

BEST APPROXIMATION SETS IN α - n -NORMED SPACE CORRESPONDING TO INTUITIONISTIC FUZZY n -NORMED LINEAR SPACE

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ABSTRACT. The aim of this paper is to present the new and interesting notion of ascending family of $\alpha - n$ -norms corresponding to an intuitionistic fuzzy n -normed linear space. The notion of best approximation sets in an $\alpha - n$ -normed space corresponding to an intuitionistic fuzzy n -normed linear space is also defined and several related results are obtained.

1. Introduction

In [7, 8], S.Gähler introduced the theory of 2-norms and n -norms on a linear space. A systematic development of n -normed linear spaces is due to S.S. Kim and Y.J. Cho [10], R. Malceski [12], A. Misiak [13] and Hendra Gunawan [9]. In [9], Hendra Gunawan and Mashadi have given a simple method of deriving an $(n - 1)$ -norm from the n -norm and shown that any n -normed space is also an $(n - 1)$ -normed space. A detailed theory of fuzzy normed linear spaces can be found in [3, 4, 5, 6, 11]. In [14], we have extended n -normed linear spaces to fuzzy n -normed linear spaces. Intuitionistic fuzzy sets have been studied in detail in [1, 2] and recently we have introduced the notion of intuitionistic fuzzy n -normed linear spaces [15].

The purpose of this paper is to generalize the notion of intuitionistic fuzzy n -normed linear spaces [15]. In this regard, the notion of ascending families of $\alpha - n$ -norms corresponding to an intuitionistic fuzzy n -normed linear space, which is a generalization of ascending families of $\alpha - n$ -norms corresponding to fuzzy n -norms [14], is introduced. Furthermore, the concept of best approximation sets in an $\alpha - n$ -normed space corresponding to an intuitionistic fuzzy n -normed linear space is considered as a generalization of best approximation sets in α - n -normed linear space [14].

2. Preliminaries

Definition 2.1. [9] Let $n \in N$ (natural numbers) and X be a real linear space of dimension $d \geq n$. A real valued function $\|\bullet, \dots, \bullet\|$ on $\underbrace{X \times \dots \times X}_n = X^n$, satisfying the following conditions is called an n -norm on X .

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- (1) $\|x_1, x_2, \dots, x_n\| = 0$ if and only if x_1, x_2, \dots, x_n are linearly dependent
- (2) $\|x_1, x_2, \dots, x_n\|$ is invariant under any permutation
- (3) $\|x_1, x_2, \dots, kx_n\| = |k| \|x_1, x_2, \dots, x_n\|$, for any $k \in \mathbb{R}$ (set of real numbers)
- (4) $\|x_1, x_2, \dots, x_{n-1}, y + z\| \leq \|x_1, x_2, \dots, x_{n-1}, y\| + \|x_1, x_2, \dots, x_{n-1}, z\|$

The pair $(X, \|\bullet, \dots, \bullet\|)$ is called an n -normed linear space.

Definition 2.2. [14] Let X be a linear space over a field F . A fuzzy subset N of $X^n \times \mathbb{R}$ (set of real numbers) is called a fuzzy n -norm on X if and only if:

- (N1) For all $t \in \mathbb{R}$ with $t \leq 0$, $N(x_1, x_2, \dots, x_n, t) = 0$
- (N2) For all $t \in \mathbb{R}$ with $t > 0$, $N(x_1, x_2, \dots, x_n, t) = 1$ if and only if x_1, x_2, \dots, x_n are linearly dependent
- (N3) $N(x_1, x_2, \dots, x_n, t)$ is invariant under any permutation of x_1, x_2, \dots, x_n
- (N4) For all $t \in \mathbb{R}$ with $t > 0$,
 $N(x_1, x_2, \dots, cx_n, t) = N(x_1, x_2, \dots, x_n, \frac{t}{|c|})$, if $c \neq 0, c \in F$ (field).
- (N5) For all $s, t \in \mathbb{R}$
 $N(x_1, x_2, \dots, x_n + x'_n, s + t) \geq \min\{N(x_1, x_2, \dots, x_n, s), N(x_1, x_2, \dots, x'_n, t)\}$
- (N6) $N(x_1, x_2, \dots, x_n, t)$ is a non-decreasing function of $t \in \mathbb{R}$
and $\lim_{t \rightarrow \infty} N(x_1, x_2, \dots, x_n, t) = 1$

Then (X, N) is called a fuzzy n -normed linear space (in short f- n -NLS).

Theorem 2.3. [14] Let (X, N) be a f- n -NLS such that that

- (N7) $N(x_1, x_2, \dots, x_n, t) > 0$ for all $t > 0$ implies x_1, x_2, \dots, x_n are linearly dependent.

Define $\|x_1, x_2, \dots, x_n\|_\alpha = \inf\{t : N(x_1, x_2, \dots, x_n, t) \geq \alpha\}, \alpha \in (0, 1)$.

Then $\{\|\bullet, \bullet, \dots, \bullet\|_\alpha : \alpha \in (0, 1)\}$ is an ascending family of n -norms on X . These n -norms will be called the α - n -norms on X corresponding to the fuzzy n -norm on X .

Definition 2.4. [15] An intuitionistic fuzzy n -normed linear space (in short i-f- n -NLS) is an object of the form

$A = \{(X, N(x_1, x_2, \dots, x_n, t), M(x_1, x_2, \dots, x_n, t)) | (x_1, x_2, \dots, x_n) \in X^n\}$,
where X is a linear space over a field F , $*$ is a continuous t -norm, \diamond is a continuous t -co-norm and N, M are fuzzy sets on $X^n \times (0, \infty)$. N and M respectively denote the degree of membership and the degree of non-membership of $(x_1, x_2, \dots, x_n, t) \in X^n \times (0, \infty)$, and satisfy the following conditions:

- (1) $N(x_1, x_2, \dots, x_n, t) + M(x_1, x_2, \dots, x_n, t) \leq 1$
- (2) $N(x_1, x_2, \dots, x_n, t) > 0$

- (3) $N(x_1, x_2, \dots, x_n, t) = 1$ if and only if x_1, x_2, \dots, x_n are linearly dependent
- (4) $N(x_1, x_2, \dots, x_n, t)$ is invariant under any permutation of x_1, x_2, \dots, x_n
- (5) $N(x_1, x_2, \dots, cx_n, t) = N(x_1, x_2, \dots, x_n, \frac{t}{|c|})$, if $c \neq 0, c \in F$ (field)
- (6) $N(x_1, x_2, \dots, x_n, s) * N(x_1, x_2, \dots, x'_n, t) \leq N(x_1, x_2, \dots, x_n + x'_n, s + t)$
- (7) $N(x_1, x_2, \dots, x_n, t) : (0, \infty) \rightarrow [0, 1]$ is continuous in t
- (8) $M(x_1, x_2, \dots, x_n, t) > 0$
- (9) $M(x_1, x_2, \dots, x_n, t) = 0$ if and only if x_1, x_2, \dots, x_n are linearly dependent
- (10) $M(x_1, x_2, \dots, x_n, t)$ is invariant under any permutation of x_1, x_2, \dots, x_n
- (11) $M(x_1, x_2, \dots, cx_n, t) = M(x_1, x_2, \dots, x_n, \frac{t}{|c|})$, if $c \neq 0, c \in F$ (field)
- (12) $M(x_1, x_2, \dots, x_n, s) \diamond M(x_1, x_2, \dots, x'_n, t) \geq M(x_1, x_2, \dots, x_n + x'_n, s + t)$
- (13) $M(x_1, x_2, \dots, x_n, t) : (0, \infty) \rightarrow [0, 1]$ is continuous in t .

3. Intuitionistic Fuzzy n -normed Linear Space

The following example illustrates the concept in Definition 2.4 .

Example 3.1. Let $(X, \|\bullet, \bullet, \dots, \bullet\|)$ be an n -normed linear space.

Define

$$N(x_1, x_2, \dots, x_n, t) = \begin{cases} \frac{t}{t + \|x_1, x_2, \dots, x_n\|}, & \text{when } t > 0, t \in R, \\ & (x_1, x_2, \dots, x_n) \in X^n \\ 0, & \text{when } t \leq 0. \end{cases}$$

and

$$M(x_1, x_2, \dots, x_n, t) = \begin{cases} \frac{\|x_1, x_2, \dots, x_n\|}{t + \|x_1, x_2, \dots, x_n\|}, & \text{when } t > 0, t \in R, \\ & (x_1, x_2, \dots, x_n) \in X^n \\ 1, & \text{when } t \leq 0. \end{cases}$$

Then

$A = \{(X, N(x_1, x_2, \dots, x_n, t), M(x_1, x_2, \dots, x_n, t)) | (x_1, x_2, \dots, x_n) \in X^n\}$
is an i-f- n -NLS.

Remark 3.2. If we define

$$N(x_1, x_2, \dots, x_n, t) = \begin{cases} \frac{t}{t + k\|x_1, x_2, \dots, x_n\|}, & \text{when } t > 0, t \in R, \\ & (x_1, x_2, \dots, x_n) \in X^n, \\ & \text{for any real constant } k > 1 \\ 0, & \text{when } t \leq 0. \end{cases}$$

and

$$M(x_1, x_2, \dots, x_n, t) = \begin{cases} \frac{\|x_1, x_2, \dots, x_n\|}{t + \|x_1, x_2, \dots, x_n\|}, & \text{when } t > 0, t \in \mathbb{R}, \\ & (x_1, x_2, \dots, x_n) \in X^n \\ 1, & \text{when } t \leq 0. \end{cases}$$

in Example 3.2, then A is an i-f- n -NLS with $N + M < 1$ for $t > 0$.

Definition 3.3. Let A be an i-f- n -NLS. Suppose that

(14) for all $t > 0$ $N(x_1, x_2, \dots, x_n, t) > 0$ implies x_1, x_2, \dots, x_n are linearly dependent. then the n -norms

$$\|x_1, x_2, \dots, x_n\|_\alpha = \inf\{t : N(x_1, x_2, \dots, x_n, t) \geq \alpha\}, \alpha \in (0, 1).$$

are called α - n -norms on X corresponding to the i-f- n -NLS A .

Theorem 3.4. $\{\|\bullet, \bullet, \dots, \bullet\|_\alpha : \alpha \in (0, 1)\}$ in the above definition is an ascending family of n -norms on X .

Proof.

$$(1) \|x_1, x_2, \dots, x_n\|_\alpha = 0$$

$$\Rightarrow \inf\{t : N(x_1, x_2, \dots, x_n, t) \geq \alpha, M(x_1, x_2, \dots, x_n, t) \leq 1 - \alpha\} = 0.$$

$$\Rightarrow \text{For all } t \in \mathbb{R}, t > 0, N(x_1, x_2, \dots, x_n, t) \geq \alpha > 0 \text{ and}$$

$$M(x_1, x_2, \dots, x_n, t) \leq 1 - \alpha < 1, \alpha \in (0, 1).$$

$$\Rightarrow \text{By (13) } x_1, x_2, \dots, x_n \text{ are linearly dependent.}$$

Conversely, assume that x_1, x_2, \dots, x_n are linearly dependent.

$$\text{By (3) and (4), } N(x_1, x_2, \dots, x_n, t) = 1 \text{ and } M(x_1, x_2, \dots, x_n, t) = 0, \\ \text{for all } t > 0.$$

$$\Rightarrow \text{For all } \alpha \in (0, 1),$$

$$\inf\{t : N(x_1, x_2, \dots, x_n, t) \geq \alpha, M(x_1, x_2, \dots, x_n, t) \leq 1 - \alpha\} = 0.$$

$$\Rightarrow \|x_1, x_2, \dots, x_n\|_\alpha = 0.$$

(2) As $N(x_1, x_2, \dots, x_n, t)$ and $M(x_1, x_2, \dots, x_n, t)$ are invariant under any permutation of x_1, x_2, \dots, x_n , it follows that $\|x_1, x_2, \dots, x_n\|_\alpha$ is invariant under any permutation.

(3) If $c \neq 0$ then,

$$\|x_1, x_2, \dots, cx_n\|_\alpha$$

$$= \inf\{s : N(x_1, x_2, \dots, cx_n, s) \geq \alpha, M(x_1, x_2, \dots, cx_n, s) \leq 1 - \alpha\}$$

$$= \inf\left\{s : N(x_1, x_2, \dots, x_n, \frac{s}{|c|}) \geq \alpha, M(x_1, x_2, \dots, x_n, \frac{s}{|c|}) \leq 1 - \alpha\right\}.$$

$$\text{Let } t = \frac{s}{|c|} \text{ then,}$$

$$\|x_1, x_2, \dots, cx_n\|_\alpha$$

$$= \inf\{|c|t : N(x_1, x_2, \dots, x_n, t) \geq \alpha, M(x_1, x_2, \dots, x_n, t) \leq 1 - \alpha\}$$

$$= |c| \inf\{t : N(x_1, x_2, \dots, x_n, t) \geq \alpha, M(x_1, x_2, \dots, x_n, t) \leq 1 - \alpha\}$$

$$= |c| \|x_1, x_2, \dots, x_n\|_\alpha.$$

If $c = 0$ then,

$$\begin{aligned} & \|x_1, x_2, \dots, cx_n\|_\alpha \\ &= \|x_1, x_2, \dots, 0\|_\alpha \\ &= 0 \\ &= 0 \|x_1, x_2, \dots, x_n\|_\alpha \\ &= |c| \|x_1, x_2, \dots, x_n\|_\alpha, \text{ for all } c \in F(\text{field}). \end{aligned}$$

$$\begin{aligned} (4) & \|x_1, x_2, \dots, x_n\|_\alpha + \|x_1, x_2, \dots, x'_n\|_\alpha \\ &= \inf\{t : N(x_1, x_2, \dots, x_n, t) \geq \alpha, M(x_1, x_2, \dots, x_n, t) \leq 1 - \alpha\} \\ &\quad + \inf\left\{s : N(x_1, x_2, \dots, x'_n, s) \geq \alpha, M(x_1, x_2, \dots, x'_n, s) \leq 1 - \alpha\right\} \\ &= \inf\{t + s : N(x_1, x_2, \dots, x_n, t) \geq \alpha, N(x_1, x_2, \dots, x'_n, s) \geq \alpha \text{ and} \\ &\quad M(x_1, x_2, \dots, x_n, t) \leq 1 - \alpha, M(x_1, x_2, \dots, x'_n, s) \leq 1 - \alpha\} \\ &\geq \inf\{t + s : N(x_1, x_2, \dots, x_n + x'_n, t + s) \geq \alpha \\ &\quad M(x_1, x_2, \dots, x_n + x'_n, t + s) \leq 1 - \alpha\} \\ &\geq \inf\left\{r : N(x_1, x_2, \dots, x_n + x'_n, r) \geq \alpha, M(x_1, x_2, \dots, x_n + x'_n, r) \leq 1 - \alpha\right\}, \\ &\quad r = t + s \\ &= \|x_1, x_2, \dots, x_n + x'_n\|_\alpha. \end{aligned}$$

Therefore,

$$\|x_1, x_2, \dots, x_n + x'_n\|_\alpha \leq \|x_1, x_2, \dots, x_n\|_\alpha + \|x_1, x_2, \dots, x'_n\|_\alpha.$$

Thus $\{\|\bullet, \bullet, \dots, \bullet\|_\alpha : \alpha \in (0, 1)\}$ is an α - n -norm on X .

Let $0 < \alpha_1 < \alpha_2$. Then

$$\begin{aligned} \|x_1, x_2, \dots, x_n\|_{\alpha_1} &= \inf\{t : N(x_1, x_2, \dots, x_n, t) \geq \alpha_1, \\ &\quad M(x_1, x_2, \dots, x_n, t) \leq 1 - \alpha_1\} \\ \|x_1, x_2, \dots, x_n\|_{\alpha_2} &= \inf\{t : N(x_1, x_2, \dots, x_n, t) \geq \alpha_2, \\ &\quad M(x_1, x_2, \dots, x_n, t) \leq 1 - \alpha_2\}. \end{aligned}$$

As $\alpha_1 < \alpha_2$,

$$\begin{aligned} & \{t : N(x_1, x_2, \dots, x_n, t) \geq \alpha_2, M(x_1, x_2, \dots, x_n, t) \leq 1 - \alpha_2\} \\ & \subset \{t : N(x_1, x_2, \dots, x_n, t) \geq \alpha_1, M(x_1, x_2, \dots, x_n, t) \leq 1 - \alpha_1\} \\ & \Rightarrow \inf\{t : N(x_1, x_2, \dots, x_n, t) \geq \alpha_2, M(x_1, x_2, \dots, x_n, t) \leq 1 - \alpha_2\} \\ & \quad \geq \inf\{t : N(x_1, x_2, \dots, x_n, t) \geq \alpha_1, M(x_1, x_2, \dots, x_n, t) \leq 1 - \alpha_1\} \\ & \Rightarrow \|x_1, x_2, \dots, x_n\|_{\alpha_2} \geq \|x_1, x_2, \dots, x_n\|_{\alpha_1}. \end{aligned}$$

Hence $\{\|\bullet, \bullet, \dots, \bullet\|_\alpha : \alpha \in (0, 1)\}$ is an ascending family of α - n -norms on X corresponding to the i-f- n -NLS A . \square

4. Best Approximation Sets in α - n -normed Space

Remark 4.1. Let $(X, \|\bullet, \bullet, \dots, \bullet\|_\alpha)$ be an α - n -normed space corresponding to the i-f- n -NLS A . Let G be an arbitrary non-empty subset of X and $x_0 \in X$. Then for every $x \in X$ and for every set $x_2, x_3, \dots, x_n \subset X \setminus G$ independent of x and x_0 , we have

$$d_{x_2, x_3, \dots, x_n}(x, G) \leq \|x - x_0, x_2, x_3, \dots, x_n\|_\alpha + d_{x_2, x_3, \dots, x_n}(x_0, G),$$

where

$$d_{x_2, x_3, \dots, x_n}(x, G) = \inf_{g \in G} \|x - g, x_2, x_3, \dots, x_n\|_\alpha. \quad (4.1)$$

For each $G \subset X$ and $x_0 \in X$.

Definition 4.2. For $x_0 \in G$ and $x_2, x_3, \dots, x_n \subset X \setminus G$ which is independent of x and x_0 , we define

$$D_{x_2, x_3, \dots, x_n}(x_0, G) = \{x \in X : d_{x_2, x_3, \dots, x_n}(x, G) = \|x - x_0, x_2, x_3, \dots, x_n\|_\alpha + d_{x_2, x_3, \dots, x_n}(x_0, G)\} \quad (4.2)$$

for any $x_2, x_3, \dots, x_n \in X \setminus G$ which are independent of x and x_0 .

We write: $P_{G, x_2, x_3, \dots, x_n}(x) = \{g_0 \in G : \|x - g_0, x_2, x_3, \dots, x_n\|_\alpha = d_{x_2, x_3, \dots, x_n}(x, G)\}$

and

$$P_{G, x_2, x_3, \dots, x_n}^{-1}(x_0) = \{x \in X : \|x - x_0, x_2, x_3, \dots, x_n\|_\alpha = d_{x_2, x_3, \dots, x_n}(x, G)\}.$$

Example 4.2. Let $X = R^3$ be a linear space over R .

Define $\|\bullet, \bullet\|$: $X \times X \rightarrow R$ by

$$\|x_1, x_2\|_1 = \max\{|a_1b_2 - a_2b_1|, |b_1c_2 - b_2c_1|, |a_1c_2 - a_2c_1|\} \text{ and} \\ \|x_1, x_2\|_2 = \frac{1}{2}\{\max\{|a_1b_2 - a_2b_1|, |b_1c_2 - b_2c_1|, |a_1c_2 - a_2c_1|\}\},$$

where $x_i = (a_i, b_i, c_i) \in R^3$; $i=1,2$.

Then $(X, \|\bullet, \bullet\|_1)$ and $(X, \|\bullet, \bullet\|_2)$ are 2-normed linear spaces.

Define $N : X \times X \times R \rightarrow [0, 1]$ by

$$N(x_1, x_2, t) = \begin{cases} 1, & \text{if } t > \|x_1, x_2\|_1 \\ 0.5, & \text{if } \|x_1, x_2\|_2 < t \leq \|x_1, x_2\|_1 \\ 0, & \text{if } t \leq \|x_1, x_2\|_2. \end{cases}$$

and $M : X \times X \times R \rightarrow [0, 1]$ by

$$M(x_1, x_2, t) = \begin{cases} 0, & \text{if } t > \|x_1, x_2\|_1 \\ 0.5, & \text{if } \|x_1, x_2\|_2 < t \leq \|x_1, x_2\|_1 \\ 1, & \text{if } t \leq \|x_1, x_2\|_2. \end{cases}$$

Then A is an i-f-2-NLS.

Define $\|x_1, x_2\|_\alpha = \inf\{t : N(x_1, x_2, t) \geq \alpha, M(x_1, x_2, t) \leq 1 - \alpha\}$, $\alpha \in (0,1)$.

The α -2-norms are given by

$$\|x_1, x_2\|_\alpha = \|x_1, x_2\|_1, \text{ when } 1 > \alpha > 0.5, \\ = \|x_1, x_2\|_2, \text{ when } 0 < \alpha \leq 0.5.$$

Let $G = \{(a, 0, 0) : a \in R\}$ be a subset of X .

Choose $x_0 = (0, 1, 1)$ and $x_2 \in K = \{(0, 0, k) : k \in R \setminus \{0\}\}$.

Then

$$D_{x_2}(x_0, G) = \{x = (0, b, 0), b \in R^+ \setminus \{0\} : d_{x_2}(x, G) = \|x - x_0, x_2\|_\alpha + d_{x_2}(x_0, G)\}$$

and

$$P_{G, x_2}(x) = \{g = (a, 0, 0) : -1 \leq a \leq 1\}.$$

Example 4.3. Let $X = R^{n+1}$ be a linear space over R .

Define $\|\bullet, \bullet, \dots, \bullet\| : X^n \rightarrow R$ by

$$\begin{aligned} \|x_1, x_2, \dots, x_n\|_1 &= \max\{\Delta_1, \Delta_2, \dots, \Delta_n\} \text{ and} \\ \|x_1, x_2, \dots, x_n\|_2 &= \frac{1}{2}\{\max\{\Delta_1, \Delta_2, \dots, \Delta_n\}\}, \end{aligned}$$

$$\text{where } \Delta_1 = \begin{vmatrix} a_{12} & a_{13} & \cdots & a_{1(n+1)} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n2} & a_{n3} & \cdots & a_{n(n+1)} \end{vmatrix}, \Delta_2 = \begin{vmatrix} a_{13} & \cdots & a_{1(n+1)} & a_{11} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n3} & \cdots & a_{n(n+1)} & a_{n1} \end{vmatrix},$$

$$\dots, \Delta_n = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \text{ and}$$

$$x_i = (a_{i1}, a_{i2}, \dots, a_{i(n+1)}) \in R^{n+1}, i = 1, 2, \dots, n.$$

Then $(X, \|\bullet, \bullet, \dots, \bullet\|_1)$ and $(X, \|\bullet, \bullet, \dots, \bullet\|_2)$ are n -normed linear spaces.

Define $N : X^n \times R \rightarrow [0, 1]$ and $M : X^n \times R \rightarrow [0, 1]$ as follows:

$$N(x_1, x_2, \dots, x_n, t) = \begin{cases} 1, & \text{if } t > \|x_1, x_2, \dots, x_n\|_1 \\ 0.5, & \text{if } \|x_1, x_2, \dots, x_n\|_2 < t \leq \|x_1, x_2, \dots, x_n\|_1 \\ 0, & \text{if } t \leq \|x_1, x_2, \dots, x_n\|_2. \end{cases}$$

and

$$M(x_1, x_2, \dots, x_n, t) = \begin{cases} 0, & \text{if } t > \|x_1, x_2, \dots, x_n\|_1 \\ 0.5, & \text{if } \|x_1, x_2, \dots, x_n\|_2 < t \leq \|x_1, x_2, \dots, x_n\|_1 \\ 1, & \text{if } t \leq \|x_1, x_2, \dots, x_n\|_2. \end{cases}$$

Then A is an i - f - n -NLS.

Define $\|x_1, x_2, \dots, x_n\|_\alpha = \inf\{t : N(x_1, x_2, \dots, x_n, t) \geq \alpha, M(x_1, x_2, \dots, x_n, t) \leq 1 - \alpha\}$,
 $\alpha \in (0, 1)$.

The $\alpha - n$ -norms are given by

$$\begin{aligned} \|x_1, x_2, \dots, x_n\|_\alpha &= \|x_1, x_2, \dots, x_n\|_1, \text{ when } 1 > \alpha > 0.5, \end{aligned}$$

$$= \|x_1, x_2, \dots, x_n\|_2, \text{ when } 0 < \alpha \leq 0.5$$

Let $G = \{(a, 0, 0, \dots, n \text{ times}) : a \in \mathbb{R}\}$ be a subset of X .

Choose $x_0 = (0, 1, 1, \dots, n \text{ times})$ and

$$x_2, x_3, \dots, x_n \in K = \left\{ (0, 0, k_3^{(i)}, \dots, k_{n+1}^{(i)}) : k_3^{(i)}, \dots, k_{n+1}^{(i)} \in \mathbb{R} \setminus \{0\} \right\}.$$

In other words, $x_2 = (0, 0, k_3^{(2)}, \dots, k_{n+1}^{(2)})$,

$$x_3 = (0, 0, k_3^{(3)}, \dots, k_{n+1}^{(3)}), \dots, x_n = (0, 0, k_3^{(n)}, \dots, k_{n+1}^{(n)})$$

Then,

$$D_{x_2, x_3, \dots, x_n}(x_0, G) = \{x = (0, b, 0, \dots, (n-1) \text{ times}), b \in \mathbb{R}^+ \setminus \{0\} : \\ d_{x_2, x_3, \dots, x_n}(x, G) = \|x - x_0, x_2, x_3, \dots, x_n\|_\alpha + d_{x_2, x_3, \dots, x_n}(x_0, G)\}$$

where $d_{x_2, x_3, \dots, x_n}(x, G) = \max\{|b|\Delta, |a|\Delta\}$,

$$\Delta = \begin{vmatrix} k_3^{(2)} & k_4^{(2)} & \dots & k_{n+1}^{(2)} \\ k_3^{(3)} & k_4^{(3)} & \dots & k_{n+1}^{(3)} \\ \vdots & \vdots & \ddots & \vdots \\ k_3^{(n)} & k_4^{(n)} & \dots & k_{n+1}^{(n)} \end{vmatrix},$$

$$\|x - x_0, x_2, \dots, x_n\|_\alpha = |b - 1|\Delta,$$

$$d_{x_2, x_3, \dots, x_n}(x_0, G) = \max\{\Delta, |a|\Delta\}$$

and also

$$P_{G, x_2, x_3, \dots, x_n}(x) = \{g = (a, 0, \dots, n \text{ times}) : -1 \leq a \leq 1\}.$$

Theorem 4.4. For $x \in D_{x_2, x_3, \dots, x_n}(x_0, G)$ and $y \in D_{x_2, x_3, \dots, x_n}(x, G)$ we have:

- (i) $\|y - x_0, x_2, x_3, \dots, x_n\|_\alpha = \|y - x, x_2, x_3, \dots, x_n\|_\alpha + \|x - x_0, x_2, x_3, \dots, x_n\|_\alpha$
- (ii) $y - x + x_0 \in D_{x_2, x_3, \dots, x_n}(x_0, G)$.

Proof. (i) Let $x \in D_{x_2, x_3, \dots, x_n}(x_0, G)$ and $y \in D_{x_2, x_3, \dots, x_n}(x, G)$.

Then by (4.2) we have

$$d_{x_2, x_3, \dots, x_n}(x, G) = \|x - x_0, x_2, x_3, \dots, x_n\|_\alpha + d_{x_2, x_3, \dots, x_n}(x_0, G),$$

$$d_{x_2, x_3, \dots, x_n}(y, G) = \|y - x, x_2, x_3, \dots, x_n\|_\alpha + d_{x_2, x_3, \dots, x_n}(x, G).$$

Now

$$\begin{aligned} & \|y - x_0, x_2, x_3, \dots, x_n\|_\alpha \\ &= \|y - x_0 - x + x, x_2, x_3, \dots, x_n\|_\alpha \\ &= \|(y - x) + (x - x_0), x_2, x_3, \dots, x_n\|_\alpha \end{aligned}$$

$$\begin{aligned}
&\leq \|y - x, x_2, x_3, \dots, x_n\|_\alpha + \|x - x_0, x_2, x_3, \dots, x_n\|_\alpha \\
&= (d_{x_2, x_3, \dots, x_n}(y, G) - d_{x_2, x_3, \dots, x_n}(x, G)) + (d_{x_2, x_3, \dots, x_n}(x, G) \\
&\quad - d_{x_2, x_3, \dots, x_n}(x_0, G)) \\
&= d_{x_2, x_3, \dots, x_n}(y, G) - d_{x_2, x_3, \dots, x_n}(x_0, G) \\
&\leq \|y - x_0, x_2, x_3, \dots, x_n\|_\alpha.
\end{aligned}$$

Therefore,

$$\|y - x_0, x_2, x_3, \dots, x_n\|_\alpha = \|y - x, x_2, x_3, \dots, x_n\|_\alpha + \|x - x_0, x_2, x_3, \dots, x_n\|_\alpha.$$

(ii) By (4.1), we have

$$\begin{aligned}
&d_{x_2, x_3, \dots, x_n}(y - x + x_0, G) \\
&\geq d_{x_2, x_3, \dots, x_n}(y, G) - \|y - (y - x + x_0), x_2, x_3, \dots, x_n\|_\alpha \\
&= d_{x_2, x_3, \dots, x_n}(y, G) - \|x - x_0, x_2, x_3, \dots, x_n\|_\alpha \\
&= (\|y - x, x_2, x_3, \dots, x_n\|_\alpha + d_{x_2, x_3, \dots, x_n}(x, G)) - \|x - x_0, x_2, x_3, \dots, x_n\|_\alpha \\
&= \|y - x, x_2, x_3, \dots, x_n\|_\alpha + (\|x - x_0, x_2, x_3, \dots, x_n\|_\alpha + d_{x_2, x_3, \dots, x_n}(x_0, G)) \\
&\quad - \|x - x_0, x_2, x_3, \dots, x_n\|_\alpha \\
&= \|y - x, x_2, x_3, \dots, x_n\|_\alpha + d_{x_2, x_3, \dots, x_n}(x_0, G) \\
&= \|(y - x + x_0) - x_0, x_2, x_3, \dots, x_n\|_\alpha + d_{x_2, x_3, \dots, x_n}(x_0, G).
\end{aligned}$$

Again it follows from (4.1) that

$$\begin{aligned}
d_{x_2, x_3, \dots, x_n}(y - x + x_0, G) &= \|(y - x + x_0) - x_0, x_2, x_3, \dots, x_n\|_\alpha + d_{x_2, x_3, \dots, x_n}(x_0, G) \\
&\Rightarrow y - x + x_0 \in D_{x_2, x_3, \dots, x_n}(x_0, G). \quad \square
\end{aligned}$$

Theorem 4.5. Let $x \in D_{x_2, x_3, \dots, x_n}(x_0, G)$. Then

- (i) $[x_0, x] = \{\lambda x_0 + (1 - \lambda)x : 0 \leq \lambda \leq 1\} \subset D_{x_2, x_3, \dots, x_n}(x_0, G)$
- (ii) $D_{x_2, x_3, \dots, x_n}(x, G) \subset D_{x_2, x_3, \dots, x_n}(x_0, G)$.

Proof. (i) Let $y = \lambda x_0 + (1 - \lambda)x$ such that $0 \leq \lambda \leq 1$.

Then

$$\begin{aligned}
&d_{x_2, x_3, \dots, x_n}(y, G) \\
&\geq d_{x_2, x_3, \dots, x_n}(x, G) - \|x - y, x_2, x_3, \dots, x_n\|_\alpha \\
&= \|x - x_0, x_2, x_3, \dots, x_n\|_\alpha + d_{x_2, x_3, \dots, x_n}(x_0, G) - \|x - y, x_2, x_3, \dots, x_n\|_\alpha \\
&= \|y - x_0, x_2, x_3, \dots, x_n\|_\alpha + d_{x_2, x_3, \dots, x_n}(x_0, G).
\end{aligned}$$

By (4.1) we have

$$\begin{aligned}
d_{x_2, x_3, \dots, x_n}(y, G) &= \|y - x_0, x_2, x_3, \dots, x_n\|_\alpha + d_{x_2, x_3, \dots, x_n}(x_0, G) \\
&\Rightarrow y \in D_{x_2, x_3, \dots, x_n}(x_0, G).
\end{aligned}$$

(ii) Let $y \in D_{x_2, x_3, \dots, x_n}(x, G)$.

Thus by (4.2) and Theorem 4.4(i),

$$\begin{aligned}
&d_{x_2, x_3, \dots, x_n}(y, G) \\
&= \|y - x, x_2, x_3, \dots, x_n\|_\alpha + d_{x_2, x_3, \dots, x_n}(x, G) \\
&= \|y - x, x_2, x_3, \dots, x_n\|_\alpha + (\|x - x_0, x_2, x_3, \dots, x_n\|_\alpha + d_{x_2, x_3, \dots, x_n}(x_0, G))
\end{aligned}$$

$$\begin{aligned}
&= \|y - x_0, x_2, x_3, \dots, x_n\|_\alpha + d_{x_2, x_3, \dots, x_n}(x_0, G) \\
&\Rightarrow y \in D_{x_2, x_3, \dots, x_n}(x_0, G).
\end{aligned}$$

Therefore $D_{x_2, x_3, \dots, x_n}(x, G) \subset D_{x_2, x_3, \dots, x_n}(x_0, G)$. \square

Theorem 4.6. Let $x_0, y_0 \in X$ and $\lambda \neq 0$. Then

- (i) $D_{x_2, x_3, \dots, x_n}(x_0, G) + y_0 = D_{x_2, x_3, \dots, x_n}(x_0 + y_0, G + y_0)$
- (ii) $D_{x_2, x_3, \dots, x_n}(x_0, \lambda G) = \lambda D_{x_2, x_3, \dots, x_n}(\frac{x_0}{\lambda}, G)$.

Proof. (i) Let $x \in D_{x_2, x_3, \dots, x_n}(x_0, G)$. Then

$$\begin{aligned}
&d_{x_2, x_3, \dots, x_n}(x + y_0, G + y_0) \\
&= d_{x_2, x_3, \dots, x_n}(x, G) \\
&= \|x - x_0, x_2, x_3, \dots, x_n\|_\alpha + d_{x_2, x_3, \dots, x_n}(x_0, G) \\
&= \|x + y_0 - (x_0 + y_0), x_2, x_3, \dots, x_n\|_\alpha + d_{x_2, x_3, \dots, x_n}(x_0, G).
\end{aligned}$$

Therefore $x + y_0 \in D_{x_2, x_3, \dots, x_n}(x_0 + y_0, G + y_0)$.

Conversely, let $y \in D_{x_2, x_3, \dots, x_n}(x_0 + y_0, G + y_0)$.

Then,

$$\begin{aligned}
&d_{x_2, x_3, \dots, x_n}(y - y_0, G) \\
&= d_{x_2, x_3, \dots, x_n}(y, G + y_0) \\
&= \|y - y_0 - x_0, x_2, x_3, \dots, x_n\|_\alpha + d_{x_2, x_3, \dots, x_n}(x_0 + y_0, G + y_0) \\
&= \|(y - y_0) - x_0, x_2, x_3, \dots, x_n\|_\alpha + d_{x_2, x_3, \dots, x_n}(x_0, G).
\end{aligned}$$

Therefore $y - y_0 \in D_{x_2, x_3, \dots, x_n}(x_0, G)$ and so

$$D_{x_2, x_3, \dots, x_n}(x_0, G) + y_0 = D_{x_2, x_3, \dots, x_n}(x_0 + y_0, G + y_0).$$

(ii) Let $x \in D_{x_2, x_3, \dots, x_n}(x_0, \lambda G)$. Then

$$\begin{aligned}
&d_{x_2, x_3, \dots, x_n}(\frac{x}{\lambda}, G) \\
&= \frac{1}{|\lambda|} d_{x_2, x_3, \dots, x_n}(x, \lambda G) \\
&= \frac{1}{|\lambda|} (\|x - x_0, x_2, x_3, \dots, x_n\|_\alpha + d_{x_2, x_3, \dots, x_n}(x_0, \lambda G)) \\
&= \|\frac{x}{\lambda} - \frac{x_0}{\lambda}, x_2, x_3, \dots, x_n\|_\alpha + d_{x_2, x_3, \dots, x_n}(\frac{x_0}{\lambda}, G).
\end{aligned}$$

Therefore $\frac{x}{\lambda} \in D_{x_2, x_3, \dots, x_n}(\frac{x_0}{\lambda}, G)$.

Conversely, let $x \in D_{x_2, x_3, \dots, x_n}(\frac{x_0}{\lambda}, G)$.

Then

$$\begin{aligned}
&d_{x_2, x_3, \dots, x_n}(\lambda x, \lambda G) \\
&= |\lambda| d_{x_2, x_3, \dots, x_n}(x, G) \\
&= |\lambda| (\|x - \frac{x_0}{\lambda}, x_2, x_3, \dots, x_n\|_\alpha + d_{x_2, x_3, \dots, x_n}(\frac{x_0}{\lambda}, G)) \\
&= \|\lambda x - x_0, x_2, x_3, \dots, x_n\|_\alpha + d_{x_2, x_3, \dots, x_n}(x_0, \lambda G).
\end{aligned}$$

Therefore $\lambda x \in D_{x_2, x_3, \dots, x_n}(x_0, \lambda G)$.

Thus $D_{x_2, x_3, \dots, x_n}(x_0, \lambda G) = \lambda D_{x_2, x_3, \dots, x_n}(\frac{x_0}{\lambda}, G)$. \square

Theorem 4.7. Let $G \subset G_1$ and $x_0 \in X$, where G_1 is a subset of X such that

$$d_{x_2, x_3, \dots, x_n}(x_0, G) = d_{x_2, x_3, \dots, x_n}(x_0, G_1). \quad (4.3)$$

Then $D_{x_2, x_3, \dots, x_n}(x_0, G_1) \subset D_{x_2, x_3, \dots, x_n}(x_0, G)$.

Proof. Let $x \in D_{x_2, x_3, \dots, x_n}(x_0, G_1)$.

By (4.3) we have,

$$\begin{aligned} d_{x_2, x_3, \dots, x_n}(x, G) &\geq d_{x_2, x_3, \dots, x_n}(x, G_1) \\ &= \|x - x_0, x_2, x_3, \dots, x_n\|_\alpha + d_{x_2, x_3, \dots, x_n}(x_0, G_1) \\ &= \|x - x_0, x_2, x_3, \dots, x_n\|_\alpha + d_{x_2, x_3, \dots, x_n}(x_0, G). \end{aligned}$$

From (4.1) it follows that

$$\begin{aligned} d_{x_2, x_3, \dots, x_n}(x, G) &= \|x - x_0, x_2, x_3, \dots, x_n\|_\alpha + d_{x_2, x_3, \dots, x_n}(x_0, G) \\ &\Rightarrow x \in D_{x_2, x_3, \dots, x_n}(x_0, G). \end{aligned}$$

Hence $D_{x_2, x_3, \dots, x_n}(x_0, G_1) \subset D_{x_2, x_3, \dots, x_n}(x_0, G)$. \square

Theorem 4.8. (i) $P_{G, x_2, x_3, \dots, x_n}(x_0) \subset P_{G, x_2, x_3, \dots, x_n}(x)$, for every $x \in D_{x_2, x_3, \dots, x_n}(x_0, G)$

(ii) $D_{x_2, x_3, \dots, x_n}(x_0, G) = P_{G, x_2, x_3, \dots, x_n}^{-1}(x_0)$ for every $x_0 \in \overline{G}$.

Proof. (i) Let $x \in D_{x_2, x_3, \dots, x_n}(x_0, G)$ and $g \in P_{G, x_2, x_3, \dots, x_n}(x)$.

Now,

$$\begin{aligned} d_{x_2, x_3, \dots, x_n}(x, G) &= \|x - x_0, x_2, x_3, \dots, x_n\|_\alpha + d_{x_2, x_3, \dots, x_n}(x_0, G) \\ &= \|x - x_0, x_2, x_3, \dots, x_n\|_\alpha + \|x_0 - g_0, x_2, x_3, \dots, x_n\|_\alpha. \end{aligned}$$

By Theorem 4.4(i) we have,

$$\begin{aligned} d_{x_2, x_3, \dots, x_n}(x, G) &= \|x - g_0, x_2, x_3, \dots, x_n\|_\alpha \\ &\Rightarrow g_0 \in P_{G, x_2, x_3, \dots, x_n}(x) \\ &\Rightarrow P_{G, x_2, x_3, \dots, x_n}(x_0) \subset P_{G, x_2, x_3, \dots, x_n}(x). \end{aligned}$$

(ii) Let $x_0 \in \overline{G}$ and $x \in D_{x_2, x_3, \dots, x_n}(x_0, G)$.

Then,

$$\begin{aligned} d_{x_2, x_3, \dots, x_n}(x, G) &= \|x - x_0, x_2, x_3, \dots, x_n\|_\alpha + d_{x_2, x_3, \dots, x_n}(x_0, G) \end{aligned}$$

$$= \|x - x_0, x_2, x_3, \dots, x_n\|_\alpha, \text{ where } x_0 \in \overline{G}.$$

$$\Rightarrow x \in P_{G, x_2, x_3, \dots, x_n}^{-1}(x_0).$$

$$\text{So, } D_{x_2, x_3, \dots, x_n}(x_0, G) \subset P_{G, x_2, x_3, \dots, x_n}^{-1}(x_0). \quad (4.4)$$

Conversely, let $x \in P_{G, x_2, x_3, \dots, x_n}^{-1}(x_0)$.

Then $x_0 \in P_{G, x_2, x_3, \dots, x_n}(x)$.

Since $x_0 \in \overline{G}$, $d_{x_2, x_3, \dots, x_n}(x_0, G) = 0$.

Hence we have,

$$d_{x_2, x_3, \dots, x_n}(x, G)$$

$$= \|x - x_0, x_2, x_3, \dots, x_n\|_\alpha + d_{x_2, x_3, \dots, x_n}(x_0, G)$$

$$\Rightarrow x \in D_{x_2, x_3, \dots, x_n}(x_0, G)$$

$$\Rightarrow P_{G, x_2, x_3, \dots, x_n}^{-1}(x_0) \subset D_{x_2, x_3, \dots, x_n}(x_0, G). \quad (4.5)$$

From (4.4) and (4.5) it follows that

$$D_{x_2, x_3, \dots, x_n}(x_0, G) = P_{G, x_2, x_3, \dots, x_n}^{-1}(x_0). \quad \square$$

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