# FUZZY RISK ANALYSIS BASED ON A NEW METHOD FOR RANKING GENERALIZED FUZZY NUMBERS

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Abstract. Fuzzy risk analysis, as a powerful tool to address uncertain information, can provide an appropriate method for risk analysis. However, the previous fuzzy risk analysis methods still have some weaknesses. To overcome the weaknesses of existing fuzzy risk analysis methods, a novel method for ranking generalized fuzzy numbers is proposed for addressing fuzzy risk analysis problems. In the proposed method, a new value of ranking score is obtained based on ordered weighted averaging (OWA) operator. The proposed method takes into consideration of the different importance of the three scoring factors defuzzified value, height and spread. Comparing to some existing methods, the new method can get more reasonable results in some situations.

## 1. **Introduction**

Risk analysis is an open issue in industrial applications. Due to a lot of subjective and objective uncertainties in risk analysis, uncertainty reasoning theories, for example fuzzy set theory [47, 11], Dempster-Shafer evidence theory [35, 13, 15, 14], generalized evidence theory [27], Z numbers [48, 25] and so on, are widely applied to risk analysis [29, 50, 17, 49, 39, 26]. Recently, Nouei *et al.* [33] proposed a method for fuzzy risk assessment by using combination of adaptive neuro-fuzzy inference system and K-means clustering. In [37], the analysis of the evolution of dengue risk was proposed. Fuzzy risk analysis, as a useful tool to address uncertainties of information, is a frequently used approach for handling risk analysis. To solve fuzzy risk analysis problems, fuzzy numbers [20] are usually used by some scholars to replace the evaluating value of the risk of each subcomponent in the production processes. Some analysis methods presented by some researchers are as follows. Chen [8] analyzed the fuzzy risk based on ranking generalized fuzzy numbers with different heights and different spreads. Wei [42] put forward a method to analyze the fuzzy risk based on interval-valued fuzzy numbers [24]. Chen and Sanguansat [10] provided a new method to process fuzzy risk analysis based on ranking generalized fuzzy numbers. Chen *et al.* [9] proposed an analysis method based on ranking generalized fuzzy numbers with different left heights and right heights. Of course, fuzzy numbers are also widely used in other applications such as supplier selection [28], decision-making [5, 38, 43].

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In fuzzy risk analysis, an important issue is to rank generalized fuzzy numbers. Recently, various methods have been put forward to rank fuzzy numbers. The centroid index ranking method with a weighting function was proposed by Yager [44]. Chu and Tsao [12] ranked the fuzzy numbers with the area between the centroid point and original point. Chen and Chen [8] ranked the generalized fuzzy numbers by considering the defuzzified values, the heights and the spreads of the generalized fuzzy numbers, which can make the result more reasonable. Their method considers that the defuzzified value and height of a generalized fuzzy number are the major factors, and the spread of the generalized fuzzy number is only a minor factor to adjust the ranking value. Chen and Sanguansat [10] considered the areas on the positive side, the areas on the negative side and the heights of the generalized fuzzy numbers to evaluate the ranking scores of the generalized fuzzy numbers. The above methods are useful for some situations. Mashinchi *et al* [32] proposed a useful method for ranking fuzzy numbers. Afterwards, Chen *et al.* [9] considered generalized fuzzy numbers with different left heights and right heights. Yu *et al.* [46] provided a method for ranking generalized fuzzy numbers in fuzzy decision making [19, 3] based on the left and right transfer coefficients and areas. Emrah *et al.* [2] presented a new method for ranking generalized trapezoidal fuzzy numbers based on the incenter and inradius of a triangle, which enables to rank crisp numbers and fuzzy numbers with the same centroid point. Madhuri *et al.* [31] ranked generalized trapezoidal fuzzy numbers based on the Circumcentre points. Wang [40] put forward a method to rank triangle and trapezoidal fuzzy numbers [18] based on the relative preference relation. Bakar and Gegoy [6] proposed a novel method for ranking fuzzy numbers, which integrated the centroid point and the spread approaches and overcomes the weaknesses of some existed methods.

Those ranking methods stated above are effective in some situations. However, these methods still have some weaknesses, such as non-discriminative, counter-intuitive problems. To overcome these weaknesses, the proposed method provides a reasonable and efficient method for ranking generalized fuzzy numbers based on OWA operator, including the generalized fuzzy numbers with different right heights and left heights in Figure 4$a$ and Figure 4$b$. In this method, with the maximum entropy method (MEM) proposed by O'Hagan [34], there different weights are assigned to three variables related to the three factors, the defuzzified value, the height and the spread, respectively. The three factors affect the ranking order of a generalized fuzzy number. In addition, a comparison of the calculation results with the existing methods to illustrate the superiority of the proposed method is made.

The remainder of this paper is organized as follows: Section 2 introduces the basic concepts of generalized fuzzy numbers, OWA operator and the maximum entropy method (MEM). In Section 3, we present an innovative method for ranking generalized fuzzy number based on OWA operator. In Section 4, the ranking results of the proposed fuzzy ranking method are compared with that of some existing methods. In Section 5, the proposed method is applied to deal with fuzzy risk analysis problems of components in industry. In Section 6, the conclusions are summarized.

## 2. **Preliminaries**

This section firstly reviews some basic concepts of generalized fuzzy numbers and arithmetic operations of generalized fuzzy numbers, then introduces the basic knowledge of OWA and the concept of maximal entropy model for calculating the weights.

**2.1. Generalized Fuzzy Numbers.** A generalized trapezoidal fuzzy number with two the same height $\omega$, namely, $\tilde{A} = (a, b, c, d; \omega, \omega)$ is described as a fuzzy set [47, 36] of the real line $R$ by a membership function [30] $\mu_{\tilde{A}}$. Several properties of $\mu_{\tilde{A}}$ are as follows:

(1) $\mu_{\tilde{A}}(x)$ maps each element $x$ in $X$ to a real interval $[0, 1]$,
(2) $\mu_{\tilde{A}}(x) = 0$ , $x \in (-\infty, a]$,
(3) $\mu_{\tilde{A}}(x)$ is strictly increasing in $[a, b]$,
(4) $\mu_{\tilde{A}}(x)$ is strictly decreasing in $[c, d]$,
(5) $\mu_{\tilde{A}}(x) = \omega$ , $x \in [b, c]$, $0 \leq \omega \leq 1$,
(6) $\mu_{\tilde{A}}(x) = 0$ , $x \in [d, +\infty)$,

where $\omega$ is a constant, $a, b, c$ and $d$ are real values, $a \leq b \leq c \leq d$. The membership function of a generalized trapezoidal fuzzy number shown in Figure 1 is defined as follows:

$$\mu_{\tilde{A}}(x) = \begin{cases} \mu^L_{\tilde{A}}(x), & a \leq x \leq b, \\ \omega_{\tilde{A}}, & b \leq x \leq c, \\ \mu^R_{\tilde{A}}(x), & c \leq x \leq d, \\ 0, & otherwise, \end{cases} \tag{1}$$

where $\mu^L_{\tilde{A}}(x)$ and $\mu^R_{\tilde{A}}(x)$ are continuous mapping function and $\omega_{\tilde{A}}$ is height, $\omega_{\tilde{A}} \in [0, 1]$. When $\omega_{\tilde{A}} = 1$, $\tilde{A}$ turns into a normal trapezoidal fuzzy number.

A generalized trapezoidal fuzzy number $\tilde{A}$ with the different left height and right height shown in Figure 2, which is given by Chen [9]. It can be denoted by $\tilde{A} = (a_1, a_2, a_3, a_4; \mu_L, \mu_R)$, where $a_1, a_2, a_3$ and $a_4$ are real values, $a_1 \leq a_2 \leq a_3 \leq a_4$. $\mu_L$ and $\mu_R$ are the left height and the right height of $\tilde{A}$, respectively, $\mu_L \in [0, 1]$ and $\mu_R \in [0, 1]$. When $-1 \leq a_1 \leq a_2 \leq a_3 \leq a_4 \leq 1$, $\tilde{A}$ is a standardized generalized trapezoidal fuzzy number. When $\mu_L = \mu_R = 1$, $\tilde{A}$ is a normal trapezoidal fuzzy number. When $a_2 = a_3$, $\tilde{A}$ is a generalized triangular fuzzy number. When $a_1 = a_2 = a_3 = a_4$, $\tilde{A}$ is called a crisp number. The membership function $f_{\tilde{A}}(x)$ of a generalized fuzzy number $\tilde{A}$ shown in Figure 2 can be defined as follows:

$$f_{\tilde{A}}(x) = \begin{cases} f^L_{\tilde{A}}(x), & a_1 \leq x \leq a_2, \\ f^T_{\tilde{A}}(x), & a_2 \leq x \leq a_3, \\ f^R_{\tilde{A}}(x), & a_3 \leq x \leq a_4, \\ 0, & otherwise, \end{cases} \tag{2}$$

where $f^L_{\tilde{A}}$ , $f^T_{\tilde{A}}$ and $f^R_{\tilde{A}}$ are continuous mapping function. $f^L_{\tilde{A}}$ is strictly increasing in $[a_1, a_2]$; $f^R_{\tilde{A}}$ is strictly decreasing in $[a_3, a_4]$. When $\mu_L > \mu_R$, $f^T_{\tilde{A}}$ is strictly decreasing in $[a_2, a_3]$, when $\mu_L < \mu_R$, $f^T_{\tilde{A}}$ is strictly increasing in $[a_2, a_3]$, otherwise $f^T_{\tilde{A}}$ is a crisp number.

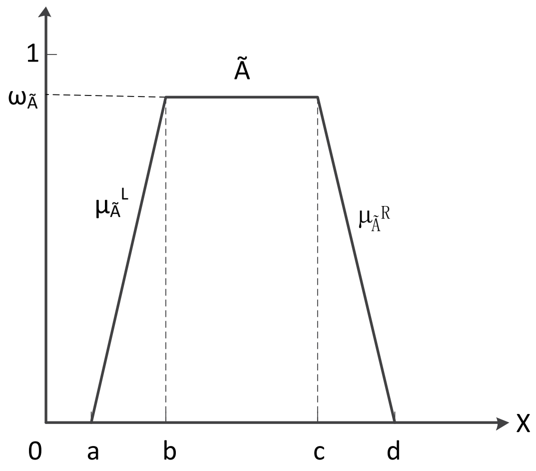


FIGURE 1. Generalized Trapezoidal Fuzzy Number $\tilde{A}$

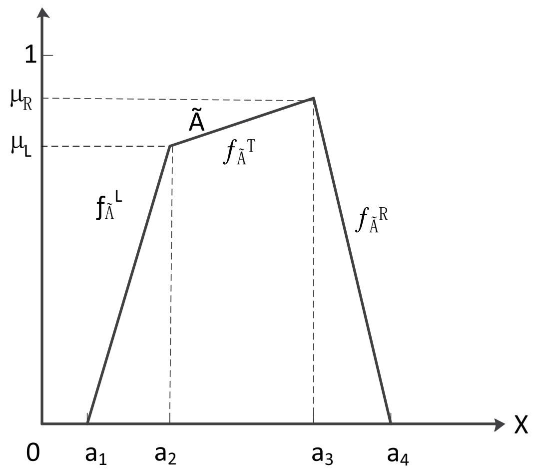


FIGURE 2. Membership Function of Trapezoidal Fuzzy Number $\tilde{A}$

2.2. **Arithmetic Operations.** In [7], Chen and Chen introduced the new arithmetic operations between two generalized fuzzy numbers. Let us consider two generalized trapezoidal fuzzy numbers with different left heights and right heights $\tilde{A}$, $\tilde{B}$. $\tilde{A} = (m_1, m_2, m_3, m_4; \mu_{\tilde{A}L}, \mu_{\tilde{A}R})$, $\tilde{B} = (n_1, n_2, n_3, n_4; \mu_{\tilde{B}L}, \mu_{\tilde{B}R})$, $m_1, m_2, m_3, m_4$, $n_1, n_2, n_3$ and $n_4$ are real values, $\mu_{\tilde{A}L}, \mu_{\tilde{A}R}$ are the left height and the right height of $\tilde{A}$, respectively and $\mu_{\tilde{B}L}, \mu_{\tilde{B}R}$ are the left height and the right height of $\tilde{B}$, respectively, $0 \leq m_1 \leq m_2 \leq m_3 \leq m_4 \leq 1$, $0 \leq n_1 \leq n_2 \leq n_3 \leq n_4 \leq 1$, $0 \leq \mu_{\tilde{A}L} \leq 1$, $0 \leq \mu_{\tilde{A}R} \leq 1$, $0 \leq \mu_{\tilde{B}L} \leq 1$ and $0 \leq \mu_{\tilde{B}R} \leq 1$. The arithmetic operations between two generalized fuzzy numbers can be reviewed as follows [7]:

(1) Generalized fuzzy numbers addition $\oplus$:

$$\begin{aligned}\tilde{A}\oplus\tilde{B} &= (m_1, m_2, m_3, m_4; \mu_{\tilde{A}L}, \mu_{\tilde{A}R})\\ &\oplus(n_1, n_2, n_3, n_4; \mu_{\tilde{B}L}, \mu_{\tilde{B}R})\\ &= (m_1+n_1, m_2+n_2, m_3+n_3,\\ &m_4+n_4; \min(\mu_{\tilde{A}L}, \mu_{\tilde{B}L}),\\ &\min(\mu_{\tilde{A}R}, \mu_{\tilde{B}R})).\end{aligned} \tag{3}$$

(2) Generalized fuzzy numbers subtraction $\ominus$:

$$\begin{aligned}\tilde{A}\ominus\tilde{B} &= (m_1, m_2, m_3, m_4; \mu_{\tilde{A}L}, \mu_{\tilde{A}R})\\ &\ominus(n_1, n_2, n_3, n_4; \mu_{\tilde{B}L}, \mu_{\tilde{B}R})\\ &= (m_1-n_4, m_2-n_3, m_3-n_2,\\ &m_4-n_1; \min(\mu_{\tilde{A}L}, \mu_{\tilde{B}L}),\\ &\min(\mu_{\tilde{A}R}, \mu_{\tilde{B}R})).\end{aligned} \tag{4}$$

(3) Generalized fuzzy numbers multiplication $\otimes$:

$$\begin{aligned}\tilde{A}\otimes\tilde{B} &= (m_1, m_2, m_3, m_4; \mu_{\tilde{A}L}, \mu_{\tilde{A}R})\\ &\otimes(n_1, n_2, n_3, n_4; \mu_{\tilde{B}L}, \mu_{\tilde{B}R})\\ &= (m_1\times n_1, m_2\times n_2, m_3\times n_3,\\ &m_4\times n_4; \min(\mu_{\tilde{A}L}, \mu_{\tilde{B}L}),\\ &\min(\mu_{\tilde{A}R}, \mu_{\tilde{B}R})).\end{aligned} \tag{5}$$

(4) Generalized fuzzy numbers division $\oslash$:

$$\begin{aligned}\tilde{A}\oslash\tilde{B} &= (m_1, m_2, m_3, m_4; \mu_{\tilde{A}L}, \mu_{\tilde{A}R})\\ &\oslash(n_1, n_2, n_3, n_4; \mu_{\tilde{B}L}, \mu_{\tilde{B}R})\\ &= (m_1/n_4, m_2/n_3, m_3/n_2,\\ &m_4/n_1; \min(\mu_{\tilde{A}L}, \mu_{\tilde{B}L}),\\ &\min(\mu_{\tilde{A}R}, \mu_{\tilde{B}R})).\end{aligned} \tag{6}$$

2.3. **Ordered Weighted Averaging (OWA) Operator.** The ordered weighted averaging operator was introduced to get a global value by aggregating certain data [22] on the real interval $[0, 1]$ by Yager [45]. The ordered weighted average operator [21] is an information integration approach between maximum and minimum operator. If vector $B$ corresponds to the ordered arguments and $W^T$ is the transpose of the weighting vector, the OWA aggregation can be expressed as follows [45]:

$$F_w = W^T B. \tag{7}$$

The elements of vector $B$ are the ordered arguments. According to a clear analysis, the three factors defuzzified value, height and spread are the main factors that affect the ranking score of a generalized fuzzy number. The ranking score can decides the ranking order of a fuzzy number. And the order of importance of the three factors of a generalized fuzzy numbers is: $x_{\tilde{A}_i} > h_{\tilde{A}_i} > STD_{\tilde{A}_i}$, $x_{\tilde{A}_i} > \mu h_{\tilde{A}_i} > \frac{\mu}{1+STD_{\tilde{A}_i}}$, that is, $x_{\tilde{A}_i}, \mu h_{\tilde{A}_i}, \frac{\mu}{1+STD_{\tilde{A}_i}}$ are three ordered arguments. Therefore, in this paper, a method for obtaining the ranking scores of fuzzy numbers based on ordered weighted averaging (OWA) operator is proposed to illustrate the reasonability and effectiveness of the proposed method.

In [45], two characterizing measures related to the weighting vector $W$ of an OWA operator were introduced. The first one is the measure of orness of the aggregation, which is shown as follows:

$$orness(W) = \frac{1}{n-1}\sum_{i=1}^{n}(n-i)w_i. \quad (8)$$

The measure of orness of an OWA operator is its aggregated value under a linear argument vector. $orness(W) \in [0,1]$ holds for any weighting vector.

The second one, the measure of entropy, is shown as follows:

$$disp(W) = -\sum_{i=1}^{n} w_i \ln w_i. \quad (9)$$

The measure of entropy is utilized to describe the degree to which $W$ considers all of the information in the aggregation.

2.4. **Maximal Entropy Method (MEM).** The determination of weights is key issue in OWA operator [23]. The Maximal entropy method (MEM) was proposed by O'Hagan [34] for obtaining OWA operator weights. In [1], a compatible weighting method that was based on other well-established academic disciplines, where the weights generated by the maximum entropy method was developed. The Maximal entropy used by this paper is different from Deng entropy [16]. In this paper, MEM is introduced to obtain the weights of the three score factors by solving the following model. The maximal entropy model is shown as follows:

$$\begin{aligned}
&Maximize\ Disp(w) = -\sum_{i=1}^{n} w_i \ln w_i,\\
&S.t.\ orness(w) = \alpha = \tfrac{1}{n-1}\sum_{i=1}^{n}(n-i)w_i, 0 \le \alpha \le 1,\\
&\sum_{i=1}^{n} w_i = 1,\\
&w_i \in [0,1], i = 1, ..., n,
\end{aligned} \quad (10)$$

when $i = 3$, the variation of the weights with orness degree is shown in Figure 3. It can be seen that if $\alpha > 0.5$, then $w_1 > w_2 > w_3$, if $\alpha < 0.5$, then $w_1 < w_2 < w_3$, if $\alpha = 1$, then $w_2 = w_3 = 0$, $w_1 = 1$ and if $\alpha = 0.5$, then $w_1 = w_2 = w_3 = \frac{1}{3}$.

## 3. The Proposed Method for Ranking Generalized Fuzzy Numbers

In this section, a novel method for ranking generalized fuzzy numbers based on OWA operator is presented. The proposed method takes into consideration of the three score factors, the defuzzified value, height and spread of a generalized fuzzy number $\tilde{A}_i$. The proposed method indicates the different importance of the three score factors by assigning three different weights to the three score factors. In [41], a method for calculating the centroid points of generalized fuzzy numbers was proposed. To obtain the ranking score, this paper extend the centroid point method to calculate the values of the defuzzified value ($x_{\tilde{A}_i}$) and the height ($h_{\tilde{A}_i}$) of a generalized fuzzy number.

Suppose that there are $n$ generalized fuzzy numbers $\tilde{A}_1, \tilde{A}_2, ..., \tilde{A}_n$ to be ranked, where $\tilde{A}_i = (x_{i1}, x_{i2}, x_{i3}, x_{i4}; \mu_{iL}, \mu_{iR})$, $-\infty < x_{i1} \le x_{i2} \le x_{i3} \le x_{i4} < \infty$, $\mu_{iL} \in [0,1], \mu_{iR} \in [0,1]$, $1 \le i \le n$, $\mu_{iL}$ and $\mu_{iR}$ denote the left height and the right height of fuzzy number $\tilde{A}$, respectively. The proposed method is shown as follows:

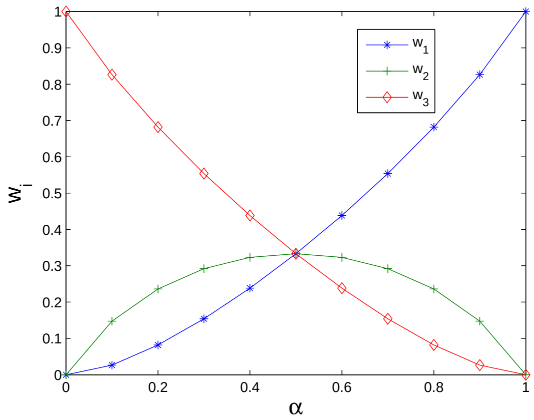


FIGURE 3. Variation of the Weights with Orness Degree

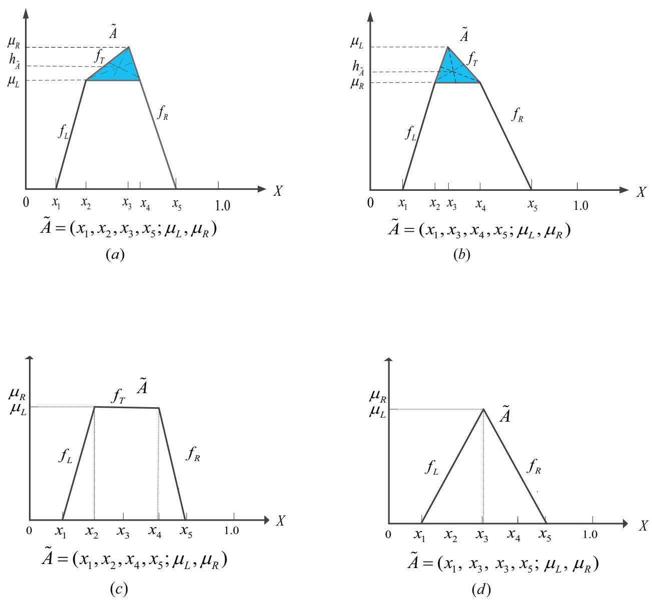


FIGURE 4. Four Kinds of Generalized Fuzzy Numbers

**Step 1:** Calculate the defuzzified value $x_{\tilde{A}_i}$ and the height $h_{\tilde{A}_i}$ of each generalized fuzzy numbers $\tilde{A}_i = (x_{i1}, x_{i2}, x_{i3}, x_{i4}; \mu_{iL}, \mu_{iR})$, shown as follows:

**Case 1:** If $\mu_{iL} = \mu_{iR}$, as is shown in Figure 4$c$ and Figure 4$d$,

$$x_{\tilde{A}_i} = \frac{\int_{x_{i1}}^{x_{i2}} x f^L_{\tilde{A}_i} dx + \int_{x_{i2}}^{x_{i3}} x f^T_{\tilde{A}_i} dx + \int_{x_{i3}}^{x_{i4}} x f^R_{\tilde{A}_i} dx}{\int_{x_{i1}}^{x_{i2}} f^L_{\tilde{A}_i} dx + \int_{x_{i2}}^{x_{i3}} f^T_{\tilde{A}_i} dx + \int_{x_{i3}}^{x_{i4}} f^R_{\tilde{A}_i} dx}, \tag{11}$$

$$h_{\tilde{A}_i} = \mu_{iL} = \mu_{iR}. \tag{12}$$

**Case 2:** If $\mu_{iL} > \mu_{iR}$, as is shown in Figure 4*b*,

$$x_{\tilde{A}_i} = \frac{\int_{x_{i1}}^{x_{i2}} x f_{\tilde{A}_i}^{L} dx + \int_{x_{i2}}^{x_{i3}} x f_{\tilde{A}_i}^{T} dx + \int_{x_{i3}}^{x_{i4}} x f_{\tilde{A}_i}^{R} dx}{\int_{x_{i1}}^{x_{i2}} f_{\tilde{A}_i}^{L} dx + \int_{x_{i2}}^{x_{i3}} f_{\tilde{A}_i}^{T} dx + \int_{x_{i3}}^{x_{i4}} f_{\tilde{A}_i}^{R} dx},$$

$$h_{\tilde{A}_i} = \frac{\int_{\mu_{iR}}^{\mu_{iL}} y g_{\tilde{A}_i}^{T} dy - \int_{\mu_{iR}}^{\mu_{iL}} y g_{\tilde{A}_i}^{L} dy}{\int_{\mu_{iR}}^{\mu_{iL}} g_{\tilde{A}_i}^{T} dy - \int_{\mu_{iR}}^{\mu_{iL}} g_{\tilde{A}_i}^{L} dy}. \tag{13}$$

**Case 3:** If $\mu_{iL} < \mu_{iR}$, as is shown in Figure 4*a*,

$$x_{\tilde{A}_i} = \frac{\int_{x_{i1}}^{x_{i2}} x f_{\tilde{A}_i}^{L} dx + \int_{x_{i2}}^{x_{i3}} x f_{\tilde{A}_i}^{T} dx + \int_{x_{i3}}^{x_{i4}} x f_{\tilde{A}_i}^{R} dx}{\int_{x_{i1}}^{x_{i2}} f_{\tilde{A}_i}^{L} dx + \int_{x_{i2}}^{x_{i3}} f_{\tilde{A}_i}^{T} dx + \int_{x_{i3}}^{x_{i4}} f_{\tilde{A}_i}^{R} dx},$$

$$h_{\tilde{A}_i} = \frac{\int_{\mu_{iL}}^{\mu_{iR}} y g_{\tilde{A}_i}^{R} dy - \int_{\mu_{iL}}^{\mu_{iR}} y g_{\tilde{A}_i}^{T} dy}{\int_{\mu_{iL}}^{\mu_{iR}} g_{\tilde{A}_i}^{R} dy - \int_{\mu_{iL}}^{\mu_{iR}} g_{\tilde{A}_i}^{T} dy}, \tag{14}$$

where $1 \le i \le n$, $g_{\tilde{A}_i}^{R}$, $g_{\tilde{A}_i}^{L}$ and $g_{\tilde{A}_i}^{T}$ are the inverse fuctions of $f_{\tilde{A}_i}^{R}$, $f_{\tilde{A}_i}^{L}$ and $f_{\tilde{A}_i}^{T}$, respectively. $x_{\tilde{A}_i}$ is the value of the centroid point on the horizontal axis of generalized fuzzy number $\tilde{A}_i$. $h_{\tilde{A}_i}$ is the value of the centroid point on the vertical axis of the positive triangle shown in the shaded parts of Figure 4*a* and Figure 4*b*.

**Step 2:** Calculate the spread $STD_{\tilde{A}_i}$ of each generalized fuzzy number $\tilde{A}_i$, shown as follows [8]:

$$STD_{\tilde{A}_i} = \sqrt{\frac{\sum_{j=1}^{4} (x_{ij} - x_{\tilde{A}_i})^2}{4-1}}, \tag{15}$$

where $x_{\tilde{A}_i} = \frac{x_{i1}+x_{i2}+x_{i3}+x_{i4}}{4}$ and $1 \le i \le n$.

**Step 3:** Construct the vector $V$ associated with the ordered arguments. According to the rigorous analysis, the value of $x_{\tilde{A}_i}$ is the most important factor that influences the ranking order of a generalized fuzzy number. The value of $h_{\tilde{A}_i}$ is the second important factor and $STD_{\tilde{A}_i}$ is the least important factor. Therefore, the ranking order of importance is: $x_{\tilde{A}_i} > h_{\tilde{A}_i} > STD_{\tilde{A}_i}$, $x_{\tilde{A}_i} > \mu h_{\tilde{A}_i} > \frac{\mu}{1+STD_{\tilde{A}_i}}$, which is consistent with the analytical geometry. The three elements $x_{\tilde{A}_i}, \mu h_{\tilde{A}_i}$ and $\frac{\mu}{1+STD_{\tilde{A}_i}}$ of vector $V$ are arranged in the order of their importance from the largest to the smallest. The vector $V$ is built as follows:

$$V = \begin{bmatrix} x_{\tilde{A}_i} \\ \mu h_{\tilde{A}_i} \\ \frac{\mu}{1+STD_{\tilde{A}_i}} \end{bmatrix}, \tag{16}$$

where $1 \le i \le n$,

$$\mu = \begin{cases} 1, & \tilde{x}_{\tilde{A}_i} \in [0, +\infty); \\ -1, & \tilde{x}_{\tilde{A}_i} \in (-\infty, 0). \end{cases}$$

**Step 4:** Calculate the transpose of weighting vector $W^T = [w_1, w_2, w_3]$ of the three elements $x_{\tilde{A}_i}, \mu h_{\tilde{A}_i}$ and $\frac{\mu}{1+STD_{\tilde{A}_i}}$ of vector $V$ based on equation (10) with $i = 3$.

The ranking order of the importance of these three factors is: $x_{\tilde{A}_i} > h_{\tilde{A}_i} > STD_{\tilde{A}_i}$, $x_{\tilde{A}_i} > \mu h_{\tilde{A}_i} > \frac{\mu}{1+STD_{\tilde{A}_i}}$. Therefore, this paper gives $x_{\tilde{A}_i}$ the maximal weight $w_1$, $\mu h_{\tilde{A}_i}$ the slightly larger weight $w_2$ and $\frac{\mu}{1+STD_{\tilde{A}_i}}$ the minimum weight $w_3$. In this situation, $w_1 > w_2 > w_3$, then $0.5 < \alpha < 1$ that can be seen in Figure 3. Generally, the value of $\alpha$ takes the middle value of the interval $(0.5, 0.9]$, that is $\alpha = 0.7$.

**Step 5:** Calculate the ranking score $Score(\tilde{A}_i)$ of each generalized fuzzy number $\tilde{A}_i$ based on equation (7), shown as follows:

$$\begin{aligned} Score(\tilde{A}_i) &= F(x_{\tilde{A}_i}, \mu h_{\tilde{A}_i}, \tfrac{\mu}{1+STD_{\tilde{A}_i}}) = W^T V \\ &= w_1 x_{\tilde{A}_i} + \mu w_2 h_{\tilde{A}_i} + w_3 \tfrac{\mu}{1+STD_{\tilde{A}_i}}, \end{aligned} \tag{17}$$

where $1 \leq i \leq n$. In the ranking score above, the vector $V$ is substituted into the vector $B$ in equation (7). The ranking score reflects or decides the ranking order of a fuzzy number. The larger the value of $Score(\tilde{A}_i)$, the better the ranking order of $\tilde{A}_i$. $x_{\tilde{A}_i}, h_{\tilde{A}_i}$ and $STD_{\tilde{A}_i}$ are three score factors that are the main score factors that affect the ranking size of a generalized fuzzy number.

**Example 3.1.** If a generalized fuzzy number $\tilde{A} = (x, x, x, x; \mu_x, \mu_x)$ is a crisp number, the defuzzified value of $\tilde{A}$ is $x$, the value of the centroid points of the positive triangle on the vertical axis ($h_{\tilde{A}}$) is $\mu_x$ and the spreads ($STD_{\tilde{A}}$) is 0. The score of the crisp number $\tilde{A}$ is shown as follows:

$$Score(\tilde{A}) = w_1 x + \mu w_2 \mu_x + \mu w_3,$$

where $\sum_{i=1}^{3} w_i = 1$.

Simple to discuss, fuzzy numbers in discuss are standardized to $[-1, 1]$. Assume that one fuzzy number is $\tilde{A}_i$, it is obvious that $-1 \leq x_{\tilde{A}_i} \leq 1$, $0 \leq h_{\tilde{A}_i} \leq 1$ and $0 \leq STD_{\tilde{A}_i} \leq 2$. Based on the stated above, in the following, we will introduce some properties of the proposed fuzzy ranking method.

**Property 1.** (*Monotonocity property*).

- *When* $h_{\tilde{A}_i} \epsilon [0, 1]$ *and* $STD_{\tilde{A}_i} \epsilon [0, 2]$, *if* $x_{\tilde{A}_1} > x_{\tilde{A}_2}$, *then* $Score(\tilde{A}_1) > Score(\tilde{A}_2)$.
- *When* $x_{\tilde{A}_i} \epsilon [-1, 1]$, $STD_{\tilde{A}_i} \epsilon [0, 2]$ *and* $x_{\tilde{A}_i} \geq 0$, *if* $h_{\tilde{A}_1} > h_{\tilde{A}_2}$, *then* $Score(\tilde{A}_1) > Score(\tilde{A}_2)$.
- *When* $x_{\tilde{A}_i} \epsilon [-1, 1]$, $STD_{\tilde{A}_i} \epsilon [0, 2]$ *and* $x_{\tilde{A}_i} < 0$, *if* $h_{\tilde{A}_1} > h_{\tilde{A}_2}$, *then* $Score(\tilde{A}_1) < Score(\tilde{A}_2)$.
- *When* $x_{\tilde{A}_i} \epsilon [-1, 1]$, $h_{\tilde{A}_i} \epsilon [0, 1]$ *and* $x_{\tilde{A}_i} \geq 0$, *if* $STD_{\tilde{A}_1} > STD_{\tilde{A}_2}$, *then* $Score(\tilde{A}_1) < Score(\tilde{A}_2)$.
- *When* $x_{\tilde{A}_i} \epsilon [-1, 1]$, $h_{\tilde{A}_i} \epsilon [0, 1]$ *and* $x_{\tilde{A}_i} < 0$, *if* $STD_{\tilde{A}_1} > STD_{\tilde{A}_2}$, *then* $Score(\tilde{A}_1) > Score(\tilde{A}_2)$.

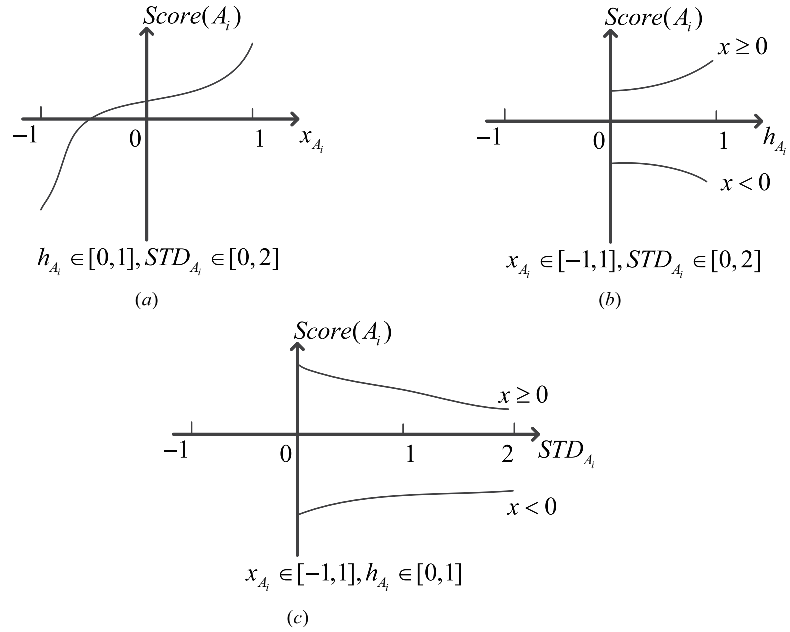


FIGURE 5. The Relationship Between Three Scoring Factors and the Ranking Score

*Proof.* According to equation (17), it is obvious that when $h_{\tilde{A}_i}\epsilon[0,1]$, $STD_{\tilde{A}_i}\epsilon[0,2]$ and $x_{\tilde{A}_1} > x_{\tilde{A}_2}$, the ranking score $Score(\tilde{A}_1) > Score(\tilde{A}_2)$; when $x_{\tilde{A}_i}\epsilon[-1,1]$, $STD_{\tilde{A}_i}\epsilon[0,2]$, $x_{\tilde{A}_i} \geq 0$ and $h_{\tilde{A}_1} > h_{\tilde{A}_2}$, the ranking score $Score(\tilde{A}_1) > Score(\tilde{A}_2)$; when $x_{\tilde{A}_i}\epsilon[-1,1]$, $STD_{\tilde{A}_i}\epsilon[0,2]$, $x_{\tilde{A}_i} < 0$ and $h_{\tilde{A}_1} > h_{\tilde{A}_2}$, the ranking score $Score(\tilde{A}_1) < Score(\tilde{A}_2)$; when $x_{\tilde{A}_i}\epsilon[-1,1]$, $h_{\tilde{A}_i}\epsilon[0,1]$, $x_{\tilde{A}_i} \geq 0$ and $STD_{\tilde{A}_1} > STD_{\tilde{A}_2}$, the ranking score $Score(\tilde{A}_1) < Score(\tilde{A}_2)$; when $x_{\tilde{A}_i}\epsilon[-1,1]$, $h_{\tilde{A}_i}\epsilon[0,1]$, $x_{\tilde{A}_i} < 0$ and $STD_{\tilde{A}_1} > STD_{\tilde{A}_2}$, the ranking score $Score(\tilde{A}_1) > Score(\tilde{A}_2)$. The relationship between three scoring factors and the ranking score is shown in Figure 5. □

**Property 2.** (*Maximum property*). *When* $x_{\tilde{A}_i} = 1, h_{\tilde{A}_i} = 1$ *and* $STD_{\tilde{A}_i} = 0$, $Score(\tilde{A}_i)$ *is maximum, namely, the fuzzy number* $\tilde{A}_i$ *has best ranking order.*

*Proof.* $x_{\tilde{A}_i} = 1, h_{\tilde{A}_i} = 1$ and $STD_{\tilde{A}_i} = 0$ are substituted into equation (17) to calculate the ranking score of the fuzzy number. And the value of $Score(\tilde{A}_i)$ is 1. Based on Figure 6, it is obvious that the maximum of equation (17) is obtained at $x_{\tilde{A}_i} = 1, h_{\tilde{A}_i} = 1$ and $STD_{\tilde{A}_i} = 0$. Therefore, a conclusion can be obtained that the value of the ranking score $Score(\tilde{A}_i)$ is maximum, namely, the fuzzy number $\tilde{A}_i$ has best ranking order, when $x_{\tilde{A}_i} = 1, h_{\tilde{A}_i} = 1$ and $STD_{\tilde{A}_i} = 0$. □

**Property 3.** (*Minimum property*). *When* $x_{\tilde{A}_i} = -1, h_{\tilde{A}_i} = 1$ *and* $STD_{\tilde{A}_i} = 2$, $Score(\tilde{A}_i)$ *is minimum, namely, the fuzzy number* $\tilde{A}_i$ *has worst ranking order.*

*Proof.* $x_{\tilde{A}_i} = -1, h_{\tilde{A}_i} = 0$ and $STD_{\tilde{A}_i} = 2$ are substituted into equation (17) to calculate the ranking score of the fuzzy number. And the value of $Score(\tilde{A}_i)$ is $-1$. Based on Figure 6, it is obvious that the minimum of equation (17) is obtained at $x_{\tilde{A}_i} = -1, h_{\tilde{A}_i} = 1$ and $STD_{\tilde{A}_i} = 2$.Therefore, a conclusion can be obtained that the value of the ranking score $Score(\tilde{A}_i)$ is minimum, namely, the fuzzy number $\tilde{A}_i$ has worst ranking order, when $x_{\tilde{A}_i} = -1, h_{\tilde{A}_i} = 1$ and $STD_{\tilde{A}_i} = 2$. □

**Property 4.** (*Completely symmetric property*). *$\tilde{A}_1$, $\tilde{A}_2$ two generalized fuzzy numbers and $\tilde{A}_1 = (x_1, x_2, x_3, x_4; \mu_L, \mu_R)$, $\tilde{A}_2 = (x'_1, x'_2, x'_3, x'_4; \mu'_L, \mu'_R) = (-x_4, -x_3, -x_2, -x_1; \mu_R, \mu_L)$, then $Score(\tilde{A}_1) = -Score(\tilde{A}_2)$, where $x_1 \neq -x_4, x_2 \neq -x_3$.*

*Proof.* If $\tilde{A}_1 = (x_1, x_2, x_3, x_4; \mu_L, \mu_R)$ and $\tilde{A}_2 = (x'_1, x'_2, x'_3, x'_4; \mu'_L, \mu'_R) = (-x_4, -x_3, -x_2, -x_1; \mu_R, \mu_L)$, based on equation (15), a result can be obtained:

$$STD_{\tilde{A}_1} = \sqrt{\frac{\sum_{j=1}^{4} (x_j - \bar{x})^2}{4-1}},$$

where

$$\bar{x} = \frac{x_1 + x_2 + x_3 + x_4}{4}.$$

Since $x'_1 = -x_4, x'_2 = -x_3, x'_3 = -x_2, x'_4 = -x_1$, the result of $STD_{\tilde{A}_2}$ is shown as follows:

$$\begin{aligned} STD_{\tilde{A}_2} &= \sqrt{\tfrac{\sum_{j=1}^{4} (x'_j - \bar{x}')^2}{4-1}} \\ &= \sqrt{\tfrac{\sum_{j=1}^{4} (-x_j + \bar{x})^2}{4-1}} \\ &= STD_{\tilde{A}_1}, \end{aligned}$$

based on equations (11-14), the values of $x_{\tilde{A}_1}$, $h_{\tilde{A}_1}$, $Score(\tilde{A}_1)$ and $Score(\tilde{A}_2)$ is shown as follows:

$$x_{\tilde{A}_1} = -x_{\tilde{A}_2}, h_{\tilde{A}_1} = h_{\tilde{A}_2},$$

$$Score(\tilde{A}_1) = w_1 x_{\tilde{A}_1} + \mu_1 w_2 h_{\tilde{A}_1} + w_3 \frac{\mu_1}{1 + STD_{\tilde{A}_1}},$$

$$Score(\tilde{A}_2) = w_1 x_{\tilde{A}_2} + \mu_2 w_2 h_{\tilde{A}_2} + w_3 \tfrac{\mu_2}{1+STD_{\tilde{A}_2}} = w_1(-x_{\tilde{A}_1}) + \mu_2 w_2 h_{\tilde{A}_1} + w_3 \tfrac{\mu_2}{1+STD_{\tilde{A}_1}}.$$

Based on equation (16), if $x_{\tilde{A}_1} > 0$, then $-x_{\tilde{A}_1} < 0$, $\mu_1 = 1$ and $\mu_2 = -1$, the value of $Score(\tilde{A}_2)$ is $-Score(\tilde{A}_1)$. If $x_{\tilde{A}_1} < 0$, then $-x_{\tilde{A}_1} > 0$, $\mu_1 = -1$ and $\mu_2 = 1$, the value of $Score(\tilde{A}_2)$ is $-Score(\tilde{A}_1)$. Therefore, a conclusion can be obtained, that is $Score(\tilde{A}_1) = -Score(\tilde{A}_2)$. □

**Property 5.** *If the generalized fuzzy number $\tilde{A} = (1, 1, 1, 1; 1, 1)$, then $Score(\tilde{A}) = 1$. If the generalized fuzzy number $\tilde{A} = (-1, -1, -1, -1; 1, 1)$, then $Score(\tilde{A}) = -1$.*

*Proof.* If $\tilde{A} = (1, 1, 1, 1; 1, 1)$, based on **Example 3.1**, conclusions can be made as follows: $x_{\tilde{A}} = 1, h_{\tilde{A}} = 1, STD_{\tilde{A}} = 0$ and $Score(\tilde{A}) = \sum_{i=1}^{3} w_i = 1$.

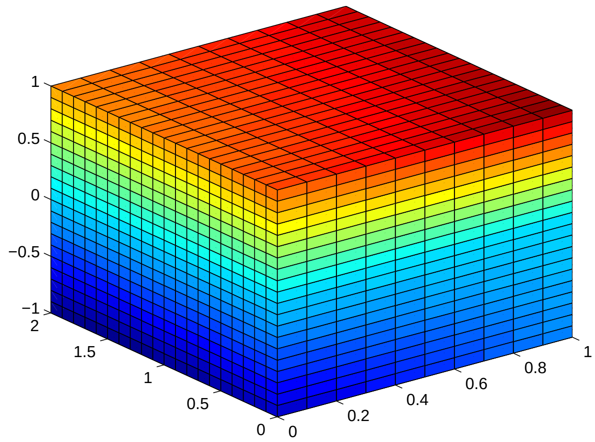


FIGURE 6. The Ranking Score of Generalized Fuzzy Number

If $\tilde{A} = (-1, -1, -1, -1; 1, 1)$, based on **Example 3.1**, conclusions can be drawn as follows: $x_{\tilde{A}} = -1, h_{\tilde{A}} = 1, STD_{\tilde{A}} = 0, u = -1$ and $Score(\tilde{A}) = -\sum_{i=1}^{3} w_i = -1$. □

## 4. Comparing the Proposed Method with Existing Methods

With the eight sets of generalized fuzzy numbers shown in Figure 7 and 8 (Set 1 - Set 8) which are frequently used as examples, the proposed method is compared with the Yager's method [44], Chu and Tsao's method [12], Chen and Chen's method [8], Chen and Sanguansat's method [10], Chen *et al.*'s method [9], Emrah *et al.*'s method [2] and Bakar and Gegoy's method [6] in this section. Besides, the other four sets of generalized fuzzy numbers (Set 9 - Set 12 shown in Figure 8) are added to this section. According to Figure 7, 8, Table 1 and Table 2, the weaknesses of the existing ranking methods and the advant ages of the proposed method can be seen:

(1) If the two fuzzy numbers are $\tilde{A}$ and $\tilde{B}$ from Set 1 in Figure 7, all of the existing fuzzy ranking methods and the proposed method get the same ranking order, $\tilde{B} > \tilde{A}$, which is consistent with the truth that the centroid point of $\tilde{B}$ on the X-axis is larger than that of $\tilde{A}$ on the X-axis, and the fact that the heights and the spreads of $\tilde{A}$ and $\tilde{B}$ are the same.

(2) If the two fuzzy numbers are $\tilde{A}$ and $\tilde{B}$ from Set 2 in Figure 7, the proposed method, Chen and Chen's method and Emrah *et al.*'s method get the same ranking order, $\tilde{B} > \tilde{A}$, which is consistent with the truth that the center of gravity of $\tilde{A}$ on the X-axis is the same as that of $\tilde{B}$, but the spread of $\tilde{A}$ is a little larger than that of $\tilde{B}$. However, Yager's method, Chu and Tsao's method, Chen and Sanguansat's method and Chen *et al.*'s method can't correctly address this situation and find the incorrect ranking order $\tilde{B} = \tilde{A}$.

(3) If the two fuzzy numbers are $\tilde{A}$ and $\tilde{B}$ from Set 3 in Figure 7, Chen and Chen's method, Emrah *et al.*'s method, Bakar and Gegoy's method and the proposed method get the same ranking order, $\tilde{B} > \tilde{A}$, which is consistent with the truth that the centroid points on the X-axis and the heights of $\tilde{A}$ and $\tilde{B}$ are the same, but the spread of $\tilde{A}$ is larger than that of $\tilde{B}$. However, the methods of Yager, Chu and Tsao, Chen and Sanguansat and Chen *et al.* get the same unreasonable ranking order, $\tilde{B} = \tilde{A}$.

(4) If the two fuzzy numbers are $\tilde{A}$ and $\tilde{B}$ from Set 4 in Figure 7, Yager's method get the wrong ranking order, $\tilde{B} = \tilde{A}$, whereas the methods of Chu and Tsao, Chen, Chen and Sanguansat, Chen *et al.* (2012), Emrah *et al.*, Bakar and Gegoy and this paper get the same ranking order, $\tilde{B} > \tilde{A}$ which is consistent with the truth that the centroid points on the X-axis and the spreads of $\tilde{A}$ and $\tilde{B}$ are the same, but the height of $\tilde{B}$ is larger than that of $\tilde{A}$.

(5) If the two fuzzy numbers are $\tilde{A}$ and $\tilde{B}$ from Set 5 in Figure 7, the methods of Chen and Chen, Chen and Sanguansat, Chen *et al.*, Emrah *et al.*, Bakar and Gegoy and this paper get the same ranking order, $\tilde{B} > \tilde{A}$, which accords with the truth that the centroid point of $\tilde{A}$ on the X-axis is larger than that of $\tilde{B}$, the spread of $\tilde{A}$ is smaller than that of $\tilde{B}$ and the heights of $\tilde{A}$ and $\tilde{B}$ are the same. However, the methods of Yager, Chu and Tsao are invalid to the crisp number.

(6) If the two fuzzy numbers are $\tilde{A}$ and $\tilde{B}$ from Set 6 in Figure 7, the proposed method and the existing method get the same ranking order, $\tilde{B} > \tilde{A}$, which is consistent with the truth that the centroid point of $\tilde{A}$ on the X-axis is larger than that of $\tilde{B}$, the spreads and the heights of $\tilde{A}$ and $\tilde{B}$ are the same.

(7) If the two fuzzy numbers are $\tilde{A}$ and $\tilde{B}$ from Set 7 in Figure 8, all of the ranking methods get the same ranking order, $\tilde{B} < \tilde{A}$, which is consistent with human intuition.

(8) If the two fuzzy numbers are $\tilde{A}$ and $\tilde{B}$ from Set 8 in Figure 8, the methods of Chu and Tsao, Chen and Chen, Chen and Sanguansat, Chen *et al.* and Bakar and Gegoy get a same incorrect ranking order, $\tilde{C} > \tilde{B} > \tilde{A}$. However, the proposed method, the methods of Yager and Emrah *et al.* get the correct ranking order, $\tilde{B} > \tilde{C} > \tilde{A}$, which is consistent with human intuition.

(9) If the two fuzzy numbers are $\tilde{A}$ and $\tilde{B}$ from Set 9 in Figure 8, the methods of Yager, Chen *et al.* and Emrah *et al.* get incorrect ranking orders, whereas the remainder of these methods and the proposed method get a right ranking order, $\tilde{B} > \tilde{A}$ which is consistent with the truth that the ranking scores of $\tilde{A}$ and $\tilde{B}$ are negative values.

(10) If the two fuzzy numbers are $\tilde{A}$ and $\tilde{B}$ from Set 10 in Figure 8, except for the proposed method and Emrah *et al.*'s method, all of the ranking methods above get the wrong ranking order, $\tilde{B} = \tilde{A} = 0.0000$. However, the proposed method and Emrah *et al.*'s method get the correct ranking order, $\tilde{B} > \tilde{A}$ which is consistent with the truth that the centroid points and the heights of $\tilde{A}$ and $\tilde{B}$ are the same, the spread of $\tilde{A}$ is larger than $\tilde{B}$.

(11) If the two fuzzy numbers are $\tilde{A}$ and $\tilde{B}$ from Set 11 in Figure 8, Emrah *et al.*'s method, Chen *et al.*'s method and the proposed method get the correct

ranking order, $\tilde{B} > \tilde{A} > \tilde{C}$, which accords with the intuition of human beings. However, Yager's method, Chu and Tsao's method, Chen and Chen's method, Chen and Sanguansat's method and Bakar and Gegoy's method get an incorrect ranking order, $\tilde{B} = \tilde{A} = \tilde{C} = 0.0000$.

(12) If the two fuzzy numbers are $\tilde{A}$ and $\tilde{B}$ from Set 12 in Figure 8, Chen *et al.*'s method and Bakar and Gegoy's method get incorrect ranking orders, whereas the proposed method gets a correct ranking order, $\tilde{A} > \tilde{B} > \tilde{C}$ which is consistent with the intuition of human beings. However, the remainder of these methods can't calculate the ranking score of generalized fuzzy numbers with different left heights and right heights, that is these methods are invalid in this case.

About all, from Figure 7 and Figure 8, Table 1 and Table 2, it can be seen that the proposed method can overcome the weaknesses of most traditional fuzzy ranking methods. In addition, the proposed method can calculate the ranking scores of generalized fuzzy numbers with different left heights and right heights that some traditional fuzzy ranking methods can't deal with, as shown in Set 12 of Figure 8. Moreover, the proposed method assigns different weights to the three scoring factors based on different importance of them, which illustrates the reasonability and generality of the proposed method.

In order to make the advantages of the proposed new method clear, we will highlight them in theory. The specific discussion is shown as follows:

- *The proposed method satisfies completely symmetric property.*

Theoretically, in the universe of discourse, the ranking values of two completely symmetric fuzzy numbers are better to be a pair of opposite numbers since that the two completely symmetric fuzzy numbers have the same distance from the origin. In accordance with the theory, the proposed method can get the reasonable result.

- *The proposed method satisfies monotonicity property.*

Theoretically, the defuzzified value and the height of a fuzzy number should be monotonically increasing and the spread should be monotonically decreasing, which is consistent with the truth that the larger the values of the defuzzified value and the height of a fuzzy number and the smaller the value of the spread, the better the ranking order. In accordance with the theory, the proposed method can get the correct result.

- *The proposed method satisfies maximum and minimum property.*

The universe of discourse is standardized in the interval $-1$ to 1, in all fuzzy numbers, fuzzy number $A = (1, 1, 1, 1; 1, 1)$ is closest to the positive axis, has the highest height and own the smallest spread. Based on the discussion of the previous monotonicity, it can be seen that the fuzzy number should get the maximum ranking value. Similarly, it can be seen that fuzzy number $B = (-1, -1, -1, -1; 1, 1)$ should get the minimum ranking value. In accordance with the theory, the proposed method can obtain the reasonable and correct result.

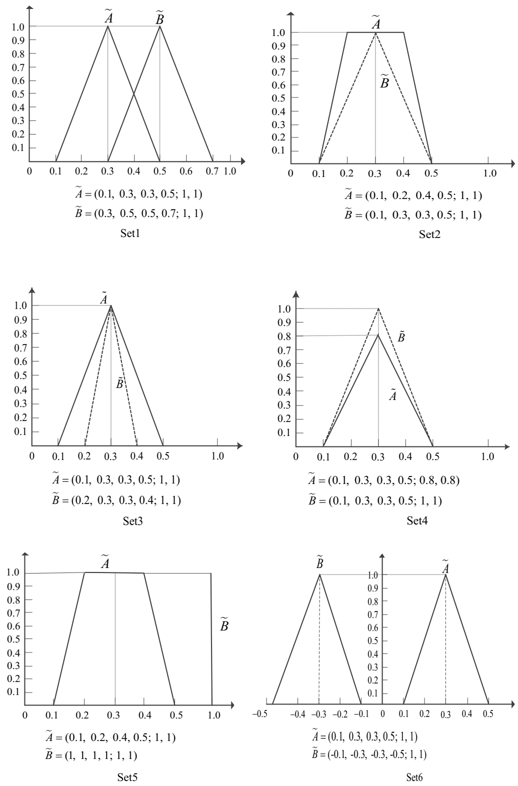


FIGURE 7. Six Sets of Generalized Fuzzy Numbers

## 5. Addressing Fuzzy Risk Analysis Problems

In this section, the proposed fuzzy ranking method is applied to handle fuzzy risk analysis problems of components in industry. Take the structure in [9] as an example. Assume that there are $n$ manufactories $M_1, M_2, ...,$ and $M_n$. Each manufactory produces the component $C_i$ which is made up of $p$ sub-components $C_{i1}, C_{i2}, ...,$ and $C_{ip}$, $1 \leq i \leq n$. In order to assess each sub-component $C_{ik}$ and obtain the probability of failure $\tilde{R_i}$ of component $C_i$, two assessing items $\tilde{R_{ik}}$ and $\tilde{W_{ik}}$ are used to evaluate each sub-component $\tilde{C_{ik}}$ [9]. $\tilde{R_{ik}}$ represents the probability of failure of $C_{ik}$ and $\tilde{W_{ik}}$ represents the severity of loss of $C_{ik}$, where $\tilde{R_{ik}}$ and $\tilde{W_{ik}}$ are generalized fuzzy numbers [9], $1 \leq k \leq 3$ and $1 \leq i \leq n$. In [4], the failure rate of fuzzy system was represented by a triangular fuzzy number. The structure for fuzzy risk analysis is shown in Figure 9.

The fuzzy risk analysis algorithm based on the proposed method is shown as follows:

**Step 1**: Calculate the probability of failure $\tilde{R_i}$ represented by generalized fuzzy numbers, where $1 \leq i \leq n$.

| methods | Set1 | | Set2 | | Set3 | |
|---|---|---|---|---|---|---|
| | $A$ | $B$ | $A$ | $B$ | $A$ | $B$ |
| Yager's methode(1978) | 0.3000 | 0.5000 | **0.3000** | **0.3000** | **0.3000** | **0.3000** |
| Chu and Tsao's method(2002) | 0.1500 | 0.2500 | **0.1500** | **0.1500** | **0.1500** | **0.1500** |
| Chen and Chen's method(2009) | 0.2579 | 0.4298 | 0.2573 | 0.2579 | 0.2579 | 0.2774 |
| Chen and Sanguansat's method(2011) | 0.3000 | 0.5000 | **0.3000** | **0.3000** | **0.3000** | **0.3000** |
| Chen et al.'s method(2012) | 0.2553 | 0.4444 | **0.2553** | **0.2553** | **0.2553** | **0.2553** |
| Emrah et al.'s method(2013) | 0.2787 | 0.4788 | 0.2622 | 0.2787 | 0.2787 | 0.2866 |
| Bakar and Gegoy's method(2014) | 0.0867 | 0.1444 | **0.1096** | **0.0867** | 0.0867 | 0.0933 |
| The proposed method | 0.5906 | 0.7014 | 0.5884 | 0.5906 | 0.5906 | 0.6006 |

| methods | Set4 | | Set5 | | Set6 | |
|---|---|---|---|---|---|---|
| | $A$ | $B$ | $A$ | $B$ | $A$ | $B$ |
| Yager's method(1978) | **0.3000** | **0.3000** | 0.3000 | N | -0.3000 | 0.3000 |
| Chu and Tsao's method(2002) | 0.1200 | 0.1500 | 0.1500 | N | -0.1500 | 0.1500 |
| Chen and Chen's method(2009) | 0.2063 | 0.2579 | 0.2537 | 1.0000 | -0.2579 | 0.2579 |
| Chen and Sanguansat's method(2011) | 0.2824 | 0.3000 | 0.3000 | 1.000 | -0.3000 | 0.3000 |
| Chen *et al.*'s method(2012) | 0.2462 | 0.2553 | 0.2553 | 1.0000 | -0.2553 | 0.2553 |
| Emrah *et al.*'s method(2013) | 0.2550 | 0.2787 | 0.2622 | 1.0000 | -0.3213 | 0.2787 |
| Bakar and Gegoy's method(2014) | 0.0715 | 0.0867 | 0.1096 | 0.3333 | -0.0867 | 0.0867 |
| The proposed method | 0.5322 | 0.5906 | 0.5884 | 1.0000 | -0.5906 | 0.5906 |

| methods | Set7 | | Set9 | | Set10 | |
|---|---|---|---|---|---|---|
| | $A$ | $B$ | $A$ | $B$ | $A$ | $B$ |
| Yager's method(1978) | 0.6000 | 0.5000 | **-0.8333** | **-0.8333** | **0.0000** | **0.0000** |
| Chu and Tsao's method(2002) | 0.2870 | 0.2619 | -0.4365 | -0.3492 | **0.0000** | **0.0000** |
| Chen and Chen's method(2009) | 0.4428 | 0.4043 | -0.7000 | -0.5600 | **0.0000** | **0.0000** |
| Chen and Sanguansat's method(2011) | 0.5750 | 0.5350 | -0.8750 | -0.8235 | **0.0000** | **0.0000** |
| Chen *et al.*'s method(2012) | 0.5111 | 0.4773 | **-0.8750** | **-0.8750** | **0.0000** | **0.0000** |
| Emrah *et al.*'s method(2013) | 0.5684 | 0.4837 | **-0.8530** | **-0.9120** | -0.0568 | -0.0378 |
| Bakar and Gegoy's method(2014) | 0.1533 | 0.1278 | -0.2315 | -0.1926 | **0.0000** | **0.0000** |
| The proposed method | 0.7430 | 0.6876 | -0.8769 | -0.8185 | 0.4110 | 0.4223 |

| methods | Set8 | | | Set11 | | |
|---|---|---|---|---|---|---|
| | $\tilde{A}$ | $\tilde{B}$ | $\tilde{C}$ | $\tilde{A}$ | $\tilde{B}$ | $\tilde{C}$ |
| Yager's method(1978) | 0.4400 | 0.5333 | 0.5250 | **0.0000** | **0.0000** | **0.0000** |
| Chu and Tsao's method(2002) | **0.2881** | **0.2624** | **0.2784** | **0.0000** | **0.0000** | **0.0000** |
| Chen and Chen's method(2009) | **0.3354** | **0.4079** | **0.4196** | **0.0000** | **0.0000** | **0.0000** |
| Chen and Sanguansat's method(2011) | **0.4500** | **0.5250** | **0.5500** | **0.0000** | **0.0000** | **0.0000** |
| Chen *et al.*'s method(2012) | **0.4000** | **0.4667** | **0.5057** | 0.2533 | 0.2687 | 0.2420 |
| Emrah *et al.*'s method(2013) | 0.4013 | 0.5063 | 0.4947 | -0.0568 | -0.0244 | -0.0721 |
| Bakar and Gegoy's method(2014) | **0.1197** | **0.1363** | **0.1452** | **0.0000** | **0.0000** | **0.0000** |
| The proposed method | 0.6506 | 0.7071 | 0.7003 | 0.4110 | 0.4157 | 0.3573 |

Note: "N" denotes the method can not calculate the ranking value of the fuzzy numbers; bold values denote incorrect ranking results.

TABLE 1. A Comparison of the Proposed Method with the Existing Fuzzy Ranking Methods in Figure 7 and Figure 8

| methods | Set12 | | |
|---|---|---|---|
| | $\tilde{A}$ | $\tilde{B}$ | $\tilde{C}$ |
| Yager's method(1978) | 0.3000 | N | N |
| Chu and Tsao's method(2002) | 0.1500 | N | N |
| Chen and Chen's method(2009) | 0.2537 | N | N |
| Chen and Sanguansat's method(2011) | 0.3000 | N | N |
| Chen *et al.*'s method(2012) | **0.2553** | **0.2687** | **0.2420** |
| Emrah *et al.*'s method(2013) | 0.2662 | N | N |
| Bakar and Gegoy's method(2014) | **0.1096** | **0.1175** | **0.1099** |
| The proposed method | 0.5884 | 0.5662 | 0.5454 |

Note: "N" denotes the method can not calculate the ranking value of the fuzzy numbers; bold values denote incorrect ranking results.

TABLE 2. A Comparison of the Proposed Method with the Existing Fuzzy Ranking Methods in Figure 8

By the fuzzy weighted mean method [8] and the generalized fuzzy numbers arithmetic operators described in equations (3-6), the assessing items $\tilde{R_{ik}}$ and $\tilde{W_{ik}}$ of sub-component $A_{ik}$ are aggregated to obtain the probability of failure $\tilde{R_i}$, shown as follows [9]:

$$\begin{aligned}\tilde{R_i} &= \sum_{k=1}^{3}(\tilde{R}_{ik}\otimes\tilde{W}_{ik})\oslash\sum_{k=1}^{3}\tilde{W}_{ik}\\ &= (r_{i1}, r_{i2}, r_{i3}, r_{i4}; \mu_{L\tilde{R}_i}, \mu_{R\tilde{R}_i}).\end{aligned}$$

This Step stated above is reviewed from Chen's paper [9].

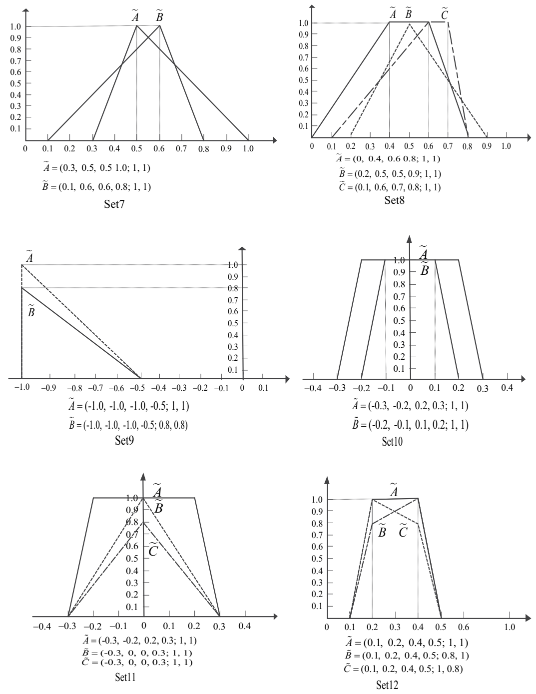


FIGURE 8. Six Sets of Generalized Fuzzy Numbers

| Manufactory | Sub-components | Severity of loss |
|---|---|---|
| $M_1$ | $C_{11}$ | $\widetilde{W}_{11} = (0.04, 0.1, 0.18, 0.23; 1.0, 1.0)$ |
| | $C_{12}$ | $\widetilde{W}_{12} = (0.58, 0.63, 0.80, 0.86; 1.0, 1.0)$ |
| | $C_{13}$ | $\widetilde{W}_{13} = (0.0, 0.0, 0.02, 0.07; 1.0, 1.0)$ |
| $M_2$ | $C_{21}$ | $\widetilde{W}_{21} = (0.04, 0.1, 0.18, 0.23; 1.0, 1.0)$ |
| | $C_{22}$ | $\widetilde{W}_{22} = (0.58, 0.63, 0.80, 0.86; 1.0, 1.0)$ |
| | $C_{23}$ | $\widetilde{W}_{23} = (0.0, 0.0, 0.02, 0.07; 1.0, 1.0)$ |
| $M_3$ | $C_{31}$ | $\widetilde{W}_{31} = (0.04, 0.1, 0.18, 0.23; 1.0, 1.0)$ |
| | $C_{32}$ | $\widetilde{W}_{32} = (0.58, 0.63, 0.80, 0.86; 1.0, 1.0)$ |
| | $C_{33}$ | $\widetilde{W}_{33} = (0.0, 0.0, 0.02, 0.07; 1.0, 1.0)$ |
| Manufactory | Sub-components | Probability of failure |
| $M_1$ | $C_{11}$ | $\widetilde{R}_{11} = (0.17, 0.22, 0.36, 0.42; 0.9, 0.9)$ |
| | $C_{12}$ | $\widetilde{R}_{12} = (0.32, 0.41, 0.58, 0.65; 0.7, 0.7)$ |
| | $C_{13}$ | $\widetilde{R}_{13} = (0.58, 0.63, 0.80, 0.86; 0.8, 0.8)$ |
| $M_2$ | $C_{21}$ | $\widetilde{R}_{21} = (0.93, 0.98, 1.0, 1.0; 0.85, 0.85)$ |
| | $C_{22}$ | $\widetilde{R}_{22} = (0.58, 0.63, 0.80, 0.86; 0.9, 0.9)$ |
| | $C_{23}$ | $\widetilde{R}_{23} = (0.32, 0.41, 0.58, 0.65; 0.9, 0.9)$ |
| $M_3$ | $C_{31}$ | $\widetilde{R}_{31} = (0.17, 0.22, 0.36, 0.42; 0.95, 0.95)$ |
| | $C_{32}$ | $\widetilde{R}_{32} = (0.72, 0.78, 0.92, 0.97; 0.8, 0.8)$ |
| | $C_{33}$ | $\widetilde{R}_{33} = (0.58, 0.63, 0.80, 0.86; 1.0, 1.0)$ |

TABLE 3. The Severity of Loss $\tilde{W}_{ik}$ and the Probability of Failure $\tilde{R}_{ik}$ of the Sub-component $C_{ik}$ [9]

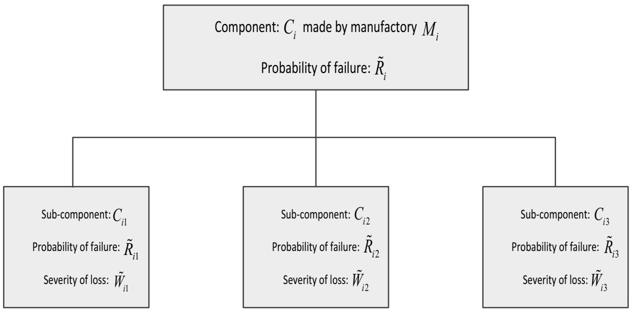


FIGURE 9. The Structure for Fuzzy Risk Analysis [9]

**Step 2**: Based on equations (11-14), calculate the defuzzified value ($x_{\tilde{R}_i}$) and height ($h_{\tilde{R}_i}$) of $\tilde{R}_i$, respectively, where $1 \le i \le n$. Based on equation (15), Calculate the spread $STD_{\tilde{R}_i}$ of $\tilde{R}_i$. Based on the Step 3 in the proposed method shown in Section 3, obtain the vector $V$.

**Step 3**: Based on the Step 4 in the proposed method shown in Section 3, calculate the transpose of weighting vector $W^T = [w_1, w_2, w_3]$ of the three elements $x_{\tilde{R}_i}, \mu h_{\tilde{R}_i}$ and $\frac{\mu}{1+STD_{\tilde{R}_i}}$ of vector $V$.

**Step 4**: Based on equation (17), calculate the ranking score $Score(\tilde{R}_i)$ of each generalized fuzzy number $\tilde{R}_i$, where $1 \le i \le n$, $\sum_{i=1}^{3} w_i = 1$. The larger the value of $Score(\tilde{R}_i)$, the higher the probability of failure of item $A_i$ made by manufactory $C_i$, where $1 \le i \le n$.

**Example 5.1.** [9] The fuzzy risk analysis process of the proposed method is illustrated in this example. Take the structure of fuzzy risk analysis in Figure 9 with $1 \le i \le 3$. Suppose that there are three manufactories $M_1, M_2$ and $M_3$ which respectively produce the components $C_1, C_2$ and $C_3$, where $C_1, C_2$ and $C_3$ are the same product. $C_i$ is made up of three sub-components $C_{i1}, C_{i2}$ and $C_{i3}$, where $1 \le i \le 3$. To obtain the probability of failure $\tilde{R}_i$, two assessing items $\tilde{W_{ik}}$ and $\tilde{R_{ik}}$ are used, where $1 \le k \le 3$, $1 \le i \le 3$.

The severity of loss $\tilde{W_{ik}}$ and the probability of failure $\tilde{R_{ik}}$ are denoted in Table 3 [9], where $1 \le i \le 3$ and $1 \le k \le 3$.

In the following, the proposed method is applied to deal with the fuzzy risk analysis issues, which is shown as follows:

**Step 1**: From equations (3-6), calculate the probability of failure $\tilde{R}_i$ by aggregating the assessing items $\tilde{R_{ik}}$ and $\tilde{W_{ik}}$ shown in Table 3, where $1 \le i \le 3$ and $1 \le k \le 3$, shown as follows [9, 8, 7]:

$$\begin{aligned}\tilde{R}_1 &= (\tilde{R_{11}} \otimes \tilde{W_{11}} \oplus \tilde{R_{12}} \otimes \tilde{W_{12}} \oplus \tilde{R_{13}} \otimes \tilde{W_{13}}) \\ &\oslash (\tilde{W_{11}} \oplus \tilde{W_{12}} \oplus \tilde{W_{13}}) \\ &= (0.1659, 0.2803, 0.7463, 1.1545; 0.7, 0.7),\end{aligned}$$

$$\begin{aligned}\tilde{R_2} &= (\tilde{R_{21}} \otimes \tilde{W_{21}} \oplus \tilde{R_{22}} \otimes \tilde{W_{22}} \oplus \tilde{R_{23}} \otimes \tilde{W_{23}}) \\ &\oslash(\tilde{W_{21}} \oplus \tilde{W_{22}} \oplus \tilde{W_{23}}) \\ &= (0.3221, 0.4949, 1.1392, 1.6373; 0.85, 0.85),\end{aligned}$$

$$\begin{aligned}\tilde{R_3} &= (\tilde{R_{31}} \otimes \tilde{W_{31}} \oplus \tilde{R_{32}} \otimes \tilde{W_{32}} \oplus \tilde{R_{33}} \otimes \tilde{W_{33}}) \\ &\oslash(\tilde{W_{31}} \oplus \tilde{W_{32}} \oplus \tilde{W_{33}}) \\ &= (0.3659, 0.5134, 1.1189, 1.5984; 0.8, 0.8).\end{aligned}$$

**Step 2**: Based on equations (11-14), calculate the defuzzified value ($x_{\tilde{R}_i}$) and height ($h_{\tilde{R}_i}$) of each generalized fuzzy numbers $\tilde{R}_i$, respectively, where $1 \leq i \leq 3$, shown as follows:

$$\begin{aligned} x_{\tilde{R}_1} &= 0.5955, \quad h_{\tilde{R}_1} = 0.9077, \\ x_{\tilde{R}_2} &= 0.7000, \quad h_{\tilde{R}_2} = 0.8500, \\ x_{\tilde{R}_3} &= 0.6877, \quad h_{\tilde{R}_3} = 0.6230. \end{aligned}$$

Calculating the spread $STD_{\tilde{R}_i}$ of each generalized fuzzy number $\tilde{R}_i$ based on equation (15), shown as follows:

$$\begin{aligned} STD_{\tilde{R}_1} &= 0.9086, \\ STD_{\tilde{R}_2} &= 0.8000, \\ STD_{\tilde{R}_3} &= 0.6375. \end{aligned}$$

Constructing the vector $V$ based on equation (16). The results are shown as follows:

$$V_1 = \begin{bmatrix} 0.5955 \\ 0.9077 \\ 0.5239 \end{bmatrix}, V_2 = \begin{bmatrix} 0.7000 \\ 0.8500 \\ 0.8000 \end{bmatrix}, V_3 = \begin{bmatrix} 0.6877 \\ 0.6230 \\ 0.6373 \end{bmatrix}.$$

**Step 3**: Calculate the transpose of weighting vector $W^T = [w_1, w_2, w_3]$ of the three elements $x_{\tilde{R}_i}, \mu h_{\tilde{R}_i}$ and $\frac{\mu}{1+STD_{\tilde{R}_i}}$ of vector $V$ based on equation (10), where $i = 3$, shown as follows:

$$W = \begin{bmatrix} 0.5540 \\ 0.2921 \\ 0.1540 \end{bmatrix}.$$

**Step 4**: Based on equations (16) and (17), calculate the ranking score $Score(\tilde{R}_i)$ of each generalized fuzzy number $\tilde{R}_i$, shown as follows:

$$Score(\tilde{R}_1) = 0.6402,$$

$$Score(\tilde{R}_2) = 0.8470,$$

$$Score(\tilde{R}_3) = 0.8351.$$

Because $Score(\tilde{R}_2) > Score(\tilde{R}_3) > Score(\tilde{R}_1)$, the ranking order of the three generalized fuzzy numbers $\tilde{R}_1$, $\tilde{R}_2$ and $\tilde{R}_3$ is $\tilde{R}_2 > \tilde{R}_3 > \tilde{R}_1$. Therefore, the order of the risk of the manufactories $M_1$, $M_2$ and $M_3$ is $M_2 > M_3 > M_1$. It means that the component $C_2$ made by the manufactory $M_2$ has the highest probability of failure, followed by $C_3$ and $C_1$, which is consistent with the human intuition.

## 6. Conclusions

Fuzzy risk analysis is a useful tool to deal with risk analysis which is a crucial topic in industry. In this paper, a new method for ranking generalized fuzzy numbers based on the OWA operator is proposed, in which the different importance of the three scoring factors that affect ranking order is taken into consideration. The centroid method is adopted to calculate the values of defuzzified value and the height of a generalized fuzzy number due to the fact that the centroid method utilized all information of fuzzy sets. The proposed method assigns the different weights to the three scoring factors of a generalized fuzzy number based on different importance of them, which improves its reasonability and generality. The proposed method can overcome the drawbacks of most traditional fuzzy ranking method. For example, the proposed method can calculate the ranking score in some situations which can't be dealt with by some existing methods, the proposed method can get the correct ranking orders in some cases which can't be correctly judged by some existing methods. In addition, an application in fuzzy risk analysis based on the proposed method is introduced. Also, the method can be applied to other domains such as decision-making according to the fact that the weights are taken into consideration in the proposed method.

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