whenever the definitive amount of claim provisions needs to be specified in financial statements. This paper proposes using the concept of the expected interval of a FN by [14] and the expected value of a FN by [8].

For a FN  $\tilde{A}$ , the *expected interval* will be symbolised as  $e_I(\tilde{A})$  where:

$$e_I(\tilde{A}) = \left[ \int_0^1 \underline{A}_\alpha d\alpha, \int_0^1 \overline{A}_\alpha d\alpha \right] \quad (3a)$$

Notice that for a linear combination of FNs,  $\sum_{i=1}^n k_i \tilde{A}_i$ ,  $k_i \in \mathfrak{R}$ :

$$e_I\left(\sum_{i=1}^n k_i \tilde{A}_i\right) = \sum_{i=1}^n k_i e_I(\tilde{A}_i) \quad (3b)$$

From the expected interval of a FN we can obtain the  $\beta$ -*expected value* of the FN  $\tilde{A}$ . Given a fixed risk aversion coefficient  $\beta$ ,  $0 \leq \beta \leq 1$ , the  $\beta$ -expected value of  $\tilde{A}$ ,  $e_V(\tilde{A}; \beta)$ , is:

$$e_V(\tilde{A}; \beta) = (1 - \beta) \int_0^1 \underline{A}_\alpha d\alpha + \beta \int_0^1 \overline{A}_\alpha d\alpha \quad (4a)$$

Again, if we consider a linear combination of FNs, it follows that:

$$e_V\left(\sum_{i=1}^n k_i \tilde{A}_i; \beta\right) = \sum_{i=1}^n k_i e_V(\tilde{A}_i; \beta) \quad (4b)$$

Let  $\tilde{A}_1, \dots, \tilde{A}_n$  be  $n$  FNs. Let  $f(\cdot)$  be a continuous function of  $n$  variables. Using Zadeh's *extension principle* ([28]) allows us to define a FN  $\tilde{B}$  induced by the FNs  $\tilde{A}_1, \dots, \tilde{A}_n$  through  $f$  as  $\tilde{B} = f(\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_n)$ . Although it is often impossible to obtain a closed expression for the membership function of  $\tilde{B}$ , in many cases it is possible to obtain its  $\alpha$ -cuts,  $B_\alpha$ , from  $A_{1\alpha}, A_{2\alpha}, \dots, A_{n\alpha}$ . Specifically, [21] demonstrates that:

$$B_\alpha = \{y = f(x_1, x_2, \dots, x_n), x_i \in A_{i\alpha}, i = 1, 2, \dots, n\} \quad (5)$$

In actuarial mathematics, many functional relationships are continuously increasing or decreasing with respect to all the variables involved, in such a way that it is easy to evaluate the  $\alpha$ -cuts of  $\tilde{B}$ . If the function  $f(\cdot)$  that induces  $\tilde{B}$  increases with respect to the first  $m$  variables, where  $m \leq n$ , and decreases with respect to the last  $n-m$  variables, [7] demonstrate that  $B_\alpha$  is:

$$\begin{aligned} B_\alpha &= [\underline{B}_\alpha, \overline{B}_\alpha] = \\ &= \left[ f(\underline{A}_{1\alpha}, \dots, \underline{A}_{m\alpha}, \overline{A}_{m+1\alpha}, \dots, \overline{A}_{n\alpha}), f(\overline{A}_{1\alpha}, \dots, \overline{A}_{m\alpha}, \underline{A}_{m+1\alpha}, \dots, \underline{A}_{n\alpha}) \right] \end{aligned} \quad (6)$$

## 2.2. Fuzzy Random Variables.

In real situations, the uncertainty comes from different sources: randomness, hazard, vagueness, inaccuracy, imprecision, etc. So, following [27], stochastic variability is described with probability theory, but other relevant types of uncertainty such as imprecision can be captured by fuzzy sets. The concept of fuzzy random variable (FRV) combines both random and fuzzy uncertainty and has been developed in several papers (see e.g. [16], [17], [24] and [29]), although there is no single definition of it. This paper uses the concept of FRV introduced in [24] because it highly suits our purposes given that the outcomes of the present value of life insurances, endowments and life annuities are FNs rather than real values due to the fuzzy nature of interest rates.

Let  $\{\Omega, \mathcal{A}\}$  be a measurable space,  $\{\mathfrak{R}, \mathcal{B}\}$  the Borel measurable space and  $F(\mathfrak{R})$  denote the set of FNs. Consider the fuzzy set valued mapping  $\tilde{\mathbf{X}}$ :

$$\begin{aligned} \tilde{\mathbf{X}} : \quad \Omega &\rightarrow F(\mathfrak{R}) \\ \forall \omega \in \Omega &\rightarrow \tilde{\mathbf{X}}(\omega) = \left\{ \left( z, \mu_{\tilde{\mathbf{X}}(\omega)}(z) \right) \right\} \in F(\mathfrak{R}) \end{aligned} \quad (7)$$

So, (7) is called a *fuzzy random variable* if:

$$\forall B \in \mathcal{B}, \forall \alpha \in [0, 1], \quad \{ \omega \in \Omega \mid X(\omega)_\alpha \cap B \neq \emptyset \} \in \mathcal{A} \quad (8)$$

where  $X(\omega)_\alpha = \{ z \in \mathfrak{R} \mid \mu_{\tilde{\mathbf{X}}(\omega)}(z) \geq \alpha \} = [\underline{X}(\omega)_\alpha, \overline{X}(\omega)_\alpha]$  are the  $\alpha$ -level sets of the FN  $\tilde{\mathbf{X}}(\omega)$ . Of course, in (7)  $z$  is a real number.

A FRV can be interpreted as a RV whose realizations are not real numbers but FNs.

It can be demonstrated that any FRV  $\tilde{\mathbf{X}}$  defines,  $\forall \alpha \in [0, 1]$ , an infima random variable (RV)  $\underline{\mathbf{X}}_\alpha$  and a suprema RV  $\overline{\mathbf{X}}_\alpha$  whose realizations are, respectively, the lower and upper extremes of the  $\alpha$ -cuts of  $\tilde{\mathbf{X}}(\omega)$ ,  $\forall \omega \in \Omega$ ,  $\underline{X}(\omega)_\alpha$ ,  $\overline{X}(\omega)_\alpha$ .

Let  $\{\Omega, \mathcal{A}, P\}$  be a probability space. Given that in our paper we will price discrete life-insurances, the following definitions refer to discrete FRVs that come from the set of elemental outcomes  $\Omega = \{\omega_i\}_{i=1, \dots, n}$  with  $P(\omega_i) = p_i$ ,  $\forall i = 1, \dots, n$ .

Let  $\tilde{\mathbf{X}}$  be a discrete FRV on  $\{\Omega, \mathcal{A}, P\}$ , with  $F_{\underline{\mathbf{X}}_\alpha}$  and  $F_{\overline{\mathbf{X}}_\alpha}$ ,  $\forall \alpha \in [0, 1]$ , the distribution functions of the RVs  $\underline{\mathbf{X}}_\alpha$  and  $\overline{\mathbf{X}}_\alpha$  being obtained from  $\tilde{\mathbf{X}}$ . Then,  $\forall \alpha$ ,

we define *the couple of the distribution functions* of the RVs infima and suprema for that membership level  $F_{\tilde{\mathbf{X}}}(x)_\alpha = \{F_{\tilde{\mathbf{X}}}(x)_\alpha, \overline{F_{\tilde{\mathbf{X}}}(x)_\alpha}\}$  as:

$$\underline{F_{\tilde{\mathbf{X}}}(x)_\alpha} = P(\overline{\mathbf{X}}_\alpha \leq x) = F_{\overline{\mathbf{X}}_\alpha}(x) \quad (9a)$$

$$\overline{F_{\tilde{\mathbf{X}}}(x)_\alpha} = P(\underline{\mathbf{X}}_\alpha \leq x) = F_{\underline{\mathbf{X}}_\alpha}(x) \quad (9b)$$

Given the probability space  $\{\Omega, \mathcal{A}, P\}$  with  $\Omega = \{\omega_i\}_{i=1, \dots, n}$  and  $P(\omega_i) = p_i, \forall i = 1, \dots, n$ , the mathematical expectation of a discrete ordinary RV is a function of its crisp realizations  $\{x_1, x_2, \dots, x_n\}$ . So, the mathematical expectation of a RV,  $\mathbf{X}$ ,  $E(\mathbf{X})$ , is a real valued function of  $n$  real variables  $E(\mathbf{X})(x_1, x_2, \dots, x_n) = \sum_{i=1}^n x_i p_i$ . If we consider, as mentioned above, that a FRV can be interpreted as a RV whose realizations are the FNs  $\tilde{X}(\omega_1), \tilde{X}(\omega_2), \dots, \tilde{X}(\omega_n)$ , the mathematical expectation of a FRV can be obtained by using Zadeh's extension principle.

Let  $\tilde{\mathbf{X}}$  be a discrete FRV on  $\{\Omega, \mathcal{A}, P\}$  whose realizations are the FNs  $\tilde{X}(\omega_1), \tilde{X}(\omega_2), \dots, \tilde{X}(\omega_n)$ . Let  $E(X)$  be the real valued mapping:

$$E(X): \begin{array}{ccc} \mathfrak{R}^n & \rightarrow & \mathfrak{R} \\ \forall (x_1, \dots, x_n) \in \mathfrak{R}^n & \rightarrow & E(X)(x_1, x_2, \dots, x_n) = \sum_{i=1}^n x_i p_i \in \mathfrak{R} \end{array} \quad (10)$$

The *mathematical expectation* of  $\tilde{\mathbf{X}}$ ,  $\tilde{E}(\tilde{\mathbf{X}})$ , is the FN induced by the FNs  $\tilde{X}(\omega_1), \tilde{X}(\omega_2), \dots, \tilde{X}(\omega_n)$  through  $E(X)$ .

Notice that in (10)  $E(X)$  is an increasing function of the outcomes  $x_1, x_2, \dots, x_n$ . So, we can use (6) to compute the extremes of the  $\alpha$ -cuts of the fuzzy mathematical expectation of  $\tilde{\mathbf{X}}$ ,  $E(\tilde{\mathbf{X}})_\alpha$ ,  $\forall \alpha \in [0, 1]$ , as:

$$E(\tilde{\mathbf{X}})_\alpha = \left[ \underline{E(\tilde{\mathbf{X}})_\alpha}, \overline{E(\tilde{\mathbf{X}})_\alpha} \right] = \left[ \sum_{i=1}^n \underline{X(\omega_i)_\alpha} p_i, \sum_{i=1}^n \overline{X(\omega_i)_\alpha} p_i \right] = [E(\underline{\mathbf{X}}_\alpha), E(\overline{\mathbf{X}}_\alpha)] \quad (11a)$$

The expected interval of  $\tilde{E}(\tilde{\mathbf{X}})$  is, from (3a):

$$e_I(\tilde{E}(\tilde{\mathbf{X}})) = \left[ \int_0^1 E(\underline{\mathbf{X}}_\alpha) d\alpha, \int_0^1 E(\overline{\mathbf{X}}_\alpha) d\alpha \right] \quad (11b)$$

Of course, the  $\beta$ -expected value of  $\tilde{E}(\tilde{\mathbf{X}})$ ,  $e_V(\tilde{E}(\tilde{\mathbf{X}}); \beta)$  is, from (4a):

$$e_V(\tilde{E}(\tilde{\mathbf{X}}); \beta) = (1 - \beta) \int_0^1 E(\underline{\mathbf{X}}_\alpha) d\alpha + \beta \int_0^1 E(\overline{\mathbf{X}}_\alpha) d\alpha \quad (11c)$$

Regarding the variance of FRVs, some authors propose fuzzy definitions, as it is the case in mathematical expectation; however, other authors such as [15] and [12] propose using scalar (crisp) values for the variance since it is a dispersion measure. This dichotomy in the definition means that one definition has to be chosen (for a

more detailed discussion of this topic see [9]). In a life-insurance context [2] propose using the concept of crisp variance by [12] that is built up from the variance of the infima and suprema RVs  $\underline{\mathbf{X}}_\alpha$  and  $\overline{\mathbf{X}}_\alpha$  of  $\widetilde{\mathbf{X}}$ .

Let  $\widetilde{\mathbf{X}}$  be a discrete FRV on  $\{\Omega, \mathbf{A}, P\}$  with infima and suprema discrete RVs  $\underline{\mathbf{X}}_\alpha$  and  $\overline{\mathbf{X}}_\alpha$ ,  $\forall \alpha \in [0, 1]$ , the *variance of  $\widetilde{\mathbf{X}}$* ,  $V(\widetilde{\mathbf{X}})$ , is the real number:

$$V(\widetilde{\mathbf{X}}) = \frac{1}{2} \int_0^1 (V(\underline{\mathbf{X}}_\alpha) + V(\overline{\mathbf{X}}_\alpha)) \, d\alpha \quad (12)$$

Of course, from this definition of the variance of a FRV we can derive a crisp standard deviation as  $D(\widetilde{\mathbf{X}}) = \sqrt{V(\widetilde{\mathbf{X}})}$ .

Notice that we use the superscript “ $\sim$ ” to symbolise fuzzy magnitudes and we write RVs with bold letters. So, the symbols corresponding to FRVs will be in bold and contain the superscript “ $\sim$ ”.

### 3. Fuzzy Discount Factor

In [2] the authors expose several ways to obtain actuarial discount rates with FNs: estimating a mean interest rate through the pricing horizon, using variable interest rates that come from a fuzzy term structure of interest rates, etc. As we have indicated above, this paper uses the first way. This is a very common way to estimate technical interest rates in actuarial practice and it is based on Fisher’s relationship between the nominal interest rate, real interest rate and anticipated inflation. [18] uses a fuzzy interest rate  $\widetilde{i}$  in this manner. Other authors like [10] or [11] consider that  $\widetilde{i}$  is a prediction based on the expected profit of the insurer’s asset portfolio.

Let the discount rate be a FN,  $\widetilde{i}$ , whose  $\alpha$ -cuts are  $i_\alpha = [\underline{i}_\alpha, \overline{i}_\alpha]$ . From  $\widetilde{i}$  we can define *the discount factor* for 1 monetary unit (m.u.) payable in  $t$  years as  $\widetilde{d}_t = (1 + \widetilde{i})^{-t}$ . Given the arithmetic of FNs and that the discount factor is a decreasing function of the interest rate, the  $\alpha$ -cuts of  $\widetilde{d}_t$  are,  $\forall \alpha \in [0, 1]$ :

$$d_{t_\alpha} = [\underline{d}_{t_\alpha}, \overline{d}_{t_\alpha}] = \left[ (1 + \overline{i}_\alpha)^{-t}, (1 + \underline{i}_\alpha)^{-t} \right] \quad (13)$$

So, if the mean interest rate is fixed as a TFN  $\widetilde{i} = (i_l, i_c, i_u)$ , the  $\alpha$ -cut representation of the discount factor is now:

$$d_{t_\alpha} = [\underline{d}_{t_\alpha}, \overline{d}_{t_\alpha}] = \left[ (1 + i_u - (i_u - i_c)\alpha)^{-t}, (1 + i_l + (i_c - i_l)\alpha)^{-t} \right] \quad (14)$$

For the purpose of this paper it will be useful to obtain the expected interval and the expected value of the discount factor to subsequently fit a crisp value for the value of life contingency liabilities. So:

$$e_I(\widetilde{d}_t) = \left[ \int_0^1 \underline{d}_{t_\alpha} \, d\alpha, \int_0^1 \overline{d}_{t_\alpha} \, d\alpha \right] =$$

$$= \left[ \int_0^1 (1 + i_u - (i_u - i_c) \alpha)^{-t} d\alpha, \int_0^1 (1 + i_l + (i_c - i_l) \alpha)^{-t} d\alpha \right] \quad (15a)$$

where, for  $t = 1$

$$e_I(\tilde{d}_t) = \left[ \frac{\ln(1 + i_u) - \ln(1 + i_c)}{i_u - i_c}, \frac{\ln(1 + i_c) - \ln(1 + i_l)}{i_c - i_l} \right] \quad (15b)$$

and for  $t \neq 1$ :

$$e_I(\tilde{d}_t) = \left[ \frac{(1 + i_c)^{-t+1} - (1 + i_u)^{-t+1}}{(t-1)(i_u - i_c)}, \frac{(1 + i_l)^{-t+1} - (1 + i_c)^{-t+1}}{(t-1)(i_c - i_l)} \right] \quad (15c)$$

Of course, from (15) it comes immediately that:

$$e_V(\tilde{d}_t; \beta) = (1 - \beta) \int_0^1 \underline{d}_{t\alpha} d\alpha + \beta \int_0^1 \overline{d}_{t\alpha} d\alpha \quad (16)$$

Notice that the  $\beta$ -expected value can be very useful in our context since the insurer can introduce the risk aversion with the value  $\beta$  intuitively. Given that actuarial evaluations must be cautious, it seems logical to state that in practical calculations  $0.5 < \beta \leq 1$ , i.e., the crisp quantification of life contingencies (e.g. to account for them in financial statements) must overestimate them.

#### 4. Fuzzy Random Value of Life Insurances

Let us consider an  $n$ -year term life insurance. The insured party aged  $x$  will receive 1 m.u. at the end of the year of his death if this occurs within the next  $n$  years. Otherwise he does not receive any quantity. Notice that our definition also includes whole life insurances for  $n = \varpi - x + 1$  where  $\varpi$  is the maximum attainable age in the considered mortality table. The space of events is  $\Omega = \{\omega_0, \omega_1, \dots, \omega_{n-1}, \omega_n\}$  where  $\omega_0 =$  "the insured survives  $n$  years (and so perceives no amount of the insurance)" and  $\omega_j =$  "the insured dies within the  $j$ th year (and so perceives the m.u. at the end of this year)",  $j = 1, 2, \dots, n$ .

From the discount function  $\tilde{d}_t$ , we can obtain the FRV *present value of an  $n$ -year life insurance* for a person aged  $x$  years  ${}_n\tilde{\mathbf{A}}_x$ . The outcomes of the present value of the life insurance are random because they depend on the insurer's death age but are also fuzzy because they are calculated with discount rates that are FNs. So, following [3], this FRV adopts the following general values, with respective probabilities  $P$ :

outcomes	P
$\tilde{d}_r$	${}_{r-1 }q_x, r = 1, \dots, n$
0	${}_n p_x$

The FRV  ${}_n\tilde{\mathbf{A}}_x$  defines,  $\forall \alpha \in [0, 1]$ , the infima and suprema RVs  $\underline{{}_n\mathbf{A}}_{x\alpha}$  and  $\overline{{}_n\mathbf{A}}_{x\alpha}$ ,  $r = 1, \dots, n$ , as:

$\underline{n\mathbf{A}_{x\alpha}}$	$\overline{n\mathbf{A}_{x\alpha}}$		
<i>outcomes</i>	<i>P</i>	<i>outcomes</i>	<i>P</i>
$\frac{d_{r\alpha}}{0}$	$r-1 q_x$ ${}_np_x$	$\frac{\bar{d}_{r\alpha}}{0}$	$r-1 q_x$ ${}_np_x$

where  ${}_r|q_x$  is the probability that the insured aged  $x$  dies within the  $r$ th year and  ${}_np_x$  is the probability that the insured survives  $n$  years. Based on the concepts defined in section 2, we can determine the following magnitudes.

**a) Mathematical expectation of  ${}_n\tilde{\mathbf{A}}_x$**

In the aforementioned paper [3], the authors state:

$$E\left({}_n\tilde{\mathbf{A}}_x\right)_\alpha = \left[ \underline{E\left({}_n\tilde{\mathbf{A}}_x\right)_\alpha}, \overline{E\left({}_n\tilde{\mathbf{A}}_x\right)_\alpha} \right], \forall \alpha \in [0, 1]$$

with:

$$\underline{E\left({}_n\tilde{\mathbf{A}}_x\right)_\alpha} = E\left(\underline{n\mathbf{A}_{x\alpha}}\right) = \sum_{r=1}^n d_{r\alpha} {}_{r-1}q_x \quad (17a)$$

$$\overline{E\left({}_n\tilde{\mathbf{A}}_x\right)_\alpha} = E\left(\overline{n\mathbf{A}_{x\alpha}}\right) = \sum_{r=1}^n \bar{d}_{r\alpha} {}_{r-1}q_x \quad (17b)$$

In our paper, we defuzzify (17a)-(17b) with the expected interval of a FN. So, from (3) it follows that:

$$\begin{aligned} e_I\left(\tilde{E}\left({}_n\tilde{\mathbf{A}}_x\right)\right) &= \left[ \int_0^1 \underline{E\left({}_n\tilde{\mathbf{A}}_x\right)_\alpha} d\alpha, \int_0^1 \overline{E\left({}_n\tilde{\mathbf{A}}_x\right)_\alpha} d\alpha \right] = \\ &= \left[ \int_0^1 \left( \sum_{r=1}^n d_{r\alpha} {}_{r-1}q_x \right) d\alpha, \int_0^1 \left( \sum_{r=1}^n \bar{d}_{r\alpha} {}_{r-1}q_x \right) d\alpha \right] = \\ &= \left[ \sum_{r=1}^n {}_{r-1}q_x \int_0^1 d_{r\alpha} d\alpha, \sum_{r=1}^n {}_{r-1}q_x \int_0^1 \bar{d}_{r\alpha} d\alpha \right] \end{aligned} \quad (18a)$$

And therefore, from (3b), it is straightforward to check that (18a) is also:

$$e_I\left(\tilde{E}\left({}_n\tilde{\mathbf{A}}_x\right)\right) = \sum_{r=1}^n {}_{r-1}q_x e_I\left(\tilde{d}_r\right) \quad (18b)$$

From this expected interval, and for a fixed risk aversion  $0 \leq \beta \leq 1$ , the  $\beta$ -expected value of the mathematical expectation is:

$$e_V\left(\tilde{E}\left({}_n\tilde{\mathbf{A}}_x\right); \beta\right) = (1 - \beta) \sum_{r=1}^n {}_{r-1}q_x \int_0^1 d_{r\alpha} d\alpha + \beta \sum_{r=1}^n {}_{r-1}q_x \int_0^1 \bar{d}_{r\alpha} d\alpha$$



Furthermore, given that in our paper the discount rate is fixed as the TFN  $\tilde{i} = (i_l, i_c, i_u)$ , considering (14) and (15b)-(15c), the above expressions are transformed as:

$$E\left({}_n\tilde{\mathbf{A}}_{\mathbf{x}}\right)_{\alpha} = \left[ \sum_{r=1}^n (1+i_u - (i_u - i_c)\alpha)^{-r} {}_{r-1|}q_x, \sum_{r=1}^n (1+i_l + (i_c - i_l)\alpha)^{-r} {}_{r-1|}q_x \right] \quad (19)$$

and:

- If  $n = 1$ :

$$e_I\left(\tilde{E}\left({}_n\tilde{\mathbf{A}}_{\mathbf{x}}\right)\right) = \left[ \frac{\ln(1+i_u) - \ln(1+i_c)}{i_u - i_c} {}_{0|}q_x, \frac{\ln(1+i_c) - \ln(1+i_l)}{i_c - i_l} {}_{0|}q_x \right] \quad (20a)$$

- If  $n \neq 1$ :

$$\begin{aligned} e_I\left(\tilde{E}\left({}_n\tilde{\mathbf{A}}_{\mathbf{x}}\right)\right) &= \\ &= \left[ \frac{\ln(1+i_u) - \ln(1+i_c)}{i_u - i_c} {}_{0|}q_x + \sum_{r=2}^n \frac{(1+i_c)^{-r+1} - (1+i_u)^{-r+1}}{(r-1)(i_u - i_c)} {}_{r-1|}q_x, \right. \\ &\quad \left. \frac{\ln(1+i_c) - \ln(1+i_l)}{i_c - i_l} {}_{0|}q_x + \sum_{r=2}^n \frac{(1+i_l)^{-r+1} - (1+i_c)^{-r+1}}{(r-1)(i_c - i_l)} {}_{r-1|}q_x \right] \quad (20b) \end{aligned}$$

### b) Variance of ${}_n\tilde{\mathbf{A}}_{\mathbf{x}}$

Following the previously cited paper [3] we must consider the variances of the infima and suprema RVs  $\underline{{}_n\mathbf{A}}_{\mathbf{x}\alpha}$  and  $\overline{{}_n\mathbf{A}}_{\mathbf{x}\alpha}$ :

$$V\left(\underline{{}_n\mathbf{A}}_{\mathbf{x}\alpha}\right) = \sum_{r=1}^n (d_{r\alpha})^2 {}_{r-1|}q_x - \left( \sum_{r=1}^n d_{r\alpha} {}_{r-1|}q_x \right)^2 \quad (21a)$$

$$V\left(\overline{{}_n\mathbf{A}}_{\mathbf{x}\alpha}\right) = \sum_{r=1}^n (\overline{d}_{r\alpha})^2 {}_{r-1|}q_x - \left( \sum_{r=1}^n \overline{d}_{r\alpha} {}_{r-1|}q_x \right)^2 \quad (21b)$$

In our case, given that we use  $\tilde{i} = (i_l, i_c, i_u)$  and by considering (14), (21a)-(21b) become:

$$V\left(\underline{{}_n\mathbf{A}}_{\mathbf{x}\alpha}\right) = \sum_{r=1}^n (1+i_u - (i_u - i_c)\alpha)^{-2r} {}_{r-1|}q_x - \left[ \sum_{r=1}^n (1+i_u - (i_u - i_c)\alpha)^{-r} {}_{r-1|}q_x \right]^2$$

$$V\left(\overline{{}_n\mathbf{A}}_{\mathbf{x}\alpha}\right) = \sum_{r=1}^n (1+i_l + (i_c - i_l)\alpha)^{-2r} {}_{r-1|}q_x - \left[ \sum_{r=1}^n (1+i_l + (i_c - i_l)\alpha)^{-r} {}_{r-1|}q_x \right]^2$$

Then, taking into account:

$$V\left({}_n\tilde{\mathbf{A}}_x\right) = \frac{1}{2} \int_0^1 [V({}_n\mathbf{A}_{x\alpha}) + V(\overline{{}_n\mathbf{A}_{x\alpha}})] d\alpha$$

The use of a triangular interest rate leads us to obtain:

$$\begin{aligned} V\left({}_n\tilde{\mathbf{A}}_x\right) &= \\ &= \frac{1}{2} \left\{ \sum_{r=1}^n \left( \frac{(1+i_c)^{-2r+1} - (1+i_u)^{-2r+1}}{i_u - i_c} + \frac{(1+i_l)^{-2r+1} - (1+i_c)^{-2r+1}}{i_c - i_l} \right) \frac{r-1|qx}{2r-1} - \right. \\ &\left. - \sum_{r=1}^n \left[ \sum_{t=1}^n \left( \frac{(1+i_c)^{-r-t+1} - (1+i_u)^{-r-t+1}}{i_u - i_c} + \frac{(1+i_l)^{-r-t+1} - (1+i_c)^{-r-t+1}}{i_c - i_l} \right) \frac{r-1|qx_{t-1}|qx}{r+t-1} \right] \right\} \end{aligned} \quad (22)$$

### c) The couple of distribution functions of ${}_n\tilde{\mathbf{A}}_x$

Following [3],  $F_{\frac{{}_n\tilde{\mathbf{A}}_x}{\alpha}}(y)$  ( $\overline{F_{\frac{{}_n\tilde{\mathbf{A}}_x}{\alpha}}}(y)$ ) can be obtained from the distribution function of the RV  $\frac{{}_n\tilde{\mathbf{A}}_x}{\alpha}$  ( $\overline{{}_n\mathbf{A}_{x\alpha}}$ ). So, in our developments, for  $r = 0, \dots, n-2$ , given that  $\tilde{i} = (i_l, i_c, i_u)$ ,  $\forall \alpha \in [0, 1]$ ,  $F_{\frac{{}_n\tilde{\mathbf{A}}_x}{\alpha}}(y) = \left\{ F_{\frac{{}_n\tilde{\mathbf{A}}_x}{\alpha}}(y), \overline{F_{\frac{{}_n\tilde{\mathbf{A}}_x}{\alpha}}}(y) \right\}$  where:

$$\begin{aligned} &F_{\frac{{}_n\tilde{\mathbf{A}}_x}{\alpha}}(y) = \\ &= \begin{cases} 0 & \text{if } y < 0 \\ npx & \text{if } 0 \leq y < (1+i_l + (i_c - i_l)\alpha)^{-n} \\ npx + \sum_{s=0}^r n-(s+1)|qx & \text{if } (1+i_l + (i_c - i_l)\alpha)^{-n+r} \leq y < (1+i_l + (i_c - i_l)\alpha)^{-n+r+1} \\ 1 & \text{if } y \geq (1+i_l + (i_c - i_l)\alpha)^{-1} \end{cases} \end{aligned} \quad (23a)$$

and:

$$\begin{aligned} &\overline{F_{\frac{{}_n\tilde{\mathbf{A}}_x}{\alpha}}}(y) = \\ &= \begin{cases} 0 & \text{if } y < 0 \\ npx & \text{if } 0 \leq y < (1+i_u - (i_u - i_c)\alpha)^{-n} \\ npx + \sum_{s=0}^r n-(s+1)|qx & \text{if } (1+i_u - (i_u - i_c)\alpha)^{-n+r} \leq y < (1+i_u - (i_u - i_c)\alpha)^{-n+r+1} \\ 1 & \text{if } y \geq (1+i_u - (i_u - i_c)\alpha)^{-1} \end{cases} \end{aligned} \quad (23b)$$

## 5. Fuzzy Random Value of Endowments

### 5.1. Pure Endowments.

Let us consider the case of an  $n$ -year pure endowment for a person aged  $x$ . In this case the insured individual will receive 1 u.m. if he survives  $n$  years and no amount otherwise. The space of events is  $\Omega = \{\omega_0, \omega_1\}$  where  $\omega_0 =$  “the insured individual survives  $n$  years (and so perceives 1 u.m.)” and  $\omega_1 =$  “the insured dies within the next  $n$  years (and so he does not receive the insured amount)”. In [4] it is formulated the FRV *present value of the pure endowment* associated with a

person aged  $x$  years  $\tilde{\mathbf{A}}_{x:\bar{n}|}$ . This FRV adopts as general values the following FNs, with respective probabilities  $P$ :

<i>outcomes</i>	P
0	$1 - {}_n p_x$
$\tilde{d}_n$	${}_n p_x$

The FRV defines,  $\forall \alpha \in [0, 1]$ , the infima and suprema RVs  $\underline{\mathbf{A}}_{x:\bar{n}|\alpha}$  and  $\overline{\mathbf{A}}_{x:\bar{n}|\alpha}$  whose realizations are, respectively, the lower and upper extremes of the  $\alpha$ -cuts of the FNs that this FRV takes as values:

$\underline{\mathbf{A}}_{x:\bar{n} \alpha}$		$\overline{\mathbf{A}}_{x:\bar{n} \alpha}$	
<i>outcomes</i>	P	<i>outcomes</i>	P
0	$1 - {}_n p_x$	0	$1 - {}_n p_x$
$\underline{d}_{n\alpha}$	${}_n p_x$	$\overline{d}_{n\alpha}$	${}_n p_x$

From these infima and suprema RVs the following magnitudes are obtained.

**a) Mathematical expectation of  $\tilde{\mathbf{A}}_{x:\bar{n}|}$**

In [4] the authors state:

$$E\left(\tilde{\mathbf{A}}_{x:\bar{n}|}\right)_\alpha = \left[ E\left(\underline{\mathbf{A}}_{x:\bar{n}|\alpha}\right), E\left(\overline{\mathbf{A}}_{x:\bar{n}|\alpha}\right) \right], \forall \alpha \in [0, 1]$$

with:

$$E\left(\underline{\mathbf{A}}_{x:\bar{n}|\alpha}\right) = E\left(\underline{\mathbf{A}}_{x:\bar{n}|\alpha}\right) = \underline{d}_{n\alpha} {}_n p_x \tag{24a}$$

and

$$E\left(\overline{\mathbf{A}}_{x:\bar{n}|\alpha}\right) = E\left(\overline{\mathbf{A}}_{x:\bar{n}|\alpha}\right) = \overline{d}_{n\alpha} {}_n p_x \tag{24b}$$

So, using the concept of expected interval described in section 2.1, the expected interval linked to this kind of life contingency liability is, from (3):

$$\begin{aligned} e_I\left(\tilde{E}\left(\tilde{\mathbf{A}}_{x:\bar{n}|}\right)\right) &= \left[ \int_0^1 E\left(\underline{\mathbf{A}}_{x:\bar{n}|\alpha}\right) d\alpha, \int_0^1 E\left(\overline{\mathbf{A}}_{x:\bar{n}|\alpha}\right) d\alpha \right] = \\ &= \left[ \int_0^1 \underline{d}_{n\alpha} {}_n p_x d\alpha, \int_0^1 \overline{d}_{n\alpha} {}_n p_x d\alpha \right] = \left[ {}_n p_x \int_0^1 \underline{d}_{n\alpha} d\alpha, {}_n p_x \int_0^1 \overline{d}_{n\alpha} d\alpha \right] \end{aligned} \tag{25}$$

That is, following (3b), it is easy to check that (25) is also:

$$e_I \left( \tilde{E} \left( \tilde{\mathbf{A}}_{\mathbf{x}:\bar{\mathbf{n}}|} \right) \right) = {}_n p_x e_I \left( \tilde{d}_n \right)$$

From this expected interval and using (4), the  $\beta$ -expected value of the mathematical expectation can be easily obtained.

Under the hypothesis of the mean annual interest rate fixed as the TFN  $\tilde{i} = (i_l, i_c, i_u)$ , and considering (14), the mathematical expectation is transformed into:

$$E \left( \tilde{\mathbf{A}}_{\mathbf{x}:\bar{\mathbf{n}}|} \right)_\alpha = \left[ (1 + i_u - (i_u - i_c) \alpha)^{-n} {}_n p_x, (1 + i_l + (i_c - i_l) \alpha)^{-n} {}_n p_x \right] \quad (26)$$

So, by using (15b)-(15c), the expected interval of (26) is:

- If  $n = 1$ :

$$e_I \left( \tilde{E} \left( \tilde{\mathbf{A}}_{\mathbf{x}:\bar{\mathbf{n}}|} \right) \right) = \left[ \frac{\ln(1 + i_u) - \ln(1 + i_c)}{i_u - i_c} p_x, \frac{\ln(1 + i_c) - \ln(1 + i_l)}{i_c - i_l} p_x \right] \quad (27a)$$

- If  $n \neq 1$ :

$$\begin{aligned} e_I \left( \tilde{E} \left( \tilde{\mathbf{A}}_{\mathbf{x}:\bar{\mathbf{n}}|} \right) \right) &= \\ &= \left[ \frac{(1 + i_c)^{-n+1} - (1 + i_u)^{-n+1}}{(n-1)(i_u - i_c)} {}_n p_x, \frac{(1 + i_l)^{-n+1} - (1 + i_c)^{-n+1}}{(n-1)(i_c - i_l)} {}_n p_x \right] \end{aligned} \quad (27b)$$

### b) Variance of $\tilde{\mathbf{A}}_{\mathbf{x}:\bar{\mathbf{n}}|}$

In [4] it is stated:

$$V \left( \tilde{\mathbf{A}}_{\mathbf{x}:\bar{\mathbf{n}}|} \right) = \frac{1}{2} \int_0^1 \left[ V \left( \mathbf{A}_{\mathbf{x}:\bar{\mathbf{n}}|\alpha} \right) + V \left( \overline{\mathbf{A}}_{\mathbf{x}:\bar{\mathbf{n}}|\alpha} \right) \right] d\alpha$$

with:

$$V \left( \mathbf{A}_{\mathbf{x}:\bar{\mathbf{n}}|\alpha} \right) = (\underline{d}_{n\alpha})^2 {}_n p_x - (\underline{d}_{n\alpha} {}_n p_x)^2 = (\underline{d}_{n\alpha})^2 {}_n p_x (1 - {}_n p_x) \quad (28a)$$

$$V \left( \overline{\mathbf{A}}_{\mathbf{x}:\bar{\mathbf{n}}|\alpha} \right) = (\overline{d}_{n\alpha})^2 {}_n p_x - (\overline{d}_{n\alpha} {}_n p_x)^2 = (\overline{d}_{n\alpha})^2 {}_n p_x (1 - {}_n p_x) \quad (28b)$$

So, in our case where  $\tilde{i} = (i_l, i_c, i_u)$  and by considering (14), (28a)-(28b) become:

$$\begin{aligned} V \left( \mathbf{A}_{\mathbf{x}:\bar{\mathbf{n}}|\alpha} \right) &= (1 + i_u - (i_u - i_c) \alpha)^{-2n} {}_n p_x (1 - {}_n p_x) \\ V \left( \overline{\mathbf{A}}_{\mathbf{x}:\bar{\mathbf{n}}|\alpha} \right) &= (1 + i_l + (i_c - i_l) \alpha)^{-2n} {}_n p_x (1 - {}_n p_x) \end{aligned}$$

and then:

$$\begin{aligned}
 V\left(\tilde{\mathbf{A}}_{\mathbf{x}:\overline{n}|}\right) &= \\
 &= \frac{1}{2} {}_n p_x (1 - {}_n p_x) \int_0^1 [(1 + i_u - (i_u - i_c) \alpha)^{-2n} + (1 + i_l + (i_c - i_l) \alpha)^{-2n}] d\alpha = \\
 &= \frac{{}_n p_x (1 - {}_n p_x)}{2(2n - 1)} \left( \frac{(1 + i_c)^{-2n+1} - (1 + i_u)^{-2n+1}}{i_u - i_c} + \frac{(1 + i_l)^{-2n+1} - (1 + i_c)^{-2n+1}}{i_c - i_l} \right)
 \end{aligned} \tag{29}$$

**c) The couple of distribution functions of  $\tilde{\mathbf{A}}_{\mathbf{x}:\overline{n}|}$**

The paper [4] justifies that  $F_{\tilde{\mathbf{A}}_{\mathbf{x}:\overline{n}|}\alpha}(y) = \overline{F_{\tilde{\mathbf{A}}_{\mathbf{x}:\overline{n}|}\alpha}(y)}$  can be obtained from the distribution function of the RV  $\overline{\mathbf{A}}_{\mathbf{x}:\overline{n}|}\alpha = \left(\mathbf{A}_{\mathbf{x}:\overline{n}|}\alpha\right)$ . So, in the present study, given that  $\tilde{i} = (i_l, i_c, i_u), \forall \alpha \in [0, 1], F_{\tilde{\mathbf{A}}_{\mathbf{x}:\overline{n}|}\alpha}(y) = \left\{ F_{\tilde{\mathbf{A}}_{\mathbf{x}:\overline{n}|}\alpha}(y), \overline{F_{\tilde{\mathbf{A}}_{\mathbf{x}:\overline{n}|}\alpha}(y)} \right\}$  with:

$$F_{\tilde{\mathbf{A}}_{\mathbf{x}:\overline{n}|}\alpha}(y) = \begin{cases} 0 & \text{if } y < 0 \\ 1 - {}_n p_x & \text{if } 0 \leq y < (1 + i_l + (i_c - i_l) \alpha)^{-n} \\ 1 & \text{if } y \geq (1 + i_l + (i_c - i_l) \alpha)^{-n} \end{cases} \tag{30a}$$

and:

$$\overline{F_{\tilde{\mathbf{A}}_{\mathbf{x}:\overline{n}|}\alpha}(y)} = \begin{cases} 0 & \text{if } y < 0 \\ 1 - {}_n p_x & \text{if } 0 \leq y < (1 + i_u - (i_u - i_c) \alpha)^{-n} \\ 1 & \text{if } y \geq (1 + i_u - (i_u - i_c) \alpha)^{-n} \end{cases} \tag{30b}$$

**5.2. Endowment Insurances.**

An endowment insurance is simply the addition of a  $n$ -term life insurance and a pure endowment. That is, the insured aged  $x$  will receive 1 m.u. at the end of the year of his death if this happens within the next  $n$  years. Moreover he will receive 1 m.u. if he survives  $n$  years. The space of events is  $\Omega = \{\omega_0, \omega_1, \dots, \omega_{n-1}, \omega_n\}$  where  $\omega_0 =$  “the insured survives  $n$  years (and so perceives 1 u.m. in  $n$  years)” and  $\omega_j =$  “the insured dies within the  $j$ th year (and so perceives the m.u. at the end of this year)”,  $j=1,2,\dots,n$ . We want to emphasize that, in this case, necessarily  $n > 1$  because, otherwise, if  $n = 1$  the m.u. is always satisfied at this maturity.

Following [4] from  $\tilde{d}_t$ , we built up the FRV *present value of the endowment insurance* associated with a person aged  $x$  years  $\tilde{\mathbf{A}}_{\mathbf{x}:\overline{n}|}$  which adopts the following FNs as values, with respective probabilities  $P$ :

<i>outcomes</i>	<i>P</i>
$\tilde{d}_r$	${}_{r-1}q_x, r = 1, \dots, n$
$\tilde{d}_n$	${}_n p_x$

The FRV  $\tilde{\mathbf{A}}_{\mathbf{x}:\overline{n}}|_{\alpha}$  defines,  $\forall \alpha \in [0, 1]$ , the infima and suprema RVs  $\underline{\mathbf{A}}_{\mathbf{x}:\overline{n}}|_{\alpha}$  and  $\overline{\mathbf{A}}_{\mathbf{x}:\overline{n}}|_{\alpha}$ ,  $r = 1, \dots, n$ , as:

$\underline{\mathbf{A}}_{\mathbf{x}:\overline{n}} _{\alpha}$	$\overline{\mathbf{A}}_{\mathbf{x}:\overline{n}} _{\alpha}$		
<i>outcomes</i>	<i>P</i>	<i>outcomes</i>	<i>P</i>
$\underline{d}_{r\alpha}$	$r-1 q_x$	$\overline{d}_{r\alpha}$	$r-1 q_x$
$\underline{d}_{n\alpha}$	$n p_x$	$\overline{d}_{n\alpha}$	$n p_x$

Following a similar process used for the pure endowment, we can determine the next magnitudes.

**a) Mathematical expectation of  $\tilde{\mathbf{A}}_{\mathbf{x}:\overline{n}}$**

From [4]:

$$E\left(\tilde{\mathbf{A}}_{\mathbf{x}:\overline{n}}\right)_{\alpha} = \left[ \underline{E}\left(\tilde{\mathbf{A}}_{\mathbf{x}:\overline{n}}\right)_{\alpha}, \overline{E}\left(\tilde{\mathbf{A}}_{\mathbf{x}:\overline{n}}\right)_{\alpha} \right], \forall \alpha \in [0, 1]$$

where:

$$\underline{E}\left(\tilde{\mathbf{A}}_{\mathbf{x}:\overline{n}}\right)_{\alpha} = E\left(\underline{\mathbf{A}}_{\mathbf{x}:\overline{n}}|_{\alpha}\right) = \sum_{r=1}^n \underline{d}_{r\alpha} r-1|q_x + \underline{d}_{n\alpha} n p_x \quad (31a)$$

$$\overline{E}\left(\tilde{\mathbf{A}}_{\mathbf{x}:\overline{n}}\right)_{\alpha} = E\left(\overline{\mathbf{A}}_{\mathbf{x}:\overline{n}}|_{\alpha}\right) = \sum_{r=1}^n \overline{d}_{r\alpha} r-1|q_x + \overline{d}_{n\alpha} n p_x \quad (31b)$$

Again, from (3) we obtain the expected interval of the FN defined with (31a)-(31b):

$$\begin{aligned} e_I\left(\tilde{E}\left(\tilde{\mathbf{A}}_{\mathbf{x}:\overline{n}}\right)\right) &= \left[ \int_0^1 \underline{E}\left(\tilde{\mathbf{A}}_{\mathbf{x}:\overline{n}}\right)_{\alpha} d\alpha, \int_0^1 \overline{E}\left(\tilde{\mathbf{A}}_{\mathbf{x}:\overline{n}}\right)_{\alpha} d\alpha \right] = \\ &= \left[ \int_0^1 \left( \sum_{r=1}^n \underline{d}_{r\alpha} r-1|q_x + \underline{d}_{n\alpha} n p_x \right) d\alpha, \int_0^1 \left( \sum_{r=1}^n \overline{d}_{r\alpha} r-1|q_x + \overline{d}_{n\alpha} n p_x \right) d\alpha \right] = \\ &= \left[ \sum_{r=1}^n r-1|q_x \int_0^1 \underline{d}_{r\alpha} d\alpha + n p_x \int_0^1 \underline{d}_{n\alpha} d\alpha, \sum_{r=1}^n r-1|q_x \int_0^1 \overline{d}_{r\alpha} d\alpha + n p_x \int_0^1 \overline{d}_{n\alpha} d\alpha \right] \end{aligned} \quad (32a)$$

Since this type of life contingency payment structure is the result of the addition of a life insurance and a pure endowment, (32a) can be rewritten as:

$$e_I\left(\tilde{E}\left(\tilde{\mathbf{A}}_{\mathbf{x}:\overline{n}}\right)\right) = \sum_{r=1}^n r-1|q_x e_I\left(\tilde{d}_r\right) + n p_x e_I\left(\tilde{d}_n\right) \quad (32b)$$

From this expected interval and using (4), the  $\beta$ -expected value of the mathematical expectation is straightforward to obtain.

Given that in this paper the mean annual interest rate is  $\tilde{i} = (i_l, i_c, i_u)$ , and considering (14), (31a)-(31b) result in:

$$E\left(\tilde{\mathbf{A}}_{\mathbf{x}:\overline{n}}\right)_{\alpha} = \left[ \sum_{r=1}^n (1 + i_u - (i_u - i_c)\alpha)^{-r} {}_{r-1|}q_x + (1 + i_u - (i_u - i_c)\alpha)^{-n} {}_n p_x, \right. \\ \left. \sum_{r=1}^n (1 + i_l + (i_c - i_l)\alpha)^{-r} {}_{r-1|}q_x + (1 + i_l + (i_c - i_l)\alpha)^{-n} {}_n p_x \right] \quad (33)$$

So, by using (15b)-(15c), we obtain:

$$e_I\left(\tilde{E}\left(\tilde{\mathbf{A}}_{\mathbf{x}:\overline{n}}\right)\right) = \\ = \left[ \frac{\ln(1 + i_u) - \ln(1 + i_c)}{i_u - i_c} {}_0|q_x + \sum_{r=2}^n \frac{(1 + i_c)^{-r+1} - (1 + i_u)^{-r+1}}{(r-1)(i_u - i_c)} {}_{r-1|}q_x + \frac{(1 + i_c)^{-n+1} - (1 + i_u)^{-n+1}}{(n-1)(i_u - i_c)} {}_n p_x, \right. \\ \left. \frac{\ln(1 + i_c) - \ln(1 + i_l)}{i_c - i_l} {}_0|q_x + \sum_{r=2}^n \frac{(1 + i_l)^{-r+1} - (1 + i_c)^{-r+1}}{(r-1)(i_c - i_l)} {}_{r-1|}q_x + \frac{(1 + i_l)^{-n+1} - (1 + i_c)^{-n+1}}{(n-1)(i_c - i_l)} {}_n p_x \right] \quad (34)$$

#### b) Variance of $\tilde{\mathbf{A}}_{\mathbf{x}:\overline{n}}$

In the general formulation of [4]:

$$V\left(\tilde{\mathbf{A}}_{\mathbf{x}:\overline{n}}\right) = \frac{1}{2} \int_0^1 \left[ V\left(\underline{\mathbf{A}}_{\mathbf{x}:\overline{n}}\right)_{\alpha} + V\left(\overline{\mathbf{A}}_{\mathbf{x}:\overline{n}}\right)_{\alpha} \right] d\alpha$$

where the variances of the RVs  $\underline{\mathbf{A}}_{\mathbf{x}:\overline{n}}\alpha$  and  $\overline{\mathbf{A}}_{\mathbf{x}:\overline{n}}\alpha$  are:

$$V\left(\underline{\mathbf{A}}_{\mathbf{x}:\overline{n}}\right)_{\alpha} = \sum_{r=1}^n (d_{r\alpha})^2 {}_{r-1|}q_x + (d_{n\alpha})^2 {}_n p_x - \left( \sum_{r=1}^n d_{r\alpha} {}_{r-1|}q_x + d_{n\alpha} {}_n p_x \right)^2 \quad (35a)$$

$$V\left(\overline{\mathbf{A}}_{\mathbf{x}:\overline{n}}\right)_{\alpha} = \sum_{r=1}^n (\overline{d}_{r\alpha})^2 {}_{r-1|}q_x + (\overline{d}_{n\alpha})^2 {}_n p_x - \left( \sum_{r=1}^n \overline{d}_{r\alpha} {}_{r-1|}q_x + \overline{d}_{n\alpha} {}_n p_x \right)^2 \quad (35b)$$

So, in the case where the discount rate is  $\tilde{i} = (i_l, i_c, i_u)$ , and bearing in mind (14), (35a)-(35b) become:

$$V\left(\underline{\mathbf{A}}_{\mathbf{x}:\overline{n}}\right)_{\alpha} = \sum_{r=1}^n (1 + i_u - (i_u - i_c)\alpha)^{-2r} {}_{r-1|}q_x + (1 + i_u - (i_u - i_c)\alpha)^{-2n} {}_n p_x - \\ - \left[ \sum_{r=1}^n (1 + i_u - (i_u - i_c)\alpha)^{-r} {}_{r-1|}q_x + (1 + i_u - (i_u - i_c)\alpha)^{-n} {}_n p_x \right]^2 \\ V\left(\overline{\mathbf{A}}_{\mathbf{x}:\overline{n}}\right)_{\alpha} = \sum_{r=1}^n (1 + i_l + (i_c - i_l)\alpha)^{-2r} {}_{r-1|}q_x + (1 + i_l + (i_c - i_l)\alpha)^{-2n} {}_n p_x -$$

$$- \left[ \sum_{r=1}^n (1 + i_l + (i_c - i_l) \alpha)^{-r} {}_{r-1|}q_x + (1 + i_l + (i_c - i_l) \alpha)^{-n} {}_n p_x \right]^2$$

And so, we obtain the following closed formulation for Feng's variance:

$$\begin{aligned} V(\tilde{\mathbf{A}}_{\mathbf{x}; \bar{n}}) &= \frac{1}{2} \left\{ \sum_{r=1}^n \left( \frac{(1 + i_c)^{-2r+1} - (1 + i_u)^{-2r+1}}{i_u - i_c} + \frac{(1 + i_l)^{-2r+1} - (1 + i_c)^{-2r+1}}{i_c - i_l} \right) \frac{{}_{r-1|}q_x}{2r-1} + \right. \\ &\quad \left. + \left( \frac{(1 + i_c)^{-2n+1} - (1 + i_u)^{-2n+1}}{i_u - i_c} + \frac{(1 + i_l)^{-2n+1} - (1 + i_c)^{-2n+1}}{i_c - i_l} \right) \frac{{}_n p_x}{2n-1} - \right. \\ &\quad \left. - \sum_{r=1}^n \left[ \sum_{t=1}^n \left( \frac{(1 + i_c)^{-r-t+1} - (1 + i_u)^{-r-t+1}}{i_u - i_c} + \frac{(1 + i_l)^{-r-t+1} - (1 + i_c)^{-r-t+1}}{i_c - i_l} \right) \frac{{}_{r-1|}q_x {}_{t-1|}q_x}{r+t-1} \right] - \right. \\ &\quad \left. - 2 \sum_{r=1}^n \left( \frac{(1 + i_c)^{-r-n+1} - (1 + i_u)^{-r-n+1}}{i_u - i_c} + \frac{(1 + i_l)^{-r-n+1} - (1 + i_c)^{-r-n+1}}{i_c - i_l} \right) \frac{{}_{r-1|}q_x {}_n p_x}{r+n-1} \right\} \end{aligned} \quad (36)$$

Notice that the mathematical expectation of the endowment insurance can be obtained by adding the mathematical expectation of the  $n$ -term life insurance and the pure endowment. However, this does not follow for the variance since the  $n$ -year life insurance and the pure endowment are dependent payment structures.

### c) The couple of distribution functions of $\tilde{\mathbf{A}}_{\mathbf{x}; \bar{n}}$

In [4], the couple of the distribution functions of  $\tilde{\mathbf{A}}_{\mathbf{x}; \bar{n}}$ , i.e.  $F_{\tilde{\mathbf{A}}_{\mathbf{x}; \bar{n}}}(y)_\alpha = \left\{ \overline{F_{\tilde{\mathbf{A}}_{\mathbf{x}; \bar{n}}}(y)}_\alpha, \overline{F_{\tilde{\mathbf{A}}_{\mathbf{x}; \bar{n}}}(y)}_\alpha \right\}$ ,  $\forall \alpha \in [0, 1]$ , is obtained from the RVs  $\overline{\mathbf{A}}_{\mathbf{x}; \bar{n}}_\alpha$  and  $\underline{\mathbf{A}}_{\mathbf{x}; \bar{n}}_\alpha$ , respectively. Since in the present study the fuzzy mean interest rate is given by the TFN  $\tilde{i} = (i_l, i_c, i_u)$ , it follows that, for  $r = 0, \dots, n-2$ :

$$\begin{aligned} & \overline{F_{\tilde{\mathbf{A}}_{\mathbf{x}; \bar{n}}}(y)}_\alpha = \\ & = \begin{cases} 0 & \text{if } y < (1 + i_l + (i_c - i_l) \alpha)^{-n} \\ n-1 p_x & \text{if } (1 + i_l + (i_c - i_l) \alpha)^{-n} \leq y < (1 + i_l + (i_c - i_l) \alpha)^{-n+1} \\ n-1 p_x + \sum_{s=1}^r n-(s+1) q_x & \text{if } (1 + i_l + (i_c - i_l) \alpha)^{-n+r} \leq y < (1 + i_l + (i_c - i_l) \alpha)^{-n+r+1} \\ 1 & \text{if } y \geq (1 + i_l + (i_c - i_l) \alpha)^{-1} \end{cases} \end{aligned} \quad (37a)$$

$$\begin{aligned} & \overline{F_{\tilde{\mathbf{A}}_{\mathbf{x}; \bar{n}}}(y)}_\alpha = \\ & = \begin{cases} 0 & \text{if } y < (1 + i_u - (i_u - i_c) \alpha)^{-n} \\ n-1 p_x & \text{if } (1 + i_u - (i_u - i_c) \alpha)^{-n} \leq y < (1 + i_u - (i_u - i_c) \alpha)^{-n+1} \\ n-1 p_x + \sum_{s=1}^r n-(s+1) q_x & \text{if } (1 + i_u - (i_u - i_c) \alpha)^{-n+r} \leq y < (1 + i_u - (i_u - i_c) \alpha)^{-n+r+1} \\ 1 & \text{if } y \geq (1 + i_u - (i_u - i_c) \alpha)^{-1} \end{cases} \end{aligned} \quad (37b)$$



**Numerical application**

We will analyze several endowment insurances where  $x = 45, 55, 65, 75, 85$  and  $n = 5$ . Like in [4]<sup>1</sup> (p. 169), we use the mortality table<sup>2</sup> GRM-80 and a discount rate  $\tilde{i} = (0.02, 0.03, 0.045)$ . So,  $\forall \alpha \in [0, 1]$ :

$$i_\alpha = [0.02 + 0.01\alpha, 0.045 - 0.015\alpha]$$

$$d_{t_\alpha} = \left[ (1.045 - 0.015\alpha)^{-t}, (1.02 + 0.01\alpha)^{-t} \right]$$

In Table 1 we indicate  $\tilde{\mathbf{A}}_{x:\overline{5}}$  outcomes with their  $\alpha$ -cuts, expected intervals and probabilities.

outcomes	$\alpha$ -cuts of the outcomes	expected interval	$P$
$\tilde{d}_1$	$[(1.045 - 0.015\alpha)^{-1}, (1.02 + 0.01\alpha)^{-1}]$	$[0.96387, 0.97562]$	${}_0 q_x$
$\tilde{d}_2$	$[(1.045 - 0.015\alpha)^{-2}, (1.02 + 0.01\alpha)^{-2}]$	$[0.92907, 0.95184]$	${}_1 q_x$
$\tilde{d}_3$	$[(1.045 - 0.015\alpha)^{-3}, (1.02 + 0.01\alpha)^{-3}]$	$[0.89553, 0.92864]$	${}_2 q_x$
$\tilde{d}_4$	$[(1.045 - 0.015\alpha)^{-4}, (1.02 + 0.01\alpha)^{-4}]$	$[0.86322, 0.90602]$	${}_3 q_x$
$\tilde{d}_5$	$[(1.045 - 0.015\alpha)^{-5}, (1.02 + 0.01\alpha)^{-5}]$	$[0,83210, 0,88396]$	${}_4 q_x$
$\tilde{d}_5$	$[(1.045 - 0.015\alpha)^{-5}, (1.02 + 0.01\alpha)^{-5}]$	$[0,83210, 0,88396]$	${}_5p_x$

TABLE 1. FRV Present Value of the Endowment Insurance  $\tilde{\mathbf{A}}_{x:\overline{5}}$   
where  $\tilde{i} = (0.02, 0.03, 0.045)$

Table 2 shows the 1-cut, the 0-cut, the expected interval and the  $\beta$ -expected value,  $\beta = 0.5, 0.75, 1$ , of the mathematical expectation for the FRV  $\tilde{\mathbf{A}}_{x:\overline{5}}$ . The standard deviations are also shown in this table.

$x$	$E(\tilde{\mathbf{A}}_{x:\overline{5}})_1$	$E(\tilde{\mathbf{A}}_{x:\overline{5}})_0$	$e_I(\tilde{E}(\tilde{\mathbf{A}}_{x:\overline{5}}))$	$e_V(\tilde{E}(\tilde{\mathbf{A}}_{x:\overline{5}}); \beta)$			$D(\tilde{\mathbf{A}}_{x:\overline{5}})$
				$\beta=0.5$	$\beta=0.75$	$\beta=1$	
45	0.8635	[0.8038, 0.9064]	[0.8332, 0.884]	0.8590	0.8719	0.8847	0.0089
55	0.8647	[0.8054, 0.9072]	[0.8347, 0.8858]	0.8602	0.8730	0.8858	0.0134
65	0.8673	[0.8091, 0.9090]	[0.8378, 0.8879]	0.8628	0.8754	0.8879	0.0196
75	0.8742	[0.8189, 0.9138]	[0,8462, 0.8938]	0.8700	0.8819	0.8938	0.0296
85	0.8905	[0.8420, 0.9251]	[0.8660, 0.9077]	0.8868	0.8972	0.9077	0.0408

TABLE 2. Mathematical Expectation of the Present Value and Its Expected Interval and  $\beta$ -expected Value and Standard Deviation of Priced Endowment Insurances

<sup>1</sup>[4] only develops the case where  $x=75$  years.

<sup>2</sup>Mortality tables of the Swiss male population “Grundzahlen Renten Männer”, 1980. Those tables can be obtained from Table Manager 3.0 available at <http://mort.soa.org/>.

Notice that 0-cuts inform us about all the possible values of the fuzzy expected present value (for example, for a person aged 45 years, the extreme optimistic/pessimistic scenarios of the fair value may be 0.8038/0.9064), but 1-cut quantifies the fair price of the endowment in the most feasible scenario (0.8635). Of course, it is possible to obtain intermediate scenarios considering a discrete scale for the values of  $\alpha$ . However, this procedure increases the computational cost and complicates the decision maker's interpretation of the fuzzy present value given that it is not a TFN. Following [25], we consider that a FN must be reduced to a crisp interval before its final defuzzification. Thus, we get a compromise between simplification and not losing a great amount of information. In this regard, the concept of the expected interval gives an estimate of the possible values of the fair price by considering all possible scenarios. In the case of a person aged 45, the reasonable scenarios for the fair premium (in a new contract) or the net mathematical reserve (in other cases) involve quantifying these magnitudes in a value that may oscillate between 0.8332 and 0.884. Nevertheless, neither the fuzzy expectation nor the expected interval can be considered as the final value of the policy; that is, to determine the final pure premium or the net mathematical reserve for financial statements, these magnitudes must be finally crisp. The  $\beta$ -expected value allows introducing insurer's risk aversion when fitting a final crisp quantification for the fuzzy price of insured life contingencies. For example, when  $\beta = 1$  (complete risk aversion) the final net premium for a person aged 45 must be 0.8847.

The standard deviation is a useful indicator for quantifying the mortality risk and therefore for including a value for risk surplus in the premiums or solvency margins in the reserves. In practice, it is common to take these magnitudes as  $k$  times the standard deviation. So, if we take  $k = 2$  as the solvency margin to compute in financial statements, for an endowment where  $x = 45$  years, that margin must be  $2 \cdot 0.0089 = 0.0178$ .

## 6. Fuzzy Random Value of Life Annuities

Now let us consider a deferred  $m$  year life annuity due of 1 m.u. with  $n$  terms for an insured party aged  $x$ . Notice that our definition also includes whole life annuities. In fact, for whole life annuities,  $n = \varpi - m + 1$  where  $\varpi$  is the maximum attainable age in the mortality table considered. The space of events is  $\Omega = \{\omega_0, \omega_1, \dots, \omega_{n-1}, \omega_n\}$  where  $\omega_0 =$  "the insured dies within the next  $m$  years (and so receives no term of the annuity)";  $\omega_j =$  "the insured dies within the  $(m + j)$ th year of the annuity (and so receives the first  $j$  terms of the annuity)",  $j = \dots, 1, 2, n-1$  and  $\omega_n =$  "the insured survives  $m+n-1$  years (and so receives all the terms of the annuity)".

Following [2], and considering the discount function  $\tilde{d}_t$ , the FRV *present value of the life annuity*,  ${}_m|_n\tilde{\mathbf{a}}_x$ , can be obtained:

<i>outcomes</i>	<i>P</i>
0	$m q_x$
$\sum_{t=m}^{m+r-1} \tilde{d}_t$	$m+r-1 q_x$ , $r = 1, \dots, n-1$
$\sum_{t=m}^{m+n-1} \tilde{d}_t$	$m+n-1p_x$

where  ${}_m q_x = 1 - {}_m p_x$ .

${}_{m|n}\tilde{\mathbf{a}}_x$  defines,  $\forall \alpha \in [0, 1]$ , the infima and suprema RVs  $\underline{{}_{m|n}\tilde{\mathbf{a}}_x}_\alpha$  and  $\overline{{}_{m|n}\tilde{\mathbf{a}}_x}_\alpha$ :

$\underline{{}_{m n}\tilde{\mathbf{a}}_x}_\alpha$	$P$	$\overline{{}_{m n}\tilde{\mathbf{a}}_x}_\alpha$	$P$
0	${}_m q_x$	0	${}_m q_x$
$\sum_{t=m}^{m+r-1} \underline{d}_{t\alpha}$	${}_{m+r-1} q_x$	$\sum_{t=m}^{m+r-1} \overline{d}_{t\alpha}$	${}_{m+r-1} q_x$
$\sum_{t=m}^{m+n-1} \underline{d}_{t\alpha}$	${}_{m+n-1} p_x$	$\sum_{t=m}^{m+n-1} \overline{d}_{t\alpha}$	${}_{m+n-1} p_x$

$r = 1, \dots, n - 1$

For this FRV, we have the following magnitudes.

**a) Mathematical expectation of  ${}_{m|n}\tilde{\mathbf{a}}_x$**

In the generic formulation of [2]:

$$E({}_{m|n}\tilde{\mathbf{a}}_x)_\alpha = \left[ \underline{E({}_{m|n}\tilde{\mathbf{a}}_x)_\alpha}, \overline{E({}_{m|n}\tilde{\mathbf{a}}_x)_\alpha} \right], \forall \alpha \in [0, 1]$$

being:

$$\begin{aligned} \underline{E({}_{m|n}\tilde{\mathbf{a}}_x)_\alpha} &= E(\underline{{}_{m|n}\tilde{\mathbf{a}}_x}_\alpha) = \\ &= \sum_{s=m}^{m+n-2} \sum_{t=m}^s \underline{d}_{t\alpha s} q_x + \sum_{t=m}^{m+n-1} \underline{d}_{t\alpha} {}_{m+n-1} p_x = \sum_{t=m}^{m+n-1} \underline{d}_{t\alpha} t p_x \end{aligned} \quad (38a)$$

$$\begin{aligned} \overline{E({}_{m|n}\tilde{\mathbf{a}}_x)_\alpha} &= E(\overline{{}_{m|n}\tilde{\mathbf{a}}_x}_\alpha) = \\ &= \sum_{s=m}^{m+n-2} \sum_{t=m}^s \overline{d}_{t\alpha s} q_x + \sum_{t=m}^{m+n-1} \overline{d}_{t\alpha} {}_{m+n-1} p_x = \sum_{t=m}^{m+n-1} \overline{d}_{t\alpha} t p_x \end{aligned} \quad (38b)$$

In the present paper, considering that the interest rate is the TFN  $\tilde{i} = (i_l, i_c, i_u)$ , it can be obtained from (14):

$$E({}_{m|n}\tilde{\mathbf{a}}_x)_\alpha = \left[ \sum_{t=m}^{m+n-1} (1 + i_u - (i_u - i_c)\alpha)^{-t} t p_x, \sum_{t=m}^{m+n-1} (1 + i_l + (i_c - i_l)\alpha)^{-t} t p_x \right] \quad (39)$$

So, with (3a) we obtain the expected interval from (38a)-(38b) as:

$$e_I(\tilde{E}({}_{m|n}\tilde{\mathbf{a}}_x)) = \left[ \sum_{t=m}^{m+n-1} t p_x \int_0^1 \underline{d}_{t\alpha} d\alpha, \sum_{t=m}^{m+n-1} t p_x \int_0^1 \overline{d}_{t\alpha} d\alpha \right] \quad (40)$$

or, alternatively, from (3b),  $e_I \left( \tilde{E} (m|n \tilde{\mathbf{a}}_x) \right) = \sum_{t=m}^{m+n-1} t p_x e_I \left( \tilde{d}_t \right)$ . Using (15b)-(15c) in (40), the above expressions results in:

- If  $m = 0$

$$\begin{aligned} e_I \left( \tilde{E} (m|n \tilde{\mathbf{a}}_x) \right) &= \\ &= \left[ 1 + \frac{\ln(1+i_u) - \ln(1+i_c)}{i_u - i_c} p_x + \sum_{t=2}^{n-1} \frac{(1+i_c)^{-t+1} - (1+i_u)^{-t+1}}{(t-1)(i_u - i_c)} t p_x, \right. \\ &\quad \left. 1 + \frac{\ln(1+i_c) - \ln(1+i_l)}{i_c - i_l} p_x + \sum_{t=2}^{n-1} \frac{(1+i_l)^{-t+1} - (1+i_c)^{-t+1}}{(t-1)(i_c - i_l)} t p_x \right] \end{aligned} \quad (41a)$$

- If  $m = 1$ :

$$\begin{aligned} e_I \left( \tilde{E} (m|n \tilde{\mathbf{a}}_x) \right) &= \\ &= \left[ \frac{\ln(1+i_u) - \ln(1+i_c)}{i_u - i_c} p_x + \sum_{t=2}^n \frac{(1+i_c)^{-t+1} - (1+i_u)^{-t+1}}{(t-1)(i_u - i_c)} t p_x, \right. \\ &\quad \left. \frac{\ln(1+i_c) - \ln(1+i_l)}{i_c - i_l} p_x + \sum_{t=2}^n \frac{(1+i_l)^{-t+1} - (1+i_c)^{-t+1}}{(t-1)(i_c - i_l)} t p_x \right] \end{aligned} \quad (41b)$$

- If  $m \geq 2$ :

$$\begin{aligned} e_I \left( \tilde{E} (m|n \tilde{\mathbf{a}}_x) \right) &= \left[ \sum_{t=m}^{m+n-1} \frac{(1+i_c)^{-t+1} - (1+i_u)^{-t+1}}{(t-1)(i_u - i_c)} t p_x, \right. \\ &\quad \left. \sum_{t=m}^{m+n-1} \frac{(1+i_l)^{-t+1} - (1+i_c)^{-t+1}}{(t-1)(i_c - i_l)} t p_x \right] \end{aligned} \quad (41c)$$

## b) Variance of $m|n \tilde{\mathbf{a}}_x$

From [2]:

$$V(m|n \tilde{\mathbf{a}}_x) = \frac{1}{2} \int_0^1 \left[ V(\underline{m|n \tilde{\mathbf{a}}_x}) + V(\overline{m|n \tilde{\mathbf{a}}_x}) \right] d\alpha$$

The variances of the RVs  $\underline{m|n \tilde{\mathbf{a}}_x}$  and  $\overline{m|n \tilde{\mathbf{a}}_x}$  are:

$$\begin{aligned} V(\underline{m|n \tilde{\mathbf{a}}_x}) &= \sum_{s=m}^{m+n-2} \left[ \sum_{t=m}^s d_{t\alpha} \right]^2 s! q_x + \left[ \sum_{t=m}^{m+n-1} d_{t\alpha} \right]^2 m+n-1 p_x - \\ &\quad - \left[ \sum_{t=m}^{m+n-1} d_{t\alpha} t p_x \right]^2 \end{aligned} \quad (42a)$$

$$V(\overline{m|n\ddot{\mathbf{a}}_x}) = \sum_{s=m}^{m+n-2} \left[ \sum_{t=m}^s \overline{d}_{t\alpha} \right]^2 {}_s|q_x + \left[ \sum_{t=m}^{m+n-1} \overline{d}_{t\alpha} \right]^2 {}_{m+n-1}p_x - \left[ \sum_{t=m}^{m+n-1} \overline{d}_{t\alpha} {}_t p_x \right]^2 \quad (42b)$$

So, with the fuzzy mean annual interest rate  $\tilde{i} = (i_l, i_c, i_u)$  and taking into account (14), the expressions in (42) are transformed into:

$$V(\underline{m|n\ddot{\mathbf{a}}_x}) = \sum_{s=m}^{m+n-2} \left[ \sum_{t=m}^s (1 + i_u - (i_u - i_c)\alpha)^{-t} \right]^2 {}_s|q_x + \left[ \sum_{t=m}^{m+n-1} (1 + i_u - (i_u - i_c)\alpha)^{-t} \right]^2 {}_{m+n-1}p_x - \left[ \sum_{t=m}^{m+n-1} (1 + i_u - (i_u - i_c)\alpha)^{-t} {}_t p_x \right]^2$$

$$V(\overline{m|n\ddot{\mathbf{a}}_x}) = \sum_{s=m}^{m+n-2} \left[ \sum_{t=m}^s (1 + i_l + (i_c - i_l)\alpha)^{-t} \right]^2 {}_s|q_x + \left[ \sum_{t=m}^{m+n-1} (1 + i_l + (i_c - i_l)\alpha)^{-t} \right]^2 {}_{m+n-1}p_x - \left[ \sum_{t=m}^{m+n-1} (1 + i_l + (i_c - i_l)\alpha)^{-t} {}_t p_x \right]^2$$

For the final expression of the variance we distinguish two cases:

- If  $m = 0$  or  $m = 1$ :

$$V(m|n\ddot{\mathbf{a}}_x) = \frac{1}{2} \left\{ \left( \frac{\ln(1 + i_u) - \ln(1 + i_c)}{i_u - i_c} + \frac{\ln(1 + i_c) - \ln(1 + i_l)}{i_c - i_l} \right) {}_1 p_x {}_1 q_x + \sum_{t=2}^{n-1} \left( \frac{(1 + i_c)^{-2t+1} - (1 + i_u)^{-2t+1}}{i_u - i_l} + \frac{(1 + i_l)^{-2t+1} - (1 + i_c)^{-2t+1}}{i_c - i_l} \right) \frac{{}_t p_x {}_t q_x}{2t-1} + 2 \sum_{t=2}^{n-1} \sum_{j=1}^{t-1} \left( \frac{(1 + i_c)^{-t-j+1} - (1 + i_u)^{-t-j+1}}{i_u - i_c} + \frac{(1 + i_l)^{-t-j+1} - (1 + i_c)^{-t-j+1}}{i_c - i_l} \right) \frac{{}_t p_x {}_j q_x}{t+j-1} \right\} \quad (43a)$$

- If  $m \geq 2$ :

$$V(m|n\ddot{\mathbf{a}}_x) = \frac{1}{2} \left\{ \sum_{t=m}^{m+n-1} \left( \frac{(1 + i_c)^{-2t+1} - (1 + i_u)^{-2t+1}}{i_u - i_c} + \frac{(1 + i_l)^{-2t+1} - (1 + i_c)^{-2t+1}}{i_c - i_l} \right) \frac{{}_t p_x {}_t q_x}{2t-1} + 2 \sum_{t=m+1}^{m+n-1} \sum_{j=m}^{t-1} \left( \frac{(1 + i_c)^{-t-j+1} - (1 + i_u)^{-t-j+1}}{i_u - i_c} + \frac{(1 + i_l)^{-t-j+1} - (1 + i_c)^{-t-j+1}}{i_c - i_l} \right) \frac{{}_t p_x {}_j q_x}{t+j-1} \right\} \quad (43b)$$

### c) The couple of distribution functions of ${}_m|n\tilde{\mathbf{a}}_x$

Taking into account the developments of [2], and following the same process that has been described in the previous sections, it is easy to check that when the mean interest rate is the TFN  $\tilde{i} = (i_l, i_c, i_u)$  the couple of the distribution functions of  ${}_m|n\tilde{\mathbf{a}}_x$ ,  $F_{{}_m|n\tilde{\mathbf{a}}_x}(y) = \left\{ \overline{F_{{}_m|n\tilde{\mathbf{a}}_x}(y)}_\alpha, \underline{F_{{}_m|n\tilde{\mathbf{a}}_x}(y)}_\alpha \right\}$ ,  $\forall \alpha \in [0, 1]$ , is for  $r = 0, \dots, n-1$ :

$$\begin{aligned} & \overline{F_{{}_m|n\tilde{\mathbf{a}}_x}(y)}_\alpha = \\ & = \begin{cases} 0 & \text{if } y < 0 \\ m q_x & \text{if } 0 \leq y < (1 + i_l + (i_c - i_l) \alpha)^{-m} \\ m q_x + \sum_{s=m}^{m+r-1} s | q_x & \text{if } \sum_{t=m}^{m+r-1} (1 + i_l + (i_c - i_l) \alpha)^{-t} \leq y < \sum_{t=m}^{m+r} (1 + i_l + (i_c - i_l) \alpha)^{-t} \\ 1 & \text{if } y \geq \sum_{t=m}^{m+n-1} (1 + i_l + (i_c - i_l) \alpha)^{-t} \end{cases} \end{aligned} \quad (44a)$$

and:

$$\begin{aligned} & \underline{F_{{}_m|n\tilde{\mathbf{a}}_x}(y)}_\alpha = \\ & = \begin{cases} 0 & \text{if } y < 0 \\ m q_x & \text{if } 0 \leq y < (1 + i_u - (i_u - i_c) \alpha)^{-m} \\ m q_x + \sum_{s=m}^{m+r-1} s | q_x & \text{if } \sum_{t=m}^{m+r-1} (1 + i_u - (i_u - i_c) \alpha)^{-t} \leq y < \sum_{t=m}^{m+r} (1 + i_u - (i_u - i_c) \alpha)^{-t} \\ 1 & \text{if } y \geq \sum_{t=m}^{m+n-1} (1 + i_u - (i_u - i_c) \alpha)^{-t} \end{cases} \end{aligned} \quad (44b)$$

### Numerical application

We will analyze several temporal annuities where  $x = 57, 62, 67, 72$ ,  $m = 3$  and  $n = 10$ . Like in [2]<sup>3</sup> (p. 34), we use the mortality table GRM-80 and the discount rate  $\tilde{i} = (0.03, 0.05, 0.055)$ .

Table 3 shows  ${}_3|_{10}\tilde{\mathbf{a}}_x$  outcomes with their  $\alpha$ -cuts and probabilities.

With these outcomes, the  $\alpha$ -cuts of the mathematical expectation of the FRV present value of the annuity are, from (39):

$$E({}_3|_{10}\tilde{\mathbf{a}}_x)_\alpha = \left[ \sum_{t=3}^{12} (1.055 - 0.005\alpha)^{-t} {}_t p_x, \sum_{t=3}^{12} (1.03 + 0.02\alpha)^{-t} {}_t p_x \right]$$

whereas the expected interval is, from (41c):

$$e_I(\tilde{E}({}_3|_{10}\tilde{\mathbf{a}}_x)) = \left[ \sum_{t=3}^{12} \frac{1.05^{-t+1} - 1.055^{-t+1}}{(t-1)0.005} {}_t p_x, \sum_{t=3}^{12} \frac{1.03^{-t+1} - 1.05^{-t+1}}{(t-1)0.02} {}_t p_x \right]$$

Table 4 shows the 1-cut, the 0-cut, the expected interval and the  $\beta$ -expected value,  $\beta = 0.5, 0.75, 1$ , of the mathematical expectation for the FRVs present value of the annuities. It also shows the standard deviations.

<sup>3</sup>[4] only develops the case where  $x = 62$  years.

outcomes	$\alpha$ -cuts of the outcomes	$P$
0	$[0, 0]$	$3q_x$
$\tilde{d}_3$	$\left[ (1.055 - 0.005\alpha)^{-3}, (1.03 + 0.02\alpha)^{-3} \right]$	$3 q_x$
$\sum_{t=3}^4 \tilde{d}_t$	$\left[ \sum_{t=3}^4 (1.055 - 0.005\alpha)^{-t}, \sum_{t=3}^4 (1.03 + 0.02\alpha)^{-t} \right]$	$4 q_x$
...	...	...
$\sum_{t=3}^{11} \tilde{d}_t$	$\left[ \sum_{t=3}^{11} (1.055 - 0.005\alpha)^{-t}, \sum_{t=3}^{11} (1.03 + 0.02\alpha)^{-t} \right]$	$11 q_x$
$\sum_{t=3}^{12} \tilde{d}_t$	$\left[ \sum_{t=3}^{12} (1.055 - 0.005\alpha)^{-t}, \sum_{t=3}^{12} (1.03 + 0.02\alpha)^{-t} \right]$	$12p_x$

TABLE 3. FRV Present Value of the Annuity  ${}_3|_{10}\tilde{\mathbf{a}}_x$  with  $\tilde{i} = (0.03, 0.05, 0.055)$

$x$	$E({}_3 _{10}\tilde{\mathbf{a}}_x)_1$	$E({}_3 _{10}\tilde{\mathbf{a}}_x)_0$	$e_I(\tilde{E}({}_3 _{10}\tilde{\mathbf{a}}_x))$	$e_V(\tilde{E}({}_3 _{10}\tilde{\mathbf{a}}_x); \beta)$			$D({}_3 _{10}\tilde{\mathbf{a}}_x)$
				$\beta=0.5$	$\beta=0.75$	$\beta=1$	
57	6.896	[6.238, 7.385]	[6.560, 7.137]	6.848	6.992	7.137	1.688
62	6.600	[5.977, 7.063]	[6.282, 6.828]	6.555	6.691	6.828	1.994
67	6.132	[5.562, 6.555]	[5.841, 6.340]	6.091	6.215	6.340	2.338
72	5.435	[4.944, 5.800]	[5.184, 5.615]	5.400	5.507	5.615	2.654

TABLE 4. Mathematical Expectation of the Present Value and Its Expected Interval and  $\beta$ -expected Value and Standard Deviation of Priced Annuities

In this case, the interpretation of the expected interval for an annuity with  $x = 57$  lead us to fix a fair price for a new contract of this kind (or a net mathematical reserve in other cases) of between 6.560 and 7.137. By using the  $\beta$ -expected value, we find that the net premium (or net reserves, depending on the situation) must be 6.992 for a moderate risk aversion ( $\beta = 0.75$ ). The solvency margin that comes by using the standard deviation may be fixed in  $k$  times as 1.688.

### 7. Conclusions and Extensions

This paper extends the results of [2], [3] and [4] to triangular fuzzy technical interest rates (which are very common in the literature) for several types of life insurance payments. Firstly, it develops closed formulas for the defuzzified value of the fuzzy present values, i.e. their expected intervals and their expected values. In our opinion, those values can be very useful for the actuary when a concrete value of net provisions, pure premiums, etc. must be defined. We opt to use the defuzzifying

method of [8] in the light of the results in [20], who point out the desirable properties of that defuzzifying method for decision making in a fuzzy-random context.

Likewise, this paper also develops analytical expressions for the variances in the present insurance values, which may be of interest when quantifying stabilizing reserves or premium recharges.

Furthermore, we have obtained the distribution functions of the life-insurance structures, which may be useful when simulating the value of a life-insurance portfolio in a similar way to [23] or [1] but by using fuzzy interest rates, as in [2], [3] and [4].

Finally, a possible extension of our paper would be to consider the possibility of periodical premiums. Another continuation of this research would be to use variable discount rates in insurance pricing, which may come, for example, from a non-flat temporal structure of interest rates. Lastly, introducing fuzzy uncertainty in the fitted probabilities of death, in the way suggested by [26], is also a natural continuation of our paper.

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J. DE ANDRÉS-SÁNCHEZ\*, SOCIAL AND BUSINESS RESEARCH LABORATORY, DEPARTMENT OF BUSINESS MANAGEMENT, ROVIRA I VIRGILI UNIVERSITY, SPAIN  
*E-mail address:* [jorge.deandres@urv.net](mailto:jorge.deandres@urv.net)

L. GONZÁLEZ-VILA PUCHADES, DEPARTMENT OF MATHEMATICS FOR ECONOMICS, FINANCE AND ACTUARIAL SCIENCE, UNIVERSITY OF BARCELONA, SPAIN  
*E-mail address:* [lgonzalezv@ub.edu](mailto:lgonzalezv@ub.edu)

\*CORRESPONDING AUTHOR