A TRANSITION FROM TWO-PERSON ZERO-SUM GAMES TO COOPERATIVE GAMES WITH FUZZY PAYOFFS

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Abstract. In this paper, we deal with games with fuzzy payoffs. We proved that players who are playing a zero-sum game with fuzzy payoffs against Nature are able to increase their joint payoff, and hence their individual payoffs by cooperating. It is shown that, a cooperative game with the fuzzy characteristic function can be constructed via the optimal game values of the zero-sum games with fuzzy payoffs against Nature at which players' combine their strategies and act like a single player. It is also proven that, the fuzzy characteristic function that is constructed in this way satisfies the superadditivity condition. Thus we considered a transition from two-person zero-sum games with fuzzy payoffs to cooperative games with fuzzy payoffs. The fair allocation of the maximum payoff (game value) of this cooperative game among players is done using the Shapley vector.

1. Introduction

Game theory has been used as a powerful analytical tool for such decision making problems of the organizations or competitive systems [9, 10, 19]. When a game theoretic approach is used as a resolution method for decision making problems, it is important to examine which solution concept we should employ, and the corresponding computational methods for obtained the solutions are also indispensable for implementing the results of the examination.

The results of analysis and resolution of decision making problems are not always appropriate and suitable for the real life problems if parameters of mathematical models for the decision making problems are determined without considering the uncertainty and the imprecision likely to occur in the competitive systems. Therefore, taking into the uncertainty and the imprecision of information of the decision making problem in the competitive systems and the ambiguity in the decision makers' judgments, analysts of the decision making problem may be requested to formulate the mathematical models under fuzzy environments.

With the develop of the fuzzy theory [8, 16, 17, 18, 24, 25], ambiguous events which are not probability events can be represented as fuzzy sets. As a result, the ambiguity in decision makers' judgments and the uncertainty, as well as, the

Received: September 2017; Revised: February 2018; Accepted: September 2018

Key words and phrases: Cooperative game, Fuzzy set, Fuzzy number, Shapley vector, Superadditivity, Zero-sum games

This paper is the extended version of presented paper in "13th Algebraic Hyperstructures and Applications Conference (AHA2017)" which was held during 24-27 July 2017 at Yildiz Technical University, Istanbul, Turkey.

imprecision of information in competitive systems can be treated explicitly in optimization problems with a single decision maker.

Research of game theory in fuzzy environments has been accumulating since the mid 1970s. In noncooperative fuzzy games, ambiguity for a player's choice of a strategy, vagueness of preference for a payoff and imprecision of payoff representation have been represented as a fuzzy sets. Cooperative fuzzy games, games with fuzzy coalitions, mean that players are admitted to participating partially in a coalition and games with fuzzy payoffs have been also considered.

Butnariu was the first to study two-person noncooperative games in a fuzzy environment [5], claiming that all of one player's strategies are not equally possible and the grade of membership of a strategy is dependent on the behavior of the opponent. He also considered the case where the set of strategies of the player could be seen as a fuzzy set. Buckley analyzed the behavior of decision makers using two-person fuzzy games similar to Butnariu's [4].

Campos examined maximin problems of the two-person zero-sum fuzzy games, in which the elements of the payoff matrix were represented as fuzzy numbers, and employed the fuzzy linear programming methods in order to compute the maximin solutions [6]. Later extended by Nishizaki and Sakawa for the multiobjective situation [14, 15, 20, 21]. In the literature there are many models of the two-person zero-sum fuzzy games with fuzzy payoffs [3, 7, 11, 12, 13].

The research of cooperative fuzzy games began with introducing fuzzy coalitions. Aubin and Butnariu have been studying cooperative fuzzy games independently from about the same time. Aubin investigated the core and Shapley value [23] for n-person cooperative games with fuzzy coalitions in his book [1], after he had published some articles on the related topics [1, 2].

This paper is related to the research fields both of zero-sum games with fuzzy payoff and cooperative games with fuzzy payoff. We proved that players who are playing a zero-sum game with fuzzy payoffs against nature are able to increase their joint payoff, and hence their individual payoffs by cooperating. It is shown that, a cooperative game with the fuzzy characteristic function can be constructed via the optimal game values of the zero-sum games with fuzzy payoffs against nature at which players' combine their strategies and act like a single player. It is also proven that, the fuzzy characteristic function that is constructed in this way satisfies the superadditivity condition. Thus we considered a transition from two-person zero-sum games with fuzzy payoffs to cooperative games with fuzzy payoffs. The fair allocation of the maximum payoff (game value) of this cooperative game among players is done using the Shapley vector.

2. A Transition from Two-person Zero-sum Games with Fuzzy Payoffs to Cooperative Games with Fuzzy Payoffs

Definition 2.1. (Zero-sum game with fuzzy payoffs): When Player I chooses a pure strategy $i \in I$ and Player II chooses a pure strategy $j \in J$, let $\widetilde{a}_{ij}$ be fuzzy payoff for Player I. The fuzzy payoff $\widetilde{a}_{ij}$ is represented by the triangular fuzzy number

$$\widetilde{a}_{ij} = \left((a_{ij})_l, a_{ij}, (a_{ij})_u\right) \tag{1}$$

where a_{ij} is a mean value, $(a_{ij})_l$ is a left spread and $(a_{ij})_u$ is a right spread.

The two-person zero-sum fuzzy game can be represented as a fuzzy payoff matrix

$$\widetilde{A} = \begin{bmatrix} \widetilde{a}_{11} & ... & \widetilde{a}_{1n} \\ ... & ... & ... \\ \widetilde{a}_{m1} & ... & \widetilde{a}_{mn} \end{bmatrix}. \tag{2}$$

The game defined by (2) is called a two-person zero-sum game with fuzzy payoffs.

When each of the players chooses a strategy, a payoff for each of them is represented as a fuzzy number, but an outcome of the game has a zero-sum structure such that, when one player receives a gain the order player suffers an equal loss [22].

Let Player A's strategies are A_i, $(i = 1, ..., m)$, Player B's strategies are B_j, $(j = 1, ..., n)$, Nature's strategies are N_k, $(k = 1, ..., l)$ and

$$H_A(A_i, N_k) = \widetilde{a}_{ik}, \ i = 1, ..., m, \ k = 1, ..., l \tag{3}$$

and

$$H_B(B_j, N_k) = \widetilde{b}_{jk}, \ j = 1, ..., n, \ k = 1, ..., l \tag{4}$$

be payoff matrices of Player A and B, respectively.

Let the probability in which A chooses be A_i x_i, $(i = 1, ..., m)$, the probability in which B chooses be B_j y_j, $(j = 1, ..., n)$, the probability in which D chooses be D_k z_k, $(k = 1, ..., l)$ and the payoff functions of the players A and B be

$$H_A(x, z) = \sum_i \sum_k x_i \widetilde{a}_{ik} z_k \tag{5}$$

and

$$H_B(y, z) = \sum_j \sum_k y_j \widetilde{b}_{jk} z_k, \tag{6}$$

respectively. Here $x_i, y_j, z_k \geq 0$, $i = 1, ..., m$, $j = 1, ..., n$, $k = 1, ..., l$ and

$$\sum_{i=1}^{m} x_i = 1, \ \sum_{j=1}^{n} y_j = 1, \ \sum_{k=1}^{l} z_k = 1. \tag{7}$$

Also each of the vectors

$$x = (x_1, ..., x_m), \ y = (y_1, ..., y_n) \text{ and } z = (z_1, ..., z_l) \tag{8}$$

represents one probability distribution.

Let

$$\max_x \min_z H_A(x, z) = H_A\left(x^0, z^A\right) = \widetilde{v}(\{A\}) \tag{9}$$

and

$$\max_y \min_z H_B(y, z) = H_B\left(y^0, z^B\right) = \widetilde{v}(\{B\}). \tag{10}$$

Hence $\left(x^0, z^A\right)$ and $\left(y^0, z^B\right)$, refer the equilibrium solutions Player A and Player B play in the game against Nature, respectively. We have the following inequalities since the player who changes the equilibrium solution loses. When the Nature player changes its strategy, namely changing the equilibrium, the gain of the Nature decreases while the gain of the opponents increase.

$$H_A\left(x^0, z^A\right) \leq H_A\left(x^0, z\right) \tag{11}$$

and

$$H_B\left(y^0, z^B\right) \leq H_B\left(y^0, z\right) \tag{12}$$

The ranking here can change according to the preference of the decision maker. The decision maker, according to the ranking of prefers

(Mean value, Right spread, Left spread),
(Mean value, Left spread, Right spread),
(Right spread, Mean value, Left spread),
(Right spread, Left spread, Mean value),
(Left spread, Mean value, Right spread),
(Left spread, Right spread, Mean value),

any of those ranking. When player D (Nature) uses D_k strategy, let player A play A_i and player B play B_j. In this case, coalition $A \cup B$ gets the value of $\widetilde{a}_{ik} + \widetilde{b}_{jk}$.

Let the probability to be played for the strategies A_i, B_j and D_k, respectively, be x_i, y_j and z_k. Thus, the payoff function of the coalition is

$$H_{A\cup B}\left(x\Theta y, z\right) = \sum_i \sum_j \sum_k x_i y_j \left(\widetilde{a}_{ik} + \widetilde{b}_{jk}\right) z_k. \tag{13}$$

Here the vector $x\Theta y = (x_1 y_1, ..., x_i y_j, ..., x_m y_n)$ is the probability distribution vector.

That is,

$$x_i \geq 0,\ y_j \geq 0 \Rightarrow x_i y_j \geq 0, \text{ and } \sum_i^m \sum_j^n x_i y_j = \sum_i^m x_i \sum_j^n y_j = 1 \text{ , for } \forall i, j. \tag{14}$$

Let the maximum gain that player A and B would get against Nature be

$$\max_{x\Theta y} \min_z H_{A\cup B}\left(x\Theta y, z\right) = H_{A\cup B}\left(x^*\Theta y^*, z^*\right) = \widetilde{v}\left(\{A\} \cup \{B\}\right). \tag{15}$$

Here $(x^*\Theta y^*, z^*)$ is the equilibrium solution of the game. The player who changes the equilibrium loses.

S is a coalition and let the mixed vector of the coalition strategies of S be w, then the maximum gain that coalition S would get against Nature is

$$\max_w \min_z H_S\left(w, z\right) = \widetilde{v}\left(\{S\}\right). \tag{16}$$

Definition 2.2. (The characteristic function of the cooperative game): As shown above, the gain player A and B would get against Nature are

$$\max_x \min_z H_A\left(x, z\right) = \widetilde{v}\left(\{A\}\right) \tag{17}$$

$$\max_y \min_z H_B\left(y, z\right) = \widetilde{v}\left(\{B\}\right) \tag{18}$$

respectively. Let the vector

$$x\Theta y = (x_1 y_1, ..., x_i y_j, ..., x_m y_n) \tag{19}$$

be probability distribution vector, then the value of the game that players A and B can be obtained. Similarly, let the mixed vector of the $S \subset I$ coalition be w. In this case, the gain that $S \subset I$ coalition would obtain is

$$\max_w \min_z H_S(w, z) = \widetilde{v}(\{S\}). \tag{20}$$

Hence the optimal value of the game of the zero-sum games played against Nature determines the $\widetilde{v}$ fuzzy characteristic function of the cooperative games with fuzzy payoffs.

Theorem 2.3. *$\widetilde{v}$ fuzzy characteristic function cooperative game is a supperadditive game, that is;*

$$\widetilde{v}(\{A\} \cup \{B\}) \geq \widetilde{v}(\{A\}) + \widetilde{v}(\{B\}). \tag{21}$$

Proof. Here whichever ranking criteria the decision maker prefers, the theory is valid under the condition in which the decision maker uses the same ranking in all the steps of the operations.

If the coalition changes the equilibrium the gain of the coalition decreases because the equilibrium solution of the game is $(x^*\Theta y^*, z^*)$. If Nature changes the equilibrium, the gain of the coalition increases.

Then,

$$H_{A\cup B}\left(x^0\Theta y^0, z^*\right) \leq H_{A\cup B}\left(x^*\Theta y^*, z^*\right) = \widetilde{v}(\{A\} \cup \{B\}) \leq H_{A\cup B}\left(x^*\Theta y^*, z\right),$$

$$\begin{aligned} H_{A\cup B}\left(x^*\Theta y^*, z\right) &= \sum_i \sum_j \sum_k x_i^* y_j^* \left(\widetilde{a}_{ik} + \widetilde{b}_{jk}\right) z_k \geq \sum_i \sum_j \sum_k x_i^0 y_j^0 \left(\widetilde{a}_{ik} + \widetilde{b}_{jk}\right) z_k \\ &= \sum_k \left[\sum_i \sum_j x_i^0 y_j^0 \widetilde{a}_{ik} + \sum_i \sum_j x_i^0 y_j^0 \widetilde{b}_{jk}\right] z_k \\ &= \sum_k \left[\sum_i x_i^0 \widetilde{a}_{ik} \left(\sum_j y_j^0\right) + \sum_j y_j^0 \widetilde{b}_{jk} \left(\sum_i x_i^0\right)\right] z_k \\ &= \sum_k \left[\sum_i x_i^0 \widetilde{a}_{ik} + \sum_j y_j^0 \widetilde{b}_{jk}\right] z_k \\ &= \sum_k \sum_i x_i^0 \widetilde{a}_{ik} z_k + \sum_k \sum_j y_j^0 \widetilde{b}_{jk} z_k \\ &= H_A\left(x^0, z\right) + H_B\left(y^0, z\right), \end{aligned}$$

$$H_{A\cup B}\left(x^*\Theta y^*, z\right) \geq H_A\left(x^0, z\right) + H_B\left(y^0, z\right).$$

If we make the minimum of the both sides of the inequality according to z, then

$$\begin{aligned} \min_z H_{A\cup B}\left(x^*\Theta y^*, z\right) &\geq \min_z\{H_A\left(x^0, z\right) + H_B\left(y^0, z\right)\} \\ &\geq \min_z H_A\left(x^0, z\right) + \min_z H_B\left(y^0, z\right) \end{aligned}$$

$$\widetilde{v}(\{A\} \cup \{B\}) \geq \widetilde{v}(\{A\}) + \widetilde{v}(\{B\}).$$

This proves that the $\widetilde{v}$ fuzzy characteristic function cooperative game is a supperadditive. □

3. The Fair Share Among Players of the Maximum Gain by Zero-sum Games with Fuzzy Payoffs Against Nature

In this section, we study the fair share of the maximum profit by zero-sum games with fuzzy payoffs against Nature at the rate that players contribute. The fair share of the maximum gain by the coalition among the players is done using Shapley vector.

Let the set of the players against Nature be $I = \{A, B, C\}$ and the strategies of A be A_i $(i = 1, ..., m)$, the strategies of B be B_j $(j = 1, ..., n)$, the strategies of C be C_k $(k = 1, ..., l)$ and the strategies of Nature be D_r $(r = 1, ..., s)$. The gain that they would get against Nature together or alone is

$$\begin{array}{l} \widetilde{v}(\{\varnothing\}), \widetilde{v}(\{A\}), \widetilde{v}(\{B\}), \widetilde{v}(\{C\}), \widetilde{v}(\{A,B\}), \\ \widetilde{v}(\{A,C\}), \widetilde{v}(\{B,C\}), \widetilde{v}(\{A,B,C\}). \end{array} \tag{22}$$

The fair share of the maximum gain by the coalition among the players will be done using the Shapley vector. Here, for the fuzzy value of $\widetilde{v}(\{A, B, C\})$ according to the v_l left spread, v medium and v_u right spread values are

$$\begin{aligned} \phi_i(v_l) &= \sum_{i \in S} \frac{(n-|S|)!}{n!} \left[v_l(S) - v_l(S \setminus \{i\}) \right], \\ \phi_i(v) &= \sum_{i \in S} \frac{(n-|S|)!}{n!} \left[v(S) - v(S \setminus \{i\}) \right], \\ \phi_i(v_u) &= \sum_{i \in S} \frac{(n-|S|)!}{n!} \left[v_u(S) - v_u(S \setminus \{i\}) \right] \end{aligned} \tag{23}$$

respectively. Shapley vectors are calculated as follows.

The share of the player A is

$$\begin{aligned} \phi_A(v_l) = {} & \frac{1}{3}\left[v_l(\{A\}) - v_l(\{\varnothing\})\right] + \frac{1}{6}\left[v_l(\{A,B\}) - v_l(\{B\})\right] + \\ & \frac{1}{6}\left[v_l(\{A,C\}) - v_l(\{C\})\right] + \frac{1}{3}\left[v_l(\{A,B,C\}) - v_l(\{B,C\})\right], \end{aligned}$$

$$\begin{aligned} \phi_A(v) = {} & \frac{1}{3}\left[v(\{A\}) - v(\{\varnothing\})\right] + \frac{1}{6}\left[v(\{A,B\}) - v(\{B\})\right] + \\ & \frac{1}{6}\left[v(\{A,C\}) - v(\{C\})\right] + \frac{1}{3}\left[v(\{A,B,C\}) - v(\{B,C\})\right], \end{aligned}$$

$$\begin{aligned} \phi_A(v_u) = {} & \frac{1}{3}\left[v_u(\{A\}) - v_u(\{\varnothing\})\right] + \frac{1}{6}\left[v_u(\{A,B\}) - v_u(\{B\})\right] + \\ & \frac{1}{6}\left[v_u(\{A,C\}) - v_u(\{C\})\right] + \frac{1}{3}\left[v_u(\{A,B,C\}) - v_u(\{B,C\})\right]. \end{aligned}$$

The share of the player B is

$$\begin{aligned} \phi_B(v_l) = {} & \frac{1}{3}\left[v_l(\{B\}) - v_l(\{\varnothing\})\right] + \frac{1}{6}\left[v_l(\{A,B\}) - v_l(\{A\})\right] + \\ & \frac{1}{6}\left[v_l(\{B,C\}) - v_l(\{C\})\right] + \frac{1}{3}\left[v_l(\{A,B,C\}) - v_l(\{A,C\})\right], \end{aligned}$$

$$\begin{aligned}\phi_B(v) &= \frac{1}{3}[v(\{B\}) - v(\{\varnothing\})] + \frac{1}{6}[v(\{A,B\}) - v(\{A\})] + \\ &\frac{1}{6}[v(\{B,C\}) - v(\{C\})] + \frac{1}{3}[v(\{A,B,C\}) - v(\{A,C\})],\end{aligned}$$

$$\begin{aligned}\phi_B(v_u) &= \frac{1}{3}[v_u(\{B\}) - v_u(\{\varnothing\})] + \frac{1}{6}[v_u(\{A,B\}) - v_u(\{A\})] + \\ &\frac{1}{6}[v_u(\{B,C\}) - v_u(\{C\})] + \frac{1}{3}[v_u(\{A,B,C\}) - v_u(\{A,C\})].\end{aligned}$$

The share of the player C is

$$\begin{aligned}\phi_C(v_l) &= \frac{1}{3}[v_l(\{C\}) - v_l(\{\varnothing\})] + \frac{1}{6}[v_l(\{A,C\}) - v_l(\{A\})] + \\ &\frac{1}{6}[v_l(\{B,C\}) - v_l(\{B\})] + \frac{1}{3}[v_l(\{A,B,C\}) - v_l(\{A,B\})],\end{aligned}$$

$$\begin{aligned}\phi_C(v) &= \frac{1}{3}[v(\{C\}) - v(\{\varnothing\})] + \frac{1}{6}[v(\{A,C\}) - v(\{A\})] + \\ &\frac{1}{6}[v(\{B,C\}) - v(\{B\})] + \frac{1}{3}[v(\{A,B,C\}) - v(\{A,B\})],\end{aligned}$$

$$\begin{aligned}\phi_C(v_u) &= \frac{1}{3}[v_u(\{C\}) - v_u(\{\varnothing\})] + \frac{1}{6}[v_u(\{A,C\}) - v_u(\{A\})] + \\ &\frac{1}{6}[v_u(\{B,C\}) - v_u(\{B\})] + \frac{1}{3}[v_u(\{A,B,C\}) - v_u(\{A,B\})].\end{aligned}$$

Example 3.1. Let the strategies of the player A be A_1, A_2, A_3 the strategies of the player B be B_1, B_2 the strategies of the player C be C_1, C_2 the strategies of the player Nature be D_1, D_2, D_3, D_4 and let the payoff matrices of the players A, B and C against Nature be

$$(A,D) = \begin{pmatrix} (5.9,6,6.1) & (6.8,7,7.1) & (2.8,3,3.1) & (8.8,9,9.1) \\ (8.7,9,9.2) & (8.8,9,9.3) & (6.9,7,7.1) & (6.9,7,7.2) \\ (1.7,2,2.2) & (5.8,6,6.2) & (6.8,7,7.1) & (6.7,7,7.1) \end{pmatrix}, \tag{24}$$

$$(B,D) = \begin{pmatrix} (6.8,7,7.1) & (7.9,8,8.2) & (7.9,8,8.2) & (3.9,4,4.2) \\ (5.7,6,6.2) & (7.8,8,8.1) & (5.8,6,6.1) & (2.8,3,3.1) \end{pmatrix}, \tag{25}$$

$$(C,D) = \begin{pmatrix} (1.9,2,2.2) & (7.9,8,8.2) & (3.8,4,4.1) & (0.9,1,1.2) \\ (2.8,3,3.1) & (0.7,1,1.2) & (8.8,9,9.1) & (4.8,5,5.2) \end{pmatrix}, \tag{26}$$

respectively.

Let the ranking preference of the decision maker be (medium, right spread, left spread). When players A, B and C play individually against Nature, the optimal values of the games are respectively,

$$\widetilde{v}(\{A\}) = (6.9, 7, 7.1), \tag{27}$$

$$\widetilde{v}(\{B\}) = (3.9, 4, 4.2), \tag{28}$$

$$\widetilde{v}(\{C\}) = (2.5, 2.75, 2.875). \tag{29}$$

The game matrix when players A and B set coalition against Nature is

$$(A \cup B, D) = \begin{pmatrix} (12.7, 13, 13.2) & (14.7, 15, 15.3) & (10.7, 11, 11.3) & (12.7, 13, 13.3) \\ (11.6, 12, 12.3) & (14.6, 15, 15.2) & (8.6, 9, 9.2) & (11.6, 12, 12.2) \\ (15.5, 16, 16.3) & (16.7, 17, 17.5) & (14.8, 15, 15.3) & (10.8, 11, 11.4) \\ (14.4, 15, 15.4) & (16.6, 17, 17.4) & (12.7, 13, 13.2) & (9.7, 10, 10.3) \\ (8.5, 9, 9.3) & (13.7, 14, 14.4) & (14.7, 15, 15.3) & (10.6, 11, 11.3) \\ (7.4, 8, 8.4) & (13.6, 14, 14.3) & (12.6, 13, 13.2) & (9.5, 10, 10.2) \end{pmatrix} \quad (30)$$

Here, pure strategies correspond to the rows and the columns of matrix for $A_1B_1, A_1B_2, A_2B_1, A_2B_2, A_3B_1, A_3B_2$ and D_1, D_2, D_3, D_4, respectively.

The game matrix when players A and C set coalition against Nature is

$$(A \cup C, D) = \begin{pmatrix} (7.8, 8, 8.3) & (14.7, 15, 15.3) & (6.6, 7, 7.2) & (9.7, 10, 10.3) \\ (8.7, 9, 9.2) & (7.5, 8, 8.3) & (11.6, 12, 12.2) & (13.6, 14, 14.3) \\ (10.6, 11, 11.4) & (16.7, 17, 17.5) & (10.7, 11, 11.2) & (7.8, 8, 8.4) \\ (11.5, 12, 12.3) & (9.5, 10, 10.5) & (15.7, 16, 16.2) & (11.7, 12, 12.4) \\ (3.6, 4, 4.4) & (13.7, 14, 14.4) & (10.6, 11, 11.2) & (7.6, 8, 8.3) \\ (4.5, 5, 5.3) & (6.5, 7, 7.4) & (15.6, 16, 16.2) & (11.5, 12, 12.3) \end{pmatrix} \quad (31)$$

Here, pure strategies correspond to the rows and the columns of matrix for $A_1C_1, A_1C_2, A_2C_1, A_2C_2, A_3C_1, A_3C_2$ and D_1, D_2, D_3, D_4, respectively.

The game matrix when players B and C set coalition against Nature is

$$(B \cup C, D) = \begin{pmatrix} (8.7, 9, 9.3) & (15.8, 16, 16.4) & (11.7, 12, 12.3) & (4.8, 5, 5.4) \\ (9.6, 10, 10.2) & (8.6, 9, 9.4) & (16.7, 17, 17.3) & (8.7, 9, 9.4) \\ (7.6, 8, 8.4) & (15.7, 16, 16.3) & (9.6, 10, 10.2) & (3.7, 4, 4.3) \\ (8.5, 9, 9.3) & (8.5, 9, 9.3) & (14.6, 15, 15.2) & (7.6, 8, 8.3) \end{pmatrix} \quad (32)$$

Here, pure strategies correspond to the rows and the columns of matrix for $B_1C_1, B_1C_2, B_2C_1, B_2C_2$ and D_1, D_2, D_3, D_4, respectively.

The game matrix when players A, B and C set coalition against Nature is

$$(A \cup B \cup C, D) = \begin{pmatrix} (14.6, 15, 15.4) & (22.6, 23, 23.5) & (14.5, 15, 15.4) & (13.6, 14, 14.5) \\ (15.6, 16, 16.3) & (15.4, 16, 16.5) & (19.5, 20, 20.4) & (17.5, 18, 18.5) \\ (13.5, 14, 14.5) & (22.5, 23, 23.4) & (12.4, 13, 13.3) & (12.5, 13, 13.4) \\ (14.4, 15, 15.4) & (15.3, 16, 16.4) & (17.4, 18, 18.3) & (16.4, 17, 17.4) \\ (17.4, 18, 18.5) & (24.6, 25, 25.7) & (18.6, 19, 19.4) & (11.7, 12, 12.6) \\ (18.3, 19, 19.4) & (17.4, 18, 18.7) & (23.6, 24, 24.4) & (15.6, 16, 16.6) \\ (16.3, 17, 17.6) & (24.5, 25, 25.6) & (16.5, 17, 17.3) & (10.6, 11, 11.5) \\ (17.2, 18, 18.5) & (17.3, 18, 18.6) & (21.5, 22, 22.3) & (14.5, 15, 15.5) \\ (10.4, 11, 11.5) & (21.6, 22, 22.6) & (18.5, 19, 19.4) & (11.5, 12, 12.5) \\ (11.3, 12, 12.4) & (14.4, 15, 15.6) & (23.5, 24, 24.4) & (15.4, 16, 16.5) \\ (9.3, 10, 10.6) & (21.5, 22, 22.5) & (16.4, 17, 17.3) & (10.4, 11, 11.4) \\ (10.2, 11, 11.5) & (14.3, 15, 15.5) & (21.4, 22, 22.3) & (14.3, 15, 15.4) \end{pmatrix} \quad (33)$$

Here, pure strategies correspond to the rows and the columns of matrix for $A_1B_1C_1, A_1B_1C_2, A_1B_2C_1, A_1B_2C_2, A_2B_1C_1, A_2B_1C_2, A_2B_2C_1, A_2B_2C_2, A_3B_1C_1, A_3B_1C_2, A_3B_2C_1, A_3B_2C_2$, and D_1, D_2, D_3, D_4, respectively.

The value of the game when players A and B set coalition from the matrix $(A \cup B, D)$ is

$$\widetilde{v}(\{A,B\}) = (12.06666, 12.33333, 12.63334). \tag{34}$$

Similarly, the value of the game when players A and C set coalition from the matrix $(A \cup C, D)$ is

$$\widetilde{v}(\{A,C\}) = (10.89534, 11.34884, 11.65818), \tag{35}$$

and from the matrix $(B \cup C, D)$ the value of the game when players B and C set coalition is

$$\widetilde{v}(\{B,C\}) = (8.6, 9, 9.4), \tag{36}$$

from the matrix $(A \cup B \cup C, D)$ the value of the game when players A, B and C set coalition is

$$\widetilde{v}(\{A,B,C\}) = (16.47906, 17.06977, 17.41167). \tag{37}$$

The mixed probabilities vector of this coalition which obtains the maximum value is

$$\begin{aligned}
x_1y_1z_1 &= 0,\ x_1y_1z_2 = 0.62789,\ x_1y_2z_1 = 0,\ x_1y_2z_2 = 0,\\
x_2y_1z_1 &= 0.04651,\ x_2y_1z_2 = 0.3256,\ x_2y_2z_1 = 0,\ x_2y_2z_2 = 0,\\
x_3y_1z_1 &= 0,\ x_3y_1z_2 = 0,\ x_3y_2z_1 = 0,\ x_3y_2z_2 = 0.
\end{aligned}$$

Then,

$$x^* = (0.62789, 0.37211, 0),\ y^* = (1, 0),\ z^* = (0.04651, 0.95349) \tag{38}$$

the characteristic function of the fuzzy cooperative game that is set with the coalition of the players A, B and C which play games with fuzzy payoffs against Nature is

$$\begin{array}{l}
\widetilde{v}(\{\varnothing\}) = (0,0,0),\ \widetilde{v}(\{A\}) = (6.9, 7, 7.1)\ ,\\
\widetilde{v}(\{B\}) = (3.9, 4, 4.2)\ , \widetilde{v}(\{C\}) = (2.5, 2.75, 2.875),\\
\widetilde{v}(\{A,B\}) = (12.06666, 12.33333, 12.63334),\\
\widetilde{v}(\{A,C\}) = (10.89534, 11.34884, 11.65818),\\
\widetilde{v}(\{B,C\}) = (8.6, 9, 9.4),\\
\widetilde{v}(\{A,B,C\}) = (16.47906, 17.06977, 17.41167).
\end{array} \tag{39}$$

As understand from above, each player obtain additive profit from the coalitions they set. the sum of the values of the game that players A, B and C obtained individually is

$$\widetilde{v}(\{A\}) + \widetilde{v}(\{B\}) + \widetilde{v}(\{C\}) = (13.3, 13.75, 14.175). \tag{40}$$

When they play together against Nature profit is

$$\widetilde{v}(\{A,B,C\}) = (16.47906, 17.06977, 17.41167). \tag{41}$$

Hence, from the coalition additive profit

$$\widetilde{v}(\{A,B,C\}) - [\widetilde{v}(\{A\}) + \widetilde{v}(\{B\}) + \widetilde{v}(\{C\})] = (2.30406, 3.31977, 4.11167)$$

is obtained.

The shares of the player A, B and C, with the help of the Shapley vector

$$\phi_A(\widetilde{v}) = (7.68668667, 7.845285, 7.906643333),$$

$$\phi_B(\widetilde{v}) = (5.039016667, 5.170865, 5.327553333),$$

$$\phi_C(\widetilde{v}) = (3.753356667, 4.05362, 4.177473333)$$

are obtained. However, the gain when the players A, B and C play individually would be

$$\widetilde{v}(\{A\}) = (6.9, 7, 7.1),$$
$$\widetilde{v}(\{B\}) = (3.9, 4, 4.2),$$
$$\widetilde{v}(\{C\}) = (2.5, 2.75, 2.875),$$

respectively. Here for the individually gain increase the players will obtain from the coalition for A is

$$\phi_A(\widetilde{v}) - \widetilde{v}(\{A\}) = (0.586686667, 0.845285, 1.006643333),$$

for B is

$$\phi_B(\widetilde{v}) - \widetilde{v}(\{B\}) = (0.839016667, 1.170865, 1.427553333),$$

and for C is

$$\phi_C(\widetilde{v}) - \widetilde{v}(\{C\}) = (0.878356667, 1.30362, 1.677473333).$$

4. Conclusion

In this paper, we have considered games with fuzzy payoffs. We proved that players who are playing a zero-sum game with fuzzy payoffs against Nature are able to increase their joint payoff, and hence their individual payoffs by cooperating. It is shown that, a cooperative game with the fuzzy characteristic function can be constructed via the optimal game values of the zero-sum games with fuzzy payoffs against Nature at which players' combine their strategies and act like a single player. It is also proven that, the fuzzy characteristic function that is constructed in this way satisfies the superadditivity condition. Thus we considered a transition from two-person zero-sum games with fuzzy payoffs to cooperative games with fuzzy payoffs. The fair allocation of the maximum payoff (game value) of this cooperative game among players is done using the Shapley vector.

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