

## MULTIPERIOD CREDIBILITIC MEAN SEMI-ABSOLUTE DEVIATION PORTFOLIO SELECTION

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**ABSTRACT.** In this paper, we discuss a multiperiod portfolio selection problem with fuzzy returns. We present a new credibilitic multiperiod mean semi-absolute deviation portfolio selection with some real factors including transaction costs, borrowing constraints, entropy constraints, threshold constraints and risk control. In the proposed model, we quantify the investment return and risk associated with the return rate on a risky asset by its credibilitic expected value and semi- absolute deviation. Since the proposed model is a nonlinear dynamic optimization problem with path dependence, we design a novel forward dynamic programming method to solve it. Finally, we provide a numerical example to demonstrate the performance of the designed algorithm and the application of the proposed model.

### 1. Introduction

The first mathematical formulation of selecting a portfolio in the framework of risk-return trade-off was provided by Markowitz [24], who combines probability theory and optimization theory to model the behavior of the investors. This classical mean-variance model is valid if the return is multivariate normally distributed and the investor is averse to risk and always prefers lower risk, or it is valid if for any given return which is multivariate distributed, the investor has a quadratic utility function (Ingersoll [10]). In contrast to the quadratic Markowitz model, Konno and Yamazaki [14] proposed the first linear model for portfolio selection model. This model uses the absolute deviation to measure the risk of the investment and it essentially gives the same results as the mean variance model when the assets are multivariate-normally distributed. Simaan [25] provided a thorough comparison of the mean variance model and the mean absolute deviation model. Some authors have recently proposed using downside risk to measure the risk of the investment (Speranza [26], Lien and Tse [18], Estrada [5]). Speranza [26] used the semi-absolute deviation to measure the risk and formulated a mean semi-absolute portfolio selection model. Lien and Tse [18] compare the hedging effectiveness of currency futures with respect to currency options on the basis of the lower partial moments, as opposed to the two-sided risk measure. Notice that a downside risk

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measure would also help investors make proper decisions when the returns are non-normally distributed, as is the case in emerging market data and for international portfolio selection (Stevenson [27], Butler and Joaquin [2]).

Probability theory is a major tool used for analyzing uncertainty in portfolio selection. They view the uncertainty associated with risky assets as random uncertainty. The basic assumption of these portfolio optimization models under the framework of probability theory is that the situation of financial markets in future can be correctly reflected by asset data in the past. However, if there is not enough historical data, it is more reasonable to assume them as fuzzy variables. Fuzzy portfolio selection has been undertaken in the literature such as Wang and Zhu [31], Terol et al. [28], Fang et al. [6], Vercher et al. [30], Zhang et al. [42, 44], Huang [8], Li et al. [17] and Liu and Liu [20] defined the expected value and variance for measuring the portfolio return and the risk, respectively.

Though possibility measure has been widely used in portfolio selection, it has limitation. One great limitation is that possibility measure is not self-dual. Huang [8] find that two fuzzy events with different occurring chances may have the same possibility value. In addition, whenever the possibility value of a portfolio return greater than a target value is lower than 1, the possibility value of the opposite event is the maximum value of 1; or whenever the possibility value of a portfolio return less than or equal to a target value is lower than 1, the possibility value of the opposite event is the maximum value of 1. These results are quite awkward and will confuse the decision maker. Thus, Huang studied the risk and return of a portfolio investment based on the self-dual credibility measure. Several definitions of risk have been proposed according to people's different understanding towards risk. Within the framework of credibility theory, several models for fuzzy portfolio selection were proposed such as, Huang [9] presented multi-period mean-risk index portfolio selection model, Li et al. [17] proposed cross-entropy minimization model and so forth, Zhang and Liu [40] proposed a credibilitic multi-period mean-variance portfolio selection model.

The long-term investment horizon is of greater importance in practice. It is heavily discussed in recent literature (see e.g., Li and Ng [16]; Zhu et al. [45]; Gülpınar and Rustem [7]; Çelikyurt and Özekici [4]; Calafiore [3]; Yan et al. [34, 33]; Yu et al. [35, 36]; Wu and Li [32]; Li and Li [15]; Zhang et al [41, 39]; Zhang and Zhang [43]; Bodnar et al. [1]), to the best of our knowledge, a closed-form solution is not available in the general case up to now. Closed-form solutions are presented only under the assumption of independence, ie. Li and Ng [16] used dynamic programming approach to deal with the multiperiod mean variance portfolio selection problem by using the idea of embedding the problem in a tractable auxiliary problem. Then, they obtained breakthrough result, that is, the optimal mean-variance portfolio policy and the efficient frontier; Zhu et al. [45] incorporated a control of the probability of bankruptcy in the generalized mean variance formulation for multi-period portfolio optimization; Yu et al. [35, 36] discussed a dynamic portfolio optimization problem with risk control for the absolute deviation model; Wu and Li [32] investigate a non-self-financing portfolio optimization problem under the framework of multi-period meanCvariance with Markov regime switching and

a stochastic cash flow; Li and Li [15] represented a multi-period portfolio optimization problem for asset liability management of an investor who intends to control the probability of bankruptcy before reaching the end of an investment horizon. For more general models, these authors use different hybrid algorithms to solve different models, ie. van Binsbergen and Brandt [29] compared the numerical performance of value function iterations with portfolio weight iterations in the context of the simulation-based dynamic programming approach ; Mansini et al. [23] presented multiperiod mean CVaR portfolio selection model; Gülpınar and Rustem [7] extend the multi-period mean variance optimization framework to worst-case design with multiple rival return and risk scenarios; Yan et al. [34, 33] proposed a hybrid genetic algorithm with particle swarm optimizer to solve a class of multi-period semi-variance portfolio selection with a four-factor futures price model and a multi-period semi-variance portfolio selection; Zhang et al [41, 39], and Liu et al. [21, 22] respectively proposed genetic algorithm, hybrid intelligent algorithm and differential evolution algorithm to solve several kinds of multiperiod fuzzy portfolio selection models; Zhang and Zhang [43] proposed the discrete approximate iteration method to solve the multiperiod fuzzy portfolio selection model with cardinality constraints; Köksalan and Şakar [13] consider expected return, conditional value at risk, and liquidity criteria in a multi-period portfolio optimization setting modeled by stochastic programming. The literatures mentioned were proposed on the framework of probability and possibility theory.

The contribution of this work is as follows. We originally propose a new multiperiod credibilitic mean semi-absolute deviation portfolio selection models with real factors including borrowing constraints, transaction costs, entropy constraints and threshold constraints. We design a forward dynamic programming method for solution. The method can solve the general multiperiod portfolio selection models which is open-loop dynamic optimization problems.

This paper is organized as follows. In Section 2, the definitions of the credibilitic mean, the credibilitic semi-absolute deviation, and some properties are respectively introduced. In Section 3, the borrowing constraints, transaction costs, entropy constraints and threshold constraints are formulated into the multiperiod portfolio. A new multiperiod credibilitic mean semi-absolute deviation portfolio selection model is constructed. A new forward dynamic programming method is proposed to solve it in Section 4. In Section 5, a numerical example is presented to illustrate the modeling idea and the effectiveness of the designed algorithm. Finally, some conclusions are given in Section 6.

## 2. Preliminaries

Let  $\xi$  be a fuzzy variable with membership function  $\mu$ . For any  $x \in \mathfrak{R}$ ,  $\mu(x)$  represents the possibility that  $\xi$  takes value  $x$ . Hence, it is also called the possibility distribution. For any set  $B$ , the possibility measure and necessity measure of  $\xi \in B$  were respectively defined by Zadeh [37, 38] as

$$Pos\{\xi \in B\} = \sup_{x \in B} \mu(x), \quad (1)$$

$$Nec\{\xi \in B\} = 1 - \sup_{x \in B^c} \mu(x). \quad (2)$$

It is proved that both possibility measure and necessity measure satisfy the properties of normality, nonnegativity and monotonicity. However, neither of them is self-dual. Since the self-duality is intuitive and important in real problems, Liu and Liu [20] defined a credibility measure as the average of possibility measure and necessity measure

$$Cr\{\xi \in B\} = \frac{1}{2}(\sup_{x \in B} \mu(x) + 1 - \sup_{x \in B^c} \mu(x)). \quad (3)$$

where  $Cr$  is self-dual measure and satisfies  $Cr\{\xi \leq r\} + Cr\{\xi \geq r\} = 1$ .

**Definition 2.1.** (Liu and Liu [20]). Let  $\xi$  be a fuzzy variable. Then the expected value is defined as

$$E[\xi] = \int_0^\infty Cr\{\xi \geq r\}dr - \int_{-\infty}^0 Cr\{\xi \leq r\}dr. \quad (4)$$

Based on Definition 2.1, Liu [19] deduced the following two theorems.

**Theorem 2.2.** (Liu [19]) Let  $\xi$  be a fuzzy variable with finite expected value. Then, for any real numbers  $\lambda$  and  $\mu$ , it holds that

$$E(\lambda\xi + \mu) = \lambda E(\xi) + \mu. \quad (5)$$

**Theorem 2.3.** (Liu [19]) Let  $\xi$  and  $\eta$  be independent fuzzy variables with finite expected values. Then, for any real numbers  $\lambda$  and  $\mu$ , it holds that

$$E(\lambda\xi + \mu\eta) = \lambda E(\xi) + \mu E(\eta). \quad (6)$$

**Definition 2.4.** Let  $\xi$  be a fuzzy variable with finite expected value  $e$ . Then its credibilitic semi- absolute deviation of  $\xi$  is defined by

$$SAD(\xi) = E[|(\xi - e)^-|] = \int_0^\infty Cr\{|(\xi - e)^-| \geq r\}dr. \quad (7)$$

$$\text{where } (\xi - e)^- = \begin{cases} \xi - e & \text{if } \xi \leq e \\ 0 & \text{if } \xi > e \end{cases}$$

From Definition 2.4, we can get the following form.

$$\begin{aligned} Cr\{|(\xi - e)^-| \geq r\} &= Cr\{-(\xi - e) \geq r\} \\ &= Cr\{\xi - e \leq -r\} = Cr\{\xi \leq e - r\}. \end{aligned} \quad (8)$$

Thus, the credibilitic semi-absolute deviation of  $\xi$  can be expressed as follows:

$$SAD(\xi) = \int_0^\infty Cr\{\xi \leq e - r\}dr. \quad (9)$$

**Theorem 2.5.** Let  $\xi$  be a fuzzy variable with finite expected value  $e$ . Then for any nonnegative real numbers  $\lambda$ , it holds

$$SAD(\lambda\xi) = \lambda SAD(\xi). \quad (10)$$

*Proof.* From Definition 2.4, it follows that

$$\begin{aligned}
 SAD(\lambda\xi) &= E[|(\lambda\xi - \lambda e)^-|] = \int_0^\infty Cr\{ |(\lambda\xi - \lambda e)^-| \geq r\} dr \\
 &= \int_0^\infty Cr\{ |(\xi - e)^-| \geq \frac{r}{\lambda}\} dr \\
 &= \lambda \int_0^\infty Cr\{ |(\xi - e)^-| \geq t\} dt \\
 &= \lambda SAD(\xi).
 \end{aligned}$$

Thus, the proof of Theorem 2.5 is ended.  $\square$

**Theorem 2.6.** *Let  $\xi$  be a fuzzy variable with finite expected value  $e$ . Then for any nonnegative real numbers  $\lambda$  and real number  $v$ , it holds*

$$SAD(\lambda\xi + v) = \lambda SAD(\xi). \quad (11)$$

*Proof.* From Theorem 2.5, it follows that

$$SAD(\lambda\xi + v) = E[|(\lambda\xi + v - \lambda e - v)^-|] = E[|(\lambda\xi - \lambda e)^-|] = \lambda SAD(\xi).$$

Thus, the proof of Theorem 2.6 is ended.  $\square$

**Theorem 2.7.** *Let  $\xi$  and  $\eta$  be independent fuzzy variables with finite expected values. Then, for any nonnegative real numbers  $\lambda$  and  $\mu$ , it holds that*

$$SAD(\lambda\xi + \mu\eta) = \lambda SAD(\xi) + \mu SAD(\eta). \quad (12)$$

*Proof.*

$$\begin{aligned}
 SAD(\lambda\xi + \mu\eta) &= E[|(\lambda\xi + \mu\eta - \lambda E(\xi) - \mu E(\eta))^-|] \\
 &= \int_0^\infty Cr\{ |(\lambda\xi + \mu\eta - \lambda E(\xi) - \mu E(\eta))^-| \geq r\} dr \\
 &= \int_0^\infty Cr\{ -(\lambda\xi - \lambda E(\xi)) - (\mu\eta - \mu E(\eta)) \geq r\} dr \\
 &= \int_0^\infty Cr\{ \lambda\xi + \mu\eta \leq \lambda E(\xi) + \mu E(\eta) - r\} dr \\
 &= \int_0^\infty \lambda Cr\{ |(\xi - E(\xi))^-| \geq r\} dr + \int_0^\infty \mu Cr\{ |(\eta - E(\eta))^-| \geq r\} dr \\
 &= \lambda SAD(\xi) + \mu SAD(\eta).
 \end{aligned} \quad (13)$$

Thus, the proof of Theorem 2.7 is ended.  $\square$

If  $\xi = (a, \alpha, \beta)$  be a triangular fuzzy number, then its membership function  $\mu_\xi(x)$  can be described as:

$$\mu_\xi(x) = \begin{cases} 1 + \frac{x-a}{\alpha}, & x \in [a - \alpha, a] \\ 1 + \frac{a-x}{\beta}, & x \in [a, a + \beta] \\ 0, & \text{otherwise.} \end{cases} \quad (14)$$

From Eq.(3), the credibility of the event  $\{\xi \leq r\}$  is as follows:

$$Cr\{\xi \leq r\} = \begin{cases} 0, & \text{if } r \leq a - \alpha \\ \frac{r-a+\alpha}{2\alpha}, & \text{if } a - \alpha \leq r \leq a \\ \frac{r-a+\beta}{2\beta}, & \text{if } a \leq r \leq a + \beta \\ 1, & \text{otherwise.} \end{cases} \quad (15)$$

**Theorem 2.8.** *Let  $\xi = (a, \alpha, \beta)$  be a triangular fuzzy number. Then, the credibilitic expected value of  $\xi$  can be given by:*

$$E(\xi) = a + \frac{\beta - \alpha}{4} \quad (16)$$

*Proof.* From the Definition 2.1 and Eq.(15), it follows that

$$\begin{aligned} E[\xi] &= \int_0^\infty Cr\{\xi \geq r\}dr - \int_{-\infty}^0 Cr\{\xi \leq r\}dr \\ &= \int_0^\infty (1 - Cr\{\xi \leq r\})dr \\ &= \int_0^{a-\alpha} 1dr + \int_{a-\alpha}^a \left(1 - \frac{r-a+\alpha}{2\alpha}\right)dr + \int_a^{a+\beta} \left(1 - \frac{r-a+\beta}{2\beta}\right)dr + 0 \\ &= a + \frac{\beta - \alpha}{4}. \end{aligned}$$

Thus, the proof of Theorem 2.8 is ended.  $\square$

**Theorem 2.9.** *Let  $\xi = (a, \alpha, \beta)$  be a triangular fuzzy number. Then, the credibilitic semi-absolute deviation of  $\xi$  can be given by*

$$SAD(\xi) = \frac{39\beta^2 + 18\alpha\beta + 7\alpha^2}{256\beta}. \quad (17)$$

*Proof.* From Eq.(15), it follows that

$$\begin{aligned} &Cr\left\{\xi \leq a + \frac{\beta - \alpha}{4} - r\right\} \\ &= \begin{cases} 0, & \text{if } a + \frac{\beta - \alpha}{4} - r \leq a - \alpha \\ \frac{a + \frac{\beta - \alpha}{4} - r - a + \alpha}{2\alpha}, & \text{if } a - \alpha \leq a + \frac{\beta - \alpha}{4} - r \leq a \\ \frac{a + \frac{\beta - \alpha}{4} - r - a + \beta}{2\beta}, & \text{if } a \leq a + \frac{\beta - \alpha}{4} - r \leq a + \beta \\ 1, & \text{otherwise} \end{cases} \\ &= \begin{cases} 0, & \text{if } r \geq \frac{\beta + 3\alpha}{4} \\ \frac{\beta + 3\alpha - 4r}{8\alpha}, & \text{if } \frac{\beta - \alpha}{4} \leq r \leq \frac{\beta + 3\alpha}{4} \\ \frac{5\beta - \alpha - r}{8\beta}, & \text{if } 0 \leq r \leq \frac{\beta - \alpha}{4} \\ 1, & \text{if } r \leq 0. \end{cases} \end{aligned} \quad (18)$$

From Eq.(18), it follows that

$$\begin{aligned}
 SAD(\xi) &= \int_0^\infty Cr\{\xi \leq e - r\}dr \\
 &= \int_0^{\frac{\beta-\alpha}{4}} \frac{5\beta - \alpha - r}{8\beta} dr + \int_{\frac{\beta-\alpha}{4}}^{\frac{\beta+3\alpha}{4}} \frac{\beta + 3\alpha - 4r}{8\alpha} dr \\
 &= \frac{(5\beta - \alpha)(\beta - \alpha)}{32\beta} - \frac{(\beta - \alpha)^2}{256\beta} + \frac{\beta + 3\alpha}{8} - \frac{(2\beta + 2\alpha)4\alpha}{64\alpha} \\
 &= \frac{39\beta^2 + 18\alpha\beta + 7\alpha^2}{256\beta}.
 \end{aligned}$$

Thus, the proof of Theorem 2.9 is ended.  $\square$

### 3. The Multiperiod Credibilitic Portfolio Selection Model

Assume that there are  $n$  risky assets and one risk-free asset in financial market for trading. An investor wants to allocate his initial wealth  $W_1$  among  $n + 1$  assets at the beginning of period 1, and obtains the final wealth at the end of period  $T$ . He can reallocate his wealth among the  $n$  risky assets at the beginning of each of the following  $T$  consecutive investment periods. Suppose that the return rates of the  $n$  risky assets at each period are denoted as triangular fuzzy variables, and the returns of portfolios among different periods are independent of each other. For the sake of description, let us first introduce the following notations:

$x_{it}$  the investment proportion of risky asset  $i$  at period  $t$ ;

$x_{i0}$  the initial investment proportion of risky asset  $i$  at period 0;

$x_t$  the portfolio at period  $t$ , where  $x_t = (x_{1t}, x_{2t}, \dots, x_{nt}, x_{ft})$ ;

$x_{ft}$  the investment proportion of risk-free asset at period  $t$ , where  $x_{ft} = 1 - \sum_{i=1}^n x_{it}$

$x_{ft}^b$  the lower bound of the investment proportion of risk-free asset at period  $t$ , where  $x_{ft} \geq x_{ft}^b$ ;

$R_{it}$  the return of risky asset  $i$  at period  $t$ ;

$r_{pt}$  the return rate of the portfolio  $x_t$  at period  $t$ ;

$r_{bt}$  the borrowing rate of the risk-free asset at period  $t$ ;

$r_{lt}$  the lending rate of the risk-free asset at period  $t$ ;

$l_{it}$  the lower bound constraints of  $x_{it}$ ;

$u_{it}$  the upper bound constraints of  $x_{it}$ ;

$r_{Nt}$  the net return rate of the portfolio  $x_t$  at period  $t$ ;

$W_t$  the crisp form of the holding wealth at the beginning of period  $t$ ;

$H_t$  the given entropy constraint at period  $t$ ;

$c_{it}$  the unit transaction cost of risky asset  $i$  at period  $t$ ;

**3.1. Return, Risk and Entropy Constraints.** In this section, we employ the credibilitic mean value of the net return on the portfolio at each period to measure the return of portfolio. The risk on the return rate of portfolio at each period is quantified by the credibilitic semi-absolute deviation. The return rate of security  $i$

at period  $t$ ,  $R_{it} = (a_{it}, \alpha_{it}, \beta_{it})$ , is triangular fuzzy variable for all  $i = 1, \dots, n$  and  $t = 1, \dots, T$ .

The credibilitic mean value of the portfolio  $x_t = (x_{1t}, x_{2t}, \dots, x_{nt})$  at period  $t$  can be expressed as

$$\begin{aligned} r_{pt} &= \sum_{i=1}^n E(R_{it})x_{it} + r_{ft}(1 - \sum_{i=1}^n x_{it}) \\ &= \sum_{i=1}^n (a_{it} + \frac{\beta_{it} - \alpha_{it}}{4})x_{it} + r_{ft}(1 - \sum_{i=1}^n x_{it}), t = 1, \dots, T. \end{aligned} \quad (19)$$

where  $E(R_{it}) = r_{it} = (a_{it} + \frac{\beta_{it} - \alpha_{it}}{4})$ ,  $x_{ft} = 1 - \sum_{i=1}^n x_{it}$ ,  $r_{ft} = \begin{cases} r_{lt}, & 1 - \sum_{i=1}^n x_{it} \geq 0 \\ r_{bt}, & 1 - \sum_{i=1}^n x_{it} \leq 0 \end{cases}$ ,

$r_{bt} \geq r_{lt}$ . When  $x_{ft} \geq 0$ , it denotes that lending is allowed on the risk-free asset; When  $x_{ft} \leq 0$ , it represents that borrowing is allowed on the risk-free asset.

We assume in the sequel that the transaction costs at period  $t$  is a  $V$  shape function of difference between the  $t$ th period portfolio  $x_t = (x_{1t}, x_{2t}, \dots, x_{nt})$  and the  $t-1$ th period portfolio  $x_{t-1} = (x_{1(t-1)}, x_{2(t-1)}, \dots, x_{n(t-1)})$ . That's to say, the transaction cost for asset  $i$  at period  $t$  can be expressed by

$$C_{it} = c_{it} |x_{it} - x_{i(t-1)}|. \quad (20)$$

where  $c_{it}$  is the unit transaction cost of risky asset  $i$  at period  $t$ .

Hence, the total transaction costs of the portfolio  $x_t = (x_{1t}, x_{2t}, \dots, x_{nt})$  at period  $t$  can be represented as

$$C_t = \sum_{i=1}^n c_{it} |x_{it} - x_{i(t-1)}|, t = 1, \dots, T. \quad (21)$$

Thus, the net return rate of the portfolio  $x_t$  at period  $t$  can be denoted as

$$\begin{aligned} r_{Nt} &= \sum_{i=1}^n (a_{it} + \frac{\beta_{it} - \alpha_{it}}{4})x_{it} + r_{ft}(1 - \sum_{i=1}^n x_{it}) - \sum_{i=1}^n c_{it} |x_{it} - x_{i(t-1)}|, \\ &t = 1, \dots, T. \end{aligned} \quad (22)$$

Then, the crisp form of the holding wealth at the beginning of the period  $t$  can be written as

$$\begin{aligned} W_{t+1} &= W_t(1 + r_{Nt}) \\ &= W_t(1 + \sum_{i=1}^n (a_{it} + \frac{\beta_{it} - \alpha_{it}}{4})x_{it} + r_{ft}(1 - \sum_{i=1}^n x_{it}) - \sum_{i=1}^n c_{it} |x_{it} - x_{i(t-1)}|), \\ &t = 1, \dots, T. \end{aligned} \quad (23)$$

Derived from Eq.(18), the semi-absolute deviation of the portfolio  $x_t$  can be expressed as

$$SAD_t(x_t) = \sum_{i=1}^n SAD(R_{it})x_{it} = \sum_{i=1}^n \frac{39\beta_{it}^2 + 18\alpha_{it}\beta_{it} + 7\alpha_{it}^2}{256\beta_{it}}x_{it}. \quad (24)$$



The main characteristic of this model is that the risk of a portfolio is measured by the semi-absolute deviation of the return rate of assets instead of the variance. Much attention has been focused on this risk function because the related portfolio optimization problem can be converted into a scalar parametric linear programming problem which can be easily implemented. Simplicity and computational robustness are perceived as one of the most important advantages of the mean absolute deviation model.

In order to satisfy the requisition of decentralized investment, entropy constraints will be developed to measure the diversification degree of portfolio. Before introducing the entropy, let us first review the existing proportion entropy, which is employed to reflect the diversification degree of single-period portfolio selection problem in Fang et al. [6], Kapur [12] and Jana et al. [11]. The entropy of the portfolio  $x_t$  can be expressed as follows:

$$En(x_t) = - \sum_{i=1}^n x_{it} \ln x_{it}. \quad (25)$$

From Eq. (25), we have  $x_{it} \geq 0 (i = 1, 2, \dots, n)$ , that is, every asset must be chosen for constructing a portfolio. Note that when  $x_{1t} = x_{2t} = \dots = \frac{1}{n}$ , Eq. (25) takes its maximum value. In other words, the diversification degree of the portfolio is the maximum. However, in the practical investment management, investors often may not hope to distribute their wealth among every asset for constructing an extremely diversified portfolio. In particular, when investors forecast the rate of return on asset  $i$ ,  $R_{it}$ , is less than the risk-free return rate, investors may not support investment in asset  $i$ , namely,  $x_{it} = 0$ .

Threshold constraints limit the amount of capital to be invested in each stock and prevent very small investments in any stock. The threshold constraints of multiperiod portfolio selection can be expressed as

$$0 \leq x_{it} \leq u_{it}. \quad (26)$$

where 0 and  $u_{it}$  are respectively the lower and upper bounds constraints of  $x_{it}$ .

For a rational investor, he/she wishes not only to maximize expected return but also to minimize the risk which is measured by the variance of the rate of return on a portfolio. So he/she must make a tradeoff between the two objectives. Let  $(1 - \theta)$  and  $\theta$  be the weights associated with criteria  $r_{Nt}$  and  $SAD_t(x_t)$  respectively. Then the investor attempts to maximize

$$\begin{aligned} F_t(r_{Nt}, SAD_t(x_t)) = & (1 - \theta) \left( \sum_{i=1}^n \left( a_{it} + \frac{\beta_{it} - \alpha_{it}}{4} \right) x_{it} + r_{ft} \left( 1 - \sum_{i=1}^n x_{it} \right) \right. \\ & \left. - \sum_{i=1}^n c_{it} |x_{it} - x_{i(t-1)}| \right) + \theta \sum_{i=1}^n \frac{39\beta_{it}^2 + 18\alpha_{it}\beta_{it} + 7\alpha_{it}^2}{256\beta_{it}} x_{it}. \end{aligned} \quad (27)$$

Here the parameter  $\theta$  can be interpreted as the risk aversion factor of the investor. The greater the factor  $\theta$  is, the more risk aversion the investor has. In this paper, we assume that the investor is of risk aversion, i.e.,  $0 \leq \theta \leq 1$ .

**3.2. The Basic Multiperiod Portfolio Optimization Models.** When the investors can give a tolerable level of risk at period  $t$ , and want to maximize the terminal wealth at the given level of risk, we have the multiperiod credibilitic mean semi-absolute deviation model as follows:

$$\begin{aligned}
& \max \sum_{t=1}^T ((1-\theta) \left( \sum_{i=1}^n \left( a_{it} + \frac{\beta_{it} - \alpha_{it}}{4} \right) x_{it} + r_{ft} \left( 1 - \sum_{i=1}^n x_{it} \right) \right. \\
& \left. - \sum_{i=1}^n c_{it} |x_{it} - x_{i(t-1)}| \right) - \theta \sum_{i=1}^n \frac{39\beta_{it}^2 + 18\alpha_{it}\beta_{it} + 7\alpha_{it}^2}{256\beta_{it}} x_{it}). \\
& s.t. \begin{cases} W_{t+1} = \left( 1 + \left( \sum_{i=1}^n \left( a_{it} + \frac{\beta_{it} - \alpha_{it}}{4} \right) x_{it} + r_{ft} \left( 1 - \sum_{i=1}^n x_{it} \right) \right. \right. \\ \quad \left. \left. - \sum_{i=1}^n c_{it} |x_{it} - x_{i(t-1)}| \right) \right) W_t & (a) \\ 1 - \sum_{i=1}^n x_{it} \geq x_{ft}^b & (b) \\ - \sum_{i=1}^n x_{it} \ln x_{it} \leq H_t & (c) \\ 0 \leq x_{it} \leq u_{it}, i = 1, \dots, n, t = 1, \dots, T & (d). \end{cases} \quad (28)
\end{aligned}$$

where  $H_t$  denotes the maximum entropy the investors can tolerate. Constraint (a) denotes the wealth accumulation constraint; constraint (b) indicates the investment proportion of risk-free asset at period  $t$  must exceed the given lower bound  $x_{ft}^b$ ; constraint (c) states the entropy of the portfolio  $x_t$  cant exceed the given value  $H_t$ ; constraint (d) represents threshold constraints of  $x_{it}$ .

Let  $y_{it} = |x_{it} - x_{i(t-1)}|$ . Then the Model (28) can be turned into as follows.

$$\begin{aligned}
& \max \sum_{t=1}^T ((1-\theta) \left( \sum_{i=1}^n \left( a_{it} + \frac{\beta_{it} - \alpha_{it}}{4} \right) x_{it} + r_{ft} \left( 1 - \sum_{i=1}^n x_{it} \right) \right. \\
& \left. - \sum_{i=1}^n c_{it} y_{it} \right) - \theta \sum_{i=1}^n \frac{39\beta_{it}^2 + 18\alpha_{it}\beta_{it} + 7\alpha_{it}^2}{256\beta_{it}} x_{it}).
\end{aligned}$$

$$\text{s.t.} \left\{ \begin{array}{l}
 W_{t+1} = (1 + (\sum_{i=1}^n (a_{it} + \frac{\beta_{it} - \alpha_{it}}{4})x_{it} + r_{ft}(1 - \sum_{i=1}^n x_{it}) \\
 \quad - \sum_{i=1}^n c_{it}y_{it}))W_t \\
 1 - \sum_{i=1}^n x_{it} \geq x_{ft}^b \\
 y_{it} \geq x_{it} - x_{i(t-1)} \\
 y_{it} \geq -(x_{it} - x_{i(t-1)}) \\
 - \sum_{i=1}^n x_{it} \ln x_{it} \leq H_t \\
 0 \leq x_{it} \leq u_{it}, i = 1, \dots, n, t = 1, \dots, T.
 \end{array} \right. \quad (29)$$

#### 4. Solution Algorithm

In this section, a forward dynamic programming method is proposed to solve the Model (29).

**4.1. Solution of the Sub-problem of Period t.** The sub-problem of period  $t$  of the Model (29) can be transformed into

$$\begin{aligned}
 & \max((1 - \theta)(\sum_{i=1}^n (a_{it} + \frac{\beta_{it} - \alpha_{it}}{4})x_{it} + r_{ft}(1 - \sum_{i=1}^n x_{it}) \\
 & \quad - \sum_{i=1}^n c_{it}y_{it}) - \theta \sum_{i=1}^n \frac{39\beta_{it}^2 + 18\alpha_{it}\beta_{it} + 7\alpha_{it}^2}{256\beta_{it}}x_{it}). \\
 \text{s.t.} \left\{ \begin{array}{l}
 W_{t+1} = (1 + (\sum_{i=1}^n (a_{it} + \frac{\beta_{it} - \alpha_{it}}{4})x_{it} + r_{ft}(1 - \sum_{i=1}^n x_{it}) \\
 \quad - \sum_{i=1}^n c_{it}y_{it}))W_t \\
 1 - \sum_{i=1}^n x_{it} \geq x_{ft}^b \\
 y_{it} \geq x_{it} - x_{i(t-1)} \\
 y_{it} \geq -(x_{it} - x_{i(t-1)}) \\
 - \sum_{i=1}^n x_{it} \ln x_{it} \geq H_t \\
 0 \leq x_{it} \leq u_{it}, i = 1, \dots, n.
 \end{array} \right. \quad (30)
 \end{aligned}$$

$H_t^{max}$ (the biggest value of  $H_t$ ) and  $H_t^{min}$ (the smallest value of  $H_t$ ) can be respectively obtained as follows.

When  $H_t$  is more than  $\max\{-\sum_{i=1}^n x_{it} \ln x_{it}\}$ , the nonlinear inequality  $-\sum_{i=1}^n x_{it} \ln x_{it} \geq H_t$  of Model (30) will not true. When  $x_{it} = \frac{1-x_{ft}}{n}$ ,

$$H_t^{max} = -(1-x_{ft}^b) \ln \frac{1-x_{ft}^b}{n}. \quad (31)$$

If investor does not invest any wealth on risk assets, that is  $x_{it} = 0$ . Thus  $H_t^{min} = 0$ .

The Model (30) is a linear optimal problem with an additional nonlinear constraint. For such problems there are no special standard algorithms. Of course, one could treat this problem with general methods of nonlinear optimization, but this would lead to inefficient algorithms. In this paper, we propose a sequence linear programming (SLP) method to solve the Model (30).

**Algorithm 4.1.** The sequence linear programming method:

**Step 1:** Let  $F(x_t) = ((1-\theta)(\sum_{i=1}^n (a_{it} + \frac{\beta_{it}-\alpha_{it}}{4})x_{it} + r_{ft}(1 - \sum_{i=1}^n x_{it}) - \sum_{i=1}^n c_{it}y_{it}) - \theta \sum_{i=1}^n \frac{39\beta_{it}^2 + 18\alpha_{it}\beta_{it} + 7\alpha_{it}^2}{256\beta_{it}}x_{it})$ , let  $x_t^0$  be original feasible solution of the Model (30), and let  $F(x_t^0)$  be objective function value of the  $x_t^0$ . Solve the linear programs:  
(SLP<sub>0</sub>)

$$\begin{aligned} & \max((1-\theta)(\sum_{i=1}^n (a_{it} + \frac{\beta_{it}-\alpha_{it}}{4})x_{it} + r_{ft}(1 - \sum_{i=1}^n x_{it}) \\ & - \sum_{i=1}^n c_{it}y_{it}) - \theta \sum_{i=1}^n \frac{39\beta_{it}^2 + 18\alpha_{it}\beta_{it} + 7\alpha_{it}^2}{256\beta_{it}}x_{it}). \\ & \left. \begin{aligned} & W_{t+1} = (1 + (\sum_{i=1}^n (a_{it} + \frac{\beta_{it}-\alpha_{it}}{4})x_{it} + r_{ft}(1 - \sum_{i=1}^n x_{it}) \\ & - \sum_{i=1}^n c_{it}y_{it}))W_t \\ & 1 - \sum_{i=1}^n x_{it} \geq x_{ft}^b \\ & y_{it} \geq x_{it} - x_{i(t-1)} \\ & y_{it} \geq -(x_{it} - x_{i(t-1)}) \\ & 0 \leq x_{it} \leq u_{it}, i = 1, \dots, n, t = 1, \dots, T \\ & - \sum_{i=1}^n x_{it}^0 \ln x_{it}^0 - \sum_{i=1}^n (\sum_{i=1}^n \ln x_{it}^0 + 1)(x_{it} - x_{it}^0) \geq H_t. \end{aligned} \right\} \quad (32) \end{aligned}$$

The optimal solution  $x_t^1$  of the Model (32) and the optimal objective function value  $F(x_t^1)$  of  $x_t^1$  can be obtained solving the Model (32) by the Simplex method. If  $|F(x_t^1) - F(x_t^0)| \leq \varepsilon, \varepsilon \leq 10^{-6}$ . Stop, then  $x_t^* = x_t^1, F(x_t^*) = F(x_t^1)$ , which  $x_t^*$  is optimal solution of the Model (29), and  $F(x_t^*)$  is the optimal objective function value of  $x_t^*$ . Otherwise turn Step 2.

**Step2:** When  $k = m$  ( $m \geq 1$  and  $m \in Z^+$ ). Let  $x_t^m = (x_{1t}^m, \dots, x_{nt}^m)'$  be the optimal solution of  $m$ th iteration subproblem formulation of Model (30), and let  $F(x_t^m)$  be objective function value of the  $x_t^m$ . Solve the linear programs:  
( $SLP_m$ )

$$\begin{aligned}
 & \max((1 - \theta) \left( \sum_{i=1}^n \left( a_{it} + \frac{\beta_{it} - \alpha_{it}}{4} \right) x_{it} + r_{ft} \left( 1 - \sum_{i=1}^n x_{it} \right) \right. \\
 & \quad \left. - \sum_{i=1}^n c_{it} y_{it} \right) - \theta \sum_{i=1}^n \frac{39\beta_{it}^2 + 18\alpha_{it}\beta_{it} + 7\alpha_{it}^2}{256\beta_{it}} x_{it}). \\
 & \text{s.t.} \begin{cases} W_{t+1} = \left( 1 + \left( \sum_{i=1}^n \left( a_{it} + \frac{\beta_{it} - \alpha_{it}}{4} \right) x_{it} + r_{ft} \left( 1 - \sum_{i=1}^n x_{it} \right) \right. \right. \\ \quad \left. \left. - \sum_{i=1}^n c_{it} y_{it} \right) \right) W_t \\ 1 - \sum_{i=1}^n x_{it} \geq x_{ft}^b \\ y_{it} \geq x_{it} - x_{i(t-1)} \\ y_{it} \geq -(x_{it} - x_{i(t-1)}) \\ 0 \leq x_{it} \leq u_{it}, i = 1, \dots, n, t = 1, \dots, T \\ - \sum_{i=1}^n x_{it}^m \ln x_{it}^m - \sum_{i=1}^n \left( \sum_{i=1}^n \ln x_{it}^m + 1 \right) (x_{it} - x_{it}^m) \geq H_t. \end{cases} \quad (33)
 \end{aligned}$$

The optimal solution  $x_t^{m+1}$  of the Model (33) and the optimal objective function value  $F(x_t^{m+1})$  of  $x_t^{m+1}$  can be obtained solving the Model (33) by the Simplex method.

**Step3:** If  $|F(x_t^{m+1}) - F(x_t^m)| \leq \varepsilon, \varepsilon \leq 10^{-6}$ . Stop, then  $x_t^* = x_t^{m+1}, F(x_t^*) = F(x_t^{m+1})$ . Otherwise  $k = m + 1$ , then turn Step 2.

**4.2. The Forward Dynamic Programming Method.** In the following section, we provide the detailed procedure of the forward dynamic programming method for finding optimal solutions to the Model (29). The procedure of the algorithm can be showed as follows:

**Algorithm 4.2.** The forward dynamic programming method:

**Step1:** When  $t = 1, W_1$  and  $x_0 = (x_{10}, \dots, x_{n0})'$  have been given, the Model (30) can be turned into as follows:

$$\begin{aligned}
 & \max((1 - \theta) \left( \sum_{i=1}^n \left( a_{i1} + \frac{\beta_{i1} - \alpha_{i1}}{4} \right) x_{i1} + r_{f1} \left( 1 - \sum_{i=1}^n x_{i1} \right) \right. \\
 & \quad \left. - \sum_{i=1}^n c_{i1} y_{i1} \right) - \theta \sum_{i=1}^n \frac{39\beta_{i1}^2 + 18\alpha_{i1}\beta_{i1} + 7\alpha_{i1}^2}{256\beta_{i1}} x_{i1}).
 \end{aligned}$$

$$s.t. \begin{cases} W_2 = (1 + (\sum_{i=1}^n (a_{i1} + \frac{\beta_{i1} - \alpha_{i1}}{4})x_{i1} + r_{f1}(1 - \sum_{i=1}^n x_{i1}) \\ - \sum_{i=1}^n c_{i1}y_{i1}))W_1 \\ 1 - \sum_{i=1}^n x_{i1} \geq x_{f1}^b \\ y_{i1} \geq x_{i1} - x_{i0} \\ y_{i1} \geq -(x_{i1} - x_{i0}) \\ - \sum_{i=1}^n x_{i1} \ln x_{i1} \geq H_1 \\ 0 \leq x_{i1} \leq u_{i1}, i = 1, \dots, n. \end{cases} \quad (34)$$

The optimal solution of period  $t = 1$ ,  $x_1^* = (x_{11}^*, \dots, x_{n1}^*)'$  can be obtained solving the Model (34) by Algorithm 4.1. At the same time,  $(1 - \theta)(\sum_{i=1}^n (a_{i1} + \frac{\beta_{i1} - \alpha_{i1}}{4})x_{i1}^* + r_{f1}(1 - \sum_{i=1}^n x_{i1}^*) - \sum_{i=1}^n c_{i1}y_{i1}^*) - \theta \sum_{i=1}^n \frac{39\beta_{i1}^2 + 18\alpha_{i1}\beta_{i1} + 7\alpha_{i1}^2}{256\beta_{i1}}x_{i1}^*$  can be obtained.

**Step2:** When  $t = m$  ( $m \geq 1$  and  $m \in Z^+$ ), and  $x_m^* = (x_{1m}^*, \dots, x_{nm}^*)'$  have been obtained, the Model (30) can be turned into as follows:

$$\begin{aligned} & \max((1 - \theta)(\sum_{i=1}^n (a_{i(m+1)} + \frac{\beta_{i(m+1)} - \alpha_{i(m+1)}}{4})x_{i(m+1)} + r_{f(m+1)}(1 - \sum_{i=1}^n x_{i(m+1)}) \\ & - \sum_{i=1}^n c_{i(m+1)}y_{i(m+1)}) - \theta \sum_{i=1}^n \frac{39\beta_{i(m+1)}^2 + 18\alpha_{i(m+1)}\beta_{i(m+1)} + 7\alpha_{i(m+1)}^2}{256\beta_{i(m+1)}}x_{i(m+1)}). \\ & s.t. \begin{cases} W_{m+2} = (1 + (\sum_{i=1}^n (a_{i(m+1)} + \frac{\beta_{i(m+1)} - \alpha_{i(m+1)}}{4})x_{i(m+1)} \\ + r_{f(m+1)}(1 - \sum_{i=1}^n x_{i(m+1)}) - \sum_{i=1}^n c_{i(m+1)}y_{i(m+1)}))W_{m+1} \\ 1 - \sum_{i=1}^n x_{i(m+1)} \geq x_{f(m+1)}^b \\ y_{i(m+1)} \geq x_{i(m+1)} - x_{im}^* \\ y_{i(m+1)} \geq -(x_{i(m+1)} - x_{im}^*) \\ - \sum_{i=1}^n x_{i(m+1)} \ln x_{i(m+1)} \geq H_{m+1} \\ 0 \leq x_{i(m+1)} \leq u_{i(m+1)}, i = 1, \dots, n. \end{cases} \quad (35) \end{aligned}$$

The optimal solution of period  $t = m + 1$ ,  $(x_{m+1}^* = (x_{1(m+1)}^*, \dots, x_{n(m+1)}^*)')$  can be obtained solving the Model (35) by the Algorithm4.1. At the same time,

$(1-\theta)(\sum_{i=1}^n (a_{i(m+1)} + \frac{\beta_{i(m+1)} - \alpha_{i(m+1)}}{4})x_{i(m+1)}^* + r_{f(m+1)}(1 - \sum_{i=1}^n x_{i(m+1)}^*) - \sum_{i=1}^n c_{i(m+1)} y_{i(m+1)}^*) - \theta \sum_{i=1}^n \frac{39\beta_{i(m+1)}^2 + 18\alpha_{i(m+1)}\beta_{i(m+1)} + 7\alpha_{i(m+1)}^2}{256\beta_{i(m+1)}} x_{i(m+1)}^*$  can be obtained.

**Step3:** If  $t = T$ , then the maximization of the terminal utility  $(1-\theta)(\sum_{i=1}^n (a_{iT} + \frac{\beta_{iT} - \alpha_{iT}}{4})x_{iT}^* + r_{fT}(1 - \sum_{i=1}^n x_{iT}^*) - \sum_{i=1}^n c_{iT}y_{iT}^*) - \theta \sum_{i=1}^n \frac{39\beta_{iT}^2 + 18\alpha_{iT}\beta_{iT} + 7\alpha_{iT}^2}{256\beta_{iT}} x_{iT}^*$  can be obtained.

The global optimal solutions of the Model (34) and Model (35) can be obtained by the Algorithm 4.1. So, the global optimal solution of the Model (29) can also be obtained by the Algorithm 4.2.

## 5. Numerical Example

In this section, a numerical example is given to express the idea of the proposed model. Assume that an investor chooses thirty stocks from Shanghai Stock Exchange for his investment. The stocks codes are respectively  $S_1, \dots, S_{30}$ . He intends to make five periods of investment with initial wealth  $W_1 = 1$  and his wealth can be adjusted at the beginning of each period. He/she assumes that the returns, risk and turnover rates of the thirty stocks at each period are represented as trapezoidal fuzzy numbers. We collect historical data of them from April 2006 to March 2015 and set every three months as a period to handle the historical data. By using the simple estimation method in Vercher et al. [30] to handle their historical data, the triangular possibility distributions of the return rates of assets at each period can be obtained as shown in Appendix .

Suppose that the transaction costs of assets of the two periods investment take the same value  $c_{it} = 0.003(i = 1, \dots, 30; t = 1, \dots, 5)$ , the lower bound of the investment proportion of risk-free asset  $x_{ft}^b = -0.5$ , the borrowing rate of the risk-free asset  $r_{bt} = 0.017$ , the lending rate of the risk-free asset  $r_{lt} = 0.009, t = 1, \dots, 5$ , the lower  $l_{it} = 0$  and upper bound constraints  $u_{it} = 0.2(i = 1, \dots, 30; t = 1, \dots, 5)$ .

As illustrations, the following numerical examples are given to show the effectiveness of the proposed model and the discrete approximate iteration method. The algorithm was programmed by C++ language and run on a personal computer with Pentium Dual CPU, 4GHz and 8GB RAM.

The smallest value and biggest value of  $H_t$  can be respectively obtained as follows.

$$H_t^{min} = 0, H_t^{max} = -(1 - x_{ft}^b) \ln \frac{1 - x_{ft}^b}{n} = -(1 + 0.5) \ln \frac{1 + 0.5}{30} = 4.4935984$$

When the preference parameter  $\theta = 0.5$ , the entropy constraints  $H_t$  is 0.5, the optimal solution of Model (29) can be obtained as follows by the forward dynamic programming method; When the entropy constraints  $H_t$  is 0.5, the preference parameter  $\theta = 0, 0.1, \dots, 1$ , the optimal terminal wealth can be obtained as follows by solving the Model (29); When the preference parameter  $\theta = 0.5$ , the entropy constraints  $H_t$  is gotten 0, 0.1,  $\dots$ , 4.4 between the interval values of  $[0, 4.4936]$ , the optimal terminal wealth can be obtained as follows by solving the Model (29).

If  $\theta = 0.5, H_t = 0.5$ , the optimal solution of Model (29) will be obtained as the Table 1 using the forward dynamic programming method.

asset $i$ $t$	The optimal investment proportions								
	1	asset1 0.2	asset8 0.2	asset12 0.1	asset13 0.2	asset17 0.2	asset19 0.2	asset24 0.2	asset28 0.2
2	asset1 0.2	asset8 0.1	asset12 0.2	asset13 0.2	asset15 0.2	asset17 0.2	asset18 0.2	asset28 0.2	risk-free asset -0.5
3	asset1 0.2	asset12 0.2	asset13 0.2	asset15 0.2	asset17 0.2	asset18 0.2	asset26 0.1	asset28 0.2	risk-free asset -0.5
4	asset1 0.2	asset8 0.1	asset12 0.2	asset13 0.2	asset15 0.2	asset17 0.2	asset18 0.2	asset28 0.2	risk-free asset -0.5
5	asset1 0.2	asset8 0.1	asset12 0.2	asset13 0.2	asset15 0.2	asset17 0.2	asset18 0.2	asset28 0.2	risk-free asset -0.5

TABLE 1. The Optimal Solution When  $\theta = 0.5, H_t = 0.5$

When  $\theta = 0.5, H_t = 0.5$ , the optimal investment strategy at period 1 is  $x_{11} = 0.2, x_{81} = 0.2, x_{121} = 0.1, x_{131} = 0.2, x_{171} = 0.2, x_{191} = 0.2, x_{241} = 0.2, x_{281} = 0.2, x_{f1} = -0.5$  and being the rest of variables equal to zero, which means investor should allocate his initial wealth on asset 1, asset 8, asset 12, asset 13, asset 17, asset 19, asset 24, asset 28, risk-free asset and otherwise asset by the proportions of 20%, 20%, 10%, 20%, 20%, 20%, 20%, 20%, 20%, 20%, -50% and being the rest of variables equal to zero among the thirty stocks, respectively. From Table 1, the optimal investment strategy at period 2, period 3, period 4 and period 5 can also be obtained. In this case, the available terminal wealth is 2.2319.

When  $\theta = 0, 0.05, 0.1, \dots, 1, H_t = 0.5$ , the terminal wealth of Model (29) will be obtained as the Table 2 using the forward dynamic programming method.

$\theta$	0	0.05	0.1	0.15	0.2	0.25	0.30	0.35	0.4	0.45
$W_6$	2.2410	2.2410	2.2410	2.2410	2.2404	2.2402	2.2402	2.2402	2.2373	2.2324
$\theta$	0.5	0.55	0.6	0.65	0.7	0.75	0.8	0.85	0.9	0.95
$W_6$	2.2319	2.2291	2.1993	2.1612	2.1512	2.1275	2.1147	1.9406	1.3111	1.1390
$\theta$	1									
$W_6$	1.0850									

TABLE 2. the Optimal Terminal Wealth of the Portfolio When  $\theta = 0, 0.05, 0.1, \dots, 1, H_t = 0.5$

Where  $W_6$  is denoted the terminal wealth.

In the used data sets, the experiments in this paper correspond to the values of  $\theta$  in the interval  $[0, 1]$ . It can be seen that, as will be seen in Table 2, the terminal wealth is the same, when  $0 \leq \theta \leq 0.15$ , and  $0.2 \leq \theta \leq 0.35$ ; the terminal wealth and risk becomes smaller, when preference coefficient  $\theta$  which  $0.3 \leq \theta \leq 1$ , become smaller, which reflects the influence of preference coefficient  $\theta$  on portfolio selection.

When  $\theta = 0.5, H_t = 0, 0.2, 0.4, \dots, 4.4$ , of Model (29) will be obtained as the Table 3 using the forward dynamic programming method.



$H_t$	0	0.2	0.4	0.6	0.8	1	1.2	1.4	1.6	1.8
$W_6$	2.2319	2.2319	2.2319	2.2319	2.2319	2.2319	2.2319	2.2319	2.2319	2.2319
$H_t$	2	2.2	2.4	2.6	2.8	3	3.2	3.4	3.6	3.8
$W_6$	2.2319	2.2319	2.2319	2.2298	2.2223	2.2100	2.1930	2.1732	2.1491	2.1182
$H_t$	4	4.2	4.4							
$W_6$	2.0767	2.0180	1.9274							

TABLE 3. The Optimal Terminal Wealth of the Portfolio When  $\theta = 0.5, H_t = 0, 0.2, 0.4, \dots, 4.4$

Where  $W_6$  is denoted the terminal wealth of the portfolio. From Table 3, the Figure1 can be obtained as follows:

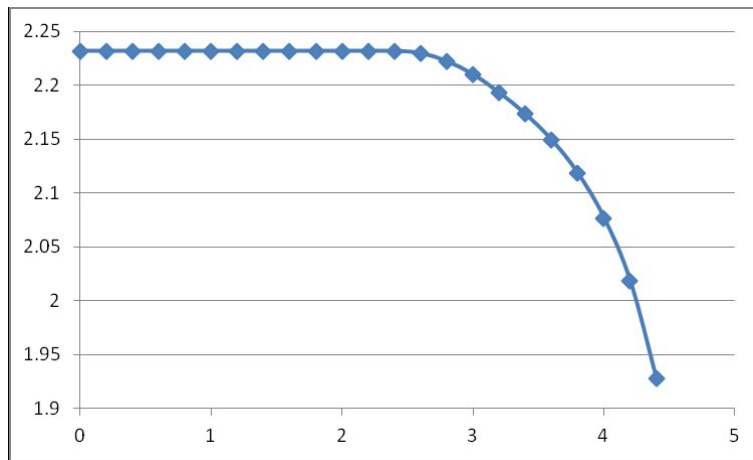


FIGURE 1. the Terminal Wealth of  $\theta = 0.5, H_t = 0, 0.2, 0.4, \dots, 4.4$

In the used data sets, the experiments in this paper correspond to the values of  $H_t$  in the interval  $[0, 4.4]$ . It can be seen that, as will be seen in Fig.1, the terminal wealth is the same, when  $0 \leq H_t \leq 2.4$ ; the terminal wealth and risk becomes smaller, when preference coefficient  $H_t$  which  $2.6 \leq H_t \leq 4.4$ , become smaller, which reflects the influence of preference coefficient  $H_t$  on portfolio selection.

### 6. Conclusions

In this paper, we consider the multi-period portfolio selection problem in fuzzy environment. We use the credibilitic mean value and the semi-absolute deviation to measure the return and the risk of the multiperiod portfolio, respectively. A new multi-period portfolio optimization models with transaction cost, borrowing constraints, entropy constraints and threshold constraints is proposed. Based on the credibilitic theories, the proposed model is transformed into a dynamic optimization problems with path dependence. A novel forward dynamic programming method is designed to obtain the optimal portfolio strategy. Finally, an example is given

to illustrate the behavior of the proposed model and the designed algorithm using real data from the Shanghai Stock Exchange.

As future work, we are planning to add more real constraints in financial market into multi-period portfolio optimization problem. We also intend to compare other heuristic algorithms with the designed algorithm to illustrate its advantage on solving complex multiperiod portfolio selection models.

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## 7. Appendix

The codes of thirty stocks are respectively  $S_1$  (600000),  $S_2$  (600005),  $S_3$  (600015),  $S_4$ (600016),  $S_5$  (600019),  $S_6$  (600028),  $S_7$  (600030),  $S_8$  (600036),  $S_9$  (600048),  $S_{10}$ (600050),  $S_{11}$  (600104),  $S_{12}$  (600362),  $S_{13}$  (600519),  $S_{14}$  (600900),  $S_{15}$  (601088),  $S_{16}$  (601111),  $S_{17}$  (601166),  $S_{18}$  (601168),  $S_{19}$  (601318),  $S_{20}$  (601328),  $S_{21}$  (601390),  $S_{22}$  (601398),  $S_{23}$  (601600),  $S_{24}$  (601601),  $S_{25}$  (601628),  $S_{26}$  (601857),  $S_{27}$  (601919),  $S_{28}$  (601939),  $S_{29}$  (601988),  $S_{30}$  (601998) . The triangular possibility distributions of the return rates of assets at each period can be obtained as shown in Table 4 to Table 13.

asset t	asset 1			asset 2			asset 3		
1	0.1430	0.1049	0.1156	0.0750	0.0657	0.1664	0.1083	0.0832	0.060
2	0.1449	0.0881	0.1136	0.0813	0.0708	0.1600	0.1085	0.0681	0.0603
3	0.1458	0.0800	0.1127	0.0857	0.0666	0.1556	0.1139	0.0725	0.0548
4	0.1516	0.0620	0.1070	0.0930	0.0579	0.1483	0.1152	0.0560	0.0540
5	0.1532	0.0609	0.1054	0.1053	0.0662	0.1359	0.1172	0.0570	0.0516

TABLE 4. The Fuzzy Return Rates on Assets of Five Periods Investment

asset t	asset 4			asset 5			asset 6		
1	0.1172	0.0731	0.0813	0.0801	0.0791	0.0616	0.1064	0.1635	0.0616
2	0.1203	0.0743	0.0782	0.0847	0.0772	0.0571	0.1073	0.1634	0.0608
3	0.1255	0.0749	0.0730	0.0900	0.0503	0.0517	0.1083	0.1093	0.0598
4	0.1274	0.0733	0.0710	0.0906	0.0507	0.0512	0.1091	0.0763	0.0590
5	0.1289	0.0633	0.0700	0.0926	0.0495	0.0492	0.1129	0.0727	0.0551

TABLE 5. The Fuzzy Return Rates on Assets of Five Periods Investment

asset t	asset 7			asset 8			asset 9		
1	0.0798	0.0562	0.1694	0.1238	0.0815	0.1023	0.0639	0.1522	0.0951
2	0.0907	0.0643	0.1750	0.1259	0.0760	0.1003	0.0790	0.0673	0.0866
3	0.0992	0.0555	0.1500	0.1277	0.0765	0.0985	0.0818	0.0606	0.0839
4	0.1029	0.0551	0.1462	0.1383	0.0538	0.0878	0.0861	0.0645	0.0800
5	0.1069	0.0534	0.1423	0.1457	0.0612	0.0805	0.0884	0.0650	0.0773

TABLE 6. The Fuzzy Return Rates on Assets of Five Periods Investment

asset t	asset 10			asset 11			asset 12		
1	0.0377	0.0325	0.0414	0.0575	0.0403	0.1282	0.1243	0.1170	0.1840
2	0.0410	0.0292	0.0379	0.0592	0.0403	0.1264	0.1303	0.1007	0.1781
3	0.0469	0.0318	0.0321	0.0669	0.0374	0.1188	0.1380	0.0827	0.1704
4	0.0480	0.0314	0.0309	0.0724	0.0400	0.1133	0.1491	0.0843	0.1593
5	0.0492	0.0318	0.0298	0.0741	0.0453	0.1116	0.1540	0.0752	0.1544

TABLE 7. The Fuzzy Return Rates on Assets of Five Periods Investment

asset t	asset 13			asset 14			asset 15		
1	0.2049	0.1244	0.1331	0.0254	0.1000	0.0743	0.0893	0.2079	0.1463
2	0.2102	0.1182	0.1277	0.0667	0.0604	0.0443	0.1518	0.1015	0.0859
3	0.2194	0.1236	0.1186	0.0700	0.0563	0.0411	0.1538	0.1033	0.0840
4	0.2225	0.1248	0.1154	0.0716	0.0500	0.0395	0.1565	0.0534	0.0812
5	0.2238	0.1029	0.1142	0.0731	0.0399	0.0379	0.1600	0.0553	0.0778

TABLE 8. The Fuzzy Return Rates on Assets of Five Periods Investment

asset t	asset 16			asset 17			asset 18		
1	0.0615	0.0622	0.4819	0.1665	0.1188	0.0375	0.0675	0.1157	0.0586
2	0.0625	0.2795	0.2306	0.1550	0.0984	0.0772	0.1183	0.1137	0.4232
3	0.0656	0.0514	0.2276	0.1553	0.0893	0.0769	0.1349	0.1165	0.4066
4	0.0747	0.0460	0.2185	0.1575	0.0844	0.0747	0.1467	0.1225	0.3947
5	0.0835	0.0506	0.2096	0.1579	0.0535	0.0744	0.1664	0.1122	0.3750

TABLE 9. The Fuzzy Return Rates on Assets of Five Periods Investment

asset t	asset 19			asset 20			asset 21		
1	0.2500	0.0736	0.0896	0.0825	0.1559	0.0853	0.0536	0.0817	0.0403
2	0.0916	0.0716	0.0634	0.1217	0.0633	0.0619	0.0704	0.0544	0.1220
3	0.0928	0.0708	0.0622	0.1218	0.0620	0.0618	0.0838	0.0666	0.1087
4	0.0940	0.0500	0.0610	0.1243	0.0516	0.0593	0.0880	0.0681	0.1045
5	0.0952	0.0472	0.0600	0.1269	0.0267	0.0568	0.0917	0.0703	0.1007

TABLE 10. The Fuzzy Return Rates on Assets of Five Periods Investment

asset t	asset 22			asset 23			asset 24		
1	0.1083	0.1197	0.0684	0.0413	0.0149	0.0749	0.1430	0.0458	0.0141
2	0.1170	0.0618	0.0739	0.0454	0.1137	0.2040	0.0947	0.0637	0.0393
3	0.1200	0.0623	0.0708	0.0531	0.1209	0.1963	0.0984	0.0628	0.0355
4	0.1235	0.0656	0.0674	0.0574	0.0930	0.1920	0.1005	0.0548	0.0334
5	0.1254	0.0488	0.0654	0.0718	0.0716	0.1777	0.1021	0.0493	0.0319

TABLE 11. The Fuzzy Return Rates on Assets of Five Periods Investment

asset t	asset 25			asset 26			asset 27		
1	0.0589	0.1432	0.1024	0.0783	0.1712	0.1096	0.0690	0.0047	0.1634
2	0.1021	0.0652	0.0591	0.1276	0.0906	0.1263	0.0438	0.1623	0.1828
3	0.1037	0.0567	0.0574	0.1329	0.0949	0.1210	0.0506	0.1526	0.1760
4	0.1044	0.0314	0.0567	0.1432	0.0812	0.1105	0.0562	0.1232	0.1694
5	0.1090	0.0243	0.0521	0.1445	0.0777	0.1093	0.0619	0.0511	0.1647

TABLE 12. The Fuzzy Return Rates on Assets of Five Periods Investment

asset t	asset 25			asset 26			asset 27		
1	0.1551	0.1128	0.0498	0.0994	0.1233	0.0677	0.0674	0.1355	0.0854
2	0.1382	0.0789	0.0969	0.1123	0.0641	0.0498	0.1037	0.0636	0.0438
3	0.1395	0.0679	0.0956	0.1134	0.0648	0.0488	0.1048	0.0645	0.0426
4	0.1426	0.0379	0.0924	0.1157	0.0416	0.0464	0.1060	0.0574	0.0414
5	0.1470	0.0364	0.0880	0.1175	0.0312	0.0446	0.1061	0.0350	0.0413

TABLE 13. The Fuzzy Return Rates on Assets of Five Periods Investment

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