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Coverings, matchings and paired domination in fuzzy graphs using strong arcs

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Abstract

The concepts of covering and matching in fuzzy graphs using strong arcs are introduced and obtained the relationship between them analogous to Gallai's results in graphs. The notion of paired domination in fuzzy graphs using strong arcs is also studied. The strong paired domination number γ_{spr} of complete fuzzy graph and complete bipartite fuzzy graph is determined and obtained bounds for the strong paired domination number of fuzzy graphs. An upper bound for the strong paired domination number of fuzzy graphs in terms of strong independence number is also obtained.

Keywords: Fuzzy graph, Strong arcs, Weight of arcs, Strong domination, Strong paired domination.

1 Introduction

Fuzzy graphs were introduced by Rosenfeld [26]. Rosenfeld has described the fuzzy analogue of several graph theoretic concepts like paths, cycles, trees and connectedness and established some of their properties [26]. Bhutani and Rosenfeld have introduced the concept of strong arcs [8]. Several works on fuzzy graphs are also done by Akram, Samanta, Dudek, Davvaz, Sunitha R.A. Borzooei, Hossein Rashmanlou and Pal [1, 2, 3, 4, 5, 6, 9, 16, 23, 24, 25, 27, 28, 33]. Paired domination in graphs was introduced by Haynes and Slater [13]. If a node in a set D is taken as the location of a guard capable of protecting each node in its closed neighborhood, then for domination a guard protects itself, and for total domination, each guard must be protected by another guard. For paired-domination the guards' locations must be selected as adjacent pairs of nodes so that each guard is assigned one other and they are "designated as backups" for each other.

Nagoor gani and Chandrasekaran discussed domination and independent domination in fuzzy graphs using strong arcs [20]. The coverings in fuzzy graphs were introduced by Somasundaram [31], who defined node covering and arc covering in fuzzy graphs using effective arcs and using scalar cardinality.

In this paper, the concepts of covering and matching in fuzzy graphs using strong arcs are introduced and obtained the relationship between them analogous to Gallai's results in graphs. The notion of paired domination in fuzzy graphs using strong arcs is also introduced and obtained some results using this parameter.

This paper is organized as follows. Section 2 contains preliminaries and in Section 3, coverings and matchings in fuzzy graphs using strong arcs are introduced. The concepts of strong node cover [Definition 3.1], strong independence number [Definition 3.6], strong arc cover [Definition 3.10] and strong matching [Definition 3.15] in fuzzy graphs using strong arcs are introduced and obtained the relationship between them [Theorem 3.21] analogous to Gallai's results in graphs. The notion of perfect strong matching in fuzzy graphs is defined [Definition 3.22]. Finally in Section 4, the notion of paired domination in fuzzy graphs using strong arcs is studied [Definition 4.3]. The strong paired domination number γ_{spr} of complete fuzzy graph [Proposition 4.7] and complete bipartite fuzzy graph [Proposition 4.8] is determined. It is established that in a non trivial strong fuzzy graph $G : (V, \sigma, \mu)$ without isolated nodes, every strong

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paired dominating set is a strong total dominating set [Theorem 4.17]. A necessary and sufficient condition is obtained so that the strong paired domination number will be the order of the fuzzy graph [heorem 4.22]. An upper bound for the strong paired domination number of fuzzy graphs in terms of strong independence number is also obtained [Theorem 4.31].

2 Preliminaries

We summarize briefly some basic definitions in fuzzy graphs which are presented in [7, 8, 15, 19, 20, 21, 26, 29, 30, 32].

A **fuzzy graph** is denoted by $G : (V, \sigma, \mu)$, where V is a node set, σ and μ are mappings defined as $\sigma : V \to [0, 1]$ and $\mu : V \times V \to [0, 1]$, where σ and μ represent the membership values of a node and an arc respectively. For any fuzzy graph, $\mu(x, y) \leq min\{\sigma(x), \sigma(y)\}$. We consider fuzzy graph G with no loops and assume that V is finite and nonempty, μ is reflexive (i.e.,$\mu(x, x) = \sigma(x)$, for all x) and symmetric (i.e., $\mu(x, y) = \mu(y, x)$, for all (x, y)). In all the examples σ is chosen suitably. Also, we denote the underlying crisp graph by $G^* : (\sigma^*, \mu^*)$ where $\sigma^* = \{u \in V : \sigma(u) > 0\}$ and $\mu^* = \{(u, v) \in V \times V : \mu(u, v) > 0\}$. Throughout we assume that $\sigma^* = V$. The fuzzy graph $H : (\tau, \nu)$ is said to be a **partial fuzzy subgraph** of $G : (V, \sigma, \mu)$ if $\nu \subseteq \mu$ and $\tau \subseteq \sigma$. In particular we call $H : (\tau, \nu)$ a **fuzzy subgraph** of $G : (V, \sigma, \mu)$ if $\tau(u) = \sigma(u)$ for all $u \in \tau^*$ and $\nu(u, v) = \mu(u, v)$ for all $(u, v) \in \nu^*$. A fuzzy graph $G : (V, \sigma, \mu)$ is called **trivial** if $|\sigma^*| = 1$. Two nodes u and v in a fuzzy graph G are said to be **adjacent** (**neighbors**) if $\mu(u, v) > 0$. The set of all neighbors of u is denoted by $N(u)$.

An arc (u, v) of a fuzzy graph $G : (V, \sigma, \mu)$ with $\mu(u, v) > 0$ is called a **weakest arc** of G if (u, v) is an arc with minimum $\mu(u, v)$.

A **path** P of length n is a sequence of distinct nodes $u_0, u_1, ..., u_n$ such that $\mu(u_{i-1}, u_i) > 0, i = 1, 2, ..., n$ and the degree of membership of a weakest arc is defined as its strength. If $u_0 = u_n$ and $n \geq 3$ then P is called a **cycle** and P is called a **fuzzy cycle**, if it contains more than one weakest arc. The **strength of a cycle** is the strength of the weakest arc in it. The **strength of connectedness** between two nodes x and y is defined as the maximum of the strengths of all paths between x and y and is denoted by $CONN_G(x, y)$.

A fuzzy graph $G : (V, \sigma, \mu)$ is **connected** if for every x, y in $\sigma^*, CONN_G(x, y) > 0$.

An arc (u, v) of a fuzzy graph is called an **effective arc** if $\mu(u, v) = \sigma(u) \wedge \sigma(v)$. Then u and v are called effective neighbors. The set of all effective neighbors of u is called **effective neighborhood** of u and is denoted by $EN(u)$.
A fuzzy graph $G : (V, \sigma, \mu)$ is said to be **complete** if $\mu(u, v) = \sigma(u) \wedge \sigma(v)$, for all $u, v \in \sigma^*$.

The **order** p and **size** q of a fuzzy graph $G : (V, \sigma, \mu)$ are defined to be $p = \sum_{x \in V} \sigma(x)$ and $q = \sum_{(x,y) \in V \times V} \mu(x, y)$.

Let $G : (V, \sigma, \mu)$ be a fuzzy graph and $S \subseteq V$. Then the **scalar cardinality** of S is defined to be $\sum_{v \in S} \sigma(v)$ and it is denoted by $|S|_s$. Let p denotes the scalar cardinality of V, also called the order of G.

The **complement** of a fuzzy graph $G : (V, \sigma, \mu)$, denoted by $\overline{G}$ is defined to be $\overline{G} = (V, \sigma, \overline{\mu})$ where $\overline{\mu}(x, y) = \sigma(x) \wedge \sigma(y) - \mu(x, y)$ for all $x, y \in V$ [34].

An **arc** of a fuzzy graph $G : (V, \sigma, \mu)$ is called **strong** if its weight is at least as great as the strength of connectedness of its end nodes when it is deleted. A fuzzy graph G is called a **strong fuzzy graph** if each arc in G is a strong arc. Depending on $CONN_G(x, y)$ of an arc (x, y) in a fuzzy graph G, Mathew and Sunitha, [32] defined three different types of arcs. Note that $CONN_{G-(x,y)}(x, y)$ is the the strength of connectedness between x and y in the fuzzy graph obtained from G by deleting the arc (x, y). An arc (x, y) in G is $\alpha-$ **strong** if $\mu(x, y) > CONN_{G-(x,y)}(x, y)$. An arc (x, y) in G is $\beta-$ **strong** if $\mu(x, y) = CONN_{G-(x,y)}(x, y)$. An arc (x, y) in G is $\delta-$ **arc** if $\mu(x, y) < CONN_{G-(x,y)}(x, y)$.
Thus an arc (x, y) is a strong arc if it is either $\alpha-$ strong or $\beta-$ strong. Also y is called **strong neighbor** of x if arc (x,y) is strong. The set of all strong neighbors of x is called the **strong neighborhood** of x and is denoted by $N_s(x)$. The **closed strong neighborhood** $N_s[x]$ is defined as $N_s[x] = N_s(x) \cup \{x\}$. A path P is called **strong path** if P contains only strong arcs.

A fuzzy graph $G : (V, \sigma, \mu)$ is said to be **bipartite** [30] if the vertex set V can be partitioned into two non empty sets V_1 and V_2 such that $\mu(v_1, v_2) = 0$ if $v_1, v_2 \in V_1$ or $v_1, v_2 \in V_2$. Further if $\mu(u, v) = \sigma(u) \bigwedge \sigma(v)$ for all $u \in V_1$ and $v \in V_2$ then G is called a **complete bipartite graph** and is denoted by K_{σ_1,σ_2}, where σ_1 and σ_2 are respectively the restrictions of σ to V_1 and V_2.

A **node** u is said to be **isolated** if $\mu(u, v) = 0$ for all $v \neq u$.

3 Coverings and Matchings Using Strong Arcs

In this Section, the concepts of covering and matching in fuzzy graphs using strong arcs are discussed. Recall the notion of covering and matching in graphs. A vertex and an incident edge are said to **cover each other** in a graph $G : (V, E)$. A **vertex cover** in G is a set of vertices that cover all edges of G. The minimum number of vertices in a vertex cover of G is the **vertex covering number** $\alpha_o(G)$ of G. A set of **vertices** in G is **independent** if no two vertices in the set are adjacent. The **vertex independence number** or (the independence number) $\beta_o(G)$ of G is the maximum cardinality of an independent set of vertices in G [10].

An **edge cover** of a graph $G:(V,E)$ without isolated nodes is a set of edges of G that covers all vertices of G. The **edge covering number** $\alpha_1(G)$ of G is the minimum cardinality of an edge cover of G. A set of **edges** in G is **independent** if no two edges in the set are adjacent. A **matching** M in G means an independent set M of edges in G. If $e=(u,v)\in M$, then we say that M matches u to v. A matching of maximum cardinality is called a **maximum matching**. The **edge independence number** or matching number $\beta_1(G)$ of G is the cardinality of a maximum matching. A matching M is called a perfect matching if M matches every vertex of G to some vertex of G, [10].

The concept of coverings in fuzzy graphs were introduced by Somasundaram. The author defined node covering and arc covering in fuzzy graphs using effective arcs and scalar cardinality. According to the author, a **node cover** in a fuzzy graph $G:(V,\sigma,\mu)$ is a subset D of V such that for every effective arc $e=(u,v)$, at least one of u,v is in D. The minimum scalar cardinality of a node cover of G is called the **covering number** of G and is denoted by $\alpha_o(G)$ or simply α_o. A set X of effective arcs of G is called an **arc cover** of G if every node of G is an end node of an arc in X. The minimum scalar cardinality of an arc cover G is called the **arc covering number** of G and is denoted by $\alpha_1(G)$ or simply α_1, [31].

Somasundaram [31] defined the notion of coverings in a smaller domain of effective arcs and using scalar cardinality. According to this definition, for any fuzzy graph without effective arcs the node cover is an empty set. Note that each effective arc is strong, but a strong arc need not be effective, [29]. Hence this definition groups only a very small class of fuzzy graphs. But the following situation motivates us to modify the definition of covering in fuzzy graphs using strong arcs.

Suppose the government has decided to fix security cameras in the cross roads of a city. The task is to determine a choice and a placement of cameras minimizing the overall cost. How best can such a security system is established? This problem can be represented by a fuzzy graph as follows. Let the nodes represent the cross roads of the city and roads connecting the cross roads represent the arcs. The weights of the arcs represent the cost of the camera that is needed in order to guarantee the adequate visibility. The bigger the length of the road segment, the bigger the cost of the camera that is needed for offering good quality images. Then the problem is equivalent to find a minimum node cover in this fuzzy graph.

Definition 3.1. *Let $G:(V,\sigma,\mu)$ be a fuzzy graph. A node and an incident strong arc are said to* ***strong cover*** *each other. A* ***strong node cover*** *in a fuzzy graph G is a set D of nodes that strong cover all strong arcs of G. The* ***fuzzy weight*** *of a* ***strong node cover*** *D is defined as $W(D)=\sum_{u\in D}\mu(u,v)$, where $\mu(u,v)$ is the minimum of the membership values (weights) of strong arcs incident on u. The* ***strong node covering number*** *of a fuzzy graph G is defined as the minimum fuzzy weight of strong node covers of G and it is denoted by $\alpha_{S_o}(G)$ or simply α_{S_o}. A minimum strong node cover in a fuzzy graph G is a strong node cover of minimum fuzzy weight.*

Now, we determine α_{S_o} of complete fuzzy graph, complete bipartite fuzzy graph and fuzzy cycle.

Proposition 3.2. *If $G:(V,\sigma,\mu)$ is a complete fuzzy graph, then $\alpha_{S_o}(G)=(n-1)\mu(u,v)$, where n is the number of nodes and $\mu(u,v)$ is the weight of a weakest arc in G.*

Proof. Since $G:(V,\sigma,\mu)$ is a complete fuzzy graph, all arcs are strong, [8] and each node is adjacent to all other nodes. Hence any set containing $(n-1)$ nodes forms a strong node cover of G. Let u be a node in G having minimum membership value (node weight). Let $v_1,v_2,\cdots,v_{n-1}$ be the nodes adjacent to u. Then the $(n-1)$ arcs $(u,v_1),(u,v_2),\cdots,(u,v_{n-1})$ are all weakest arcs of G with strength of each arc equal to $\mu(u,v)$ where $v\in\{v_1,v_2,\cdots,v_{n-1}\}$. Hence this set $D=\{v_1,v_2,\cdots,v_{n-1}\}$ of $(n-1)$ nodes forms a minimum strong node cover of G with

$$W(D)=\sum_{v_i\in D}\mu(u,v_i)=\mu(u,v_1)+\mu(u,v_2)+\cdots+\mu(u,v_{n-1})$$

where $\mu(u,v_i), i=1,2,\cdots,(n-1)$ is the minimum of the weights of strong arcs incident on v_i. Therefore

$$W(D)=\underbrace{\mu(u,v)+\mu(u,v)+\cdots+\mu(u,v)}_{(n-1)times}=(n-1)\mu(u,v)$$

where $\mu(u,v)$ is the weight of a weakest arc in G. Hence $\alpha_{S_o}(G)=(n-1)\mu(u,v)$. □

Proposition 3.3. *For a complete bipartite fuzzy graph K_{σ_1,σ_2} with partite sets V_1 and V_2,*

$$\alpha_{S_o}(K_{\sigma_1,\sigma_2})=\bigwedge\{W(V_1),W(V_2)\}$$

Proof. In K_{σ_1,σ_2}, all arcs are strong, [29, 30]. Also each node in V_1 is adjacent with all nodes in V_2 and vice-versa. Note that the set of all arcs of K_{σ_1,σ_2} is the set of all arcs incident on each node of V_1 or the set of all arcs incident on each node of V_2. Hence the strong node covers in K_{σ_1,σ_2} are V_1, V_2 and $V_1 \cup V_2$. Clearly $W(V_1 \cup V_2)$ is greater than $W(V_1)$ and $W(V_2)$. Hence

$$\alpha_{S_o}(K_{\sigma_1,\sigma_2}) = \bigwedge \{W(V_1), W(V_2)\}$$ □

Proposition 3.4. *If $G : (V, \sigma, \mu)$ is a fuzzy cycle such that G^* is a cycle. Then,*

$$\alpha_{S_o}(G) = \bigwedge \left\{W(D) : D \text{ is a strong node cover in } G \text{ with } |D| \geq \lceil \frac{n}{2} \rceil \right\}$$

and n is the number of nodes in G.

Proof. In a fuzzy cycle $G : (V, \sigma, \mu)$, every arc is strong, [8]. Also, the number of nodes in the strong node cover of both G and G^* are the same because each arc in both graphs is strong. Now, the strong node covering number of G^* is $\lceil \frac{n}{2} \rceil$, [10]. Hence the minimum number of nodes in a strong node cover of G is $\lceil \frac{n}{2} \rceil$. Hence the result follows. □

Definition 3.5. [20] *Two nodes in a fuzzy graph $G : (V, \sigma, \mu)$ are said to be* ***strongly independent*** *if there is no strong arc between them. A set of nodes in G is* ***strong independent*** *if any two nodes in the set are strongly independent.*

Definition 3.6. *The* ***fuzzy weight*** *of a* ***strong independent set*** *D in a fuzzy graph $G : (V, \sigma, \mu)$ is defined as $W(D) = \sum_{u \in D} \mu(u, v)$, where $\mu(u, v)$ is the minimum of the membership values (weights) of strong arcs incident on u. The* ***strong independence number*** *of a fuzzy graph G is defined as the maximum fuzzy weight of strong independent sets of nodes in G and it is denoted by $\beta_{S_o}(G)$ or simply β_{S_o}. A maximum strong independent set in a fuzzy graph G is a strong independent set of maximum fuzzy weight.*

Now, we determine β_{S_o} of complete fuzzy graph, complete bipartite fuzzy graph and fuzzy cycle.

Proposition 3.7. *If $G : (V, \sigma, \mu)$ is a complete fuzzy graph, then $\beta_{S_o}(G) = \mu(u, v)$, where $\mu(u, v)$ is the membership value (weight) of a weakest arc in G.*

Proof. Since $G : (V, \sigma, \mu)$ is a complete fuzzy graph, all arcs are strong, [8] and each node is adjacent to all other nodes. Hence $D = \{u\}$ is the only strong independent set for each $u \in \sigma^*$. Hence the result follows. □

Proposition 3.8. *For a complete bipartite fuzzy graph K_{σ_1,σ_2} with partite sets V_1 and V_2,*

$$\beta_{S_o}(K_{\sigma_1,\sigma_2}) = \bigvee \{W(V_1), W(V_2)\}$$

Proof. In K_{σ_1,σ_2}, all arcs are strong, [29, 30]. Also each node in V_1 is adjacent with all nodes in V_2 and vice-versa. Hence the strong independent sets in K_{σ_1,σ_2} are V_1 and V_2. Hence the result follows. □

Proposition 3.9. *If $G : (V, \sigma, \mu)$ is a fuzzy cycle such that G^* is a cycle. Then,*

$$\beta_{S_o}(G) = \bigvee \left\{W(D) : D \text{ is a strong independent set of nodes in } G \text{ with } |D| \leq \lfloor \frac{n}{2} \rfloor \right\}$$

Proof. In a fuzzy cycle $G : (V, \sigma, \mu)$, every arc is strong, [8]. Also, the number of nodes in a strong independent set of both G and G^* are the same because each arc in both graphs is strong. Now, the strong independence number of G^* is $\lfloor \frac{n}{2} \rfloor$, [10]. Hence the maximum number of nodes in a strong independent set of G is $\lfloor \frac{n}{2} \rfloor$. Hence the result follows. □

Definition 3.10. *Let $G : (V, \sigma, \mu)$ be a fuzzy graph without isolated nodes. A* ***strong arc cover*** *of G is a set X of strong arcs of G that strong covers all nodes of G. The* ***fuzzy weight*** *of a* ***strong arc cover*** *X is defined as $W(X) = \sum_{(u,v) \in X} \mu(u, v)$. The* ***strong arc covering number*** *of a fuzzy graph G is defined as the minimum fuzzy weight of strong arc covers of G and it is denoted by $\alpha_{S_1}(G)$ or simply α_{S_1}. A minimum strong arc cover of a fuzzy graph G is a strong arc cover of minimum fuzzy weight.*

Now, we determine α_{S_1} of complete fuzzy graph, complete bipartite fuzzy graph and fuzzy cycle.

Proposition 3.11. *If $G : (V, \sigma, \mu)$ is a complete fuzzy graph, then,*

$$\alpha_{S_1}(G) = \bigwedge \left\{W(D) : D \text{ is a strong arc cover in } G \text{ with } |D| \geq \lceil \frac{n}{2} \rceil \right\}$$

where n is the number of nodes in G.

Proof. Since $G : (V, \sigma, \mu)$ is a complete fuzzy graph, all arcs are strong, [8] and each node is adjacent to all other nodes. Also, the number of arcs in a strong arc cover of both G and G^* are the same because each arc in both graphs is strong. Now, the strong arc covering number of G^* is $\lceil \frac{n}{2} \rceil$, [10]. Hence the minimum number of arcs in a strong arc cover of G is $\lceil \frac{n}{2} \rceil$. Hence the result follows. □

Remark 3.12. *Since every complete fuzzy graph contains at most one $\alpha-$ strong arc,* [32]*, every strong arc cover in a complete fuzzy graph G contains only $\beta-$ strong arcs or contains at most one $\alpha-$ strong arc and other arcs are $\beta-$ strong.*

Proposition 3.13. *For a complete bipartite fuzzy graph K_{σ_1,σ_2} with partite sets V_1 and V_2, $\alpha_{S_1}(K_{\sigma_1,\sigma_2}) =$*

$$\bigwedge \left\{ W(D) : D \text{ is a strong arc cover in } K_{\sigma_1,\sigma_2} \text{ with } |D| \geq \bigvee \{|V_1|, |V_2|\} \right\}$$

Proof. In K_{σ_1,σ_2}, all arcs are strong, [29, 30]. Also each node in V_1 is adjacent with all nodes in V_2 and vice-versa. Also, the number of arcs in a strong arc cover of both K_{σ_1,σ_2} and $K^*_{\sigma_1,\sigma_2}$ are the same because each arc in both graphs is strong. Now, the strong arc covering number of $K^*_{\sigma_1,\sigma_2}$ is $\bigvee \{|V_1|, \{|V_2|\}$, [10]. Hence the minimum number of arcs in a strong arc cover of K_{σ_1,σ_2} is $\bigvee \{|V_1|, \{|V_2|\}$. Hence the result follows. □

Proposition 3.14. *If $G : (V, \sigma, \mu)$ is a fuzzy cycle such that G^* is a cycle. Then*

$$\alpha_{S_1}(G) = \bigwedge \left\{ W(D) : D \text{ is a strong arc cover in } G \text{ with } |D| \geq \lceil \frac{n}{2} \rceil \right\}$$

Proof. In a fuzzy cycle $G : (V, \sigma, \mu)$, every arc is strong, [8]. Also, the number of arcs in a strong arc cover of both G and G^* are the same because each arc in both graphs is strong. Now, the strong arc covering number of G^* is $\lceil \frac{n}{2} \rceil$, [10]. Hence the minimum number of arcs in a strong arc cover of G is $\lceil \frac{n}{2} \rceil$. Hence the result follows. □

Definition 3.15. [22] *Let $G : (V, \sigma, \mu)$ be a fuzzy graph. A set M of strong arcs in G such that no two arcs in M have a common node is called a **strong independent set** of arcs or a **strong matching** in G.*

Definition 3.16. *Let M be a strong matching in a fuzzy graph $G : (V, \sigma, \mu)$. If $e = (u, v) \in M$, then we say that M strongly matches u to v. The **fuzzy weight** of a **strong matching** is defined as $W(M) = \sum_{(u,v)\in M} \mu(u, v)$. The **strong arc independence number** or **strong matching number** of a fuzzy graph G is defined as the maximum fuzzy weight of strong matchings of G and it is denoted by $\beta_{S_1}(G)$ or simply β_{S_1}. A maximum strong matching in a fuzzy graph G is a strong matching of maximum fuzzy weight.*

Now, we determine β_{S_1} of complete fuzzy graph, complete bipartite fuzzy graph and fuzzy cycle.

Proposition 3.17. *If $G : (V, \sigma, \mu)$ is a complete fuzzy graph, then*

$$\beta_{S_1}(G) = \bigvee \left\{ W(M) : M \text{ is a strong matching with } |M| \leq \lfloor \frac{n}{2} \rfloor \right\}$$

where n is the number of nodes in G.

Proof. Since $G : (V, \sigma, \mu)$ is a complete fuzzy graph, all arcs are strong, [8] and each node is adjacent to all other nodes. Also, the number of arcs in a strong matching of both G and G^* are the same because each arc in both graphs is strong. Now, the strong matching number of G^* is $\lfloor \frac{n}{2} \rfloor$, [10]. Hence the maximum number of arcs in a strong matching of G is $\lfloor \frac{n}{2} \rfloor$. Hence the result follows. □

Proposition 3.18. *For a complete bipartite fuzzy graph K_{σ_1,σ_2} with partite sets V_1 and V_2, $\beta_{S_1}(K_{\sigma_1,\sigma_2}) =$*

$$\bigvee \left\{ W(M) : M \text{ is a strong matching in } K_{\sigma_1,\sigma_2} \text{ with} |M| \leq \bigwedge \{|V_1|, |V_2|\} \right\}$$

Proof. In K_{σ_1,σ_2}, all arcs are strong, [29, 30]. Also each node in V_1 is adjacent with all nodes in V_2 and vice-versa. Also, the number of arcs in a strong matching of both K_{σ_1,σ_2} and $K^*_{\sigma_1,\sigma_2}$ are the same because each arc in both graphs is strong. Now, the strong matching number of $K^*_{\sigma_1,\sigma_2}$ is $\bigwedge \{|V_1|, \{|V_2|\}$, [10]. Hence the maximum number of arcs in a strong matching of K_{σ_1,σ_2} is $\bigwedge \{|V_1|, \{|V_2|\}$. Hence the result follows. □

Proposition 3.19. *If $G : (V, \sigma, \mu)$ is a fuzzy cycle such that G^* is a cycle. Then,*

$$\beta_{S_1}(G) = \bigvee \left\{ W(M) : M \text{ is a strong matching in } G \text{ with} |M| \leq \lfloor \frac{n}{2} \rfloor \right\}$$

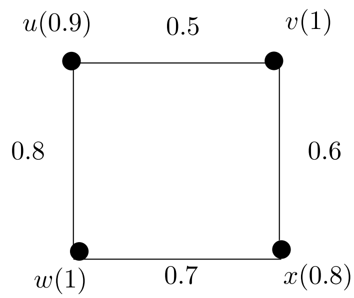

Figure 1: Illustration of Strong Coverings in Fuzzy Graphs

Proof. In a fuzzy cycle $G : (V, \sigma, \mu)$, every arc is strong, [8]. Also, the number of arcs in a strong matching of both G and G^* are the same because each arc in both graphs is strong. Now, the strong matching number of G^* is $\lfloor \frac{n}{2} \rfloor$, [10]. Hence the maximum number of arcs in a strong matching of G is $\lfloor \frac{n}{2} \rfloor$. Hence the result follows. □

Example 3.20. *Consider the fuzzy graph in Figure 1. In this fuzzy graph, strong arcs are $(u, w), (w, x)$ and (x, v). The strong node covers in this fuzzy graph are as follows.*
$D_1 = \{w, x\}$, $D_2 = \{v, w\}$, $D_3 = \{u, x\}$
$D_4 = \{u, v, w\}$, $D_5 = \{u, v, x\}$, $D_6 = \{v, w, x\}$, $D_7 = \{u, w, x\}$,
$D_8 = \{u, v, w, x\}$.
$W(D_1) = 0.7 + 0.6 = 1.3$ *and* $W(D_2) = 0.6 + 0.7 = 1.3$.
$W(D_3) = 0.8 + 0.6 = 1.4$, $W(D_4) = 0.8 + 0.6 + 0.7 = 2.1$.
$W(D_5) = 0.8 + 0.6 + 0.6 = 2.0$.
$W(D_6) = 0.6 + 0.7 + 0.6 = 1.9$.
$W(D_7) = 0.8 + 0.7 + 0.6 = 2.1$. $W(D_8) = 0.8 + 0.6 + 0.7 + 0.6 = 2.7$. *Among these fuzzy weights minimum fuzzy weight is 1.3. Hence* $\alpha_{S_o} = 1.3$. *The strong independent sets are* $D_1 = \{u, v\}, D_2 = \{u, x\}$ *and* $D_3 = \{v, w\}$. *Note that any set containing 3 or 4 nodes is not strong independent.*
$W(D_1) = 0.8 + 0.6 = 1.4$, $W(D_2) = 0.8 + 0.6 = 1.4$.
$W(D_3) = 0.6+0.7 = 1.3$. *The maximum fuzzy weight among these is 1.4. Hence* $\beta_{S_o} = 1.4$. *The set* $M = \{(u, w), (v, x)\}$ *is the only strong arc cover and strong matching in G (Figure 1) and* $W(M) = 0.8 + 0.6 = 1.4$. *Hence* $\alpha_{S_1} = \beta_{S_1} = 1.4$.

The next theorem describes the result analogous to Gallai's identities in graphs, [10].

Theorem 3.21. *For every fuzzy graph $G : (V, \sigma, \mu)$ of order p containing no isolated nodes,*
1. $\alpha_{S_o} + \beta_{S_o} = W(V) \leq p$
2. $\alpha_{S_1} + \beta_{S_1} \leq p$

Proof. Let $G : (V, \sigma, \mu)$ be a fuzzy graph of order p containing no isolated nodes. Let $\alpha_{S_o} = W(N_{S_o})$ where N_{S_o} is a minimum strong node cover of G. Then $V - N_{S_o}$ is a strong independent set of nodes. That is, nodes in $V - N_{S_o}$ are incident on no strong arcs of G. Therefore $\beta_{S_o} \geq W(V - N_{S_o}) = W(V) - \alpha_{S_o}$. Hence

$$\alpha_{S_o} + \beta_{S_o} \geq W(V) \tag{1}$$

Let $\beta_{S_o} = W(M_{S_o})$ where, M_{S_o} is a maximum strong independent set of nodes in G. That is, no two nodes in M_{S_o} are adjacent to each other by a strong arc and thus nodes in $V - M_{S_o}$ strong cover all the strong arcs of G. Hence $V - M_{S_o}$ is a strong node cover of G and α_{S_o} is the minimum fuzzy weight of such strong node covers. Hence $\alpha_{S_o} \leq W(V - M_{S_o}) = W(V) - \beta_{S_o}$. Hence

$$\alpha_{S_o} + \beta_{S_o} \leq W(V) \tag{2}$$

From (1) and (2), $\alpha_{S_o} + \beta_{S_o} = W(V)$.
Next, we have $W(V) \leq p$ always by the definition of $W(V)$.
Hence $\alpha_{S_o} + \beta_{S_o} = W(V) \leq p$.
The second inequality follows directly since the weights of strong arcs are considered for determining α_{S_1} and β_{S_1} and p is the sum of the node weights. □

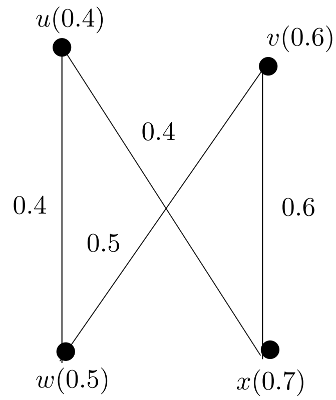

Figure 2: Perfect Strong Matching

Definition 3.22. *Let $G : (V, \sigma, \mu)$ be a fuzzy graph and let M be a strong matching in G. Then M is called* ***perfect strong matching*** *if M strongly matches every node of G to some node of G.*

Example 3.23. *Consider the fuzzy graph in Figure 2. In this fuzzy graph, all arcs are strong. The sets $M_1 = \{(u, w), (v, x)\}$ and $M_2 = \{(u, x), (v, w)\}$ are perfect strong matchings with fuzzy weights $W(M_1) = 0.4 + 0.6 = 1.0$ and $W(M_2) = 0.4 + 0.5 = 0.9$.*

Remark 3.24. *In the case of graphs, every perfect matching is a maximum matching,* [10]*. But this is not true in the case of fuzzy graphs. In fuzzy graphs a perfect strong matching M is not always a maximum strong matching.*

Example 3.25. *In Figure 2 of Example 3.23, both M_1 and M_2 are perfect strong matchings. But M_2 is not a maximum strong matching.*

4 Paired Domination in Fuzzy Graphs Using Strong Arcs

Paired domination was introduced by Haynes and Slater in 1995, [13]. It is also studied in [11, 12, 13, 14]. In a graph $G : (V, E)$, a set $S \subseteq V$ is a paired dominating set if S is a dominating set and the induced subgraph $\langle S \rangle$ has a perfect matching, [13]. The paired-domination number $\gamma_{pr}(G)$ is the minimum cardinality of a paired dominating set S in G. Domination in fuzzy graphs was first introduced by Somasudaram and Somasundaram, [30]. Also, Nagoorgani discussed domination in fuzzy graphs using strong arcs, [20]. In this Section, the concept of paired domination in fuzzy graphs using strong arcs based on perfect strong matchings is introduced and established some classic results.

Definition 4.1. [20] *A node v in a fuzzy graph $G : (V, \sigma, \mu)$ is said to* ***strongly dominate*** *itself and each of its strong neighbors, that is, v strongly dominates the nodes in $N_s[v]$. A set D of nodes of G is a* ***strong dominating set*** *of G if every node of $V(G) - D$ is a strong neighbor of some node in D.*

Definition 4.2. [17] *The* ***fuzzy weight*** *of a strong dominating set D is defined as $W(D) = \sum_{u \in D} \mu(u, v)$, where $\mu(u, v)$ is the minimum of the membership values (weights) of strong arcs incident on u. The* ***strong domination number*** *of a fuzzy graph G is defined as the minimum fuzzy weight of strong dominating sets of G and it is denoted by $\gamma_s(G)$ or simply γ_s. A* ***minimum strong dominating set*** *in a fuzzy graph G is a strong dominating set of minimum fuzzy weight.*

Definition 4.3. *Let $G : (V, \sigma, \mu)$ be a fuzzy graph. A set $D \subseteq V$ of nodes is said to be a* ***strong paired dominating set*** *if D is a strong dominating set and the induced fuzzy subgraph $\langle D \rangle$ has a perfect strong matching. The* ***fuzzy weight*** *of a strong paired dominating set is defined as $W(D) = \sum_{u \in D} \mu(u, v)$, where $\mu(u, v)$ is the minimum of the weights of strong arcs incident on u. The* ***strong paired domination number*** *of a fuzzy graph G is defined as the minimum fuzzy weight of strong paired dominating sets of G and it is denoted by $\gamma_{spr}(G)$ or simply γ_{spr}.*
A ***minimum strong paired dominating set*** *in a fuzzy graph G is a strong paired dominating set of minimum fuzzy weight.*

Remark 4.4. *Note that strong paired domination require that there be no isolated nodes in a fuzzy graph $G : (V, \sigma, \mu)$.*

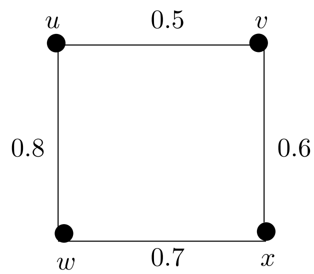

Figure 3: Illustration of strong paired domination

Remark 4.5. *Note that in a graph $G:(V,E)$, all arcs are strong. Hence for a graph $G:(V,E)$, $\gamma_{spr}(G)=\gamma_{pr}(G)$, $\alpha_{S_o}=\alpha_o$, $\beta_{S_o}=\beta_o$, $\alpha_{S_1}=\alpha_1$ and $\beta_{S_1}=\beta_1$.*

Example 4.6. *Consider the fuzzy graph in Figure 3. In this fuzzy graph, the strong arcs are (u,w), (w,x) and (x,v). The sets $D_1=\{u,v\}$, $D_2=\{w,x\}$ and $D_3=\{u,v,w,x\}$ are strong paired dominating sets and fuzzy weights of these sets are $W(D_1)=0.8+0.6=1.4$, $W(D_2)=0.7+0.6=1.3$, $W(D_3)=0.8+0.6+0.7+0.6=2.7$. Hence D_2 is a strong paired dominating set with minimum fuzzy weight. Hence strong paired domination number is $\gamma_{spr}(G)=1.3$.*

Now, we determine the strong paired domination number of several classes of fuzzy graphs.

Proposition 4.7. *If $G:(V,\sigma,\mu)$ is a complete fuzzy graph, then $\gamma_{spr}(G)=2\mu(u,v)$ where $\mu(u,v)$ is the weight of any weakest arc in G.*

Proof. Since $G:(V,\sigma,\mu)$ is a complete fuzzy graph, all arcs are strong, [8] and each node is adjacent to all other nodes. Then any set $\{u,v\}$ having two nodes say u,v in G forms a strong paired dominating set. Hence $\gamma_{spr}(G)=\mu(u,v)+\mu(u,v)=2\mu(u,v)$. □

Proposition 4.8. *For a complete bipartite fuzzy graph K_{σ_1,σ_2}, $\gamma_{spr}(K_{\sigma_1,\sigma_2})=2\mu(u,v)$, where $\mu(u,v)$ is the weight of a weakest arc in K_{σ_1,σ_2}.*

Proof. In K_{σ_1,σ_2}, all arcs are strong, [29, 30]. Also each node in V_1 is adjacent with all nodes in V_2. Hence in K_{σ_1,σ_2}, a strong paired dominating set is any set containing 2 nodes, one in V_1 and other in V_2. Then the set $\{u,v\}$ of end nodes of any weakest arc (u,v) in K_{σ_1,σ_2}, $u\in V_1, v\in V_2$ forms a strong paired dominating set. Hence $\gamma_{spr}(K_{\sigma_1,\sigma_2})=\mu(u,v)+\mu(u,v)=2\mu(u,v)$. □

Now, we recall the definition of the union and join of two graphs.

Definition 4.9. [11] *Union of two graphs: Let $G_1:(V_1,E_1)$ and $G_2:(V_2,E_2)$ be two simple graphs with $V_1\cap V_2=\phi$. Then the union of G_1 and G_2 is defined as $G_1\cup G_2=(V_1\cup V_2,E_1\cup E_2)$.*

Definition 4.10. [11] *Join of two graphs: Let $G_1:(V_1,E_1)$ and $G_2:(V_2,E_2)$ be two simple graphs with $V_1\cap V_2=\phi$. Then the join of G_1 and G_2 is defined as $G_1+G_2=(V_1\cup V_2,E)$ where $E=E_1\cup E_2\cup\{e=(u,v):u\in V_1,v\in V_2\}$.*

Definition 4.11. [11] *Wheel graph: A wheel graph W_n of n vertices is a graph G which is the join of a complete graph on one node K_1 and C_{n-1}.*

Proposition 4.12. *If $G:(V,\sigma,\mu)$ is a fuzzy graph with all arcs as strong arcs such that G^* is a wheel graph, then $\gamma_{spr}(G)=2\mu(u,v)$ where $\mu(u,v)$ is the weight of any weakest arc in G.*

Proof. Since $G:(V,\sigma,\mu)$ is a fuzzy graph with all arcs are strong such that G^* is a wheel graph, the central node is adjacent to the remaining nodes by a strong arc. Hence any set containing the central node say u and any other node say v form a strong paired dominating set. Hence the result follows. □

Proposition 4.13. *If $G:(V,\sigma,\mu)$ is a fuzzy cycle such that G^* is a cycle, then each node of every strong paired dominating set is incident on a strong arc.*

Proof. In a fuzzy cycle $G:(V,\sigma,\mu)$, every arc is strong, [8]. Hence the result follows. □

The next theorem describes the relationship between the total domination and the paired domination in fuzzy graphs using strong arcs.

Definition 4.14. [16] *A set D of nodes in a fuzzy graph $G : (V, \sigma, \mu)$ is a **strong total(open) dominating set** of G if every node of G is a strong neighbor of at least one node of D.*

Remark 4.15. [16] *Note that a fuzzy graph $G : (V, \sigma, \mu)$ contains a strong total dominating set if and only if G contains no isolated nodes and further the induced fuzzy subgraph $\langle D \rangle$ contains no isolated nodes.*

Definition 4.16. [16] *The **fuzzy weight** of a strong total dominating set D is defined as $W(D) = \sum_{u \in D} \mu(u, v)$,where $\mu(u, v)$ is the minimum of the weights of strong arcs incident on u. The **strong total domination number** of a fuzzy graph $G : (V, \sigma, \mu)$ is defined as the minimum fuzzy weight of strong total dominating sets of G and it is denoted by $\gamma_{st}(G)$ or simply γ_{st}. A **minimum strong total dominating set** in a fuzzy graph G is a strong total dominating set of minimum fuzzy weight.*

Theorem 4.17. *In a non trivial strong fuzzy graph $G : (V, \sigma, \mu)$ without isolated nodes, every strong paired dominating set is a strong total dominating set.*

Proof. Let D be a strong paired dominating set in $G : (V, \sigma, \mu)$. Then D strongly dominates every node in $V - D$ and $H = \langle D \rangle$ contains a perfect strong matching. Since G is a strong fuzzy graph, every arc in H is strong in G. Also since H contains a perfect strong matching, every node in D is strongly dominated by some node in D. Hence every node in G is strongly dominated by some node in D. Therefore D is a strong total dominating set. □

Remark 4.18. *Theorem 4.17 does not hold generally as illustrated in the following example.*

Example 4.19. *In Figure 3 of Example 4.6, $D = \{u, v\}$ is a strong paired dominating set, but not a strong total dominating set since the nodes u and v have no strong neighbors in D.*

Theorem 4.20. *In any non trivial strong fuzzy graph $G : (V, \sigma, \mu)$ without isolated nodes, $\gamma_s(G) \leq \gamma_{st}(G) \leq \gamma_{spr}(G)$ and any strong paired dominating set contains an even number of nodes.*

Proof. Every strong paired dominating set is also strong total and strong dominating set by Theorem 4.17. Hence $\gamma_s(G) \leq \gamma_{st}(G) \leq \gamma_{spr}(G)$. Now, let D be any strong paired dominating set in a fuzzy graph $G : (V, \sigma, \mu)$. Since $H = \langle D \rangle$ contains a perfect strong matching, it follows that D contains an even number of nodes. □

Remark 4.21. *Note that in any non trivial fuzzy graph $G : (V, \sigma, \mu)$ of order p, $min_{(u,v) \in \mu^*} \mu(u, v) < \gamma_{spr} \leq p$, since every strong paired dominating set contains at least 2 nodes and in a connected fuzzy graph with distinct node weights, $\gamma_{spr} < p$, always.*

Theorem 4.22. *In a non trivial fuzzy graph $G : (V, \sigma, \mu)$ of order p, $\gamma_{spr} = p$ if and only if the following conditions hold.*
1. All nodes have same weight.
2. All arcs are effective arcs.
3. Each node of G has a unique strong neighbor.

Proof. Let $G : (V, \sigma, \mu)$ be a non trivial fuzzy graph of order p. If all nodes have same weight, all arcs are effective arcs and each node of G has a unique strong neighbor then obviously $\gamma_{spr} = p$, since V is the only strong paired dominating set. Conversely suppose that $\gamma_{spr} = p$. If any one of the conditions 1, 2, 3 violated, then $\gamma_{spr} < p$, a contradiction from the definition of γ_{spr}. □

Theorem 4.23. *In a non trivial fuzzy graph $G : (V, \sigma, \mu)$, if $\gamma_{spr} = p$, then the number of nodes in G is even.*

Proof. Let $G : (V, \sigma, \mu)$ be a non trivial fuzzy graph of order p. Suppose $\gamma_{spr} = p$. Then by Theorem 4.22, each node of G has a unique strong neighbor, all nodes have same weight and all arcs are effective arcs. If G contains an odd number of nodes say $2n + 1$, then G contains at least one node having two strong neighbors. But this is a contradiction, since no node of G has two strong neighbors. Hence the theorem. □

The notions of private strong neighbors and strong irredundant sets in fuzzy graphs are discussed by Nagoorgani and Vadivel, [22]. The authors used strong arcs to define these concepts which are given as follows.

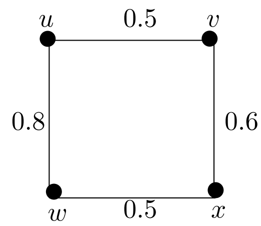

Figure 4: Private Strong Dominating Set

Definition 4.24. [22] *Let $G : (V, \sigma, \mu)$ be any fuzzy graph and D be a subset of V. A node v is said to be a **private strong neighbor** of $u \in D$ with respect to D, if $N_s[v] \cap D = \{u\}$. Furthermore, a **private strong neighbor set** of $u \in D$ with respect to D is defined to be $psn[u, D] = \{v : N_s[v] \cap D = \{u\}\}$. Stated in other words, $psn[u, D] = N_s[u] - N_s[D - \{u\}]$. If $u \in psn[u, D]$, then D is a strongly independent set of nodes. In such a case, u is said to be its own private strong neighbor.*

Definition 4.25. [22] *A set of nodes D in a fuzzy graph $G : (V, \sigma, \mu)$ is called a **strong irredundant** set of G if $psn[u, D] \neq \phi$ for every node $u \in D$. A strong irredundant set D is called a maximal strong irredundant set, if for every node $u \in V - D$, the set $D \cup \{u\}$ is not a strong irredundant set.*

Proposition 4.26. [22] *If D is a minimal strong dominating set of a fuzzy graph $G : (V, \sigma, \mu)$ then, D is a maximal strong irredundant set of G.*

Motivated by these concepts, we define open strong irruedundant sets and private strong dominating sets in fuzzy graphs as follows. Using these concepts, an upper bound is obtained for the strong paired domination number of fuzzy graphs.

Definition 4.27. *Let $G : (V, \sigma, \mu)$ be any fuzzy graph. A set $D \subseteq V$ is called an **open strong irredundant** set of G if for every node $u \in D$, $N_s(u) - N_s[D - \{u\}] \neq \phi$. In an open strong irredundant set D, every node must have a private strong neighbor outside of D, that is in $V - D$. Such a node is called an **open private strong neighbor**.*

Definition 4.28. *Let $G : (V, \sigma, \mu)$ be any fuzzy graph. A set $D \subseteq V$ is called **private strong dominating set** of G if D is a strong dominating set and an open strong irrendundant set.*

These concepts are illustrated in the following example.

Example 4.29. *Consider the fuzzy graph $G : (V, \sigma, \mu)$ in Figure 4. In this fuzzy graph, all arcs are strong. Consider the set $D = \{v, x\}$. For this set, $N_s[u] \cap D = \{u, v, w\} \cap \{v, x\} = \{v\}$. Also $N_s[w] \cap D = \{u, w, x\} \cap \{v, x\} = \{x\}$. Hence u is a private strong neighbor of v and w is a private strong neighbor of x with respect to D.*
$N_s(v) - N_s[D - v] = N_s(v) - N_s[x] = \{u, x\} - \{v, x, w\} = \{u\}$.
$N_s(x) - N_s[D - x] = N_s(x) - N_s[v] = \{v, w\} - \{u, v, x\} = \{w\}$.
That is, for every node $u \in D$, $N_s(u) - N_s[D - u] \neq \phi$. Hence D is an open strong irredundant set. Note that u and w are open private strong neighbors of v and x respectively. Also note that D is a strong dominating set of G. Hence D is a private strong dominating set of G.

Theorem 4.30. *If a fuzzy graph $G : (V, \sigma, \mu)$ has no isolated nodes, then G has a minimum strong dominating set which is open strong irredundant (private strong dominating set).*

Proof. Let D be a minimum strong dominating set in $G : (V, \sigma, \mu)$. Assume that D contains maximum number of nodes having an open private strong neighbor.
To prove that D is open strong irredundant. That is every node in D must have a private strong neighbor in $V - D$ (open private strong neighbor). If a node $u \in D$ does not have an open private strong neighbor, then it must be an isolated node of $\langle D \rangle$, since D must be a maximal strong irredundant set by Proposition 4.26. Since G has no isolated nodes and D is a strong dominating set, u must have a strong neighbor say, v in $V - D$. But in this case, the set $(D - \{u\}) \cup \{v\}$ is a minimum strong dominating set in which, node v has node u as an open private strong neighbor, contradicting the maximality of the number of nodes in D having an open private strong neighbor. Thus D must be an open strong irredundant set. □

Theorem 4.31. *For any fuzzy graph $G : (V, \sigma, \mu)$ without isolated nodes, $\gamma_{spr}(G) \leq 2\beta_{S_o}(G)$.*

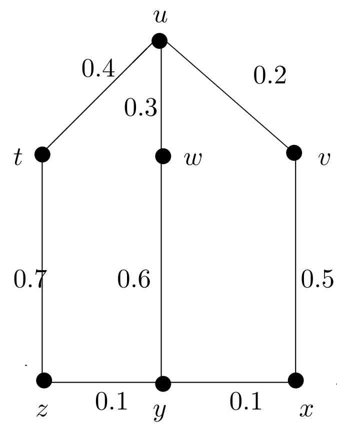

Figure 5: Illustrating that $\gamma_{spr} > \gamma_{sc}$

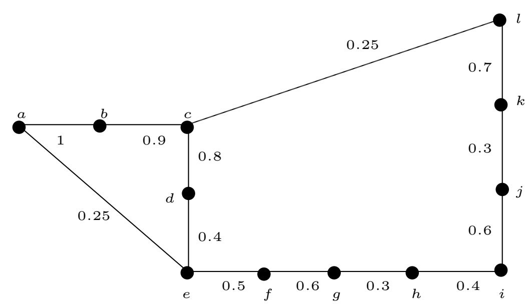

Figure 6: Illustrating that $\gamma_{spr} < \gamma_{sc}$

Proof. Let $G : (V, \sigma, \mu)$ be a fuzzy graph without isolated nodes and D be a private strong dominating set for G. Now, pair each node $v \in D$ with an open private strong neighbor which forms a strong paired dominating set with maximum fuzzy weight $2\beta_{S_o}(G)$. Hence the result. □

Remark 4.32. *The bound in Theorem 4.31 is sharp. For example, for the complete fuzzy graph K_σ, $\gamma_{spr}(K_\sigma) = 2\mu(u, v) = 2\beta_{S_o}(G)$.*

The following examples show that the strong paired domination number γ_{spr} and the strong connected domination number γ_{sc}, [18] are incomparable.

Example 4.33. *Consider the fuzzy graph in Figure 5. In this fuzzy graph, $(x, y), (y, z)$ are $\delta-$ arcs and all others are strong arcs. The set $D_1 = \{u, v, w, t\}$ is the minimum strong connected dominating set with fuzzy weight $W(D_1) = 0.2 + 0.2 + 0.3 + 0.4 = 1.1$. Hence $\gamma_{sc} = 1.1$. The set $D_2 = \{v, w, t, x, y, z\}$ is the minimum strong paired dominating set with fuzzy weight $W(D_2) = 0.2 + 0.3 + 0.4 + 0.5 + 0.6 + 0.7 = 2.7$. Hence $\gamma_{spr} = 2.7$. Hence in this fuzzy graph, $\gamma_{spr} > \gamma_{sc}$.*

Example 4.34. *Consider the fuzzy graph in Figure 6. In this fuzzy graph, $(a, e), (c, l)$ are $\delta-$ arcs and all others are strong arcs. The set $D_3 = \{b, c, d, e, g, h, j, k\}$ is the minimum strong paired dominating set with fuzzy weight $W(D_3) = 0.9 + 0.8 + 0.4 + 0.4 + 0.3 + 0.3 + 0.3 + 0.3 = 3.7$. Hence $\gamma_{spr} = 3.7$. The set $D_4 = \{b, c, d, e, f, g, h, i, j, k\}$, the set of fuzzy cutnodes of the fuzzy graph is the minimum strong connected dominating set with fuzzy weight $W(D_4) = 0.9 + 0.8 + 0.4 + 0.4 + 0.5 + 0.3 + 0.3 + 0.4 + 0.3 + 0.3 = 4.6$. Hence $\gamma_{sc} = 4.6$. Hence in this fuzzy graph, $\gamma_{spr} < \gamma_{sc}$. These examples [Figure 5 and Figure 6] show that in fuzzy graphs, γ_{spr} and γ_{sc} are incomparable.*

5 Conclusions

In this paper, coverings and matchings in fuzzy graphs using strong arcs are introduced. The concepts of strong node cover, strong independence number, strong arc cover and strong matching in fuzzy graphs using strong arcs are defined and obtained the relationship between them analogous to Gallai's results in graphs. Also the concept of paired domination in fuzzy graphs using strong arcs is introduced. The strong paired domination number γ_{spr} of complete fuzzy graph and complete bipartite fuzzy graph is determined and obtained bounds for the strong paired domination number of fuzzy graphs.

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