ALMOST S^*-COMPACTNESS IN L-TOPOLOGICAL SPACES

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ABSTRACT. In this paper, the notion of almost S^*-compactness in L-topological spaces is introduced following Shi's definition of S^*-compactness. The properties of this notion are studied and the relationship between it and other definitions of almost compactness are discussed. Several characterizations of almost S^*-compactness are also presented.

1. Introduction

The concept of compactness is one of the most important concepts in general topology. The notion of compactness in $[0,1]$-fuzzy set theory was first introduced by C. L. Chang in terms of open cover [5]. However the analogue of Tychonoff Theorem is false in Chang's compactness theory [13]. Hence Gantner, Steinlage and Warren introduced the idea of α-compactness [11], Lowen introduced the ideas of fuzzy, strong fuzzy, as well as ultra-fuzzy compactness [18, 19], Liu defined Q-compactness [16] and Wang and Zhao defined N-compactness [28, 30]. Recently Shi has introduced S^*-compactness [24]. In 1924, Alexandroff and Urysohn [1] studied the idea of almost compactness (a weak form of compactness) in topological spaces. The analogous concept in fuzzy topological spaces was first studied by Concilio and Gerla [8] and developed by A. Haydar Es [10], M.N. Mukherjee and R.P. Chakraborty [23]. However, Concilio and Gerla's definition of fuzzy almost compactness is not a good extension of the notion in general topology.

In [4], the notion of almost compactness was again generalized to [0,1]-topological spaces following Lowen's definition of compactness [19]. In [6, 15, 22], it was also generalized to L-topological spaces following Lowen's definition of fuzzy compactness, Kudri's definition of compactness, and Wang's definition of N-compactness.

In this paper, we generalize the concept of almost compactness to L-topological spaces following Shi's definition of S^*-compactness [24]. We call this concept almost S^*-compactness. We first prove several properties of almost S^*-compactness and study some characterizations. Then we discuss the relationship between the different definitions of fuzzy almost compactness in L-topological spaces.

2. Preliminaries

Throughout this paper $(L, \bigvee, \bigwedge, ')$ is a completely distributive DeMorgan algebra, X is a nonempty set and L^X is the set of all L-fuzzy sets on X. The smallest element and the largest element in L^X are denoted by $\underline{0}$ and $\underline{1}$ respectively.

Received: April 2007; Accepted: February 2008

Key words and phrases: L-topology, β_a-cover, Q_a-cover, S^*-compactness, Almost S^*-compactness.

An element a in L is called a prime element if $a \geqslant b \wedge c$ implies $a \geqslant b$ or $a \geqslant c$. a in L is called a co-prime element if a' is a prime element [12]. The set of nonunit prime elements in L is denoted by $P(L)$, the set of nonzero co-prime elements in L is denoted by $M(L)$ and the set of nonzero co-prime elements in L^X is denoted by $M(L^X)$.

The binary relation $\prec$ in L is defined as follows: for $a, b \in L$, $a \prec b$ if and only if for every subset $D \subseteq L$, the relation $b \leqslant \sup D$ always implies the existence of $d \in D$ with $a \leqslant d$ [9]. In a completely distributive DeMorgan algebra L, each element b is a sup of $\{a \in L \mid a \prec b\}$. In the sense of [17, 29], $\{a \in L \mid a \prec b\}$, denoted by $\beta(b)$, is the greatest minimal family of b. Moreover, for $b \in L$, we define $\alpha(b) = \{a \in L \mid a' \prec b'\}$ and $\alpha^*(b) = \alpha(b) \cap P(L)$.

Following [24, 27], for $a \in L$ and $A \in L^X$, we write:

$$A_{[a]} = \{x \in X \mid A(x) \geqslant a\}, \quad A_{(a)} = \{x \in X \mid a \in \beta(A(x))\},$$
$$A^{(a)} = \{x \in X \mid A(x) \nleqslant a\}.$$

An L-topological space (or L-space for short) is a pair $(X, \mathcal{T})$, where $\mathcal{T}$ is a subfamily of L^X which contains $\underline{0}$, $\underline{1}$ and is closed for any suprema and finite infima. $\mathcal{T}$ is called an L-topology on X. Each member of $\mathcal{T}$ is called an open L-set and its complement is called a closed L-set.

For a subfamily $\Phi \subseteq L^X$, $2^{(\Phi)}$ denotes the set of all finite subfamilies of Φ.

The operator ω was first introduced by R. Lowen in [19]. It was generalized to an L-fuzzy setting by T. Kubiak in [14]. The following is an equivalent form of their definition:

Definition 2.1. [14, 17, 29] For a topological space $(X, \mathcal{T})$, let $\omega_L(\mathcal{T})$ denote the family of all lower semi-continuous maps from $(X, \mathcal{T})$ to L, i.e., $\omega_L(\mathcal{T}) = \{A \in L^X \mid A^{(a)} \in \mathcal{T}, \forall a \in L\}$. Then $\omega_L(\mathcal{T})$ is an L-topology on X and we said that $(X, \omega_L(\mathcal{T}))$ is topologically generated by $(X, \mathcal{T})$.

The concept of weakly induced spaces was introduced by H.W. Martin in [20] and generalized to an L-fuzzy setting by Y.M. Liu and M.K. Luo in 1987. An equivalent form of their definition is as follows:

Definition 2.2. [17, 20, 29] An L-space $(X, \mathcal{T})$ is called weakly induced if $\forall a \in L$, $\forall A \in \mathcal{T}$, it follows that $A^{(a)} \in [\mathcal{T}]$, where $[\mathcal{T}]$ denotes the topology formed by all crisp sets in $\mathcal{T}$.

It is obvious that $(X, \omega_L(\mathcal{T}))$ is weakly induced.

Lemma 2.3. [20, 24] *Let $(X, \mathcal{T})$ be a weakly induced L-space, $a \in L$, $A \in \mathcal{T}$. Then $A_{(a)}$ is an open set in $[\mathcal{T}]$.*

Definition 2.4. $A \in L^X$ is called (1) semi-open [3] if $A \leqslant A^{\circ -}$, (2) regularly open [3] if $A^{-\circ} = A$ and (3) α-open [21] if $A \leqslant A^{\circ - \circ}$. The complement of a semi-open L-set is called semi-closed, the complement of a regularly open L-set is called regularly closed and the complement of an α-open L-set is called α-closed.

Definition 2.5. Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be two L-spaces. A map $f : (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$ is called (1) almost continuous [3] if $f_L^{\leftarrow}(G) \in \mathcal{T}_1$ for each regularly open L-set G in $(Y, \mathcal{T}_2)$, (2) weakly continuous [3] if $f_L^{\leftarrow}(G) \leqslant f_L^{\leftarrow}(G^-)^\circ$ for each open L-set G in $(Y, \mathcal{T}_2)$ and (3) strongly continuous [2] if $f_L^{\rightarrow}(G^-) \leqslant f_L^{\rightarrow}(G)$ for each L-set G in $(X, \mathcal{T}_1)$.

Definition 2.6. [25] A net S with index set D is denoted by $\{S(n) \mid n \in D\}$ or $\{S(n)\}_{n\in D}$. For $G \in L^X$, a net S is said to quasi-coincide with G if $\forall n \in D, S(n) \nleqslant G'$.

Definition 2.7. [25] Let $\alpha \in M(L)$. A net $\{S(n) \mid n \in D\}$ in L^X is called an α^--net if there exists $n_0 \in D$ such that $\forall n \geqslant n_0, V(S(n)) \leqslant \alpha$, where $V(S(n))$ denotes the height of $S(n)$. A net $\{S(n)\}_{n\in D}$ in L^X is said to be a constant α-net if the height of each $S(n)$ is a constant value α.

Obviously each constant α-net is an α^--net.

Definition 2.8. [29] Let $(X, \mathcal{T})$ be an L-space. $A \in \mathcal{T}'$ is called a closed remote neighborhood of a fuzzy point x_a if $x_a \nleqslant A$. $A \in L^X$ is called a remote neighborhood of x_a if there exists $B \in \mathcal{T}'$ such that $A \leqslant B$ and B is a closed remote neighborhood of x_a. The set of all closed remote neighborhoods of x_a and the set of all remote neighborhoods of x_a are denoted by $\eta^-(x_a)$ and $\eta(x_a)$, respectively.

It is evident that $A \in \eta(x_a)$ if and only if $A^- \in \eta^-(x_a)$.

Definition 2.9. [30] Let $A \in L^X$, $a \in M(L)$. $\Phi \subseteq \mathcal{T}'$ is called an a-remote neighborhood family (briefly a-RF) of A, if for each $x_a \leqslant A$ there is $P \in \Phi$ such that $P \in \eta^-(x_a)$. Φ is called an a^--RF of A if there exists $b \in \beta^*(a)$ such that Φ is a b-RF of A.

Definition 2.10. [6] Let $A \in L^X$, $a \in M(L)$. $\Phi \subseteq \mathcal{T}'$ is called an almost a-RF of A, if for each $x_a \leqslant A$ there is $P \in \Phi$ such that $P^\circ \in \eta(x_a)$. Φ is called an almost a^--RF of A if there exists $t \in \beta^*(a)$ such that Φ is an almost t-RF of A.

Definition 2.11. [22] Let $A \in L^X$, $r \in P(L)$. $\Omega \subseteq L^X$ is called an r-cover of A if, for each $x \in A_{[r']}$, there is $U \in \Omega$ such that $U(x) \nleqslant r$. Ω is called an r^+-cover of A if there exists $t \in \alpha^*(r)$ such that Ω is a t-cover of A.

The notion of r-cover is equivalent to the notion of r-shading in [14].

Definition 2.12. [22] Let $A \in L^X$, $r \in P(L)$. $\Omega \subseteq L^X$ is called an almost r-cover of A, if for each $x \in A_{[r']}$, there is $U \in \Omega$ such that $U^-(x) \nleqslant r$. Ω is called an almost r^+-cover of A if there exists $t \in \alpha^*(r)$ such that Ω is an almost t-cover of A.

Definition 2.13. [6] Let $(X, \mathcal{T})$ be an L-space and $G \in L^X$. Then G is called almost F-compact if for any $r \in P(L)$, each open r^+-cover of G has a finite subfamily which is an almost r^+-cover of G. $(X, \mathcal{T})$ is said to be almost F-compact if $\underline{1}$ is almost F-compact.

Definition 2.14. [24] Let $(X, \mathcal{T})$ be an L-space, $a \in M(L)$ and $G \in L^X$. A subfamily $\mathcal{U}$ of L^X is called a β_a-cover of G if for any $x \in X$ with $a \notin \beta(G'(x))$, there exists an $A \in \mathcal{U}$ such that $a \in \beta(A(x))$. A β_a-cover $\mathcal{U}$ of G is called open(regularly open, α-open, etc.) β_a-cover of G if each member of $\mathcal{U}$ is open (regularly open, α-open, etc.).

It is obvious that $\mathcal{U}$ is a β_a-cover of G if and only if for any $x \in X$ we have $a \in \beta\left(G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x)\right)$.

Definition 2.15. [24] Let $(X, \mathcal{T})$ be an L-space, $a \in M(L)$ and $G \in L^X$. A subfamily $\mathcal{U}$ of L^X is called a Q_a-cover of G if for any $x \in X$, $G(x) \nleqslant a'$, implies $\bigvee_{A \in \mathcal{U}} A(x) \geqslant a$. A Q_a-cover $\mathcal{U}$ of G is called open (regularly open, α-open, etc.) Q_a-cover of G if each member of $\mathcal{U}$ is open (regularly open, α-open, etc.).

Definition 2.16. [24] Let $(X, \mathcal{T})$ be an L-space and $G \in L^X$. G is called S^*-compact if for any $a \in M(L)$, each open β_a-cover of G has a finite subfamily $\mathcal{V}$ which is an open Q_a-cover of G. $(X, \mathcal{T})$ is said to be S^*-compact if $\underline{1}$ is S^*-compact.

In [15], Kudri and Warner introduced a notion of almost compactness based on Kudri's compactness. Since Kudri's compactness is equivalent to strong compactness in the sense of [17, 29], we call this new notion, which is defined below, almost strong compactness.

Definition 2.17. Let $(X, \mathcal{T})$ be an L-space and $G \in L^X$. Then G is called almost strongly compact if for any $r \in P(L)$, each open r-cover $\mathcal{U}$ of G has a finite subfamily $\mathcal{V}$ such that $\mathcal{V}^-$ is an r-cover of G. $(X, \mathcal{T})$ is said to be almost strongly compact if $\underline{1}$ is almost strongly compact. .

Definition 2.18. [24] Let $(X, \mathcal{T})$ be an L-space. An open L-set U is called a strongly open neighborhood of a fuzzy point x_λ, if $\lambda \in \beta(U(x))$. An L-set A is called a strong neighborhood of x_a if there exists a strongly open neighborhood B of x_a such that $B \leqslant A$.

Definition 2.19. [8] An L-space $(X, \mathcal{T})$ is said to be regular if and only if each open L-set A is a union of open L-sets whose closure is less than A.

3. Definitions and Properties of Almost S^*-compactness

Definition 3.1. Let $(X, \mathcal{T})$ be an L-space and $G \in L^X$. Then G is called almost S^*-compact if for any $a \in M(L)$, every open β_a-cover of G has a finite subfamily $\mathcal{V}$ such that $\mathcal{V}^- = \{A^- \mid A \in \mathcal{V}\}$ is a Q_a-cover of G. $(X, \mathcal{T})$ is said to be almost S^*-compact if $\underline{1}$ is almost S^*-compact.

The following theorem is obvious.

Theorem 3.2. *S^*-compactness implies almost S^*-compactness.*

Theorem 3.3. *Let $(X, \mathcal{T})$ be a regular L-space and $G \in L^X$. Then G is almost S^*-compact if and only if G is S^*-compact.*

Proof. The sufficiency is obvious. Hence we only need to prove the necessity. Let $\mathcal{A} = \{A_i\}_{i\in I}$ be an open β_a-cover of G. By regularity of $(X, \mathcal{T})$, we know that for each $i \in I$, there exists a family $\{B_{ij} \mid j \in J_i\}$ of open L-sets such that $A_i = \bigvee_{j\in J_i} B_{ij}$ and $B_{ij}^- \leqslant A_i$. Let $\mathcal{B} = \{B_{ij} \mid i \in I, j \in J_i\}$, then $\mathcal{B}$ is an open β_a-cover of G. By almost S^*-compactness of G, we know that $\mathcal{B}$ has a finite subfamily $\mathcal{C}$ such that $\mathcal{C}^- = \{C^- \mid C \in \mathcal{C}\}$ is a Q_a-cover of G. Suppose $\mathcal{C} = \{B_{ij} \mid i \in I_0, j \in J_{i0}\}$, where I_0 and J_{i0} are finite subfamilies of I and J_i respectively. Obviously, $\bigvee_{i\in I_0}\bigvee_{j\in J_{i0}} B_{ij}^- \leqslant \bigvee_{i\in I_0} A_i$, hence $\{A_i \mid i \in I_0\}$ is a finite open Q_a-cover of G. It follows that G is S^*-compact. □

Theorem 3.4. *Let $(X, \mathcal{T})$ be an L-space and $G \in L^X$. Then G is almost S^*-compact if and only if for any $a \in M(L)$, each regularly open β_a-cover of G has a finite subfamily $\mathcal{V}$ such that $\mathcal{V}^-$ is a Q_a-cover of G.*

Proof. Again, the necessity is obvious. Now, for any $a \in M(L)$, suppose that $\mathcal{U}$ is an open β_a-cover of G. Then $H = \mathcal{U}^{-\circ} = \{A^{-\circ} \mid A \in \mathcal{U}\}$ is a regularly open β_a-cover of G. So there exists a finite subfamily $\mathcal{V}$ of $\mathcal{U}$ such that $\mathcal{V}^{-\circ-} = \{A^{-\circ-} \mid A \in \mathcal{V}\}$ is a Q_a-cover of G. Since $A^{-\circ-} \leqslant A^-$ for any $A \in \mathcal{V}$, hence $\mathcal{V}^-$ is a Q_a-cover of G. This shows that G is almost S^*-compact. □

Theorem 3.5. *If both G and H are almost S^*-compact, then $G \vee H$ is almost S^*-compact.*

Proof. For any $a \in M(L)$, suppose that $\mathcal{U}$ is an open β_a-cover of $G\vee H$. Then from

$$(G \vee H)'(x) \vee \bigvee_{A\in\mathcal{U}} A(x) = \left(G'(x) \vee \bigvee_{A\in\mathcal{U}} A(x)\right) \wedge \left(H'(x) \vee \bigvee_{A\in\mathcal{U}} A(x)\right)$$

we obtain that for any $x \in X$, $a \in \beta\left(G'(x) \vee \bigvee_{A\in\mathcal{U}} A(x)\right)$ and $a \in \beta\left(H'(x) \vee \bigvee_{A\in\mathcal{U}} A(x)\right)$. So $\mathcal{U}$ is an open β_a-cover of G and H. From almost S^*-compactness of G and H, it follows that $\mathcal{U}$ has finite subfamilies $\mathcal{V}_1$ and $\mathcal{V}_2$ such that $\mathcal{V}_1^-$ is a Q_a-cover of G and $\mathcal{V}_2^-$ is a Q_a-cover of H. Hence for any $x \in X$, $a \leqslant G'(x) \vee \bigvee_{A\in\mathcal{V}_1} A^-(x)$ and $a \leqslant H'(x) \vee \bigvee_{A\in\mathcal{V}_2} A^-(x)$. Now let $\mathcal{W} = \mathcal{V}_1 \cup \mathcal{V}_2$. Then $\mathcal{W}$ is a finite subfamily of $\mathcal{U}$ and it satisfies the conditions $a \leqslant G'(x) \vee \bigvee_{A\in\mathcal{W}} A^-(x)$ and $a \leqslant H'(x) \vee \bigvee_{A\in\mathcal{W}} A^-(x)$. It follows that $a \leqslant (G \vee H)'(x) \vee \bigvee_{A\in\mathcal{W}} A^-(x)$, which implies $\mathcal{W}^-$ is a Q_a-cover of $G \vee H$. Therefore $G \vee H$ is almost S^*-compact. □

Theorem 3.6. *If G is almost S^*-compact and H is a clopen set, then $G \wedge H$ is almost S^*-compact.*

Proof. For any $a \in M(L)$, suppose that $\mathcal{U}$ is an open β_a-cover of $G \wedge H$. Then $\mathcal{U} \cup \{H'\}$ is an open β_a-cover of G. By almost S^*-compactness of G, we know that $\mathcal{U} \cup \{H'\}$ has a finite subfamily $\mathcal{V}$ such that $\mathcal{V}^-$ is a Q_a-cover of G. Take

$\mathcal{W} = \mathcal{V}\backslash\{H'\}$. Then $\mathcal{W}^-$ is a Q_a-cover of $G \wedge H$. This shows that $G \wedge H$ is almost S^*-compact. □

Theorem 3.7. *Let $f : (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$ be almost continuous. If G is almost S^*-compact in $(X, \mathcal{T}_1)$, then so is $f_L^{\to}(G)$ in $(Y, \mathcal{T}_2)$.*

Proof. For any $a \in M(L)$, suppose that $\mathcal{U} \subseteq \mathcal{T}_2$ is an open β_a-cover of $f_L^{\to}(G)$. Then $\mathcal{U}^{-\circ} = \{A^{-\circ} \mid A \in \mathcal{U}\}$ is a regularly open β_a-cover of $f_L^{\to}(G)$. For any $y \in Y$, we have that $a \in \beta\left(f_L^{\to}(G)'(y) \vee \bigvee_{A\in\mathcal{U}} A^{-\circ}(y)\right)$. Since f is almost continuous and

$$\begin{aligned} f_L^{\to}(G)'(y) \vee \bigvee_{A\in\mathcal{U}} A^{-\circ}(y) &= \bigwedge_{x\in f^{-1}(y)} \left(G'(x) \vee \bigvee_{A\in\mathcal{U}} A^{-\circ}(f(x))\right) \\ &= \bigwedge_{x\in f^{-1}(y)} \left(G'(x) \vee \bigvee_{A\in\mathcal{U}} f_L^{\leftarrow}(A^{-\circ})(x)\right), \end{aligned}$$

It follows that $f_L^{\leftarrow}(\mathcal{U}^{-\circ}) = \{f_L^{\leftarrow}(A^{-\circ}) \mid A \in \mathcal{U}\}$ is an open β_a-cover of G. By almost S^*-compactness of G, $\mathcal{U}$ has a finite subfamily $\mathcal{V}$ such that $f_L^{\leftarrow}(\mathcal{V}^{-\circ})^-$ is a Q_a-cover of G. Hence for any $y \in Y$,

$$\begin{aligned} a &\leqslant \bigwedge_{x\in f^{-1}(y)} \left(G'(x) \vee \bigvee_{A\in\mathcal{V}} f_L^{\leftarrow}(A^{-\circ})^-(x)\right) \\ &\leqslant \bigwedge_{x\in f^{-1}(y)} \left(G'(x) \vee \bigvee_{A\in\mathcal{V}} f_L^{\leftarrow}(A^{-\circ-})(x)\right) \\ &= f_L^{\to}(G)'(y) \vee \bigvee_{A\in\mathcal{V}} A^{-\circ-}(y) \\ &\leqslant f_L^{\to}(G)'(y) \vee \bigvee_{A\in\mathcal{V}} A^{-}(y). \end{aligned}$$

This shows that $\mathcal{V}^-$ is a Q_a-cover of $f_L^{\to}(G)$. Therefore $f_L^{\to}(G)$ is almost S^*-compact. □

The following theorems can be proved similarly.

Theorem 3.8. *Let $f : (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$ be weakly continuous. If G is S^*-compact in $(X, \mathcal{T}_1)$, then $f_L^{\to}(G)$ is almost S^*-compact in $(Y, \mathcal{T}_2)$.*

Theorem 3.9. *Let $f : (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$ be strongly continuous. If G is almost S^*-compact in $(X, \mathcal{T}_1)$, then $f_L^{\to}(G)$ is S^*-compact in $(Y, \mathcal{T}_2)$.*

The following theorem shows that the notion of almost S^*-compactness is a good extension of the notion of almost compactness in general topology.

Theorem 3.10. *If $(X, \mathcal{T})$ is a weakly induced L-space, then $(X, \mathcal{T})$ is almost S^*-compact if and only if $(X, [\mathcal{T}])$ is almost compact.*

Proof. Let $(X, [\mathcal{T}])$ be almost compact. For $a \in M(L)$, let $\mathcal{U}$ be an open β_a-cover of $\underline{1}$ in $(X, \mathcal{T})$. By Lemma 2.3, $\{A_{(a)} \mid A \in \mathcal{U}\}$ is an open cover of $(X, [\mathcal{T}])$. By almost compactness of $(X, [\mathcal{T}])$, we know that there exists a finite subfamily $\mathcal{V}$ of $\mathcal{U}$ such that $(\mathcal{V}_{(a)})^- = \{(A_{(a)})^- \mid A \in \mathcal{V}\}$ is a cover of $(X, [\mathcal{T}])$. For any $A \in \mathcal{V}$, by $(A_{(a)})^- \subseteq (A_{[a]})^- \subseteq (A^-)_{[a]}$ we know that $\mathcal{V}^-$ is a Q_a-cover of $\underline{1}$ in $(X, \mathcal{T})$. This shows that $(X, \mathcal{T})$ is almost S^*-compact.

Conversely let $(X, \mathcal{T})$ be almost S^*-compact and $\mathcal{W}$ be an open cover of $(X, [\mathcal{T}])$. Then for each $a \in \beta^*(1)$, $\{\chi_A \mid A \in \mathcal{W}\}$ is an open β_a-cover of $\underline{1}$ in $(X, \mathcal{T})$. By almost S^*-compactness of $(X, \mathcal{T})$, we know that there exists a finite subfamily $\mathcal{V}$ of $\mathcal{W}$ such that $\{(\chi_A)^- \mid A \in \mathcal{V}\}$ is a Q_a-cover of $\underline{1}$ in $(X, \mathcal{T})$. By $(\chi_A)^- = \chi_{A^-}$ we know that $\mathcal{V}^-$ is a cover of $(X, [\mathcal{T}])$. This shows that $(X, [\mathcal{T}])$ is almost compact. □

Corollary 3.11. *Let (X, τ) be a topological space and $(X, \omega_L(\tau))$ be generated topologically by (X, τ). Then $(X, \omega_L(\tau))$ is almost S^*-compact if and only if (X, τ) is almost compact.*

4. The Relationship between Different Definitions of Almost Compactness

In order to compare almost S^*-compactness and almost F-compactness, we first study some characterizations of almost F-compactness. The following lemma is obvious.

Lemma 4.1. *Let $(X, \mathcal{T})$ be an L-space and $G \in L^X$, $\Omega \subseteq L^X$. Then*

(1) *Ω is an r-cover of G if and only if $G'(x) \vee \bigvee_{A \in \Omega} A(x) \nleqslant r$ for any $x \in X$;*

(2) *Ω is an r^+-cover of G if and only if $\bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{A \in \Omega} A(x) \right) \nleqslant r$;*

(3) *Ω is an almost r-cover of G if and only if $G'(x) \vee \bigvee_{A \in \Omega} A^-(x) \nleqslant r$ for any $x \in X$;*

(4) *Ω is an almost r^+-cover of G if and only if $\bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{A \in \Omega} A^-(x) \right) \nleqslant r$.*

Analogous to the method in [26], the following two theorems are obtained easily from Lemma 4.1.

Theorem 4.2. *Let $(X, \mathcal{T})$ be an L-space and $G \in L^X$. Then the following conditions are equivalent.*

(1) *G is almost F-compact.*

(2) *For every subfamily $\mathcal{U} \subset \mathcal{T}$,*

$$\bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x) \right) \leqslant \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} \bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{A \in \mathcal{V}} A^-(x) \right).$$

(3) *For every subfamily $\mathcal{P} \in \mathcal{T}'$,*

$$\bigvee_{x \in X} \left(G(x) \wedge \bigwedge_{B \in \mathcal{P}} B(x) \right) \geqslant \bigwedge_{\mathcal{V} \in 2^{(\mathcal{P})}} \bigvee_{x \in X} \left(G(x) \wedge \bigwedge_{B \in \mathcal{V}} B^\circ(x) \right).$$

Theorem 4.3. *Let $(X, \mathcal{T})$ be an L-space and $G \in L^X$. Then the following conditions are equivalent:*

(1) *G is almost F-compact.*

(2) *For any $r \in L \backslash \{1\}$, each open r^+-cover of G has a finite subfamily which is an almost r^+-cover of G.*

(3) *For any* $r \in L\backslash\{1\}$, *each open* r^+*-cover of* G *has a finite subfamily which is an almost* r*-cover of* G.

(4) *For any* $r \in P(L)$, *each open* r^+*-cover of* G *has a finite subfamily which is an almost* r*-cover of* G.

(5) *For any* $r \in P(L)$ *and each open* r^+*-cover* $\mathcal{U}$ *of* G, *there exists* $b \in \alpha^*(r)$ *and a finite subfamily* $\mathcal{V}$ *such that* $\mathcal{V}$ *is an almost* b*-cover of* G.

(6) *For any* $a \in L\backslash\{0\}$ *and any* $b \in \beta(a) \setminus \{0\}$, *each open* Q_a*-cover* $\mathcal{U}$ *of* G *has a finite subfamily* $\mathcal{V}$ *such that* $\mathcal{V}^-$ *is a* Q_b*-cover of* G.

(7) *For any* $a \in M(L)$ *and any* $b \in \beta^*(a)$, *each open* Q_a*-cover* $\mathcal{U}$ *of* G *has a finite subfamily* $\mathcal{V}$ *such that* $\mathcal{V}^-$ *is a* Q_b*-cover of* G.

Theorem 4.4. *Almost* S^**-compactness implies almost* F*-compactness.*

Proof. Let G be almost S^*-compact. For each $a \in M(L)$, suppose that Φ is an open Q_a-cover of G. Then $a \leqslant G'(x) \vee \bigvee_{A\in\Phi} A(x)$ for any $x \in X$. Thus for all $b \in \beta^*(a)$ we know that Φ is an open β_b-cover of G. By almost S^*-compactness of G we know that Φ has a finite subfamily Ψ such that Ψ^- is a Q_b-cover of G. By Lemma 4.3 this implies that G is almost F-compact. □

However, as the following example shows, F-compactness does not always imply almost S^*-compactness.

Example 4.5. Let $L = [0, 1]$, $X = \{2, 3, 4, \cdots\}$ and $\mathcal{T}$ be an L-topology generated by $\Phi = \{A_n, B_n \mid n \in X\}$, where

$$A_n(x) = \begin{cases} \frac{1}{2} + \frac{1}{n}, & x = n,\\ 0, & x \neq n,\end{cases} \qquad B_n(x) = \begin{cases} \frac{1}{2} - \frac{1}{n}, & x = n,\\ 0, & x \neq n.\end{cases}$$

From

$$A'_n(x) = 1 - A_n(x) = \begin{cases} \frac{1}{2} - \frac{1}{n}, & x = n,\\ 1, & x \neq n,\end{cases} \quad \text{and } B'_n(x) = 1 - B_n(x) = \begin{cases} \frac{1}{2} + \frac{1}{n}, & x = n,\\ 1, & x \neq n,\end{cases}$$

we obtain

$$A^-_n(x) = \begin{cases} \frac{1}{2} + \frac{1}{n}, & x = n,\\ \frac{1}{2} - \frac{1}{x}, & x \neq n,\end{cases} \qquad B^-_n(x) = \frac{1}{2} - \frac{1}{x}.$$

Obviously if $a \in (0.5, 1]$, no subfamily of Φ is an open Q_a-cover of $\underline{1}$. Thus we only need to consider $a \in (0, 0.5]$. Suppose that $\mathcal{U}$ is an open Q_a-cover of $\underline{1}$. For each $b \in (0, a)$, we can take $A_m \leqslant U \in \mathcal{U}$ or $B_n \leqslant U \in \mathcal{U}$. Then $b \leqslant A^-_m(x) \leqslant U^-(x)$ or $b \leqslant B^-_n(x) \leqslant U^-(x)$ when $x \geqslant l = \dfrac{1}{0.5 - b}$ and $x \in X$. Let $I = \{x \mid x \in X$ and $x < l\}$, then I is finite. For each $x \in I$, there exists $U_x \in \mathcal{U}$ such that $b < U_x(x)$. Let $\mathcal{C} = \{U_x, x \in I\} \bigcup \{U\}$, then $\mathcal{C}$ is finite subfamily of $\mathcal{U}$ and $\mathcal{C}^-$ is a Q_b-cover of $\underline{1}$. Therefore $(X, \mathcal{T})$ is almost F-compact.

It is also clear that $\mathcal{U} = \{A_n\}_{n\in X}$ is an open $\beta_{0.5}$-cover of $\underline{1}$, but $\mathcal{U}$ has no finite subfamily $\mathcal{V}$ such that $\mathcal{V}^-$ is a $Q_{0.5}$ -cover of $\underline{1}$, hence $(X, \mathcal{T})$ is not almost S^*-compact.

Theorem 4.6. *When* $L = [0, 1]$, *almost strong compactness implies almost* S^**-compactness.*

Proof. Suppose that G is almost strongly compact and $\mathcal{U}$ is an open β_a-cover of G. Then $\mathcal{U}$ is an a-cover of G since

$$a \in \beta\left(G'(x) \vee \bigvee_{A\in\mathcal{U}} A(x)\right) \Leftrightarrow a < G'(x) \vee \bigvee_{A\in\mathcal{U}} A(x)$$
$$\Leftrightarrow G'(x) \vee \bigvee_{A\in\mathcal{U}} A(x) \nleqslant a.$$

By almost strong compactness of G we know that there exists a finite subfamily $\mathcal{V}$ of $\mathcal{U}$ such that $\mathcal{V}^- = \{A^- \mid A \in \mathcal{V}\}$ is an a-cover of G. Obviously $\mathcal{V}^-$ is a Q_a-cover of G. Therefore G is almost S^*-compact. □

However, as the following example shows, almost S^*-compactness does not always imply almost strong compactness.

Example 4.7. Let $L = [0,1]$, $X = \{2,3,4,\cdots\}$ and $\mathcal{T}$ be an L-topology generated by $\Phi = \{A_n, B_n, C_n \mid n \in X\}$, where

$$A_n(x) = \begin{cases} \frac{1}{2} - \frac{1}{n}, & x = n, \\ 0, & x \neq n, \end{cases} \quad B_n(x) = \begin{cases} \frac{1}{2} + \frac{1}{n}, & x = n, \\ \frac{1}{2}, & x \neq n, \end{cases} \quad C_n(x) = \begin{cases} \frac{1}{2}, & x = n, \\ 0, & x \neq n. \end{cases}$$

It is obvious that when $m \neq n$ we have

$$A_n \wedge A_m = C_n \wedge C_m = A_n \wedge C_m = \underline{0}, \quad B_n \wedge B_m = \underline{\frac{1}{2}}$$

and

$$A_n \wedge B_m = A_n, \; C_n \wedge B_m = C_n, \; A_n \wedge \underline{\frac{1}{2}} = A_n, \; B_n \wedge \underline{\frac{1}{2}} = \underline{\frac{1}{2}}, \; C_n \wedge \underline{\frac{1}{2}} = C_n.$$

Thus $\{A_n, B_n, C_n \mid n = 2,3,4,\cdots\} \cup \{\underline{\frac{1}{2}}\}$ is a base of $(X, \mathcal{T})$. By

$$A'_n(x) = \begin{cases} \frac{1}{2} + \frac{1}{n}, & x = n, \\ 1, & x \neq n, \end{cases} \quad B'_n(x) = \begin{cases} \frac{1}{2} - \frac{1}{n}, & x = n, \\ \frac{1}{2}, & x \neq n, \end{cases} \quad C'_n(x) = \begin{cases} \frac{1}{2}, & x = n, \\ 1, & x \neq n, \end{cases}$$

we have

$$A_n^-(x) = \frac{1}{2} - \frac{1}{x}, \; B_n^-(x) = B_n(x), \; (\underline{\frac{1}{2}})^- = \underline{\frac{1}{2}}, \; C_n^-(x) = \begin{cases} \frac{1}{2}, & x = n, \\ \frac{1}{2} - \frac{1}{x}, & x \neq n. \end{cases}$$

Obviously for any $a \in (0.5, 1]$, no subfamily of Φ is an open β_a-cover of $\underline{1}$. Thus we only need to consider $a \in (0, 0.5]$. Suppose that $\mathcal{U}$ is an open β_a-cover of $\underline{1}$. We can take $B_k \leqslant U \in \mathcal{U}$ or $\underline{\frac{1}{2}} \leqslant U \in \mathcal{U}$, then $\{U^-\}$ is a Q_a-cover of $\underline{1}$. Otherwise, $a < 0.5$. We can take $A_m \leqslant U \in \mathcal{U}$ or $C_n \leqslant U \in \mathcal{U}$, then when $x \geqslant l = \frac{1}{0.5 - a}$ and $x \in X$, we have $a \leqslant A_m^-(x) \leqslant U^-(x)$ or $a \leqslant C_n^-(x) \leqslant U^-(x)$. Let $I = \{x \mid x \in X$ and $x < l\}$, then I is finite. For each $x \in I$, there exists $U_x \in \mathcal{U}$ such that $a < U_x(x)$. Let $\mathcal{C} = \{U_x, x \in I\} \bigcup \{U\}$. Then $\mathcal{C}$ is a finite subfamily of $\mathcal{U}$ and $\mathcal{C}^-$ is a Q_a-cover of $\underline{1}$. Therefore $(X, \mathcal{T})$ is almost S^*-compact.

Now $\mathcal{U} = \{B_n\}_{n\in X}$ is a 0.5-cover of $\underline{1}$. However, for any finite subfamily $\mathcal{V}$ of $\mathcal{U}$, there exists $x \in X$ such that $\bigvee_{A\in\mathcal{V}} A^-(x) = 0.5$. So $(X, \mathcal{T})$ is not almost strongly compact.

The notion of almost N-compactness was defined in [22] as follows:

Definition 4.8. [22] Let $(X,\mathcal{T})$ be an L-space and $G\in L^X$. Then G is called almost N-compact if for any $a\in M(L)$, each a-RF Φ of G has a finite subfamily which is an almost a^--RF of G. $(X,\mathcal{T})$ is said to be almost N-compact if $\underline{1}$ is almost N-compact.

From the fact that $P^\circ\in\eta(x_a)\Leftrightarrow P^{\circ-}\in\eta^-(x_a)$, it follows that Φ is an almost a^--RF of G if and only if $\Phi^{\circ-}$ is an a^--RF of G. Hence Definition 4.8 is not a generalization of almost compactness in general topology, but of near compactness. In fact it is easily seen to be equivalent to near N-compactness as defined by Chen in [7]. In the proof of several theorems in [22], the authors have used the following fact:

$$P^\circ\in\eta(x_a)\Longleftrightarrow a\nleqslant P^\circ(x).$$

This shows that results in [22] are correct. Thus we revise the definition of the almost N-compactness as follows:

Definition 4.9. Let $(X,\mathcal{T})$ be an L-space and $G\in L^X$. G is called almost N-compact if for any $a\in M(L)$ and any a-RF Φ of G, there exists a finite subfamily Ψ of Φ and $t\in\beta^*(a)$ such that for all $x\in X$, $t\nleqslant G(x)\wedge\bigwedge_{P\in\Psi}P^\circ(x)$. $(X,\mathcal{T})$ is said to be almost N-compact if $\underline{1}$ is almost N-compact.

Theorem 4.10. *Almost N-compactness implies almost strong compactness.*

Proof. Suppose that G is almost N-compact. For any $r\in P(L)$, let $\mathcal{U}$ be an open r-cover of G. Then $\mathcal{U}'$ is an r'-RF of G. By almost N-compactness of G we know that there exist $t\in\beta^*(r')$ and a finite subfamily $\mathcal{V}$ of $\mathcal{U}$ such that $t\nleqslant G(x)\wedge\bigwedge_{A\in\mathcal{V}}A^{'\circ}(x)$. This implies that

$$G'(x)\vee\bigvee_{A\in\mathcal{V}}A^-(x)=G'(x)\vee\bigvee_{A\in\mathcal{V}}A^{'\circ'}(x)\nleqslant t'.$$

By $r\leqslant t'$ we know that $G'(x)\vee\bigvee_{A\in\mathcal{V}}A^-(x)\nleqslant r$, i.e., $\mathcal{V}^-$ is an r-cover of G. Therefore G is almost strongly compact. □

As the following example shows, almost strong compactness does not always imply almost N-compactness.

Example 4.11. Let $X=(0,1)$, $\mathcal{T}$ be a $[0,1]$-topology generated by A,B and all constant L-sets, where $A(x)=x, B(x)=1-x$. It is obvious that $A^-=A,\ B^-=B$.

For $a\in[0,1)$, suppose that $\mathcal{U}$ is an open a-cover of $\underline{1}$.

(1) If $a\geqslant 0.5$, take $x=0.5$, then $A(x)=B(x)=0.5$. In this case, there exists $U\in\mathcal{U}$ such that $U(x)>a\geqslant 0.5$, this implies that there exists a constant fuzzy set $\underline{s}\leqslant U$ such that $s>a$. Therefore $\{U^-\}$ is an a-cover of $\underline{1}$.

(2) If $a<0.5$, then we know from the structure of $\mathcal{T}$, that there exists a subfamily $\mathcal{B}$ of $\{\underline{r},\underline{r}\wedge A,\underline{r}\wedge B,\underline{r}\wedge A\wedge B\mid r\in[0,1]\}$ such that $\mathcal{B}$ is a refinement of $\mathcal{U}$ and $\mathcal{B}$ is an a-cover of $\underline{1}$. Obviously $\mathcal{B}$ has a finite subfamily $\mathcal{D}$ which is an a-cover of $\underline{1}$, hence $\mathcal{U}$ has a finite subfamily which is an a-cover of $\underline{1}$.

This shows that $(X, \mathcal{T})$ is almost strongly compact.

Let $\mathcal{U} = \{A\}$. Then $\mathcal{U}$ is a 1-RF of $\underline{1}$. But there is no $t < 1$ such that $t \nleqslant A(x) = A^\circ(x)$ for all $x \in X$. So $(X, \mathcal{T})$ is not almost N-compact.

Corollary 4.12. *When $L = [0,1]$, almost N-compactness implies almost S^*-compactness.*

5. Other Characterizations of Almost S^*-compactness

Definition 5.1. Let $\{S(n) \mid n \in D\}$ be a net in $(X, \mathcal{T})$, $x_\lambda \in M(L^X)$. x_λ is called a weak O_θ-cluster point of S, if for each strongly open neighborhood U of x_λ, S is frequently in U^-. x_λ is called a weak O_θ-limit point of S, if for each strongly open neighborhood U of x_λ, S is eventually in U^-. In this case, we also say that S weakly O_θ-converges to x_λ and write $S \xrightarrow{WO_\theta} x_\lambda$.

From [24] we know that if S weakly O-converges to x_λ then that S weakly O_θ-converges to x_λ, and if x_λ is a weak O-cluster point of S then x_λ is a weak O_θ-cluster point of S.

Theorem 5.2. *An L-set G is almost S^*-compact in $(X, \mathcal{T})$ if and only if $\forall a \in M(L)$, each constant a-net quasi-coinciding with G has a weak O_θ-cluster point $x_a \notin \beta(G')$.*

Proof. Suppose that G is almost S^*-compact. For $a \in M(L)$, let $\{S(n) \mid n \in D\}$ be a constant a-net quasi-coinciding with G. Suppose that S has no weak O_θ-cluster point $x_a \notin \beta(G')$. Then for each $x_a \notin \beta(G')$ there exists a strongly open neighborhood U_x of x_a and $n_x \in D$ such that $\forall n \geqslant n_x$, $S(n) \nleqslant U_x^-$. Let $\Phi = \{U_x \mid x_a \notin \beta(G')\}$. Then Φ is an open β_a-cover of G. Since G is almost S^*-compact, Φ has a finite subfamily $\Psi = \{U_{x^i} \mid i = 1, 2, \cdots, k\}$ such that Ψ^- is a Q_a-cover of G. Since D is a directed set, there exists $n_0 \in D$ such that $n_0 \geqslant n_{x^i}$ for each $i \leqslant k$. Thus $\forall n \geqslant n_0$, $S(n) \nleqslant \bigvee\{U_{x^i}^- \mid i = 1, 2, \cdots, k\}$. This contradicts the fact that Ψ^- is a Q_a-cover of G. Therefore S has a weak O_θ-cluster point $x_a \notin \beta(G')$.

Conversely, suppose that for each $a \in M(L)$, each constant a-net quasi-coinciding with G has a weak O_θ-cluster point $x_a \notin \beta(G')$. We prove that G is almost S^*-compact. Let Φ be an open β_a-cover of G. If for each finite subfamily Ψ of Φ, $\Psi^-$is not a Q_a-cover of G, then for each finite subfamily Ψ of Φ, there exists $S(\Psi) \in M(L^X)$ with height a such that $S(\Psi) \nleqslant G'$ and $S(\Psi) \nleqslant \bigvee \Psi^-$. Let $S = \{S(\Psi) \mid \Psi \text{ is a finite subfamily of } \Phi\}$. Then S is a constant a-net quasi-coinciding with G. Suppose that S has a weak O_θ-cluster point $x_a \notin \beta(G')$. Then for each finite subfamily Ψ of Φ, we have $x_a \notin \beta(\bigvee \Psi)$. In particular, $x_a \notin \beta(B)$ for any $B \in \Phi$. But since Φ is an open β_a-cover of G, we know that there exists $B \in \Phi$ such that $x_a \in \beta(B)$, which is in contradiction with $x_a \notin \beta(B)$. So G is almost S^*-compact. □

Theorem 5.3. *An L-set G is almost S^*-compact in $(X, \mathcal{T})$ if and only if $\forall a \in M(L)$, each a^--net quasi-coinciding with G has a weak O_θ-cluster point $x_a \notin \beta(G')$.*

Proof. The sufficiency is obvious and so we only need to prove the necessity.

Let G be almost S^*-compact, $a \in M(L)$ and $\{S(n) \mid n \in D\}$ be an a^--net quasi-coinciding with G. Then there exists $n_0 \in D$ such that $\forall n \geqslant n_0$, $S(n) \leqslant a$. Put $E = \{n \in D \mid n \geqslant n_0\}$ and

$$T = \{T(n) \mid n \in E, V\,(T(n)) = a, \text{ the support point of } T(n) \text{ is same as } S(n)\}.$$

Then T is a constant a-net quasi-coinciding with G. Let x_a be a weak O_θ-cluster point of T. It is easy to see that x_a is also a weak O_θ-cluster point of S. □

Definition 5.4. Let $A \in L^X$. The θ-closure of A is defined to be

$$cl_\theta(A) = \bigwedge\{V \mid A \leqslant V^\circ, V \in \mathcal{T}'\}.$$

The θ-interior of A is defined to be $cl_\theta(A')'$, written as $int_\theta(A)$.

The following lemmas are obvious.

Lemma 5.5. *Let $A \in L^X$, then $cl_\theta(A) \in \mathcal{T}'$, $int_\theta(A) \in \mathcal{T}$, $A^- \leqslant cl_\theta(A)$, and $int_\theta(A) \leqslant A^\circ$.*

Lemma 5.6. *If $A \in \mathcal{T}$, then $A^- = cl_\theta(A)$; If $A \in \mathcal{T}'$, then $A^\circ = int_\theta(A)$.*

Definition 5.7. An L-set A is called a Θ^C-set if $A = cl_\theta(B)$, for some $B \in L^X$. An L-set A is called Θ^O-set if $A = int_\theta(B)$, for some $B \in L^X$.

Obviously, a Θ^C-set is closed and a Θ^O-set is open.

Theorem 5.8. *An L-set G is almost S^*-compact in $(X, \mathcal{T})$ if and only if for each $a \in M(L)$ and for each family $\mathcal{U}$ of Θ^C-sets such that $\mathcal{U}^\circ$ forms a β_a-cover of G, there exists a finite subfamily $\mathcal{V}$ of $\mathcal{U}$ such that $\mathcal{V}$ is a Q_a-cover of G.*

Proof. ($\Rightarrow$) Suppose that G is almost S^*-compact. For any $a \in M(L)$, let $\mathcal{U}$ be a family of Θ^C-sets such that $\mathcal{U}^\circ$ forms a β_a-cover of G. By almost S^*-compactness of G, there exists a finite subfamily $\mathcal{V}$ of $\mathcal{U}$ such that $\mathcal{V}^{\circ-} = \{V^{\circ-} \mid V \in \mathcal{V}\}$ is a Q_a-cover of G. Now it follows from $V^{\circ-} \leqslant V$ for each $V \in \mathcal{V}$ that $\mathcal{V}$ is a Q_a-cover of G.

($\Leftarrow$) For any $a \in M(L)$, let $\mathcal{U}$ be an open β_a-cover of G. Then by Lemma 5.6, $\mathcal{U}^- = \{U^- \mid U \in \mathcal{U}\}$ is a family of Θ^C-sets. It follows from $U^{-\circ} \geqslant U$ for each $U \in \mathcal{U}$ that $\mathcal{U}^{-\circ}$ is a β_a-cover of G. Thus $\mathcal{U}$ has a finite subfamily $\mathcal{V}$ such that $\mathcal{V}^-$ is a Q_a-cover of G. So G is almost S^*-compact. □

Theorem 5.9. *An L-set G is almost S^*-compact in $(X, \mathcal{T})$ if and only if $\forall a \in M(L)$, every β_a-cover of G by Θ^O-sets has a finite subfamily $\mathcal{V}$ such that $cl_\theta(\mathcal{V})$ is a Q_a-cover of G.*

Proof. ($\Rightarrow$) Suppose that G is almost S^*-compact. For any $a \in M(L)$, let $\mathcal{U}$ be a β_a-cover of G by Θ^O-sets. Then $\mathcal{U}$ is also an open β_a-cover of G. By almost S^*-compactness of G, there exists a finite subfamily $\mathcal{V}$ of $\mathcal{U}$ such that $\{A^- \mid A \in \mathcal{V}\}$ is a Q_a-cover of G. By $A^- = cl_\theta(A)$ we know that $cl_\theta(\mathcal{V}) = \{cl_\theta(A) \mid A \in \mathcal{V}\}$ is a Q_a-cover of G.

($\Leftarrow$) For any $a \in M(L)$, let $\mathcal{U}$ be an open β_a-cover of G. It follows from Lemma 5.6 that $\mathcal{U}^{-\circ} = \{A^{-\circ} \mid A \in \mathcal{U}\}$ is a family of Θ^O-sets and it is a β_a-cover of G

since $A^{-\circ} \geqslant A$ for each $A \in \mathcal{U}$. By hypothesis, $\mathcal{U}$ has a finite subfamily $\mathcal{V}$ such that $cl_\theta(\mathcal{V}^{-\circ})$ is a Q_a-cover of G. From

$$G'(x) \vee \bigvee_{A\in\mathcal{V}} cl_\theta(A^{-\circ}) = G'(x) \vee \bigvee_{A\in\mathcal{V}} A^{-\circ-}(x) \leqslant G'(x) \vee \bigvee_{A\in\mathcal{V}} A^{-}(x),$$

we obtain that $\mathcal{V}^-$ is a Q_a-cover of G. This shows that G is almost S^*-compact. □

Definition 5.10. Let $A \in L^X$. The α-closure of A is defined by

$$cl_\alpha(A) = \bigwedge\{B \mid A \leqslant B \text{ and } B \text{ is } \alpha\text{-closed}\}.$$

$cl_\alpha(A')'$ is called the α-interior of A and denoted by $int_\alpha(A)$.

Lemma 5.11. *If A is a semi-open L-set, then $cl_\alpha(A) = A^-$.*

Proof. Obviously, $cl_\alpha(A) \leqslant A^-$. In order to prove that $A^- \leqslant cl_\alpha(A)$, suppose that $x_a \not\leqslant cl_\alpha(A)$. Then there exists an α-closed set B such that $A \leqslant B$ and $x_a \not\leqslant B$. Since A is semi-open and B is α-closed, hence $A^- \leqslant A^{\circ-} \leqslant B^{\circ-} \leqslant B^{-\circ-} \leqslant B$. This shows that $x_a \not\leqslant A^-$. Thus $A^- \leqslant cl_\alpha(A)$. □

Theorem 5.12. *An L-set G is almost S^*-compact in $(X, \mathcal{T})$ if and only if $\forall a \in M(L)$, each α-open β_a-cover $\mathcal{U}$ of G has a finite subfamily $\mathcal{V}$ such that $cl_\alpha(\mathcal{V})$ is a Q_a-cover of G.*

Proof. ($\Rightarrow$) Suppose that G is almost S^*-compact. For any $a \in M(L)$, let $\mathcal{U}$ be an α-open β_a-cover of G. Let $\mathcal{W} = \{A^{\circ-\circ} \mid A \in \mathcal{U}\}$, then $\mathcal{W}$ is an open β_a-cover of G. By almost S^*-compactness of G, there exists a finite subfamily $\mathcal{V}$ of $\mathcal{U}$ such that $\{A^{\circ-\circ-} \mid A \in \mathcal{V}\}$ is a Q_a-cover of G. Since $A^{\circ-\circ-} = A^- = cl_\alpha(A)$, $cl_\alpha(\mathcal{V}) = \{cl_\alpha(A) \mid A \in \mathcal{V}\}$ is also a Q_a-cover of G.

($\Leftarrow$) For any $a \in M(L)$, let $\mathcal{U}$ be an open β_a-cover of G. Then $\mathcal{U}$ is also an α-open β_a-cover of G. By hypothesis there exists a finite subfamily $\mathcal{V}$ of $\mathcal{U}$ such that $cl_\alpha(\mathcal{V})$ is a Q_a-cover of G. Since $cl_\alpha(A) = A^-$ for any $A \in \mathcal{V}$, G is almost S^*-compact. □

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