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Multiobjective security game with fuzzy payoffs

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Abstract

A multiobjective security game problem with fuzzy payoffs is studied in this paper. The problem is formulated as a bilevel programming problem with fuzzy coefficients. Using the idea of nearest interval approximation of fuzzy numbers, the problem is transformed into a bilevel programming problem with interval coefficients. The Karush-Kuhn-Tucker conditions is applied then to reduce the problem to an interval multiobjective single-level problem. It is shown that the solutions of this problem are obtained by solving a single-objective programming problem. Validity and applicability of the method are illustrated by a practical example.

Keywords: Security game, Bilevel programming, Fuzzy numbers, Nearest interval approximation.

1 Introduction

Game theory analyzes optimal decision making problems under some conflict interests and/or conditions. Game theory is a branch of operations research and may be used for planning and conducting military operations (for more studies see [14, 24]). Now, game theory is used extensively in analyzing psychology, philosophy, sociology, politics, economics and military sciences (e.g. see [14, 18, 21, 22, 23, 24]). The great book by J. von Neumann and O. Morgenstern in 1944 [21] is as the starting point of the mathematical theory of games. There are remarkable advances for analysis of economic problems by game theory (e.g. see [12, 19]).

In security field, the players may be security forces (defenders), on one hand, and adversaries (attackers) on the other. Game-theoretic approaches assume that these players will anticipate the opponents' moves, and act wisely. Game theory provides a mathematical approach for allocating limited security resources to maximize their effectiveness. The communication between game theory and security has been studied for the last several decades [32]. Stackelberg games are used in analysing security problems, for example, police and robbers scenario [15], computer network security [21], missile defense systems [7], and terrorism [27]. Arms inspections and border patrolling were also modeled using inspection games [6].

Research on security game extensively was made by Milind Tambe [32]. In this book, he reviewed the presented works of his research team in the security game field. They studied different approaches of solving security game problems and presented various applications of these problems in real-world. Brown et al. [6] made a study on multiobjective security games. They explained motivation of introducing multiobjective security games and proposed different methods of solving the obtained multiobjective problem. Bigdeli and Hassanpour [3] studied multiobjective security game using goal programming in another research.

Studies of fuzzy games have been made by incorporating fuzzy set theory. Fuzzy set theory is used to model decision making problems involving vagueness due to the lack of information and/or imprecision of the available information on the problem situation [35]. In the field of fuzzy games, considerable studies have been made (e.g. see [1, 2, 4, 5, 8, 9,

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20, 23, 28, 29, 30, 33]).
In real-world decision making problems, decision makers have multiple objectives. Hence, it seems natural that the game theoretic approaches handle multiple objectives simultaneously. In this field, several studies have been made (e.g. see [10, 13, 16, 23]).
In this paper, we consider fuzzy multiobjective security games in which players' payoffs are expressed as fuzzy numbers and players' pure and mixed strategies are crisp. This problem has not been considered in previouse researches, based on the best knowledge of the authors. We propose a multiobjective fuzzy bilevel model (for example, for solving fuzzy single-objective bilevel programming problem refer to [25]) for these games.
The remainder of the paper is organized as follows. In Section 2, some preliminaries, necessary notations and definitions of fuzzy sets, interval arithmetic and the KKT conditions for the linear programming problems with interval-valued objective functions are presented. In Section 3, a method is proposed to solve multiobjective security games with fuzzy payoffs. In Section 4, a numerical example is presented to illustrate the mentioned approach. Finally, conclusion is made in Section 5.

2 Preliminaries

2.1 Fuzzy Sets and Interval Arithmetic

In this subsection, we recall some notations and preliminaries of fuzzy sets according to [11, 26, 35].
Let X denote a universal set. A fuzzy subset $\tilde{a}$ of X is defined by its membership function $\mu_{\tilde{a}} : X \to [0,1]$, which assigns to each element $x \in X$ a real number $\mu_{\tilde{a}}(x)$ in the interval $[0,1]$. The value of $\mu_{\tilde{a}}(x)$ represents the grade of membership of x in $\tilde{a}$. The fuzzy subset $\tilde{a}$ can be characterized as a set of ordered pairs of elements x and grades $\mu_{\tilde{a}}(x)$, and is often written as $\tilde{a} = \{(x, \mu_{\tilde{a}}(x)) | x \in X\}$. The support of the fuzzy set $\tilde{a}$ on X, denoted by $supp(\tilde{a})$, is the set of points $x \in X$ at which $\mu_{\tilde{a}}(x)$ is positive. The fuzzy set $\tilde{a}$ on X is said to be normal if there is $x \in X$ such that $\mu_{\tilde{a}}(x) = 1$. The α-cut of the fuzzy set $\tilde{a}$, denoted by $\tilde{a}_\alpha$, is an ordinary set defined by $\tilde{a}_\alpha = \{x \,|\mu_{\tilde{a}}(x) \geq \alpha\}$ where $\alpha \in (0,1]$, and for $\alpha = 0$, $\tilde{a}_\alpha = closure \{x \,|\mu_{\tilde{a}}(x) > 0\}$. The concept of α-cut serves as an important transferer between ordinary sets and fuzzy sets. It also plays an important role in the construction of a fuzzy set by a series of ordinary sets. Using the concept of α-cut, the fuzzy set $\tilde{a}$ can be represented by $\tilde{a} = \cup_{\alpha \in [0,1]} \alpha \tilde{a}_\alpha$ where $\alpha \tilde{a}_\alpha$ denotes the algebraic product of a scalar α with the α-cut $\tilde{a}_\alpha$. The fuzzy set $\tilde{a}$ in X is said to be a convex fuzzy set if its α-cuts are convex.
A fuzzy number is a convex normalized fuzzy set of the real line $\mathbb{R}$ whose membership function is piecewise continuous. From the definition of a fuzzy number $\tilde{a}$, it is significant to note that each α-cut $\tilde{a}_\alpha$ of a fuzzy number $\tilde{a}$ is a closed interval $[a^L_\alpha, a^R_\alpha]$.
A triangular fuzzy number $\tilde{a} = (a^l, a^m, a^r)$ is a special fuzzy number, whose membership function is given by

$$\mu_{\tilde{a}}(x) = \begin{cases} (x - a^l)/(a^m - a^l) & a^l \leq x \leq a^m \\ (a^r - x)/(a^r - a^m) & a^m \leq x \leq a^r \\ 0 & \text{otherwise}, \end{cases} \tag{1}$$

where a^m is the mean of $\tilde{a}$, and a^l and a^r are the left and right extreme points of $supp(\tilde{a})$, respectively. $\tilde{a} = (a^l, a^m, a^r)$ is called a non-negative triangular fuzzy number if $a^l \geq 0$ and $a^r > 0$. Let $\tilde{a} = (a^l, a^m, a^r)$ and $\tilde{b} = (b^l, b^m, b^r)$ be two triangular fuzzy numbers. By the extension principle of Zadeh [35], the addition of $\tilde{a}$ and $\tilde{b}$ is given by

$$\tilde{a} + \tilde{b} = (a^l + b^l, a^m + b^m, a^r + b^r),$$

and the scalar multiplication of $\tilde{a}$ by the scalar $\lambda \in \mathbb{R}$ is given by

$$\lambda \tilde{a} = \begin{cases} (\lambda a^l, \lambda a^m, \lambda a^r) & \lambda \geq 0 \\ (\lambda a^r, \lambda a^m, \lambda a^l) & \lambda < 0. \end{cases}$$

The following theorem gives an interval approximation for each fuzzy number.

Theorem 2.1. [17] *Suppose $\tilde{a}$ is a fuzzy number with α-cut $[a^L_\alpha, a^R_\alpha]$. The nearest interval approximation of $\tilde{a}$ is*

$$\left[\int_0^1 a^L_\alpha d\alpha, \int_0^1 a^R_\alpha d\alpha \right].$$

The operations on intervals may be explicitly expressed as follows.
The set of all closed intervals in $\mathbb{R}$ is denoted by Ω. Let $a = \left[a^L, a^R\right]$ and $b = \left[b^L, b^R\right]$ be two intervals. Then,

$$a+b=\left[a^L+b^L,a^R+b^R\right],\ a-b=\left[a^L-b^R,a^R-b^L\right],$$

$$\lambda a=\begin{cases}\left[\lambda a^L,\lambda a^R\right] & \lambda\geq 0\\ \left[\lambda a^R,\lambda a^L\right] & \lambda<0,\end{cases}$$

where λ is a real scalar.

Definition 2.2. [34] *Let $a=\left[a^L,a^R\right]$ and $b=\left[b^L,b^R\right]$ be two intervals. The order relations $\preceq_{LR}$ and $\prec_{LR}$ between a and b are defined as*
(i) $a\preceq_{LR}b$ if and only if $a^L\leq b^L$ and $a^R\leq b^R$.
(ii) $a\prec_{LR}b$ if and only if $a\preceq_{LR}b$ and $a\neq b$.

In this definition, we do not say that an interval is larger than another; instead, we usually say that an interval is better than another.
Now, we define an order relation of the vectors with interval components.

Definition 2.3. *Let a and b be two n-dimensional vectors with interval components $a_i=\left[a_i^L,a_i^R\right]$, $b_i=\left[b_i^L,b_i^R\right]$, respectively. The order relation $\prec_{LR}$ between a and b is defined as $a\prec_{LR}b$ if and only if $a_i\preceq_{LR}b_i$ for all i, and $a_j\neq b_j$ for at least one j.*

2.2 The Karush-Kuhn-Tucker (KKT) Optimality Conditions to Interval-valued Linear Programming

In this subsection some definitions of the interval-valued functions are recalled and the KKT optimality conditions for interval-valued linear programming are stated.
Before expressing the solution concepts and preliminaries of interval-valued functions, we first illustrate the solution concept of multiobjective programming problem called efficient solution.
Consider the following multiobjective programming problem:

$$\begin{array}{l}\max\left(f_1(x),f_2(x),...,f_k(x)\right)\\ \quad x\in X.\end{array}\tag{2}$$

Definition 2.4. [31] *$x^*\in X$ is said to be an efficient solution of the problem (2) if there does not exist another $x\in X$ such that $f_i(x)\geq f_i(x^*)$ for all i and $f_j(x)\neq f_j(x^*)$ for at least one j.*

As a slightly weaker solution concept, weak efficient solution is defined as follows.

Definition 2.5. [31] *$x^*\in X$ is said to be a weak efficient solution of the problem (2) if there does not exist another $x\in X$ such that $f_i(x)>f_i(x^*)$ for all i.*

An interval-valued function can be denoted by $f(x)=[f^L(x),f^R(x)]$ where $f^L(x)\leq f^R(x)$ for any $x\in\mathbb{R}^n$.

Definition 2.6. [34] *Let $f(x)=[f^L(x),f^R(x)]$ be an interval-valued function defined on a convex set $X\subseteq\mathbb{R}^n$. Then, f is said to be LR-convex at x^* if*

$$f(\lambda x^*+(1-\lambda)x)\preceq_{LR}\lambda f(x^*)+(1-\lambda)f(x)$$

for all $\lambda\in(0,1)$ and each $x\in X$.

Proposition 2.7. [34] *Let X be a convex subset of $\mathbb{R}^n$ and f be an interval-valued function defined on X. f is LR-convex at x^* if and only if f^L and f^R are convex at x^*.*

Definition 2.8. [34] *Let f be an interval-valued function defined on $X\subseteq\mathbb{R}^n$ and $x_0=(x_1^0,...,x_n^0)\in X$ be fixed. Then,*
1) f is said to be weakly differentiable at x_0 if the real-valued functions f^L and f^R are differentiable at x_0 (in the usual sense).
2) f is weakly continuously differentiable at x_0 if the real-valued functions f^L and f^R are continuously differentiable at x_0 (i.e., all the partial derivatives of f^L and f^R exist on some neighbourhoods of x_0 and are continuous at x_0).

The concepts of limit, continuity and differentiation of interval valued functions can be seen in [34].
Consider the following interval-valued optimization problem:

$$\begin{aligned} &\min \ f(x) = [f^L(x), f^R(x)] \\ &s.t. \quad x \in X = \{x \in \mathbb{R}^n | g_i(x) \leq 0, \ i = 1, ..., m\}. \end{aligned} \tag{3}$$

where $g_i : \mathbb{R}^n \to \mathbb{R}, \ i = 1, ..., m$, are convex real-valued functions. Clearly, X is a convex set.

Definition 2.9. [34] *$x^* \in X$ is said to be an LR optimal solution of the problem (3) if there exists no $x \in X$ such that $f(x) \prec_{LR} f(x^*)$.*

We say that the real-valued constraint functions $g_i, i = 1, ..., m$, satisfy the KKT assumptions at x* if g_i's are convex on $\mathbb{R}^n$ and continuously differentiable at x^* for $i = 1, ..., m$. The KKT optimality conditions are stated as follows.

Theorem 2.10. [34] *Assume that the real constraint functions $g_i, i = 1, ..., m$, of the problem (3) satisfy the KKT assumptions at x^* and the interval-valued objective function $f : \mathbb{R}^n \to \Omega$ is LR-convex and weakly continuosly differentiable at $x^* \in X$. If there exist (Lagrange) multipliers $0 < \lambda^L, \lambda^R$ and $0 \leq \mu_i \in \mathbb{R}, i = 1, ..., m$, such that*

$$(i) \ \lambda^L \nabla f^L(x^*) + \lambda^R \nabla f^R(x^*) + \sum_{i=1}^{m} \mu_i \nabla g_i(x^*) = 0$$
$$(ii) \ \mu_i g_i(x^*) = 0, \forall i = 1, ..., m$$

then x^* is an LR optimal solution of the problem (3).

3 Multiobjective Security Game with Fuzzy Payoffs

A multiobjective security game is a game between a defender and n types of attackers. Using m identical resources, the defender want to prevent attacks by covering targets $T = \{1, 2, ..., p\}$. Note that m identical resources can be distributed continuously among the targets. The multiobjective security game model adopts the Stackelberg framework in which the defender acts first by committing to a strategy that the attackers are able to observe and then best respond. We represent defender's strategy as a coverage vector $c = (c_1, ..., c_p) \in C$ where for $k = 1, ..., p$, c_k is the amount of coverage placed on target k and represents the probability that the defender prevents successfully any attack on k. Also, we assume that the covering of each target costs the same amount of resources. Thus, it made the equivalence between the amount of resources placed on a target and the probability that the target being covered. Then, given a budget of m resources, the defender could choose to fully protect m targets. According to Stackelberg paradigm, the attackers can easily select one of the targets that are known to be unprotected. Therefore, the defender considers mixed strategies where resources are allocated to a larger set of partially protected targets. An attacker is able to observe this mixed strategy [6].
We consider the defender's strategy space as follows:

$$C = \{c = (c_1, ..., c_p) | 0 \leq c_k \leq 1, \sum_{k=1}^{p} c_k \leq m\}.$$

The mixed strategy for attacker type i is a vector $a_i = (a_i^1, a_i^2, ..., a_i^p)$, where a_i^k is the probability of attacking k. Thus attacker type i's strategy space is as follows:

$$A_i = \left\{ a_i = (a_i^1, a_i^2, \ldots, a_i^p) \, | a_i^k \geq 0, \, k \in T, \sum_{k=1}^{p} a_i^k = 1 \right\}.$$

When the defender chooses a strategy and attacker type i chooses a target for attacking, each of them obtains a payoff which indicates the amount of profit or loss. It may specifically represent some measure of loss of life or economic loss or a combination of both and other factors. These payoffs result by domain experts, and may arise from calculations based on a set of answers to a set of questions by domain experts; also, may be generated by other researchers with expertise in risk analysis. In most real-world situations, the possible values of players' payoffs are often imprecisely or ambiguously known to the experts. Therefore, it would be appropriate to interpret the experts' understanding of the parameters as fuzzy data.
Assume that $\tilde{U}_i^{c,d}(k)$ is the defender's utility if k is chosen by attacker type i and is fully covered by defender. If k is uncovered, the defender's penalty is $\tilde{U}_i^{u,d}(k)$. Similarly, the attacker's utilities are denoted by $\tilde{U}_i^{c,a}(k)$ and $\tilde{U}_i^{u,a}(k)$.
For a strategy profile $< c, a_i >$ for the game between the defender and attacker type i, the expected payoffs for both players are given by

$$\tilde{U}_i^d(c,a_i) = \sum_{k=1}^{p} a_i^k \tilde{U}_i^d(c_k,k),$$
$$\tilde{U}_i^a(c,a_i) = \sum_{k=1}^{p} a_i^k \tilde{U}_i^a(c_k,k),$$

where

$$\tilde{U}_i^d(c_k,k) = c_k\tilde{U}_i^{c,d}(k) + (1-c_k)\tilde{U}_i^{u,d}(k),$$
$$\tilde{U}_i^a(c_k,k) = c_k\tilde{U}_i^{c,a}(k) + (1-c_k)\tilde{U}_i^{u,a}(k),$$

are the payoffs received by the defender and attacker type i, respectively, if target k is attacked and is covered by c_k resources.
In this game, defender makes decision first. Defender encounters n types of attackers and wants to maximize utilities. He wants to allocate resources for covering targets. In fact the defender commits first to an optimal strategy based on the assumption that the attacker will be able to observe this strategy and then choose an optimal response. Therefore, the mentioned security game is formulated by a fuzzy bilevel programming problem as follows:

$$\begin{array}{ll} max_{c\in C}\left(\tilde{U}_1^d(c,a_1),...,\tilde{U}_n^d(c,a_n)\right) & \\ \sum_{k=1}^{p} c_k \leq m & \\ 0\leq c_k\leq 1 & k=1,...,p, \\ \left.\begin{array}{l} where\, a_i\, solves \\ \quad max_{a_i}\tilde{U}_i^a(c,a_i) \\ \quad \sum_{k=1}^{p} a_i^k = 1 \\ \quad a_i^k \geq 0 \end{array}\right\} & i=1,...,n. \end{array} \tag{4}$$

We indicate fuzzy parameters by fuzzy numbers. Now, by considering the nearest interval approximation of fuzzy numbers, the above problem is transformed into the following interval bilevel programming problem.

$$\begin{array}{ll} max_{c\in C}\left(\left[U_1^{dL}(c,a_1),U_1^{dR}(c,a_1)\right],...,\left[U_n^{dL}(c,a_n),U_n^{dR}(c,a_n)\right]\right) & \\ \sum_{k=1}^{p} c_k \leq m & \\ 0\leq c_k\leq 1 \quad k=1,...,p & \\ \left.\begin{array}{l} where\, a_i\, solves \\ \quad max_{a_i}\left[U_i^{aL}(c,a_i),U_i^{aR}(c,a_i)\right] \\ \quad \sum_{k=1}^{p} a_i^k = 1 \\ \quad a_i^k \geq 0 \end{array}\right\} & i=1,...,n \end{array} \tag{5}$$

Before explaining of solving method of the problem (5), we illustrate the solution concepts of the bilevel problem (5). Assume that feasible region of the bilevel problem (5) is

$$S = \left\{(c_1,...,c_p,a_1,...,a_n)|\, 0\leq c_k\leq 1, \sum_{k=1}^{p} c_k\leq m,\;\; a_i^k\geq 0, \sum_{k=1}^{p} a_i^k = 1\right\}.$$

The decision space of the defender is

$$S_1 = \left\{c=(c_1,...,c_p)|0\leq c_k\leq 1, \sum_{k=1}^{p} c_k\leq m\right\}.$$

For given $c\in S_1$, the attacker type i's problem can be written as

$$\begin{array}{ll} max_{a_i} & \left[U_i^{aL}(c,a_i),U_i^{aR}(c,a_i)\right] \\ s.t. & \sum_{k=1}^{p} a_i^k = 1 \\ & a_i^k\geq 0 \end{array} \tag{6}$$

For each fixed $c \in S_1$, we denote the constraint region of the attacker type i's problem by

$$S_{2i} = \left\{ a_i = \left(a_i^1, a_i^2, \ldots, a_i^p\right) | a_i^k \geq 0, \sum_{k=1}^{p} a_i^k = 1 \right\}.$$

Now, the concept of an optimal solution of the problem (6) can be introduced by the ordering relation $\preceq_{LR}$ as follows.

Definition 3.1. *For any given $c \in S_1$, a feasible solution $a_i^* \in S_{2i}$ is said to be an LR efficient solution of the problem (6) if there exists no other feasible solution $a_i \in S_{2i}$ such that*

$$\left[U_i^{aL}(c, a_i^*), U_i^{aR}(c, a_i^*)\right] \preceq_{LR} \left[U_i^{aL}(c, a_i), U_i^{aR}(c, a_i)\right].$$

We denote the set of LR efficient solutions of the problem (6) as RE_i.
For a bilevel programming problem, the solution set of the lower level problem determines the feasible region of problem (5) as follows:

$$RE_{BP} = \{(c, a_1, ..., a_n) | (c, a_1, ..., a_n) \in S,\, a_i \in RE_i\}.$$

Definition 3.2. *$(c^*, a_1^*, ..., a_n^*) \in RE_{BP}$ is said to be an LR efficient solution of problem (5) if there exists no other feasible solution $(c, a_1, ..., a_n) \in RE_{BP}$ such that $U_i^d(c^*, a_i^*) \preceq_{LR} U_i^d(c, a_i)$, for all i, and $U_j^d(c^*, a_j^*) \neq U_j^d(c, a_j)$ for at least one j.*

In the following an approach to solve the problem (5) is proposed (Ren and Wang [25] presented a method to solve single-objective bilevel programming problem with interval coefficients.). We use KKT approach to solve the problem (5), which is one of the most common approaches to solve bilevel programming problems. By this means, the bilevel programming problem can be converted to a single-level problem by replacing the lower level problem with the equivalent KKT conditions.
By fixing the policy of the defender to some policy c, the attacker type i (for $i = 1, ..., n$) solves the following problem to find his optimal response to c:

$$\begin{array}{ll} max_{a_i} & \left[U_i^{aL}(c, a_i), U_i^{aR}(c, a_i)\right] \\ s.t. & \sum\limits_{k=1}^{p} a_i^k = 1 \\ & a_i^k \geq 0 \quad k = 1, .., p. \end{array} \tag{7}$$

According to the KKT optimality conditions in interval-valued programming (Theorem 2.10), given the defender's strategy c, the optimal response of attacker type i satisfies the optimality conditions

$$\begin{array}{ll} \lambda_i^L \frac{\partial U_i^{aL}(c,a_i)}{\partial a_i^k} + \lambda_i^R \frac{\partial U_i^{aR}(c,a_i)}{\partial a_i^k} - \mu_0^i + \mu_k^i = 0 & k = 1, ..., p \\ \mu_k^i a_i^k = 0 & k = 1, ..., p \\ \mu_k^i \geq 0, \lambda_i^L, \lambda_i^R \geq 0 & k = 1, ..., p, \end{array} \tag{8}$$

where

$$\begin{array}{l} \frac{\partial U_i^{aL}(c,a_i)}{\partial a_i^k} = c_k U_i^{c,aL}(k) + (1 - c_k) U_i^{u,aL}(k), \\ \frac{\partial U_i^{aR}(c,a_i)}{\partial a_i^k} = c_k U_i^{c,aR}(k) + (1 - c_k) U_i^{u,aR}(k). \end{array}$$

Thus, considering the n types of attackers, the defender solves the following problem to maximize its own utility.

$$\begin{array}{lll} max_{c \in C} & \left(\left[U_1^{dL}(c, a_1), U_1^{dR}(c, a_1)\right], ..., \left[U_n^{dL}(c, a_n), U_n^{dR}(c, a_n)\right]\right) & \\ s.t. & \sum\limits_{k=1}^{p} c_k \leq m & \\ & 0 \leq c_k \leq 1 & k = 1, ..., p \\ & \lambda_i^L \frac{\partial U_i^{aL}(c,a_i)}{\partial a_i^k} + \lambda_i^R \frac{\partial U_i^{aR}(c,a_i)}{\partial a_i^k} - \mu_0^i + \mu_k^i = 0 & k = 1, ..., p,\ i = 1, ..., n \\ & \mu_k^i a_i^k = 0 & k = 1, ..., p,\ \ i = 1, ..., n \\ & \sum\limits_{k=1}^{p} a_i^k = 1, \quad a_i^k \geq 0 & k = 1, ..., p,\ \ i = 1, ..., n \\ & \mu_k^i \geq 0, \lambda_i^L, \lambda_i^R \geq 0 & k = 1, ..., p,\ \ i = 1, ..., n. \end{array}$$

(9)

The problem (9) is a multiobjective mathematical programming problem with interval coefficients in objective functions. To solve the problem (9) first we allocate weights to the objectives that indicate probabilities of facing each attacker with defender. Thus the defender faces the following problem:

$$\begin{array}{lll} max_{c\in C} \left[\sum\limits_{i=1}^{n} w_i U_i^{dL}(c,a_i),\ \sum\limits_{i=1}^{n} w_i U_i^{dR}(c,a_i)\right] & & \\ s.t. \quad \sum\limits_{k=1}^{p} c_k \leq m & & \\ \quad 0 \leq c_k \leq 1 & k=1,...,p & \\ \quad \lambda_i^L \frac{\partial U_i^{aL}(c,a_i)}{\partial a_i^k} + \lambda_i^R \frac{\partial U_i^{aR}(c,a_i)}{\partial a_i^k} - \mu_0^i + \mu_k^i = 0 & k=1,...,p,\ i=1,...,n & \\ \quad \mu_k^i a_i^k = 0 & k=1,...,p,\ \ i=1,...,n & \\ \quad \sum\limits_{k=1}^{p} a_i^k = 1 & i=1,...,n & \\ \quad a_i^k \geq 0,\ \mu_k^i \geq 0, \lambda_i^L, \lambda_i^R \geq 0 & k=1,...,p,\ \ i=1,...,n & \end{array} \tag{10}$$

where $w \in \left\{ w \in \mathbb{R}^n \middle| 0 \leq w_i \leq 1, \sum\limits_{i=1}^{n} w_i \leq p \right\}$.

We consider the problem (10) as a bi-objective mathematical programming problem by considering the left bound as the first objective and the right bound as the second objective. Because according to Definition (2.2), a better interval is an interval with bigger left and right bounds. We have the following problem:

$$\begin{array}{ll} max \sum\limits_{i=1}^{n} w_i U_i^{dL}(c,a_i) & \\ max \sum\limits_{i=1}^{n} w_i U_i^{dR}(c,a_i) & \\ s.t. \quad \sum\limits_{k=1}^{p} c_k \leq m & \\ \quad 0 \leq c_k \leq 1 & k=1,...,p \\ \quad \lambda_i^L \frac{\partial U_i^{aL}(c,a_i)}{\partial a_i^k} + \lambda_i^R \frac{\partial U_i^{aR}(c,a_i)}{\partial a_i^k} - \mu_0^i + \mu_k^i = 0 & k=1,...,p,\ i=1,...,n \\ \quad \mu_k^i a_i^k = 0, & k=1,...,p,\ \ i=1,...,n \\ \quad \sum\limits_{k=1}^{p} a_i^k = 1, \quad a_i^k \geq 0 & k=1,...,p,\ \ i=1,...,n \\ \quad \mu_k^i \geq 0, \lambda_i^L, \lambda_i^R \geq 0 & k=1,...,p,\ \ i=1,...,n \end{array} \tag{11}$$

Now searching for LR efficient solutions to the fuzzy bilevel programming problem turns out to solve the bi-objective optimization problem.

Theorem 3.3. *If there exist $\lambda_i^L \geq 0, \lambda_i^R \geq 0, \mu^i = (\mu_0^i, \mu_1^i, ..., \mu_p^i) \in \mathbb{R}^{p+1}$ with $\mu_1^i, ..., \mu_p^i \geq 0, i = 1, ..., n$ such that $(c^*, a_1^*, ..., a_n^*, \lambda_i^L, \lambda_i^R, \mu^i)$ is an efficient solution of the problem (11) then $(c^*, a_1^*, ..., a_n^*)$ is an LR efficient solution of the problem (5).*

Proof. Note that the lower level problem of the problem (5) is replaced with its KKT optimality conditions which are the same as the last three constraints in the problem (11). This means that the feasible point $(c^*, a_1^*, ..., a_n^*)$ in the problem (11) satisfies the constraints of the problem (5). Now, suppose that $(c^*, a_1^*, ..., a_n^*)$ is not an LR efficient solution of the problem (5). Then there exists a feasible point $(c, a_1, ..., a_n)$ such that

$$\left[U_i^{dL}(c^*,a_i^*), U_i^{dR}(c^*,a_i^*)\right] \preceq_{LR} \left[U_i^{dL}(c,a_i), U_i^{dR}(c,a_i)\right] \ \ \forall i \in \{1,...,n\}$$

and

$$\left[U_j^{dL}(c^*,a_j^*), U_j^{dR}(c^*,a_j^*)\right] \neq \left[U_j^{dL}(c,a_j), U_j^{dR}(c,a_j)\right]$$

for least one j.

This means that

$$U_i^{dL}(c^*,a_i^*) \leq U_i^{dL}(c,a_i),$$

$$U_i^{dR}(c^*,a_i^*) \leq U_i^{dR}(c,a_i),$$

for all i and

$$U_j^{dL}(c^*, a_j^*) < U_j^{dL}(c, a_j)$$

or

$$U_j^{dR}(c^*, a_j^*) < U_j^{dR}(c, a_j)$$

for least one j. Therefore,

$$\sum_{i=1}^{n} w_i U_i^{dL}(c^*, a_i^*) \leq \sum_{i=1}^{n} w_i U_i^{dL}(c, a_i),$$
$$\sum_{i=1}^{n} w_i U_i^{dR}(c^*, a_i^*) \leq \sum_{i=1}^{n} w_i U_i^{dR}(c, a_i),$$

and at least one of them is strict inequality. The above relation contradicts the assumption that $(c^*, a_1^*, ..., a_n^*)$ is efficient solution of the problem (11). □

The efficient solutions of a multiobjective optimization problem can be obtained by any suitable method such as scalarization methods [31]. We propose a new approach in this situation to solve the problem (11). According to the fact that the defender is cautious, we consider a pessimistic process of the problem (11) as the following problem,

$$\begin{array}{lll}
max \sum_{i=1}^{n} w_i U_i^{dL}(c, a_i) & & \\
s.t. \ \sum_{i=1}^{n} w_i U_i^{dL}(c, a_i) \leq \sum_{i=1}^{n} w_i U_i^{dR}(c, a_i) & & \\
\sum_{k=1}^{p} c_k \leq m & & \\
0 \leq c_k \leq 1 & & k = 1, ..., p \\
\lambda_i^L \frac{\partial U_i^{aL}(c,a_i)}{\partial a_i^k} + \lambda_i^R \frac{\partial U_i^{aR}(c,a_i)}{\partial a_i^k} - \mu_0^i + \mu_k^i = 0 & & k = 1, ..., p, \ i = 1, ..., n \\
\mu_k^i a_i^k = 0 & & k = 1, ..., p, \ i = 1, ..., n \\
\sum_{k=1}^{p} a_i^k = 1, \quad a_i^k \geq 0, \ \mu_k^i \geq 0, \lambda_i^L, \lambda_i^R \geq 0 & & k = 1, ..., p, \ i = 1, ..., n.
\end{array} \tag{12}$$

The relationships between the optimal solution of the problem (12) and the efficient solution of the problem (11) can be characterized by the following theorem.

Theorem 3.4. *1) If $(c^*, a_1^*, ..., a_n^*)$ is a unique optimal solution of the problem (12) then $(c^*, a_1^*, ..., a_n^*)$ is an efficient solution of the problem (11).*

2) If $(c^*, a_1^*, ..., a_n^*)$ is a optimal solution of the problem (12) then $(c^*, a_1^*, ..., a_n^*)$ is a weak efficient solution of the problem (11).

Proof. 1) Assume that the unique optimal solution $(c^*, a_1^*, ..., a_n^*)$ to the problem (12) is not an efficient solution of the problem (11). Then there exists $(c, a_1, ..., a_n)$ of the feasible region of (11) such that

$$\sum_{i=1}^{n} w_i U_i^{dL}(c^*, a_i^*) \leq \sum_{i=1}^{n} w_i U_i^{dL}(c, a_i), \tag{13}$$

and

$$\sum_{i=1}^{n} w_i U_i^{dR}(c^*, a_i^*) \leq \sum_{i=1}^{n} w_i U_i^{dR}(c, a_i), \tag{14}$$

and at least one of them is strict inequality.
The relation (13) contradicts the assumption that $(c^*, a_1^*, ..., a_n^*)$ is a unique optimal solution of the problem (12).

2) Assume that the optimal solution $(c^*, a_1^*, ..., a_n^*)$ to the problem (12) is not a weak efficient solution of the problem (11). Then there exists $(c, a_1, ..., a_n)$ of the feasible region of (11) such that

$$\sum_{i=1}^{n} w_i U_i^{dL}(c^*, a_i^*) < \sum_{i=1}^{n} w_i U_i^{dL}(c, a_i),$$

$$(15)$$

and

$$\sum_{i=1}^{n} w_i U_i^{dR}(c^*, a_i^*) < \sum_{i=1}^{n} w_i U_i^{dR}(c, a_i). \tag{16}$$

The relation (15) contradicts the assumption that $(c^*, a_1^*, ..., a_n^*)$ is a optimal solution of the problem (12). □

From Theorem 3.4, if the uniqueness of the optimal solution $(c^*, a_1^*, ..., a_n^*)$ for the problem (12) is not guaranteed, it is necessary to perform the efficiency test of $(c^*, a_1^*, ..., a_n^*)$. The efficiency test for $(c^*, a_1^*, ..., a_n^*)$ can be performed by solving the following nonlinear programming with the decision variables $(c, a_1, ..., a_n)$ and $\epsilon = (\epsilon_1, \epsilon_2)$.

$$\begin{aligned}
max\ & \epsilon_1 + \epsilon_2 \\
s.t.\ & \sum_{i=1}^{n} w_i U_i^{dL}(c, a_i) + \epsilon_1 = \sum_{i=1}^{n} w_i U_i^{dL}(c^*, a_i^*) \\
& \sum_{i=1}^{n} w_i U_i^{dR}(c, a_i) + \epsilon_2 = \sum_{i=1}^{n} w_i U_i^{dR}(c^*, a_i^*) \\
& \sum_{k=1}^{p} c_k \leq m \\
& 0 \leq c_k \leq 1 && k = 1, ..., p \\
& \lambda_i^L \frac{\partial U_i^{aL}(c, a_i)}{\partial a_i^k} + \lambda_i^R \frac{\partial U_i^{aR}(c, a_i)}{\partial a_i^k} - \mu_0^i + \mu_k^i = 0 && k = 1, ..., p,\ i = 1, ..., n \\
& \mu_k^i a_i^k = 0 && k = 1, ..., p,\ \ i = 1, ..., n \\
& \sum_{k=1}^{p} a_i^k = 1, \ \ a_i^k \geq 0,\ \mu_k^i \geq 0, \lambda_i^L, \lambda_i^R \geq 0 && k = 1, ..., p,\ \ i = 1, ..., n \\
& \epsilon_1 \geq 0, \epsilon_2 \geq 0
\end{aligned} \tag{17}$$

For the optimal solutions $(\bar{c}, \bar{a}_1, ..., \bar{a}_n)$ and $\bar{\epsilon} = (\bar{\epsilon}_1, \bar{\epsilon}_2)$ of this nonlinear programming problem, the following theorem holds.

Theorem 3.5. *For the optimal solution $(\bar{c}, \bar{a}_1, ..., \bar{a}_n)$ and $\bar{\epsilon} = (\bar{\epsilon}_1, \bar{\epsilon}_2)$ of the efficiency test problem (17),*
I) If $\bar{\epsilon}_1 = 0$ and $\bar{\epsilon}_2 = 0$, then $(c^*, a_1^*, ..., a_n^*)$ is an efficient solution of the problem (11).
II) If at least one $\bar{\epsilon}_l > 0, l = 1, 2$, then not $(c^*, a_1^*, ..., a_n^*)$ but $(\bar{c}, \bar{a}_1, ..., \bar{a}_n)$ is the efficient solution.

Proof. The proof of Theorem 3.5 is similar to the presented proof in [26] for efficiency test of the multiobjective optimization problems. □

4 Numerical Example

In this section we consider a practical example taken from [6] with little changes. This example considers the security in the metro system. The metro system consists of several stations and maintains daily several passengers. Usually defender (security police of metro system) faces the three adversary types: ticketless travelers, criminals, and terrorists. A significant number of the rail stations feature barrier-free entrances that do not employ static security measures such as metal detectors or turnstiles. Instead, randomized patrols and inspections are utilized in order to verify that passengers have purchased a valid ticket as well as to generally maintain security of the system. Thus, police must make decisions on how best to allocate their available security resources as well as on how frequently to visit each station. Each of the three adversary types are distinct and present a unique set of challenges which may require different responses by the security force. For example, each adversary may have different preferences over the stations they choose to target. Ticketless travelers may choose to fare evade at busier stations thinking that the larger crowds decrease the likelihood of having their ticket checked. Whereas, criminals may prefer to commit crimes at less frequented stations, as they believe the reduced crowds will result in a smaller security presence. Finally, terrorists may prefer to strike stations which hold economic or cultural significance, as they believe that such choice of targets can help achieve their political goals. Security force may also have different motivation for preventing the various adversary types. Deploying security policies that target ticketless travelers can help to recuperate a portion of this lost revenue as it implicitly encourages passengers to purchase tickets. Pursuing criminals will reduce the amount of property damage and violent crimes, increasing the overall sense of passenger safety. Finally, due to the highly sensitive nature of the information, statistics regarding the frequency and severity of any terrorist threats targeting the transit system are not made available to the public.

	station1		station2	
	covered	uncovered	covered	uncovered
defender	(3,5,6)	(-3,-2,-1)	(9,10,11)	(2,3,5)
attacker	(-2,-1,0)	(2,4,5)	(-2,-1,0)	(9,10,11)

Table 1: The Game Matrix of Defender and Attacker Type 1

	station1		station2	
	covered	uncovered	covered	uncovered
defender	(0,1,2)	(0,0,0)	(1,2,4)	(-3,-2,-1)
attacker	(-2,-1,0)	(0,1,2)	(0,0,0)	(3,5,6)

Table 2: The Game Matrix of Defender and Attacker Type 2

Thus, despite the relatively low likelihood of a terrorist attack, security measures designed to prevent and mitigate the effects of terrorism must always remain a priority, given the substantial number of lives at risk. The security force is required to simultaneously consider all of the threats posed by the different adversary types in order to design effective and robust security strategies. Thus, defending against each adversary type can be viewed as an objective for security police. While these objectives are not strictly conflicting (e.g. checking tickets at a station may lead to a reduction in crime), focusing security measures too much on one adversary may neglect the threat posed by the others. As security force has finite resources to protect all of the stations in the city, it is not possible to protect all stations against all adversaries at all times. Therefore, strategic decisions must be made such as where to allocate security resources and for how long.

Consider the mentioned security game among a defender (security force of the metro system) and the three attackers types. Suppose that there are two stations and one security resource (i.e. $n = 3, p = 2, m = 1$). The payoffs matrices of game are represented in Tables 1, 2, 3.

Assume that probability of facing the three types of attackers is the same and equals to $\frac{1}{3}$. The mathematical

	station1		station2	
	covered	uncovered	covered	uncovered
defender	(1,2,4)	(-2,-1,0)	(2,3,5)	(-3,-2,-1)
attacker	(-3,-2,-1)	(0,1,2)	(-5,-3,-2)	(2,4,5)

Table 3: The Game Matrix of Defender and Attacker Type 3

programming problem (12) is as follows:

$$
\begin{aligned}
\max\ & \tfrac{1}{3}\left\{-2.5a_1^1+6.5a_1^1c_1+2.5a_1^2+7c_2+a_1^2c_2+\right.\\
&\quad \left.+0.5a_2^1c_1-2.5a_2^2+4c_2a_2^2-1.5a_3^1+3c_1a_3^1-2.5a_3^2+5a_3^2c_2\right\}\\
s.t.\ & \\
& \tfrac{1}{3}\left\{-2.5a_1^1+6.5a_1^1c_1+2.5a_1^2+7c_2+a_1^2c_2+\right.\\
&\quad \left.+0.5a_2^1c_1-2.5a_2^2+4c_2a_2^2-1.5a_3^1+3c_1a_3^1-2.5a_3^2+5a_3^2c_2\right\}\leq\\
& \tfrac{1}{3}\left\{-1.5a_1^1+7a_1^1c_1+4a_1^2+6.5c_2a_1^2+1.5a_1^2c_1+\right.\\
&\quad \left.-1.5a_2^2+4.5c_2a_2^2-.5a_3^1+3.5c_2a_3^1-1.5a_3^2+5.5c_2a_3^2\right\}\\
& \lambda_1^L(3-4.5c_1)+\lambda_1^R(4.5-5c_1)-\mu_1^0+\mu_1^1=0\\
& \lambda_1^L(9.5-11c_2)+\lambda_1^R(10.5-11c_2)-\mu_1^0+\mu_1^2=0\\
& \lambda_2^L(.5-2c_1)+\lambda_2^R(1.5-2c_1)-\mu_2^0+\mu_2^1=0\\
& \lambda_2^L(4-4c_2)+\lambda_2^R(5.5-5.5c_2)-\mu_2^0+\mu_2^2=0\\
& \lambda_3^L(.5-3c_1)+\lambda_3^R(1.5-3c_1)-\mu_3^0+\mu_3^1=0\\
& \lambda_3^L(3-7c_2)+\lambda_3^R(4.5-7c_2)-\mu_3^0+\mu_3^2=0\\
& \mu_1^1a_1^1=0\\
& \mu_2^1a_1^2=0\\
& \mu_1^2a_2^1=0\\
& \mu_2^2a_2^2=0\\
& \mu_1^3a_3^1=0\\
& \mu_2^3a_3^2=0\\
& 0\leq c_1\leq 1\\
& 0\leq c_2\leq 1\\
& c_1+c_2\leq 1\\
& a_1^1+a_1^2=1\\
& a_2^1+a_2^2=1\\
& a_3^1+a_3^2=1\\
& a_1^1,a_1^2,a_2^1,a_2^2,a_3^1,a_3^2\geq 0\\
& \lambda_i^L,\lambda_i^R,\mu_1^i,\mu_2^i,\mu_3^i\geq 0\quad i=1,2,3
\end{aligned}
\tag{18}
$$

The optimal solution of the problem (18) obtained by Lingo software is

$$c_1=0.29, c_2=0.71, a_1^1=0, a_1^2=1, a_2^1=0, a_2^2=1, a_3^1=0, a_3^2=1.$$

This means that the defender should protect the station 2 with more presence of the security source.

5 Conclusions

In this paper, security game in the uncertain environment was considered where the security forces must balance multiple objectives. A method was presented to solve the multiobjective security game with fuzzy payoffs. First, this game was formulated as bilevel programming problem with fuzzy coefficients. By the idea of nearest interval approximation of the fuzzy numbers, the mentioned problem was rewritten as bilevel programming problem with interval coefficients. The KKT optimality conditions in lower level problem of bilevel problem were applied. By this approach, the bilevel programming problem was transformed to a single level programming problem with interval coefficients in objective functions. By solving this problem, optimal strategies of defender were obtained. It was shown that the defender's strategies in facing different types of attackers can be obtained by solving a single-objective programming problem. The main advantage of this method is computation efficiency, because the proposed method provides a single-level single-objective optimization model which can be solved easily. The proposed method in this paper can be applied to interval-valued multiobjective security games simply. Finally, Validity and applicability of the method were illustrated by a practical example in security of metro system.

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