CHARACTERIZATIONS OF (L, M)-FUZZY TOPOLOGY DEGREES

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Abstract. In this paper, characterizations of the degree to which a mapping $\mathcal{T} : L^X \longrightarrow M$ is an (L, M)-fuzzy topology are studied in detail. What is more, the degree to which an L-subset is an L-open set with respect to $\mathcal{T}$ is introduced. Based on that, the degrees to which a mapping $f : X \longrightarrow Y$ is continuous, open, closed or a quotient mapping with respect to $\mathcal{T}_X$ and $\mathcal{T}_Y$ are defined, and their characterizations are given, respectively. Besides, the relationships among the continuity degrees, the openness degrees, the closedness degrees and the quotient degrees of mappings are discussed.

1. Introduction

In 1968, fuzzy topology was first introduced by Chang in [1] and later extended by many other authors [3, 4, 8, 14]. In Chang's fuzzy topology, each open set is fuzzy, but the topology comprising those open sets is a crisp subset of the I-powerset I^X, where I is the unit interval $[0, 1]$. In 1973, J.A. Goguen [4] replaced I by a commutative GL-monoid L (which is now referred to unital commutative quantale). Then he obtained the concept of L-fuzzy topology (or L-topology for short). In 1980, Höhle [6] gave a new approach to fuzzification of topologies in a completely different direction and proposed that a fuzzy topology can be viewed as a fuzzy subset of the powerset 2^X. In 1991, Ying [27] introduced the concept of fuzzifying topology from logical point of view independently, which is similar with the thought of Höhle's fuzzy topology. In 1985, Šostak [21] and Kubiak [10] extended Höhle's fuzzy topology to an I-subset of I^X and an M-subset of L^X (which is now called an (L, M)-fuzzy topology) respectively. For the history of (L, M)-fuzzy topology, the readers can consult "Historical remark" in [5].

Continuous mappings, open mappings, closed mappings and quotient mappings are some important concepts in general topology. With the development of fuzzy mathematics, the continuity degree of mappings between (L, M)-fuzzy topological spaces (including L-fuzzifying topological spaces) was considered by A.P. Šostak in [24, 25]. Subsequently, Pang [16] introduced the continuity degree, the openness degree, the closedness degree of mappings between L-fuzzifying topological spaces in detail. And, Liang and Shi [12] discussed the relationships among the degrees of continuity, openness, closedness, compactness, connectedness in (L, M)-fuzzy

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topological spaces. In fuzzy categories, potential objects and morphisms can be endowed for some degree. Particularly, the degrees of axioms related to (L, M)-fuzzy topology and the degrees of continuous mappings between (L, M)-fuzzy topological spaces were presented by T. Kubiak and A.P. Šostak in [11, 24, 25]. Actually, they gave the definition of the degree to which a mapping $\mathcal{T} : L^X \longrightarrow M$ is an (L, M)-fuzzy topology.

The main aim of this paper is to discuss the characterizations of (L, M)-fuzzy topology degrees in detail and introduce a new definition of the degrees to which a mapping $f : X \longrightarrow Y$ is continuous, open, closed or a quotient mapping with respect to the (L, M)-fuzzy topology degrees of $\mathcal{T}_X$ and $\mathcal{T}_Y$.

This paper is organized as follows. In Section 2, some necessary concepts of general topology and (L, M)-fuzzy topology are recalled. In Section 3, characterizations of the degree to which a mapping $\mathcal{T} : L^X \longrightarrow M$ is an (L, M)-fuzzy topology are studied in detail. Additionally, the degrees to which $\mathcal{T}$ is an (L, M)-fuzzy pretopology, an M-fuzzifying topology or an M-fuzzifying pretopology are also introduced and their characterizations are given. In Section 4, the degree to which an L-subset is an L-open set with respect to $\mathcal{T}$ is proposed. Based on that, we define the degrees to which a mapping $f : X \longrightarrow Y$ is continuous, open, or closed with respect to $\mathcal{T}_X$ and $\mathcal{T}_Y$. Then we give their characterizations and discuss some properties of them. In Section 5, the degree to which a mapping $f : X \longrightarrow Y$ is a quotient mapping with respect to $\mathcal{T}_X$ and $\mathcal{T}_Y$ is defined and its characterizations are given. In Section 6, a theoretical application of (L, M)-fuzzy topology degrees is presented.

2. **Preliminaries**

Throughout this paper, unless otherwise stated, $(L, \vee, \wedge)$ and $(M, \vee, \wedge)$ are completely distributive lattices. The smallest element and the largest element in L are denoted by $\perp_L$ and $\top_L$, respectively. X is a non-empty set. We denote the set of all subsets of X by 2^X and denote the set of all L-subsets of X by L^X. The smallest element and the largest element in L^X are denoted by $\perp_{L^X}$ and $\top_{L^X}$, respectively.

We define a residual implication in L by $a \to b = \bigvee\{c \in L \mid a \wedge c \leq b\}$. Also, we denote $a \leftrightarrow b = (a \to b) \wedge (b \to a)$. Some properties of the implication operation are listed in the following lemma.

Lemma 2.1. [7] *For all $a, b, c \in L$ and $\{a_i\}_{i\in I}, \{b_i\}_{i\in I} \subseteq L$, the following statements hold.*

(1) $\top_L \to a = a$.
(2) $c \leq a \to b \Leftrightarrow a \wedge c \leq b$.
(3) $a \to b = \top_L \Leftrightarrow a \leq b$.
(4) $a \to \left(\bigwedge_{i\in I} b_i\right) = \bigwedge_{i\in I}(a \to b_i)$, *hence* $a \to b \leq a \to c$ *whenever* $b \leq c$.
(5) $\left(\bigvee_{i\in I} a_i\right) \to b = \bigwedge_{i\in I}(a_i \to b)$, *hence* $a \to c \geq b \to c$ *whenever* $a \leq b$.
(6) $(a \to c) \wedge (c \to b) \leq a \to b$.
(7) $a \to (b \to c) = (a \wedge b) \to c$.

An element a in L is called a prime element if $b \wedge c \leqslant a$ implies $b \leqslant a$ or $c \leqslant a$. An element a in L is called a co-prime element if $b \vee c \geqslant a$ implies $b \geqslant a$ or $c \geqslant a$

[2]. The set of non-unit (L removes the largest element) prime elements in L is denoted by $P(L)$. The set of non-zero (L removes the smallest element) co-prime elements in L is denoted by $J(L)$.

We say that a is wedge below b (in symbols, $a \prec b$) if for every subset $D \subseteq M$, $b \leq \bigvee D$ implies $a \leq d$ for some $d \in D$. And $a \prec^{op} b$ means that if for every subset $D \subseteq L$, $\bigwedge D \leq b$ implies $d \leq a$ for some $d \in D$ (see [19]). $\alpha(a) = \{x \in L \mid x \prec^{op} a\}$ is called the greatest maximal family of a in the sense of [26]. In a completely distributive L, $\alpha(a)$ exists for each $a \in L$, and $a = \bigwedge \alpha(a)$. Besides, $\alpha(\bigwedge_{i\in I} a_i) = \bigcup_{i\in I} \alpha(a_i)$ for any $\{a_i\}_{i\in I} \subseteq L$ and $\alpha(b) \subseteq \alpha(a)$ whenever $a, b \in L$ with $a \leq b$ (see [26]).

The forward L-power operator $f_L^{\rightarrow} : L^X \longrightarrow L^Y$ and the backward L-power operator $f_L^{\leftarrow} : L^Y \longrightarrow L^X$ induced by the mapping $f : X \longrightarrow Y$ are defined by $f_L^{\rightarrow}(A)(y) = \bigvee_{f(x)=y} A(x)$ for all $A \in L^X, y \in Y$ and $f_L^{\leftarrow}(B) = B \circ f$ for all $B \in L^Y$, respectively (see [17]).

Definition 2.2. [10, 21] An M-valued L-fuzzy topology (or (L, M)-fuzzy topology for short) on X is a mapping $\mathcal{T} : L^X \longrightarrow M$ which satisfies

(LMT1) $\mathcal{T}(\perp_{L^X}) = \mathcal{T}(\top_{L^X}) = \top_M$;
(LMT2) $\forall A_1, A_2 \in L^X$, $\mathcal{T}(A_1) \wedge \mathcal{T}(A_2) \leq \mathcal{T}(A_1 \wedge A_2)$;
(LMT3) $\forall \{A_j \mid j \in J\} \subseteq L^X$, $\bigwedge_{j\in J} \mathcal{T}(A_j) \leq \mathcal{T}(\bigvee_{j\in J} A_j)$.

The pair $(X, \mathcal{T})$ is called an (L, M)-fuzzy topological space. $\mathcal{T}(A)$ can be regarded as the degree to which A is an L-open set.

Denote $\{0, 1\} = 2$ and $[0, 1] = I$. An $(L, 2)$-fuzzy topology can be interpreted as an L-fuzzy topology defined in [4, 8, 14] (which is now called an L-topology for short), an $(2, M)$-fuzzy topology can be interpreted as an M-fuzzifying topology defined in [6, 27], an $(2, 2)$-fuzzy topology can be interpreted as a topology defined in [9, 15].

Definition 2.3. [10, 21] A mapping $f : (X, \mathcal{T}_X) \longrightarrow (Y, \mathcal{T}_Y)$ between (L, M)-fuzzy topological spaces is called continuous if it satisfies $\mathcal{T}_Y(B) \leq \mathcal{T}_X(f_L^{\leftarrow}(B))$ for all $B \in L^Y$.

Remark 2.4. If a mapping $\mathcal{T} : L^X \longrightarrow M$ only satisfies (LMT1) and (LMT3), then $\mathcal{T}$ is called an (L, M)-fuzzy pretopology [20]. The pair $(X, \mathcal{T})$ is called an (L, M)-fuzzy pretopological space.

For all $a \in M$, we define

$$\mathcal{T}_{[a]} = \{A \in L^X \mid \mathcal{T}(A) \geq a\},\ \mathcal{T}^{[a]} = \{A \in L^X \mid a \notin \alpha(\mathcal{T}(A))\}.$$

In many papers, it is shown that $\mathcal{T}$ is an (L, M)-fuzzy topology if and only if $\mathcal{T}_{[a]}$ is an L-topology for any $a \in M$. In addition, we have the following theorem.

Theorem 2.5. [28] *Let $\mathcal{T} : L^X \longrightarrow M$ be a mapping. If M is completely distributive, then the following statements are equivalent.*

(1) *$\mathcal{T}$ is an (L, M)-fuzzy topology on X.*
(2) *$\forall a \in J(M)$, $\mathcal{T}_{[a]}$ is an L-topology on X.*

(3) $\forall a \in M$, $\mathcal{T}^{[a]}$ *is an* L*-topology on* X.
(4) $\forall a \in P(M)$, $\mathcal{T}^{[a]}$ *is an* L*-topology on* X.

3. Characterizations of (L, M)-fuzzy Topology Degrees

In this section, we shall study the characterizations of (L, M)-fuzzy topology degrees which was introduced in [11, 24, 25]. Additionally, (L, M)-fuzzy pretopology degrees, M-fuzzifying topology degrees and M-fuzzifying pretopology degrees are also introduced and their characterizations are given.

Definition 3.1. [11, 24, 25] Let $\mathcal{T} : L^X \longrightarrow M$ be a mapping. Then $\mathrm{Top}(\mathcal{T})$ defined by

$$\mathrm{Top}(\mathcal{T}) = \mathcal{T}(\perp_{L^X}) \wedge \mathcal{T}(\top_{L^X}) \wedge \left[\bigwedge_{A_1, A_2 \in L^X} (\mathcal{T}(A_1) \wedge \mathcal{T}(A_2)) \to \mathcal{T}(A_1 \wedge A_2)\right] \wedge \left[\bigwedge_{\{A_j\}_{j\in J} \subseteq L^X} (\bigwedge_{j \in J} \mathcal{T}(A_j)) \to \mathcal{T}(\bigvee_{j\in J} A_j)\right]$$

is called *the degree to which* $\mathcal{T}$ *is an* (L, M)*-fuzzy topology (or the* (L, M)*-fuzzy topology degree of* $\mathcal{T}$*)*.

Remark 3.2. (1) If $\mathrm{Top}(\mathcal{T}) = \top_M$, then $\mathcal{T}(\perp_{L^X}) = \mathcal{T}(\top_{L^X}) = \top_M$, $\mathcal{T}(A_1) \wedge \mathcal{T}(A_2) \le \mathcal{T}(A_1 \wedge A_2)$ for all $A_1, A_2 \in L^X$ and $\bigwedge_{j\in J} \mathcal{T}(A_j) \le \mathcal{T}(\bigvee_{j\in J} A_j)$ for all $\{A_j\}_{j\in J} \subseteq L^X$. It is exactly the definition of (L, M)-fuzzy topology. Moreover, $\mathcal{T}$ is an (L, M)-fuzzy topology if and only if $\mathrm{Top}(\mathcal{T}) = \top_M$. However, there exist examples to show $\mathrm{Top}(\mathcal{T}) \ne \top_M$.
(2) Since one of the principal goals of this paper is to obtain the level-wise type characterizations of the (L, M)-fuzzy topology degrees, we need the completely distributive lattice instead of a general lattice structure used in [11].

Example 3.3. Let $M = [0, 1]$ and $\mathcal{T} : L^X \longrightarrow M$ be a mapping which satisfies $\mathcal{T}(A) = 0.5$ for all $A \in L^X$. By Definition 3.1, we have $\mathrm{Top}(\mathcal{T}) = 0.5$. This shows $\mathrm{Top}(\mathcal{T}) \ne 1$.

Definition 3.4. Let $\mathcal{T} : L^X \longrightarrow M$ be a mapping. Then $\mathrm{Ptop}(\mathcal{T})$ defined by

$$\mathrm{Ptop}(\mathcal{T}) = \mathcal{T}(\perp_{L^X}) \wedge \mathcal{T}(\top_{L^X}) \wedge \left[\bigwedge_{\{A_j\}_{j\in J} \subseteq L^X} (\bigwedge_{j \in J} \mathcal{T}(A_j)) \to \mathcal{T}(\bigvee_{j\in J} A_j)\right]$$

is called *the degree to which* $\mathcal{T}$ *is an* (L, M)*-fuzzy pretopology (or the* (L, M)*-fuzzy pretopology degree of* $\mathcal{T}$*)*.

Remark 3.5. If $L = \{0, 1\}$ in Definitions 3.1 and 3.4, then $\mathrm{Top}(\mathcal{T})$ (resp., $\mathrm{Ptop}(\mathcal{T})$) is called the M-fuzzifying topology degree (resp., the M-fuzzifying pretopology degree) of $\mathcal{T}$.

From properties of the implication operation in Lemma 2.1, the following lemma is straightforward.

Lemma 3.6. *Let $\mathcal{T} : L^X \longrightarrow M$ be a mapping. For any $a \in M$, $a \leq \mathrm{Top}(\mathcal{T})$ if and only if $a \leq \mathcal{T}(\perp_{L^X})$, $a \leq \mathcal{T}(\top_{L^X})$, $\mathcal{T}(A_1) \wedge \mathcal{T}(A_2) \wedge a \leq \mathcal{T}(A_1 \wedge A_2)$ for all $A_1, A_2 \in L^X$ and $(\bigwedge_{j\in J} \mathcal{T}(A_j)) \wedge a \leq \mathcal{T}(\bigvee_{j\in J} A_j)$ for all $\{A_j\}_{j\in J} \subseteq L^X$.*

By Lemma 3.6, we can easily obtain the following theorem.

Theorem 3.7. *Let $\mathcal{T} : L^X \longrightarrow M$ be a mapping. Then*

$$\mathrm{Top}(\mathcal{T}) = \bigvee \left\{ a \in M \mid \begin{array}{c} a \leq \mathcal{T}(\perp_{L^X}), a \leq \mathcal{T}(\top_{L^X}), \\ \mathcal{T}(A_1) \wedge \mathcal{T}(A_2) \wedge a \leq \mathcal{T}(A_1 \wedge A_2), \forall A_1, A_2 \in L^X, \\ (\bigwedge_{j\in J} \mathcal{T}(A_j)) \wedge a \leq \mathcal{T}(\bigvee_{j\in J} A_j), \forall \{A_j\}_{j\in J} \subseteq L^X \end{array} \right\}.$$

The next two theorems give other characterizations of (L, M)-fuzzy topology degrees by means of its two kinds of cut sets.

Theorem 3.8. *Let $\mathcal{T} : L^X \longrightarrow M$ be a mapping. Then*

$$\mathrm{Top}(\mathcal{T}) = \bigvee \left\{ a \in M \mid \forall b \leq a, \mathcal{T}_{[b]} \textit{ is an } L\textit{-topology} \right\}.$$

Proof. Let RHS denote the right side of equality.

Suppose that $a \leq \mathcal{T}(\perp_{L^X}), a \leq \mathcal{T}(\top_{L^X})$, $\mathcal{T}(A_1) \wedge \mathcal{T}(A_2) \wedge a \leq \mathcal{T}(A_1 \wedge A_2)$, $(\bigwedge_{j\in J} \mathcal{T}(A_j)) \wedge a \leq \mathcal{T}(\bigvee_{j\in J} A_j)$ for all $A_1, A_2 \in L^X$ and for all $\{A_j\}_{j\in J} \subseteq L^X$. For any $b \leq a$, $A_1, A_2 \in \mathcal{T}_{[b]}$ and $\{A_j\}_{j\in J} \subseteq \mathcal{T}_{[b]}$, we have $\mathcal{T}(\perp_{L^X}) \geq b$, $\mathcal{T}(\top_{L^X}) \geq b$, $\mathcal{T}(A_1 \wedge A_2) \geq \mathcal{T}(A_1) \wedge \mathcal{T}(A_2) \wedge a \geq b$ and $\mathcal{T}(\bigvee_{j\in J} A_j) \geq (\bigwedge_{j\in J} \mathcal{T}(A_j)) \wedge a \geq b$. This shows $\perp_{L^X}, \top_{L^X} \in \mathcal{T}_{[b]}$, $A_1 \wedge A_2 \in \mathcal{T}_{[b]}$ and $\bigvee_{j\in J} A_j \in \mathcal{T}_{[b]}$. Combing this with Theorem 3.7, we know $\mathrm{Top}(\mathcal{T}) \leq RHS$.

Conversely, assume that $\mathcal{T}_{[b]}$ is an L-topology for any $b \leq a$. Let $b = a$. Then $\perp_{L^X}, \top_{L^X} \in \mathcal{T}_{[a]}$, which means $\mathcal{T}(\perp_{L^X}) \geq a$, $\mathcal{T}(\top_{L^X}) \geq a$. For any $A_1, A_2 \in L^X$, let $b = \mathcal{T}(A_1) \wedge \mathcal{T}(A_2) \wedge a$. Then $b \leq a$ and $A_1, A_2 \in \mathcal{T}_{[b]}$. Thus $A_1 \wedge A_2 \in \mathcal{T}_{[b]}$, i.e., $\mathcal{T}(A_1 \wedge A_2) \geq \mathcal{T}(A_1) \wedge \mathcal{T}(A_2) \wedge a$. For any $\{A_j\}_{j\in J} \subseteq L^X$, let $b = (\bigwedge_{j\in J} \mathcal{T}(A_j)) \wedge a$. Then $b \leq a$ and $\{A_j\}_{j\in J} \subseteq \mathcal{T}_{[b]}$. Thus $\bigvee_{j\in J} A_j \in \mathcal{T}_{[b]}$, i.e., $\mathcal{T}(\bigvee_{j\in J} A_j) \geq (\bigwedge_{j\in J} \mathcal{T}(A_j)) \wedge a$. Combine this with Theorem 3.7, we get $\mathrm{Top}(\mathcal{T}) \geq RHS$. □

Theorem 3.9. *Let $\mathcal{T} : L^X \longrightarrow M$ be a mapping. Then*

$$\mathrm{Top}(\mathcal{T}) = \bigvee \left\{ a \in M \mid \forall b \notin \alpha(a), \mathcal{T}^{[b]} \textit{ is an } L\textit{-topology} \right\}.$$

Proof. Let RHS denote the right side of equality.

Suppose that $a \leq \mathcal{T}(\perp_{L^X}), a \leq \mathcal{T}(\top_{L^X})$, $\mathcal{T}(A_1) \wedge \mathcal{T}(A_2) \wedge a \leq \mathcal{T}(A_1 \wedge A_2)$, $(\bigwedge_{j\in J} \mathcal{T}(A_j)) \wedge a \leq \mathcal{T}(\bigvee_{j\in J} A_j)$ for all $A_1, A_2 \in L^X$ and for all $\{A_j\}_{j\in J} \subseteq L^X$. For any $b \notin \alpha(a)$, $A_1, A_2 \in \mathcal{T}^{[b]}$ and $\{A_j\}_{j\in J} \subseteq \mathcal{T}^{[b]}$, we have $b \notin \alpha(a) \cup \alpha(\mathcal{T}(A_1)) \cup \alpha(\mathcal{T}(A_2))$ and $b \notin \alpha(a) \cup (\bigcup_{j\in J} \alpha(\mathcal{T}(A_j)))$. Note that $\alpha(a) \cup \alpha(\mathcal{T}(A_1)) \cup \alpha(\mathcal{T}(A_2)) = \alpha(a \wedge \mathcal{T}(A_1) \wedge \mathcal{T}(A_2)) \supseteq \alpha(\mathcal{T}(A_1 \wedge A_2))$ and $\alpha(a) \cup (\bigcup_{j\in J} \alpha(\mathcal{T}(A_j))) = \alpha(a \wedge (\bigwedge_{j\in J} \mathcal{T}(A_j))) \supseteq \alpha(\mathcal{T}(\bigvee_{j\in J} A_j))$. Then $b \notin \alpha(\mathcal{T}(A_1 \wedge A_2))$ and $b \notin \alpha(\mathcal{T}(\bigvee_{j\in J} A_j))$. This shows $A_1 \wedge A_2 \in \mathcal{T}^{[b]}$ and $\bigvee_{j\in J} A_j \in \mathcal{T}^{[b]}$. Since $\alpha(a) \supseteq \alpha(\mathcal{T}(\perp_{L^X}))$ and $\alpha(a) \supseteq \alpha(\mathcal{T}(\top_{L^X}))$, we know $b \notin \alpha(\mathcal{T}(\perp_{L^X}))$ and $b \notin \alpha(\mathcal{T}(\top_{L^X}))$, which shows $\perp_{L^X}, \top_{L^X} \in \mathcal{T}^{[b]}$. Combing this with Theorem 3.7, we know $\mathrm{Top}(\mathcal{T}) \leq RHS$.

Conversely, assume that $\mathcal{T}^{[b]}$ is an L-topology for any $b \notin \alpha(a)$. Take any $b \notin \alpha(a)$, we have $\perp_{L^X}, \top_{L^X} \in \mathcal{T}^{[b]}$, i.e., $b \notin \alpha(\mathcal{T}(\perp_{L^X})), b \notin \alpha(\mathcal{T}(\top_{L^X}))$. By the arbitrariness of b, we obtain $\alpha(\mathcal{T}(\perp_{L^X})) \subseteq \alpha(a)$ and $\alpha(\mathcal{T}(\top_{L^X})) \subseteq \alpha(a)$, i.e., $a \leq \mathcal{T}(\perp_{L^X})$ and $a \leq \mathcal{T}(\top_{L^X})$. For any $A_1, A_2 \in L^X$, take any $b \notin \alpha(\mathcal{T}(A_1) \wedge \mathcal{T}(A_2) \wedge a)$. By $\alpha(\mathcal{T}(A_1) \wedge \mathcal{T}(A_2) \wedge a) = \alpha(\mathcal{T}(A_1)) \cup \alpha(\mathcal{T}(A_2)) \cup \alpha(a)$, we know $b \notin \alpha(a)$, $b \notin \alpha(\mathcal{T}(A_1))$, $b \notin \alpha(\mathcal{T}(A_2))$, which means $A_1 \in \mathcal{T}^{[b]}$ and $A_2 \in \mathcal{T}^{[b]}$. Since $\mathcal{T}^{[b]}$ is an L-topology, we have $A_1 \wedge A_2 \in \mathcal{T}^{[b]}$, i.e., $b \notin \alpha(\mathcal{T}(A_1 \wedge A_2))$. By the arbitrariness of b, we obtain $\alpha(\mathcal{T}(A_1 \wedge A_2)) \subseteq \alpha(\mathcal{T}(A_1) \wedge \mathcal{T}(A_2) \wedge a)$, i.e., $\mathcal{T}(A_1) \wedge \mathcal{T}(A_2) \wedge a \leq \mathcal{T}(A_1 \wedge A_2)$. Similarly, we can prove $(\bigwedge_{j \in J} \mathcal{T}(A_j)) \wedge a \leq \mathcal{T}(\bigvee_{j \in J} A_j)$. Combine this with Theorem 3.7, we get $\mathrm{Top}(\mathcal{T}) \geq RHS$. □

Analogously, characterizations of (L, M)-fuzzy pretopology degrees, M-fuzzifying topology degrees and M-fuzzifying pretopology degrees are given in the following theorem and corollaries.

Theorem 3.10. *Let $\mathcal{T} : L^X \longrightarrow M$ be a mapping. Then*

$$\text{(1) } \mathrm{Ptop}(\mathcal{T}) = \bigvee \left\{ a \in M \mid \begin{array}{c} a \leq \mathcal{T}(\perp_{L^X}), a \leq \mathcal{T}(\top_{L^X}), \\ (\bigwedge_{j \in J} \mathcal{T}(A_j)) \wedge a \leq \mathcal{T}(\bigvee_{j \in J} A_j), \forall \{A_j\}_{j \in J} \subseteq L^X \end{array} \right\}.$$

$$\text{(2) } \mathrm{Ptop}(\mathcal{T}) = \bigvee \left\{ a \in M \mid \forall b \leq a, \mathcal{T}_{[b]} \textit{ is an } L\textit{-pretopology} \right\}.$$

$$\text{(3) } \mathrm{Ptop}(\mathcal{T}) = \bigvee \left\{ a \in M \mid \forall b \notin \alpha(a), \mathcal{T}^{[b]} \textit{ is an } L\textit{-pretopology} \right\}.$$

Corollary 3.11. *Let $\mathcal{T} : 2^X \longrightarrow M$ be a mapping. Then*

$$\text{(1) } \mathrm{Top}(\mathcal{T}) = \bigvee \left\{ a \in M \mid \begin{array}{l} a \leq \mathcal{T}(\emptyset), a \leq \mathcal{T}(X), \\ \mathcal{T}(A_1) \wedge \mathcal{T}(A_2) \wedge a \leq \mathcal{T}(A_1 \cap A_2)\ \forall A_1, A_2 \in 2^X \\ (\bigwedge_{j \in J} \mathcal{T}(A_j)) \wedge a \leq \mathcal{T}(\bigcup_{j \in J} A_j)\ \forall \{A_j\}_{j \in J} \subseteq 2^X \end{array} \right\}.$$

$$\text{(2) } \mathrm{Top}(\mathcal{T}) = \bigvee \left\{ a \in M \mid \forall b \leq a, \mathcal{T}_{[b]} \textit{ is a topology} \right\}.$$

$$\text{(3) } \mathrm{Top}(\mathcal{T}) = \bigvee \left\{ a \in M \mid \forall b \notin \alpha(a), \mathcal{T}^{[b]} \textit{ is a topology} \right\}.$$

Corollary 3.12. *Let $\mathcal{T} : 2^X \longrightarrow M$ be a mapping. Then*

$$\text{(1) } \mathrm{Ptop}(\mathcal{T}) = \bigvee \left\{ a \in M \mid \begin{array}{c} a \leq \mathcal{T}(\emptyset), a \leq \mathcal{T}(X), \\ (\bigwedge_{j \in J} \mathcal{T}(A_i)) \wedge a \leq \mathcal{T}(\bigcup_{j \in J} A_j), \forall \{A_j\}_{j \in J} \subseteq 2^X \end{array} \right\}.$$

$$\text{(2) } \mathrm{Ptop}(\mathcal{T}) = \bigvee \left\{ a \in M \mid \forall b \leq a, \mathcal{T}_{[b]} \textit{ is a pretopology} \right\}.$$

$$\text{(3) } \mathrm{Ptop}(\mathcal{T}) = \bigvee \left\{ a \in M \mid \forall b \notin \alpha(a), \mathcal{T}^{[b]} \textit{ is a pretopology} \right\}.$$

$\mathrm{Top}(\mathcal{T})$ can also be treated as a mapping $\mathrm{Top} : M^{L^X} \longrightarrow M$ defined by $\mathcal{T} \mapsto \mathrm{Top}(\mathcal{T})$. Then we have the following theorem.

Theorem 3.13. *Let* $\{\mathcal{T}_\lambda \,|\, \mathcal{T}_\lambda : L^X \longrightarrow M\}_{\lambda\in\Lambda}$ *be a family of mappings. Then* $\bigwedge_{\lambda\in\Lambda} \mathrm{Top}(\mathcal{T}_\lambda) \le \mathrm{Top}\left(\bigwedge_{\lambda\in\Lambda} \mathcal{T}_\lambda\right)$.

Proof. By Lemma 2.1, we have

$$
\begin{aligned}
\mathrm{Top}(\bigwedge_{\lambda\in\Lambda} \mathcal{T}_\lambda) = {} & \bigwedge_{\lambda\in\Lambda} \mathcal{T}_\lambda(\bot_{L^X}) \wedge \bigwedge_{\lambda\in\Lambda} \mathcal{T}_\lambda(\top_{L^X}) \wedge \\
& \left[\bigwedge_{\{A_j\}_{j\in J}\subseteq L^X} (\bigwedge_{j\in J}\bigwedge_{\lambda\in\Lambda} \mathcal{T}_\lambda(A_j)) \to \bigwedge_{\lambda\in\Lambda} \mathcal{T}_\lambda(\bigvee_{j\in J} A_j)\right] \wedge \\
& \left[\bigwedge_{A_1,A_2\in L^X} (\bigwedge_{\lambda\in\Lambda} \mathcal{T}_\lambda(A_1) \wedge \bigwedge_{\lambda\in\Lambda} \mathcal{T}_\lambda(A_2)) \to \bigwedge_{\lambda\in\Lambda} \mathcal{T}_\lambda(A_1\wedge A_2)\right] \\
= {} & \bigwedge_{\lambda\in\Lambda} \mathcal{T}_\lambda(\bot_{L^X}) \wedge \bigwedge_{\lambda\in\Lambda} \mathscr{T}_\lambda(\top_{L^X}) \wedge \\
& \left[\bigwedge_{\{A_j\}_{j\in J}\subseteq L^X} \bigwedge_{\lambda\in\Lambda} (\bigwedge_{j\in J}\bigwedge_{\lambda\in\Lambda} \mathcal{T}_\lambda(A_j)) \to \mathcal{T}_\lambda(\bigvee_{j\in J} A_j)\right] \wedge \\
& \left[\bigwedge_{A_1,A_2\in L^X} \bigwedge_{\lambda\in\Lambda} (\bigwedge_{\lambda\in\Lambda} \mathcal{T}_\lambda(A_1) \wedge \bigwedge_{\lambda\in\Lambda} \mathcal{T}_\lambda(A_2)) \to \mathcal{T}_\lambda(A_1\wedge A_2)\right] \\
\ge {} & \bigwedge_{\lambda\in\Lambda} \mathcal{T}_\lambda(\bot_{L^X}) \wedge \bigwedge_{\lambda\in\Lambda} \mathcal{T}_\lambda(\top_{L^X}) \wedge \\
& \left[\bigwedge_{\lambda\in\Lambda} \bigwedge_{\{A_j\}_{j\in J}\subseteq L^X} (\bigwedge_{j\in J} \mathcal{T}_\lambda(A_j)) \to \mathcal{T}_\lambda(\bigvee_{j\in J} A_j)\right] \wedge \\
& \left[\bigwedge_{\lambda\in\Lambda} \bigwedge_{A_1,A_2\in L^X} (\mathcal{T}_\lambda(A_1) \wedge \mathcal{T}_\lambda(A_2)) \to \mathcal{T}_\lambda(A_1\wedge A_2))\right] \\
= {} & \bigwedge_{\lambda\in\Lambda} \mathrm{Top}(\mathcal{T}_\lambda).
\end{aligned}
$$

□

4. Degrees of Continuity, Openness and Closedness of Mappings and Their Characterizations

In this section, the degree to which an L-subset is an L-open set with respect to a mapping $\mathcal{T} : L^X \longrightarrow M$ is proposed. Based on this, we define the degrees to which a mapping $f : (X, \mathcal{T}_X) \longrightarrow (Y, \mathcal{T}_Y)$ is continuous, open or closed with respect to $\mathcal{T}_X$ and $\mathcal{T}_Y$. Then we give their characterizations and discuss some properties about them.

Definition 4.1. Given a mapping $\mathcal{T} : L^X \longrightarrow M$ and let $\mathrm{Top}(\mathcal{T})$ denote the (L,M)-fuzzy topology degree of $\mathcal{T}$. For each $A \in L^X$, $\mathcal{O}_\mathcal{T}(A)$ defined by

$$\mathcal{O}_\mathcal{T}(A) = \mathrm{Top}(\mathcal{T}) \wedge \mathcal{T}(A)$$

is called *the degree to which A is an open L-set with respect to $\mathcal{T}$ (or the open L-set degree of A with respect to $\mathcal{T}$).*

Remark 4.2. If $\mathrm{Top}(\mathcal{T}) = \top_M$, which means $\mathcal{T}$ is an (L, M)-fuzzy topology, then $\mathcal{O}_{\mathcal{T}}(A) = \mathcal{T}(A)$, which can be regarded as a generalization of $\mathcal{T}(A)$.

Proposition 4.3. *Given a mapping $\mathcal{T} : L^X \longrightarrow M$. Then*
(1) $\mathcal{O}_{\mathcal{T}}(A_1) \wedge \mathcal{O}_{\mathcal{T}}(A_2) \leq \mathcal{O}_{\mathcal{T}}(A_1 \wedge A_2)$, $\forall A_1, A_2 \in L^X$.
(2) $\bigwedge_{j\in J} \mathcal{O}_{\mathcal{T}}(A_j) \leq \mathcal{O}_{\mathcal{T}}(\bigvee_{j\in J} A_j)$, $\forall \{A_j \mid j \in J\} \subseteq L^X$.

If $\mathcal{O}_{\mathcal{T}}(A)$ is regarded as a mapping $\mathcal{O}_{\mathcal{T}} : L^X \longrightarrow M$ defined by $A \mapsto \mathcal{O}_{\mathcal{T}}(A)$, then $\mathcal{O}_{\mathcal{T}}$ satisfies the conditions (LMT2) and (LMT3) in Definition 2.2.

Proof. (1) By Definition 4.1, it suffices to show that $\mathrm{Top}(\mathcal{T}) \wedge \mathcal{T}(A_1) \wedge \mathcal{T}(A_2) \leq \mathrm{Top}(\mathcal{T})$ and $\mathrm{Top}(\mathcal{T}) \wedge \mathcal{T}(A_1) \wedge \mathcal{T}(A_2) \leq \mathcal{T}(A_1 \wedge A_2)$. By Definition 3.1 and Lemma 2.1, we have $\mathrm{Top}(\mathcal{T}) \wedge \mathcal{T}(A_1) \wedge \mathcal{T}(A_2) \leq \big((\mathcal{T}(A_1) \wedge \mathcal{T}(A_2)) \to \mathcal{T}(A_1 \wedge A_2)\big) \wedge \mathcal{T}(A_1) \wedge \mathcal{T}(A_2) \leq \mathcal{T}(A_1 \wedge A_2)$.
(2) is similar with (1). □

In what follows, characterizations of L-open set degrees will be given. The proofs are similar to those of Theorems 3.7-3.9 and omitted here.

Theorem 4.4. *Given a mapping $\mathcal{T} : L^X \longrightarrow M$. Then*

$$(1)\ \mathcal{O}_{\mathcal{T}}(A) = \bigvee \left\{ a \in M \mid \begin{array}{c} a \leq \mathcal{T}(A), a \leq \mathcal{T}(\perp_{L^X}), a \leq \mathcal{T}(\top_{L^X}), \\ \mathcal{T}(A_1) \wedge \mathcal{T}(A_2) \wedge a \leq \mathcal{T}(A_1 \wedge A_2)\ \forall A_1, A_2 \in L^X \\ (\bigwedge\limits_{j\in J} \mathcal{T}(A_j)) \wedge a \leq \mathcal{T}(\bigvee\limits_{j\in J} A_j)\ \forall \{A_j\}_{j\in J} \subseteq L^X \end{array} \right\}.$$

$$(2)\ \mathcal{O}_{\mathcal{T}}(A) = \bigvee \left\{ a \in M \mid \forall b \leq a, \mathcal{T}_{[b]} \textit{ is an } L\textit{-topology and } A \in \mathcal{T}_{[b]} \right\}.$$

$$(3)\ \mathcal{O}_{\mathcal{T}}(A) = \bigvee \left\{ a \in M \mid \forall b \notin \alpha(a), \mathcal{T}^{[b]} \textit{ is an } L\textit{-topology and } A \in \mathcal{T}^{[b]} \right\}.$$

In [11], T. Kubiak and A.P. Šostak gave a fuzzification of the continuity degree of mappings from the view of fuzzy category. However they did not consider other problems about it. Now, we shall give a new definition of the degree to which a mapping $f : (X, \mathcal{T}_X) \longrightarrow (Y, \mathcal{T}_Y)$ is continuous, open or closed with respect to the (L, M)-fuzzy topology degrees of $\mathcal{T}_X$ and $\mathcal{T}_Y$, which is different from that in [11]. Besides, we will discuss some characterizations and properties about them in detail.

In the following contents, we suppose that $(L, \vee, \wedge, ')$ is a completely distributive lattice with an order-reversing involution $'$ (i.e. a completely distributive De Morgan algebra). L^X is also a completely distributive De Morgan algebra when it inherits the structure of L in a natural way, by defining $\vee, \wedge, \leq, '$ pointwisely. For all $A \in L^X$, A' denotes the complementary set of A.

Definition 4.5. Given mappings $\mathcal{T}_X : L^X \longrightarrow M$ and $\mathcal{T}_Y : L^Y \longrightarrow M$. Let $f : X \longrightarrow Y$ be a mapping. Then

a) *the degree to which f is a continuous mapping with respect to $\mathcal{T}_X$ and $\mathcal{T}_Y$ (or the continuity degree of f with respect to $\mathcal{T}_X$ and $\mathcal{T}_Y$)* is defined by

$$\mathrm{Cont}(f) = \bigwedge_{B \in L^Y} \Big(\mathcal{O}_{\mathcal{T}_Y}(B) \to \mathcal{O}_{\mathcal{T}_X}(f_L^{\leftarrow}(B)) \Big).$$

b) *the degree to which f is an open mapping with respect to $\mathcal{T}_X$ and $\mathcal{T}_Y$ (or the openness degree of f with respect to $\mathcal{T}_X$ and $\mathcal{T}_Y$)* is defined by

$$\mathrm{Open}(f) = \bigwedge_{A \in L^X} \Big(\mathcal{O}_{\mathcal{T}_X}(A) \to \mathcal{O}_{\mathcal{T}_Y}(f_L^{\rightarrow}(A)) \Big).$$

c) *the degree to which f is a closed mapping with respect to $\mathcal{T}_X$ and $\mathcal{T}_Y$ (or the closedness degree of f with respect to $\mathcal{T}_X$ and $\mathcal{T}_Y$)* is defined by

$$\mathrm{Clos}(f) = \bigwedge_{A \in L^X} \Big(\mathcal{O}_{\mathcal{T}_X}(A') \to \mathcal{O}_{\mathcal{T}_Y}(f_L^{\rightarrow}(A)') \Big).$$

Remark 4.6. The continuity degree, the openness degree and the closedness degree of f with respect to $\mathcal{T}_X$ and $\mathcal{T}_Y$ defined in Definition 4.5 can be viewed as generalizations of them defined in [12, 16].

Characterizations of the continuity degrees of f with respect to $\mathcal{T}_X$ and $\mathcal{T}_Y$ will be given in the following theorem.

Theorem 4.7. *Given mappings $\mathcal{T}_X : L^X \longrightarrow M$ and $\mathcal{T}_Y : L^Y \longrightarrow M$. Let $f : X \longrightarrow Y$ be a mapping. Then*

$$(1)\ \mathrm{Cont}(f) = \bigvee \left\{ a \in M \mid \begin{array}{l} \mathrm{Top}(\mathcal{T}_Y) \wedge \mathcal{T}_Y(B) \wedge a \leq \mathrm{Top}(\mathcal{T}_X) \wedge \mathcal{T}_X(f_L^{\leftarrow}(B)), \\ \forall B \in L^Y \end{array} \right\}.$$

$$(2)\ \mathrm{Cont}(f) = \bigvee \left\{ a \in M \mid \begin{array}{l} \forall b \leq \mathrm{Top}(\mathcal{T}_Y) \wedge a, \forall B \in (\mathcal{T}_Y)_{[b]}, \\ b \leq \mathrm{Top}(\mathcal{T}_X), f_L^{\leftarrow}(B) \in (\mathcal{T}_X)_{[b]} \end{array} \right\}.$$

$$(3)\ \mathrm{Cont}(f) = \bigvee \left\{ a \in M \mid \begin{array}{l} \forall b \notin \alpha(\mathrm{Top}(\mathcal{T}_Y) \wedge a), \forall B \in (\mathcal{T}_Y)^{[b]}, \\ b \notin \alpha(\mathrm{Top}(\mathcal{T}_X)), f_L^{\leftarrow}(B) \in (\mathcal{T}_X)^{[b]} \end{array} \right\}.$$

Proof. (1) For any $a \in M$, $a \leq \mathrm{Cont}(f)$ if and only if $a \wedge \mathcal{O}_{\mathcal{T}_Y}(B) \leq \mathcal{O}_{\mathcal{T}_X}(f_L^{\leftarrow}(B))$ for all $B \in L^Y$, if and only if $\mathrm{Top}(\mathcal{T}_Y) \wedge \mathcal{T}_Y(B) \wedge a \leq \mathrm{Top}(\mathcal{T}_X) \wedge \mathcal{T}_X(f_L^{\leftarrow}(B))$ for all $B \in L^Y$.

(2) Let RHS denote the right side of equality. Suppose that $\mathrm{Top}(\mathcal{T}_Y) \wedge \mathcal{T}_Y(B) \wedge a \leq \mathrm{Top}(\mathcal{T}_X) \wedge \mathcal{T}_X(f_L^{\leftarrow}(B))$ for all $B \in L^Y$. For any $b \leq \mathrm{Top}(\mathcal{T}_Y) \wedge a$ and $B \in (\mathcal{T}_Y)_{[b]}$, i.e., $b \leq \mathcal{T}_Y(B)$, we have $b \leq \mathrm{Top}(\mathcal{T}_X) \wedge \mathcal{T}_X(f_L^{\leftarrow}(B))$. Thus $b \leq \mathrm{Top}(\mathcal{T}_X)$ and $b \leq \mathcal{T}_X(f_L^{\leftarrow}(B))$, i.e., $f_L^{\leftarrow}(B) \in (\mathcal{T}_X)_{[b]}$. By the result of (1), we know $\mathrm{Cont}(f) \leq RHS$.

Conversely, take any $b \leq \mathrm{Top}(\mathcal{T}_Y) \wedge \mathcal{T}_Y(B) \wedge a$. Then $b \leq \mathrm{Top}(\mathcal{T}_Y) \wedge a$ and $b \leq \mathcal{T}_Y(B)$, i.e., $B \in (\mathcal{T}_Y)_{[b]}$. Thus $\mathrm{Top}(\mathcal{T}_X) \geq b$ and $f_L^{\leftarrow}(B) \in (\mathcal{T}_X)_{[b]}$, i.e., $\mathcal{T}_X(f_L^{\leftarrow}(B)) \geq b$. This implies $\mathrm{Top}(\mathcal{T}_X) \wedge \mathcal{T}_X(f_L^{\leftarrow}(B)) \geq b$. By the arbitrariness of b, we obtain $\mathrm{Top}(\mathcal{T}_Y) \wedge \mathcal{T}_Y(B) \wedge a \leq \mathrm{Top}(\mathcal{T}_X) \wedge \mathcal{T}_X(f_L^{\leftarrow}(B))$. Combining this with (1), we get $\mathrm{Cont}(f) \geq RHS$.

(3) Let RHS denote the right side of equality. Suppose that $\mathrm{Top}(\mathcal{T}_Y)\wedge\mathcal{T}_Y(B)\wedge a\leq \mathrm{Top}(\mathcal{T}_X)\wedge\mathcal{T}_X(f_L^{\leftarrow}(B))$ for all $B\in L^Y$. For any $b\notin\alpha(\mathrm{Top}(\mathcal{T}_Y)\wedge a)$ and $B\in(\mathcal{T}_Y)^{[b]}$, i.e., $b\notin\alpha(\mathcal{T}_Y(B))$, we have $b\notin\alpha(\mathcal{T}_Y(B))\cup\alpha(\mathrm{Top}(\mathcal{T}_Y)\wedge a)$. Note that $\alpha(\mathcal{T}_Y(B))\cup\alpha(\mathrm{Top}(\mathcal{T}_Y)\wedge a)=\alpha(\mathrm{Top}(\mathcal{T}_Y)\wedge\mathcal{T}_Y(B)\wedge a)\supseteq\alpha(\mathrm{Top}(\mathcal{T}_X)\wedge\mathcal{T}_X(f_L^{\leftarrow}(B)))=\alpha(\mathrm{Top}(\mathcal{T}_X))\cup\alpha(\mathcal{T}_X(f_L^{\leftarrow}(B)))$. This implies $b\notin\alpha(\mathrm{Top}(\mathcal{T}_X))$ and $b\notin\alpha(\mathcal{T}_X(f_L^{\leftarrow}(B)))$, i.e., $f_L^{\leftarrow}(B)\in(\mathcal{T}_X)^{[b]}$. By the result of (1), we know $\mathrm{Cont}(f)\leq RHS$.

Conversely, in order to prove $\mathrm{Top}(\mathcal{T}_Y)\wedge\mathcal{T}_Y(B)\wedge a\leq\mathrm{Top}(\mathcal{T}_X)\wedge\mathcal{T}_X(f_L^{\leftarrow}(B))$, take any $b\notin\alpha(\mathrm{Top}(\mathcal{T}_Y)\wedge\mathcal{T}_Y(B)\wedge a)$. By $\alpha(\mathrm{Top}(\mathcal{T}_Y)\wedge\mathcal{T}_Y(B)\wedge a)=\alpha(\mathrm{Top}(\mathcal{T}_Y)\wedge a)\cup\alpha(\mathcal{T}_Y(B))$, we have $b\notin\alpha(\mathrm{Top}(\mathcal{T}_Y)\wedge a)$ and $b\notin\alpha(\mathcal{T}_Y(B))$, i.e., $B\in(\mathcal{T}_Y)^{[b]}$. Thus $b\notin\alpha(\mathrm{Top}(\mathcal{T}_X))$ and $f_L^{\leftarrow}(B)\in(\mathcal{T}_X)^{[b]}$, i.e., $b\notin\alpha(\mathcal{T}_X(f_L^{\leftarrow}(B)))$. This implies $b\notin\alpha(\mathrm{Top}(\mathcal{T}_X))\cup\alpha(\mathcal{T}_X(f_L^{\leftarrow}(B)))=\alpha\left(\mathrm{Top}(\mathcal{T}_X)\wedge\mathcal{T}_X(f_L^{\leftarrow}(B))\right)$. By the arbitrariness of b, we obtain $\mathrm{Top}(\mathcal{T}_Y)\wedge\mathcal{T}_Y(B)\wedge a\leq\mathrm{Top}(\mathcal{T}_X)\wedge\mathcal{T}_X(f_L^{\leftarrow}(B))$. Combining this with (1), we get $\mathrm{Cont}(f)\geq RHS$. □

In what follows, characterizations of the openness degrees and the closedness degrees of f with respect to $\mathcal{T}_X$ and $\mathcal{T}_Y$ will be presented. The proofs are similar to those of Theorem 4.7 and omitted here.

Theorem 4.8. *Given mappings $\mathcal{T}_X: L^X\longrightarrow M$ and $\mathcal{T}_Y: L^Y\longrightarrow M$. Let $f:X\longrightarrow Y$ be a mapping. Then*

$$(1)\ \mathrm{Open}(f)=\bigvee\left\{a\in M\mid \begin{array}{l}\mathrm{Top}(\mathcal{T}_X)\wedge\mathcal{T}_X(A)\wedge a\leq\mathrm{Top}(\mathcal{T}_Y)\wedge\mathcal{T}_Y(f_L^{\rightarrow}(A)),\\ \forall A\in L^X\end{array}\right\}.$$

$$(2)\ \mathrm{Open}(f)=\bigvee\left\{a\in M\mid \begin{array}{l}\forall b\leq\mathrm{Top}(\mathcal{T}_X)\wedge a,\forall A\in(\mathcal{T}_X)_{[b]},\\ b\leq\mathrm{Top}(\mathcal{T}_Y), f_L^{\rightarrow}(A)\in(\mathcal{T}_Y)_{[b]}\end{array}\right\}.$$

$$(3)\ \mathrm{Open}(f)=\bigvee\left\{a\in M\mid \begin{array}{l}\forall b\notin\alpha(\mathrm{Top}(\mathcal{T}_X)\wedge a),\forall A\in(\mathcal{T}_X)^{[b]},\\ b\notin\alpha(\mathrm{Top}(\mathcal{T}_Y)), f_L^{\rightarrow}(A)\in(\mathcal{T}_Y)^{[b]}\end{array}\right\}.$$

Theorem 4.9. *Given mappings $\mathcal{T}_X: L^X\longrightarrow M$ and $\mathcal{T}_Y: L^Y\longrightarrow M$. Let $f:X\longrightarrow Y$ be a mapping. Then*

$$(1)\ \mathrm{Clos}(f)=\bigvee\left\{a\in M\mid \begin{array}{l}\mathrm{Top}(\mathcal{T}_X)\wedge\mathcal{T}_X(A')\wedge a\leq\mathrm{Top}(\mathcal{T}_Y)\wedge\mathcal{T}_Y(f_L^{\rightarrow}(A)'),\\ \forall A\in L^X\end{array}\right\}.$$

$$(2)\ \mathrm{Clos}(f)=\bigvee\left\{a\in M\mid \begin{array}{l}\forall b\leq\mathrm{Top}(\mathcal{T}_X)\wedge a,\forall A'\in(\mathcal{T}_Y)_{[b]},\\ b\leq\mathrm{Top}(\mathcal{T}_Y), f_L^{\rightarrow}(A)'\in(\mathcal{T}_Y)_{[b]}\end{array}\right\}.$$

$$(3)\ \mathrm{Clos}(f)=\bigvee\left\{a\in M\mid \begin{array}{l}\forall b\notin\alpha(\mathrm{Top}(\mathcal{T}_X)\wedge a),\forall A'\in(\mathcal{T}_X)^{[b]},\\ b\notin\alpha(\mathrm{Top}(\mathcal{T}_Y)), f_L^{\rightarrow}(A)'\in(\mathcal{T}_Y)^{[b]}\end{array}\right\}.$$

Next, we shall discuss some properties of the continuity degrees, the openness degrees, the closedness degrees of f with respect to $\mathcal{T}_X$ and $\mathcal{T}_Y$.

Proposition 4.10. *Given a mapping $\mathcal{T}_X: L^X\longrightarrow M$. Let $id: X\longrightarrow X$ be the identity mapping. Then* $\mathrm{Cont}(id)=\mathrm{Open}(id)=\mathrm{Clos}(id)=\top_M$.

Proof. Obviously. □

Proposition 4.11. *Given mappings $\mathcal{T}_X: L^X\longrightarrow M$, $\mathcal{T}_Y: L^Y\longrightarrow M$ and $\mathcal{T}_Z: L^Z\longrightarrow M$. Let $f:X\longrightarrow Y$ and $g:Y\longrightarrow Z$ be mappings. Then*

(1) $\text{Cont}(f) \wedge \text{Cont}(g) \leq \text{Cont}(g \circ f)$.
(2) $\text{Open}(f) \wedge \text{Open}(g) \leq \text{Open}(g \circ f)$.
(3) $\text{Clos}(f) \wedge \text{Clos}(g) \leq \text{Clos}(g \circ f)$.

Proof. We only prove (1), (2) and (3) can be proved similarly. Since $(g \circ f)_L^{\leftarrow}(D) = f_L^{\leftarrow}(g_L^{\leftarrow}(D))$ for all $D \in L^Z$, by Definition 4.5 and Lemma 2.1 (6), we have

$$\begin{aligned}
&\text{Cont}(f) \wedge \text{Cont}(g)\\
&= \bigwedge_{B \in L^Y} \Big(\mathcal{O}_{\mathcal{T}_Y}(B) \to \mathcal{O}_{\mathcal{T}_X}(f_L^{\leftarrow}(B))\Big) \wedge \bigwedge_{C \in L^Z} \Big(\mathcal{O}_{\mathcal{T}_Z}(C) \to \mathcal{O}_{\mathcal{T}_Y}(g_L^{\leftarrow}(C))\Big)\\
&\leq \bigwedge_{D \in L^Z} \Big(\mathcal{O}_{\mathcal{T}_Y}(g_L^{\leftarrow}(D)) \to \mathcal{O}_{\mathcal{T}_X}(f_L^{\leftarrow}(g_L^{\leftarrow}(D)))\Big) \wedge \bigwedge_{C \in L^Z} \Big(\mathcal{O}_{\mathcal{T}_Z}(C) \to \mathcal{O}_{\mathcal{T}_Y}(g_L^{\leftarrow}(C))\Big)\\
&= \bigwedge_{D \in L^Z} \Big(\mathcal{O}_{\mathcal{T}_Y}(g_L^{\leftarrow}(D)) \to \mathcal{O}_{\mathcal{T}_X}((g \circ f)_L^{\leftarrow}(D))\Big) \wedge \bigwedge_{C \in L^Z} \Big(\mathcal{O}_{\mathcal{T}_Z}(C) \to \mathcal{O}_{\mathcal{T}_Y}(g_L^{\leftarrow}(C))\Big)\\
&= \bigwedge_{C \in L^Z} \Big((\mathcal{O}_{\mathcal{T}_Y}(g_L^{\leftarrow}(C)) \to \mathcal{O}_{\mathcal{T}_X}((g \circ f)_L^{\leftarrow}(C))) \wedge (\mathcal{O}_{\mathcal{T}_Z}(C) \to \mathcal{O}_{\mathcal{T}_Y}(g_L^{\leftarrow}(C)))\Big)\\
&\leq \bigwedge_{C \in L^Z} \Big(\mathcal{O}_{\mathcal{T}_Z}(C) \to \mathcal{O}_{\mathcal{T}_X}((g \circ f)_L^{\leftarrow}(C))\Big)\\
&= \text{Cont}(g \circ f).
\end{aligned}$$

□

Before discussing other properties about the continuity degrees, the openness degrees, the closedness degrees of f with respect to $\mathcal{T}_X$ and $\mathcal{T}_Y$, we need the following lemma.

Lemma 4.12. *Given mappings $\mathcal{T}_X : L^X \longrightarrow M$ and $\mathcal{T}_Y : L^Y \longrightarrow M$. Let $f : X \longrightarrow Y$ be a mapping. Then*

$$\text{Cont}(f) = \bigwedge_{D \in L^Y} \Big(\mathcal{O}_{\mathcal{T}_Y}(D') \to \mathcal{O}_{\mathcal{T}_X}(f_L^{\leftarrow}(D)')\Big).$$

Proof. Since $f_L^{\leftarrow}(D)' = f_L^{\leftarrow}(D')$ for all $D \in L^Y$, we have

$$\begin{aligned}
\text{Cont}(f) &= \bigwedge_{B \in L^Y} \Big(\mathcal{O}_{\mathcal{T}_Y}(B) \to \mathcal{O}_{\mathcal{T}_X}(f_L^{\leftarrow}(B))\Big)\\
&= \bigwedge_{D \in L^Y} \Big(\mathcal{O}_{\mathcal{T}_Y}(D') \to \mathcal{O}_{\mathcal{T}_X}(f_L^{\leftarrow}(D'))\Big)\\
&= \bigwedge_{D \in L^Y} \Big(\mathcal{O}_{\mathcal{T}_Y}(D') \to \mathcal{O}_{\mathcal{T}_X}(f_L^{\leftarrow}(D)')\Big).
\end{aligned}$$

The reason of the second equality is that D' replace B in the first equality and $D' \in L^Y$ if and only if $D \in L^Y$. □

Proposition 4.13. *Given mappings $\mathcal{T}_X : L^X \longrightarrow M$, $\mathcal{T}_Y : L^Y \longrightarrow M$ and $\mathcal{T}_Z : L^Z \longrightarrow M$. Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be mappings. If f is surjective, then*
(1) $\text{Open}(g \circ f) \wedge \text{Cont}(f) \leq \text{Open}(g)$.
(2) $\text{Clos}(g \circ f) \wedge \text{Cont}(f) \leq \text{Clos}(g)$.

Proof. (1) Since f is surjective, we know $f_L^{\rightarrow}(f_L^{\leftarrow}(D)) = D$ for all $D \in L^Y$. Then $(g \circ f)_L^{\rightarrow}(f_L^{\leftarrow}(D)) = g_L^{\rightarrow}(f_L^{\rightarrow}(f_L^{\leftarrow}(D))) = g_L^{\rightarrow}(D)$. By Lemma 2.1 (6), we have

$$
\begin{aligned}
&\mathrm{Open}(g\circ f)\wedge \mathrm{Cont}(f)\\
=&\bigwedge_{A\in L^X}\Big(\mathcal{O}_{\mathcal{T}_X}(A)\to \mathcal{O}_{\mathcal{T}_Z}((g\circ f)_L^{\rightarrow}(A))\Big)\wedge \bigwedge_{B\in L^Y}\Big(\mathcal{O}_{\mathcal{T}_Y}(B)\to \mathcal{O}_{\mathcal{T}_X}(f_L^{\leftarrow}(B))\Big)\\
\leq&\bigwedge_{D\in L^Y}\Big(\mathcal{O}_{\mathcal{T}_X}(f_L^{\leftarrow}(D))\to \mathcal{O}_{\mathcal{T}_Z}((g\circ f)_L^{\rightarrow}(f_L^{\leftarrow}(D)))\Big)\wedge \bigwedge_{B\in L^Y}\Big(\mathcal{O}_{\mathcal{T}_Y}(B)\to \mathcal{O}_{\mathcal{T}_X}(f_L^{\leftarrow}(B))\Big)\\
=&\bigwedge_{D\in L^Y}\Big(\mathcal{O}_{\mathcal{T}_X}(f_L^{\leftarrow}(D))\to \mathcal{O}_{\mathcal{T}_Z}(g_L^{\rightarrow}(D))\Big)\wedge \bigwedge_{B\in L^Y}\Big(\mathcal{O}_{\mathcal{T}_Y}(B)\to \mathcal{O}_{\mathcal{T}_X}(f_L^{\leftarrow}(B))\Big)\\
=&\bigwedge_{B\in L^Y}\Big((\mathcal{O}_{\mathcal{T}_X}(f_L^{\leftarrow}(B))\to \mathcal{O}_{\mathcal{T}_Z}(g_L^{\rightarrow}(B)))\wedge(\mathcal{O}_{\mathcal{T}_Y}(B)\to \mathcal{O}_{\mathcal{T}_X}(f_L^{\leftarrow}(B)))\Big)\\
\leq&\bigwedge_{B\in L^Y}\Big(\mathcal{O}_{\mathcal{T}_Y}(B)\to \mathcal{O}_{\mathcal{T}_Z}(g_L^{\rightarrow}(B))\Big)=\mathrm{Open}(g).
\end{aligned}
$$

(2) By Lemmas 4.12 and 2.1 (6), we have

$$
\begin{aligned}
&\mathrm{Clos}(g\circ f)\wedge \mathrm{Cont}(f)\\
=&\bigwedge_{A\in L^X}\Big(\mathcal{O}_{\mathcal{T}_X}(A')\to \mathcal{O}_{\mathcal{T}_Z}((g\circ f)_L^{\rightarrow}(A)')\Big)\wedge \bigwedge_{D\in L^Y}\Big(\mathcal{O}_{\mathcal{T}_Y}(D')\to \mathcal{O}_{\mathcal{T}_X}(f_L^{\leftarrow}(D)')\Big)\\
\leq&\bigwedge_{D\in L^Y}\Big(\mathcal{O}_{\mathcal{T}_X}(f_L^{\leftarrow}(D)')\to \mathcal{O}_{\mathcal{T}_Z}((g\circ f)_L^{\rightarrow}(f_L^{\leftarrow}(D))')\Big)\wedge\\
&\bigwedge_{D\in L^Y}\Big(\mathcal{O}_{\mathcal{T}_Y}(D')\to \mathcal{O}_{\mathcal{T}_X}(f_L^{\leftarrow}(D)')\Big)\\
=&\bigwedge_{D\in L^Y}\Big((\mathcal{O}_{\mathcal{T}_X}(f_L^{\leftarrow}(D)')\to \mathcal{O}_{\mathcal{T}_Z}(g_L^{\rightarrow}(D)'))\wedge(\mathcal{O}_{\mathcal{T}_Y}(D')\to \mathcal{O}_{\mathcal{T}_X}(f_L^{\leftarrow}(D)'))\Big)\\
\leq&\bigwedge_{D\in L^Y}\Big(\mathcal{O}_{\mathcal{T}_Y}(D')\to \mathcal{O}_{\mathcal{T}_Z}(g_L^{\rightarrow}(D)')\Big)=\mathrm{Clos}(g).
\end{aligned}
$$

□

Proposition 4.14. *Given mappings $\mathcal{T}_X : L^X \longrightarrow M$, $\mathcal{T}_Y : L^Y \longrightarrow M$ and $\mathcal{T}_Z : L^Z \longrightarrow M$. Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be mappings. If g is injective, then*
(1) $\mathrm{Open}(g\circ f)\wedge \mathrm{Cont}(g) \leq \mathrm{Open}(f)$.
(2) $\mathrm{Clos}(g\circ f)\wedge \mathrm{Cont}(g) \leq \mathrm{Clos}(f)$.

Proof. (1) Since g is injective, we have $g_L^{\leftarrow}(g_L^{\rightarrow}(B)) = B$ for all $B \in L^Y$. Then $g_L^{\leftarrow}((g\circ f)_L^{\rightarrow}(D)) = f_L^{\rightarrow}(D)$ for all $D \in L^X$. By Lemma 2.1 (6), we have

$$
\begin{aligned}
&\mathrm{Open}(g\circ f)\wedge \mathrm{Cont}(g)\\
=&\bigwedge_{A\in L^X}\Big(\mathcal{O}_{\mathcal{T}_X}(A)\to \mathcal{O}_{\mathcal{T}_Z}((g\circ f)_L^{\rightarrow}(A))\Big)\wedge \bigwedge_{C\in L^Z}\Big(\mathcal{O}_{\mathcal{T}_Z}(C)\to \mathcal{O}_{\mathcal{T}_Y}(g_L^{\leftarrow}(C))\Big)\\
\leq&\bigwedge_{A\in L^X}\Big(\mathcal{O}_{\mathcal{T}_X}(A)\to \mathcal{O}_{\mathcal{T}_Z}((g\circ f)_L^{\rightarrow}(A))\Big)\wedge\\
&\bigwedge_{D\in L^X}\Big(\mathcal{O}_{\mathcal{T}_Z}((g\circ f)_L^{\rightarrow}(D))\to \mathcal{O}_{\mathcal{T}_Y}(g_L^{\leftarrow}((g\circ f)_L^{\rightarrow}(D)))\Big)\\
=&\bigwedge_{A\in L^X}\Big(\mathcal{O}_{\mathcal{T}_X}(A)\to \mathcal{O}_{\mathcal{T}_Z}(g_L^{\rightarrow}(f_L^{\rightarrow}(A))\Big)\wedge\\
&\bigwedge_{D\in L^X}\Big(\mathcal{O}_{\mathcal{T}_Z}((g_L^{\rightarrow}(f_L^{\rightarrow}(D))\to \mathcal{O}_{\mathcal{T}_Y}(f_L^{\rightarrow}(D))\Big)\\
=&\bigwedge_{A\in L^X}\Big((\mathcal{O}_{\mathcal{T}_X}(A)\to \mathcal{O}_{\mathcal{T}_Z}(g_L^{\rightarrow}(f_L^{\rightarrow}(A)))\wedge(\mathcal{O}_{\mathcal{T}_Z}((g_L^{\rightarrow}(f_L^{\rightarrow}(A))\to \mathcal{O}_{\mathcal{T}_Y}(f_L^{\rightarrow}(A)))\Big)\\
\leq&\bigwedge_{A\in L^X}\Big(\mathcal{O}_{\mathcal{T}_X}(A)\to \mathcal{O}_{\mathcal{T}_Y}(f_L^{\rightarrow}(A))\Big)=\mathrm{Open}(f).
\end{aligned}
$$

(2) By Lemma 4.12, it is not difficult to be proved. □

Definition 4.15. Given mappings $\mathcal{T}_X : L^X \longrightarrow M$ and $\mathcal{T}_Y : L^Y \longrightarrow M$. Let $f : X \longrightarrow Y$ be a bijective mapping. Then *the homeomorphism degree of f with respect to $\mathcal{T}_X$ and $\mathcal{T}_Y$* is defined by

$$\mathrm{Home}(f) = \mathrm{Cont}(f) \wedge \mathrm{Cont}(f^{-1}),$$

where f^{-1} is the inverse mapping of f.

Lemma 4.16. *Given mappings $\mathcal{T}_X : L^X \longrightarrow M$ and $\mathcal{T}_Y : L^Y \longrightarrow M$. Let $f : X \longrightarrow Y$ be a bijective mapping. Then*

$$\mathrm{Cont}(f^{-1}) = \mathrm{Open}(f) = \mathrm{Clos}(f).$$

Proof. Since f is bijective, we have $(f^{-1})_L^{\leftarrow}(A) = f_L^{\rightarrow}(A)$ and $f_L^{\rightarrow}(A') = f_L^{\rightarrow}(A)'$ for all $A \in L^X$. Then

$$\begin{aligned}\mathrm{Cont}(f^{-1}) &= \bigwedge_{A\in L^X} \Big(\mathcal{O}_{\mathcal{T}_X}(A) \to \mathcal{O}_{\mathcal{T}_Y}((f^{-1})_L^{\leftarrow}(A))\Big)\\ &= \bigwedge_{A\in L^X} \Big(\mathcal{O}_{\mathcal{T}_X}(A) \to \mathcal{O}_{\mathcal{T}_Y}(f_L^{\rightarrow}(A))\Big)\\ &= \mathrm{Open}(f).\end{aligned}$$

Further, we get

$$\begin{aligned}\mathrm{Open}(f) &= \bigwedge_{A\in L^X} \Big(\mathcal{O}_{\mathcal{T}_X}(A) \to \mathcal{O}_{\mathcal{T}_Y}(f_L^{\rightarrow}(A))\Big)\\ &= \bigwedge_{A\in L^X} \Big(\mathcal{O}_{\mathcal{T}_X}(A') \to \mathcal{O}_{\mathcal{T}_Y}(f_L^{\rightarrow}(A'))\Big)\\ &= \bigwedge_{A\in L^X} \Big(\mathcal{O}_{\mathcal{T}_X}(A') \to \mathcal{O}_{\mathcal{T}_Y}(f_L^{\rightarrow}(A)')\Big) = \mathrm{Clos}(f).\end{aligned}$$

□

By Lemma 4.16 and Proposition 4.11, we can easily have the following results.

Proposition 4.17. *Given mappings $\mathcal{T}_X : L^X \longrightarrow M$, $\mathcal{T}_Y : L^Y \longrightarrow M$ and $\mathcal{T}_Z : L^Z \longrightarrow M$. Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be bijective mappings. Then*

(1) $\mathrm{Home}(f) = \mathrm{Cont}(f) \wedge \mathrm{Cont}(f^{-1}) = \mathrm{Cont}(f) \wedge \mathrm{Open}(f) = \mathrm{Cont}(f) \wedge \mathrm{Clos}(f)$.

(2) $\mathrm{Home}(f) \wedge \mathrm{Home}(g) \le \mathrm{Home}(g \circ f)$.

Lemma 4.18. *Given mappings $\mathcal{T}_X : L^X \longrightarrow M$ and $\mathcal{T}_Y : L^Y \longrightarrow M$. Let $f : X \longrightarrow Y$ be a bijective mapping. Then*

(1) $\mathrm{Cont}(f) = \bigwedge_{A\in L^X} \Big(\mathcal{O}_{\mathcal{T}_Y}(f_L^{\rightarrow}(A)) \to \mathcal{O}_{\mathcal{T}_X}(A)\Big)$.

(2) $\mathrm{Open}(f) = \bigwedge_{B\in L^Y} \Big(\mathcal{O}_{\mathcal{T}_X}(f_L^{\leftarrow}(B)) \to \mathcal{O}_{\mathcal{T}_Y}(B)\Big)$.

Proof. We only prove (1). The proof of (2) is similar to that of (1). Since f is bijective, we have $f_L^{\leftarrow}(f_L^{\rightarrow}(A)) = A$ for all $A \in L^X$ and $f_L^{\rightarrow}(f_L^{\leftarrow}(B)) = B$ for all $B \in L^Y$. Then

$$\begin{aligned}
& \bigwedge_{A\in L^X} \Big(\mathcal{O}_{\mathcal{T}_Y}(f_L^{\rightarrow}(A)) \to \mathcal{O}_{\mathcal{T}_X}(A)\Big) \\
= & \bigwedge_{A\in L^X} \Big(\mathcal{O}_{\mathcal{T}_Y}(f_L^{\rightarrow}(A)) \to \mathcal{O}_{\mathcal{T}_X}(f_L^{\leftarrow}(f_L^{\rightarrow}(A)))\Big) \\
\geq & \bigwedge_{B\in L^Y} \Big(\mathcal{O}_{\mathcal{T}_Y}(B) \to \mathcal{O}_{\mathcal{T}_X}(f_L^{\leftarrow}(B))\Big) = Cont(f) \\
= & \bigwedge_{B\in L^Y} \Big(\mathcal{O}_{\mathcal{T}_Y}(f_L^{\rightarrow}(f_L^{\leftarrow}(B))) \to \mathcal{O}_{\mathcal{T}_X}(f_L^{\leftarrow}(B))\Big) \\
\geq & \bigwedge_{A\in L^X} \Big(\mathcal{O}_{\mathcal{T}_Y}(f_L^{\rightarrow}(A)) \to \mathcal{O}_{\mathcal{T}_X}(A)\Big).
\end{aligned}$$

Hence $\mathrm{Cont}(f) = \bigwedge_{A\in L^X} \Big(\mathcal{O}_{\mathcal{T}_Y}(f_L^{\rightarrow}(A)) \to \mathcal{O}_{\mathcal{T}_X}(A)\Big)$. □

By Lemma 4.18 and Proposition 4.17, the following proposition is straightforward.

Proposition 4.19. *Given mappings $\mathcal{T}_X : L^X \longrightarrow M$ and $\mathcal{T}_Y : L^Y \longrightarrow M$. Let $f : X \longrightarrow Y$ be a bijective mapping. Then*
(1) $\mathrm{Home}(f) = \bigwedge_{A\in L^X} \Big(\mathcal{O}_{\mathcal{T}_X}(A) \leftrightarrow \mathcal{O}_{\mathcal{T}_Y}(f_L^{\rightarrow}(A))\Big)$.
(2) $\mathrm{Home}(f) = \bigwedge_{B\in L^Y} \Big(\mathcal{O}_{\mathcal{T}_Y}(B) \leftrightarrow \mathcal{O}_{\mathcal{T}_X}(f_L^{\leftarrow}(B))\Big)$.

5. **Degrees of Quotient of Mappings and Their Characterizations**

In this section, the degree to which a mapping $f : (X, \mathcal{T}_X) \longrightarrow (Y, \mathcal{T}_Y)$ is a quotient mapping with respect to $\mathcal{T}_X$ and $\mathcal{T}_Y$ is defined and its characterizations are given. What's more, the relationships among the quotient degree, the continuity degree, the openness degree and the closedness degree of f are discussed.

Definition 5.1. Given mappings $\mathcal{T}_X : L^X \longrightarrow M$ and $\mathcal{T}_Y : L^Y \longrightarrow M$. Let $f : X \longrightarrow Y$ be a surjective mapping. Then $\mathrm{Quot}(f)$ defined by

$$\mathrm{Quot}(f) = \bigwedge_{B\in L^Y} \Big(\mathcal{O}_{\mathcal{T}_Y}(B) \leftrightarrow \mathcal{O}_{\mathcal{T}_X}(f_L^{\leftarrow}(B))\Big)$$

is called *the degree to which f is a quotient mapping with respect to $\mathcal{T}_X$ and $\mathcal{T}_Y$ (or the quotient degree of f with respect to $\mathcal{T}_X$ and $\mathcal{T}_Y$).*

In the following theorem, characterizations of the quotient degrees of f with respect to $\mathcal{T}_X$ and $\mathcal{T}_Y$ will be given.

Theorem 5.2. *Given mappings $\mathcal{T}_X : L^X \longrightarrow M$ and $\mathcal{T}_Y : L^Y \longrightarrow M$. Let $f : X \longrightarrow Y$ be a surjective mapping. Then*

$$(1)\ \mathrm{Quot}(f) = \bigvee\left\{a\in M \;\middle|\; \begin{array}{l}\mathrm{Top}(\mathcal{T}_Y)\wedge \mathcal{T}_Y(B)\wedge a\le \mathrm{Top}(\mathcal{T}_X)\wedge \mathcal{T}_X(f_L^{\leftarrow}(B)),\\ \mathrm{Top}(\mathcal{T}_X)\wedge \mathcal{T}_X(f_L^{\leftarrow}(B))\wedge a\le \mathrm{Top}(\mathcal{T}_Y)\wedge \mathcal{T}_Y(B),\\ \forall B\in L^Y\end{array}\right\}.$$

$$(2)\ \mathrm{Quot}(f) = \bigvee\left\{a\in M \;\middle|\; \begin{array}{l}\forall b\le \mathrm{Top}(\mathcal{T}_Y)\wedge a, \forall B\in (\mathcal{T}_Y)_{[b]},\\ b\le \mathrm{Top}(\mathcal{T}_X), f_L^{\leftarrow}(B)\in (\mathcal{T}_X)_{[b]},\\ \forall c\le \mathrm{Top}(\mathcal{T}_X)\wedge a, \forall f_L^{\leftarrow}(B)\in (\mathcal{T}_X)_{[c]},\\ c\le \mathrm{Top}(\mathcal{T}_Y), B\in (\mathcal{T}_Y)_{[c]}\end{array}\right\}.$$

$$(3)\ \mathrm{Quot}(f) = \bigvee\left\{a\in M \;\middle|\; \begin{array}{l}\forall b\notin \alpha(\mathrm{Top}(\mathcal{T}_Y)\wedge a), \forall B\in (\mathcal{T}_Y)^{[b]},\\ b\notin \alpha(\mathrm{Top}(\mathcal{T}_X)), f_L^{\leftarrow}(B)\in (\mathcal{T}_X)^{[b]},\\ \forall c\notin \alpha(\mathrm{Top}(\mathcal{T}_X)\wedge a), \forall f_L^{\leftarrow}(B)\in (\mathcal{T}_X)^{[c]},\\ c\notin \alpha(\mathrm{Top}(\mathcal{T}_Y)), B\in (\mathcal{T}_Y)^{[c]}\end{array}\right\}.$$

Proof. The proofs are similar to those of Theorem 4.7 and omitted here. □

Proposition 5.3. *Given mappings $\mathcal{T}_X : L^X \longrightarrow M$ and $\mathcal{T}_Y : L^Y \longrightarrow M$. Let $f : X \longrightarrow Y$ be a surjective mapping. Then*
(1) $\mathrm{Quot}(f) = \bigwedge_{B\in L^Y}\Big(\mathcal{O}_{\mathcal{T}_Y}(B') \leftrightarrow \mathcal{O}_{\mathcal{T}_X}(f_L^{\leftarrow}(B)')\Big)$.
(2) $\mathrm{Quot}(f) \le \mathrm{Cont}(f)$.

Proof. Obviously. □

Next we discuss relationships among $\mathrm{Quot}(f)$, $\mathrm{Cont}(f)$, $\mathrm{Open}(f)$ and $\mathrm{Clos}(f)$.

Proposition 5.4. *Given mappings $\mathcal{T}_X : L^X \longrightarrow M$ and $\mathcal{T}_Y : L^Y \longrightarrow M$. Let $f : X \longrightarrow Y$ be a surjective mapping. Then*
(1) $\mathrm{Cont}(f)\wedge \mathrm{Open}(f) \le \mathrm{Quot}(f)$.
(2) $\mathrm{Cont}(f)\wedge \mathrm{Clos}(f) \le \mathrm{Quot}(f)$.

Proof. (1) since f is surjective, we have $f_L^{\rightarrow}(f_L^{\leftarrow}(D)) = D$ for all $D\in L^Y$. Then

$$\begin{aligned}
&\mathrm{Cont}(f)\wedge \mathrm{Open}(f)\\
=&\bigwedge_{B\in L^Y}\Big(\mathcal{O}_{\mathcal{T}_Y}(B)\to \mathcal{O}_{\mathcal{T}_X}(f_L^{\leftarrow}(B))\Big)\wedge \bigwedge_{A\in L^X}\Big(\mathcal{O}_{\mathcal{T}_X}(A)\to \mathcal{O}_{\mathcal{T}_Y}(f_L^{\rightarrow}(A))\Big)\\
\le&\bigwedge_{B\in L^Y}\Big(\mathcal{O}_{\mathcal{T}_Y}(B)\to \mathcal{O}_{\mathcal{T}_X}(f_L^{\leftarrow}(B))\Big)\wedge \bigwedge_{D\in L^Y}\Big(\mathcal{O}_{\mathcal{T}_X}(f_L^{\leftarrow}(D))\to \mathcal{O}_{\mathcal{T}_Y}(f_L^{\rightarrow}(f_L^{\leftarrow}(D)))\Big)\\
=&\bigwedge_{B\in L^Y}\Big(\mathcal{O}_{\mathcal{T}_Y}(B)\to \mathcal{O}_{\mathcal{T}_X}(f_L^{\leftarrow}(B))\Big)\wedge \bigwedge_{D\in L^Y}\Big(\mathcal{O}_{\mathcal{T}_X}(f_L^{\leftarrow}(D))\to \mathcal{O}_{\mathcal{T}_Y}(D)\Big)\\
=&\bigwedge_{B\in L^Y}\Big(\big(\mathcal{O}_{\mathcal{T}_Y}(B)\to \mathcal{O}_{\mathcal{T}_X}(f_L^{\leftarrow}(B))\big)\wedge\big(\mathcal{O}_{\mathcal{T}_X}(f_L^{\leftarrow}(B))\to \mathcal{O}_{\mathcal{T}_Y}(B)\big)\Big)\\
=&\bigwedge_{B\in L^Y}\Big(\mathcal{O}_{\mathcal{T}_Y}(B)\leftrightarrow \mathcal{O}_{\mathcal{T}_X}(f_L^{\leftarrow}(B))\Big) = \mathrm{Quot}(f).
\end{aligned}$$

(2) By Lemma 4.12 and Proposition 5.3 (1), it can be proved easily. □

Proposition 5.5. *Given mappings $\mathcal{T}_X : L^X \longrightarrow M$, $\mathcal{T}_Y : L^Y \longrightarrow M$ and $\mathcal{T}_Z : L^Z \longrightarrow M$. Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be surjective mappings. Then*
(1) $\mathrm{Quot}(f) \wedge \mathrm{Quot}(g) \leq \mathrm{Quot}(g \circ f)$.
(2) $\mathrm{Quot}(g \circ f) \wedge \mathrm{Cont}(f) \wedge \mathrm{Cont}(g) \leq \mathrm{Quot}(g)$.

Proof. (1) By Lemma 2.1, we have

$$\begin{aligned}
&\mathrm{Quot}(f) \wedge \mathrm{Quot}(g)\\
= &\bigwedge_{B \in L^Y} \Big(\mathcal{O}_{\mathcal{T}_Y}(B) \leftrightarrow \mathcal{O}_{\mathcal{T}_X}(f_L^{\leftarrow}(B))\Big) \wedge \bigwedge_{C \in L^Z} \Big(\mathcal{O}_{\mathcal{T}_Z}(C) \leftrightarrow \mathcal{O}_{\mathcal{T}_Y}(g_L^{\leftarrow}(C))\Big)\\
\leq &\bigwedge_{D \in L^Z} \Big(\mathcal{O}_{\mathcal{T}_Y}(g_L^{\leftarrow}(D)) \leftrightarrow \mathcal{O}_{\mathcal{T}_X}(f_L^{\leftarrow}(g_L^{\leftarrow}(D)))\Big) \wedge \bigwedge_{C \in L^Z} \Big(\mathcal{O}_{\mathcal{T}_Z}(C) \leftrightarrow \mathcal{O}_{\mathcal{T}_Y}(g_L^{\leftarrow}(C))\Big)\\
= &\bigwedge_{D \in L^Z} \Big(\mathcal{O}_{\mathcal{T}_Y}(g_L^{\leftarrow}(D)) \leftrightarrow \mathcal{O}_{\mathcal{T}_X}((g \circ f)_L^{\leftarrow}(D))\Big) \wedge \bigwedge_{C \in L^Z} \Big(\mathcal{O}_{\mathcal{T}_Z}(C) \leftrightarrow \mathcal{O}_{\mathcal{T}_Y}(g_L^{\leftarrow}(C))\Big)\\
= &\bigwedge_{C \in L^Z} \Big((\mathcal{O}_{\mathcal{T}_Y}(g_L^{\leftarrow}(C)) \leftrightarrow \mathcal{O}_{\mathcal{T}_X}((g \circ f)_L^{\leftarrow}(C))) \wedge (\mathcal{O}_{\mathcal{T}_Z}(C) \leftrightarrow \mathcal{O}_{\mathcal{T}_Y}(g_L^{\leftarrow}(C)))\Big)\\
\leq &\bigwedge_{C \in L^Z} \Big(\mathcal{O}_{\mathcal{T}_Z}(C) \leftrightarrow \mathcal{O}_{\mathcal{T}_X}((g \circ f)_L^{\leftarrow}(C))\Big) = \mathrm{Quot}(g \circ f).
\end{aligned}$$

(2) Firstly, we know

$$\begin{aligned}
\mathrm{Quot}(g \circ f) &= \bigwedge_{C \in L^Z} \Big(\mathcal{O}_{\mathcal{T}_Z}(C) \leftrightarrow \mathcal{O}_{\mathcal{T}_X}((g \circ f)_L^{\leftarrow}(C))\Big)\\
&\leq \bigwedge_{C \in L^Z} \Big(\mathcal{O}_{\mathcal{T}_X}((g \circ f)_L^{\leftarrow}(C)) \to \mathcal{O}_{\mathcal{T}_Z}(C)\Big),
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Cont}(f) &= \bigwedge_{B \in L^Y} \Big(\mathcal{O}_{\mathcal{T}_Y}(B) \to \mathcal{O}_{\mathcal{T}_X}(f_L^{\leftarrow}(B))\Big)\\
&\leq \bigwedge_{D \in L^Z} \Big(\mathcal{O}_{\mathcal{T}_Y}(g_L^{\leftarrow}(D)) \to \mathcal{O}_{\mathcal{T}_X}(f_L^{\leftarrow}(g_L^{\leftarrow}(D)))\Big)\\
&= \bigwedge_{D \in L^Z} \Big(\mathcal{O}_{\mathcal{T}_Y}(g_L^{\leftarrow}(D)) \to \mathcal{O}_{\mathcal{T}_X}((g \circ f)_L^{\leftarrow}(D))\Big).
\end{aligned}$$

Then

$$\mathrm{Quot}(g \circ f) \wedge \mathrm{Cont}(f) \leq \bigwedge_{D \in L^Z} \Big(\mathcal{O}_{\mathcal{T}_Y}(g_L^{\leftarrow}(D)) \to \mathcal{O}_{\mathcal{T}_Z}(D)\Big).$$

Hence

$$\begin{aligned}
&\mathrm{Quot}(g \circ f) \wedge \mathrm{Cont}(f) \wedge \mathrm{Cont}(g)\\
\leq &\bigwedge_{D \in L^Z} \Big(\mathcal{O}_{\mathcal{T}_Y}(g_L^{\leftarrow}(D)) \to \mathcal{O}_{\mathcal{T}_Z}(D)\Big) \wedge \bigwedge_{C \in L^Z} \Big(\mathcal{O}_{\mathcal{T}_Z}(C) \to \mathcal{O}_{\mathcal{T}_Y}(g_L^{\leftarrow}(C))\Big)\\
= &\bigwedge_{C \in L^Z} \Big((\mathcal{O}_{\mathcal{T}_Y}(g_L^{\leftarrow}(C)) \to \mathcal{O}_{\mathcal{T}_Z}(C)) \wedge (\mathcal{O}_{\mathcal{T}_Z}(C) \to \mathcal{O}_{\mathcal{T}_Y}(g_L^{\leftarrow}(C)))\Big)\\
= &\bigwedge_{C \in L^Z} \Big(\mathcal{O}_{\mathcal{T}_Z}(C) \leftrightarrow \mathcal{O}_{\mathcal{T}_Y}(g_L^{\leftarrow}(C))\Big)\\
= &\mathrm{Quot}(g).
\end{aligned}$$

□

6. **An Application of (L, M)-fuzzy Topology Degrees**

In this section, we will introduce a theoretical application of (L, M)-fuzzy topology degrees. As we know, in general topology, if A is compact with respect to a crisp topology $\mathcal{T}_X$ and $f : X \longrightarrow Y$ is continuous, then $f(A)$ is compact with respect to a crisp topology $\mathcal{T}_Y$. If we endow some degree to each topology, do we have similar conclusion? to answer this question, we need to recall some notions.

Throughout this section, both L and M denote completely distributive lattices with order-reversing involution (i.e., completely distributive De Morgan algebras). For a non-empty set X, L^X is also a completely distributive De Morgan algebra when it inherits the structure of L in a natural way, by defining $\vee, \wedge, \leq, '$ pointwisely.

For a subfamily $\Phi \subseteq L^X$, $2^{(\Phi)}$ denotes the set of all finite subfamilies of Φ.

Definition 6.1. [22, 23] An L-fuzzy inclusion on X is a mapping $\widetilde{\subseteq} : L^X \times L^X \longrightarrow L$ defined by $\forall A, B \in L^X$,

$$\widetilde{\subseteq}(A, B) = \bigwedge_{x \in X} \left(A'(x) \vee B(x)\right).$$

In the sequel, we shall write $[A \widetilde{\subseteq} B]$ instead of $\widetilde{\subseteq}(A, B)$.

Lemma 6.2. [18] *Let $f : X \longrightarrow Y$ be a mapping. Then for any $A \in L^X$ and $\mathscr{B} \subseteq L^Y$,*

$$\bigwedge_{x \in X} \left(A'(x) \vee \bigvee_{B \in \mathscr{B}} f_L^{\leftarrow}(B)(x) \right) = \bigwedge_{y \in Y} \left(f_L^{\rightarrow}(A)'(y) \vee \bigvee_{B \in \mathscr{B}} B(y) \right).$$

In an L-topological space $(X, \mathcal{T})$, an L-set $A \in L^X$ is said to be fuzzy compact [18] if for every family $\mathscr{U}$ of open L-sets, we have $[A \widetilde{\subseteq} \bigvee \mathscr{U}] \leq \bigvee_{\mathscr{V} \in 2^{(\mathscr{U})}} [A \widetilde{\subseteq} \bigvee \mathscr{V}]$.

According to the above definition, we know that an L-fuzzy set is either compact or not. Let $\mathcal{T} : L^X \longrightarrow M$ be a mapping. If $\mathcal{T}$ is an (L, M)-fuzzy topology, then the fuzzy compactness degree [13] of $A \in L^X$ is defined by

$$cd(A) = \bigwedge_{\mathscr{U} \subseteq L^X} \left\{ \left(\bigwedge_{U \in \mathscr{U}} \mathcal{T}(U) \wedge \left[A \widetilde{\subseteq} \bigvee \mathscr{U} \right] \right) \to \bigvee_{\mathscr{V} \in 2^{(\mathscr{U})}} \left[A \widetilde{\subseteq} \bigvee \mathscr{V} \right] \right\}.$$

Now, if $\mathcal{T}$ is an (L, M)-fuzzy topology to some degree, by Definitions 3.1 and 4.1, the open L-set degree of $A \in L^X$ with respect to $\mathcal{T}$ is defined by $\mathcal{O}_{\mathcal{T}}(A) = \mathrm{Top}(\mathcal{T}) \wedge \mathcal{T}(A)$. For any family $\mathscr{U} \subseteq L^X$, we define the open L-set degree of $\mathscr{U}$ by

$$\mathcal{O}_{\mathcal{T}}(\mathscr{U}) = \mathrm{Top}(\mathcal{T}) \wedge \bigwedge_{U \in \mathscr{U}} \mathcal{T}(U) = \bigwedge_{U \in \mathscr{U}} \mathcal{O}_{\mathcal{T}}(U).$$

Then the fuzzy compactness degree of an L-set with respect to $\mathcal{T}$ is defined as follows.

Definition 6.3. Let $\mathcal{T} : L^X \longrightarrow M$ be a mapping. For any $A \in L^X$, $cd(A)_{\mathcal{T}}$ defined by

$$\begin{aligned} cd_{\mathcal{T}}(A) &= \bigwedge_{\mathscr{U}\subseteq L^X} \left\{ \left(\bigwedge_{U\in\mathscr{U}} \mathcal{O}_{\mathcal{T}}(U) \wedge \left[A \widetilde{\subseteq} \bigvee \mathscr{U}\right] \right) \to \bigvee_{\mathscr{V}\in 2^{(\mathscr{U})}} \left[A \widetilde{\subseteq} \bigvee \mathscr{V}\right] \right\} \\ &= \bigwedge_{\mathscr{U}\subseteq L^X} \left(\bigwedge_{U\in\mathscr{U}} \mathcal{O}_{\mathcal{T}}(U) \wedge \bigwedge_{x\in X} \left(A'(x) \vee \bigvee_{U\in\mathscr{U}} U(x) \right) \right) \\ &\quad \to \bigvee_{\mathscr{V}\in 2^{(\mathscr{U})}} \bigwedge_{x\in X} \left(A'(x) \vee \bigvee_{V\in\mathscr{V}} V(x) \right). \end{aligned}$$

is called the fuzzy compactness degree of G with respect to $\mathcal{T}$.

By Lemma 2.1 (7), $cd(A)_{\mathcal{T}}$ can also be characterized as follows.

Theorem 6.4. *Let $\mathcal{T} : L^X \longrightarrow M$ be a mapping. and $A \in L^X$. Then*

$$cd_{\mathcal{T}}(A) = \bigwedge_{\mathscr{U}\subseteq L^X} \left\{ \bigwedge_{U\in\mathscr{U}} \mathcal{O}_{\mathcal{T}}(U) \to \left(\left[A \widetilde{\subseteq} \bigvee \mathscr{U}\right] \to \bigvee_{\mathscr{V}\in 2^{(\mathscr{U})}} \left[A \widetilde{\subseteq} \bigvee \mathscr{V}\right] \right) \right\}$$

Given mappings $\mathcal{T}_X : L^X \longrightarrow M$ and $\mathcal{T}_Y : L^Y \longrightarrow M$. Let $f : X \longrightarrow Y$ be a mapping. From Definition 4.5, we know the continuity degree of f with respect to $\mathcal{T}_X$ and $\mathcal{T}_Y$ is defined by

$$\mathrm{Cont}(f) = \bigwedge_{B\in L^Y} \Big(\mathcal{O}_{\mathcal{T}_Y}(B) \to \mathcal{O}_{\mathcal{T}_X}(f_L^{\leftarrow}(B)) \Big).$$

Now, we shall answer the above question in the following theorem.

Theorem 6.5. *Given mappings $\mathcal{T}_X : L^X \longrightarrow M$ and $\mathcal{T}_Y : L^Y \longrightarrow M$. Let $f : X \longrightarrow Y$ be a mapping. Then $cd_{\mathcal{T}_X}(A) \wedge \mathrm{Cont}(f) \leq cd_{\mathcal{T}_Y}(f_L^{\rightarrow}(A))$ for all $A \in L^X$.*

Proof. Since the wedge below relation $\prec$ in a completely distributive lattice has the property: $a \prec \bigwedge_{i\in I} b_i$ implies $a \prec b_i$ for every $i \in I$. Further $a \leq b_i$ for every $i \in I$.

Take any $a \in M$ with $a \prec cd_{\mathcal{T}_X}(A) \wedge \mathrm{Cont}(f)$. Then

$$a \prec \mathrm{Cont}(f) = \bigwedge_{B\in L^Y} \Big(\mathcal{O}_{\mathcal{T}_Y}(B) \to \mathcal{O}_{\mathcal{T}_X}(f_L^{\leftarrow}(B)) \Big),$$

and

$$a \prec cd_{\mathcal{T}_X}(A) = \bigwedge_{\mathscr{U}\subseteq L^X} \left\{ \left(\bigwedge_{U\in\mathscr{U}} \mathcal{O}_{\mathcal{T}_X}(U) \wedge \left[A \widetilde{\subseteq} \bigvee \mathscr{U}\right] \right) \to \bigvee_{\mathscr{V}\in 2^{(\mathscr{U})}} \left[A \widetilde{\subseteq} \bigvee \mathscr{V}\right] \right\}.$$

For any $B \in L^Y$ and $\mathscr{U} \subseteq L^X$, we have

$$a \leq \mathcal{O}_{\mathcal{T}_Y}(B) \to \mathcal{O}_{\mathcal{T}_X}(f_L^{\leftarrow}(B)),$$

and

$$a \leq \left(\bigwedge_{U\in\mathscr{U}} \mathcal{O}_{\mathcal{T}_X}(U) \wedge \bigwedge_{x\in X} \left(A'(x) \vee \bigvee_{U\in\mathscr{U}} U(x) \right) \right) \to \bigvee_{\mathscr{V}\in 2^{(\mathscr{U})}} \bigwedge_{x\in X} \left(A'(x) \vee \bigvee_{V\in\mathscr{V}} V(x) \right).$$

By Lemma 2.1 (2), we know

$$a \wedge \mathcal{O}_{\mathcal{T}_Y}(B) \leq \mathcal{O}_{\mathcal{T}_X}(f_L^{\leftarrow}(B)), \tag{1}$$

and

$$a \wedge \bigwedge_{U \in \mathscr{U}} \mathcal{O}_{\mathcal{T}_X}(U) \wedge \bigwedge_{x \in X} \left(A'(x) \vee \bigvee_{U \in \mathscr{U}} U(x) \right) \leq \bigvee_{\mathscr{V} \in 2^{(\mathscr{U})}} \bigwedge_{x \in X} \left(A'(x) \vee \bigvee_{V \in \mathscr{V}} V(x) \right). \tag{2}$$

Suppose that

$$\begin{aligned}
& cd_{\mathcal{T}_Y}(f_L^{\rightarrow}(A)) \\
= & \bigwedge_{\mathscr{W} \subseteq L^Y} \left\{ \left(\bigwedge_{W \in \mathscr{W}} \mathcal{O}_{\mathcal{T}_Y}(W) \wedge \left[f_L^{\rightarrow}(A) \widetilde{\subseteq} \bigvee \mathscr{W} \right] \right) \to \bigvee_{\mathscr{D} \in 2^{(\mathscr{W})}} \left[f_L^{\rightarrow}(A) \widetilde{\subseteq} \bigvee \mathscr{D} \right] \right\} \\
= & \bigwedge_{\mathscr{W} \subseteq L^Y} \left(\bigwedge_{W \in \mathscr{W}} \mathcal{O}_{\mathcal{T}_Y}(W) \wedge \bigwedge_{y \in Y} \left(f_L^{\rightarrow}(A)'(y) \vee \bigvee_{W \in \mathscr{W}} W(y) \right) \right) \\
& \quad \to \bigvee_{\mathscr{D} \in 2^{(\mathscr{W})}} \bigwedge_{y \in Y} \left(f_L^{\rightarrow}(A)'(y) \vee \bigvee_{D \in \mathscr{D}} D(y) \right).
\end{aligned}$$

For any $\mathscr{W} \subseteq L^Y$, let $f^{\leftarrow}(\mathscr{W}) = \{ f_L^{\leftarrow}(W) \mid W \in \mathscr{W} \} \subseteq L^X$. Then

$$\begin{aligned}
& a \wedge \bigwedge_{W \in \mathscr{W}} \mathcal{O}_{\mathcal{T}_Y}(W) \wedge \bigwedge_{y \in Y} \left(f_L^{\rightarrow}(A)'(y) \vee \bigvee_{W \in \mathscr{W}} W(y) \right) \\
\leq & \ a \wedge \bigwedge_{W \in \mathscr{W}} \mathcal{O}_{\mathcal{T}_X}(f_L^{\leftarrow}(W)) \wedge \bigwedge_{y \in Y} \left(f_L^{\rightarrow}(A)'(y) \vee \bigvee_{W \in \mathscr{W}} W(y) \right) \text{ (by (1))} \\
= & \ a \wedge \bigwedge_{W \in \mathscr{W}} \mathcal{O}_{\mathcal{T}_X}(f_L^{\leftarrow}(W)) \wedge \bigwedge_{x \in X} \left(A'(x) \vee \bigvee_{W \in \mathscr{W}} f_L^{\leftarrow}(W)(x) \right) \text{ (by Lemma 6.2)} \\
= & \ a \wedge \bigwedge_{U \in f^{\leftarrow}(\mathscr{W})} \mathcal{O}_{\mathcal{T}_X}(U) \wedge \bigwedge_{x \in X} \left(A'(x) \vee \bigvee_{U \in f^{\leftarrow}(\mathscr{W})} U(x) \right) \\
\leq & \bigvee_{\mathscr{V} \in 2^{(f^{\leftarrow}(\mathscr{W}))}} \bigwedge_{x \in X} \left(A'(x) \vee \bigvee_{V \in \mathscr{V}} V(x) \right) \text{ (by (2))} \\
= & \bigvee_{\mathscr{D} \in 2^{(\mathscr{W})}} \bigwedge_{x \in X} \left(A'(x) \vee \bigvee_{D \in \mathscr{D}} f_L^{\leftarrow}(D)(x) \right) \\
= & \bigvee_{\mathscr{D} \in 2^{(\mathscr{W})}} \bigwedge_{y \in Y} \left(f_L^{\rightarrow}(A)'(y) \vee \bigvee_{D \in \mathscr{D}} D(y) \right) \text{ (by Lemma 6.2).}
\end{aligned}$$

Hence

$$\begin{aligned}
a \leq & \left(\bigwedge_{W \in \mathscr{W}} \mathcal{O}_{\mathcal{T}_Y}(W) \wedge \bigwedge_{y \in Y} \left(f_L^{\rightarrow}(A)'(y) \vee \bigvee_{W \in \mathscr{W}} W(y) \right) \right) \\
& \quad \to \bigvee_{\mathscr{D} \in 2^{(\mathscr{W})}} \bigwedge_{y \in Y} \left(f_L^{\rightarrow}(A)'(y) \vee \bigvee_{D \in \mathscr{D}} D(y) \right).
\end{aligned}$$

Therefore

$$a \leq \bigwedge_{\mathscr{W} \subseteq L^Y} \left\{ \left(\bigwedge_{W \in \mathscr{W}} \mathcal{O}_{\mathcal{T}_Y}(W) \wedge \left[f_L^{\rightarrow}(A) \widetilde{\subseteq} \bigvee \mathscr{W} \right] \right) \rightarrow \bigvee_{\mathscr{D} \in 2^{(\mathscr{W})}} \left[f_L^{\rightarrow}(A) \widetilde{\subseteq} \bigvee \mathscr{D} \right] \right\}$$
$$= cd_{\mathcal{T}_Y}(f_L^{\rightarrow}(A)).$$

By the arbitrariness of a, we obtain $cd_{\mathcal{T}_X}(A) \wedge \mathrm{Cont}(f) \leq cd_{\mathcal{T}_Y}(f_L^{\rightarrow}(A))$. □

7. Conclusions

In this paper, the degree to which a mapping $\mathcal{T} : L^X \longrightarrow M$ is an (L, M)-fuzzy topology and the degree to which an L-subset is an L-open set with respect to $\mathcal{T}$ were introduced. Based on these, we defined the degrees to which a mapping $f : (X, \mathcal{T}_X) \longrightarrow (Y, \mathcal{T}_Y)$ is continuous, open, closed or quotient with respect to $\mathcal{T}_X$ and $\mathcal{T}_Y$. Also, we not only gave their characterizations but also discussed the relationships among them.

Motivated by these, many properties and results in general topology can be reconsidered under the (L, M)-fuzzy topology degrees, such as connectedness, separation axioms etc., which will be the subjects of our future researches.

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