Iranian Journal of Fuzzy Systems

Volume 16, Number 2, (2019), pp. 31-43

University of Sistan and Baluchestan

Incidence cuts and connectivity in fuzzy incidence graphs

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Abstract

Fuzzy incidence graphs can be used as models for nondeterministic interconnection networks having extra node-edge relationships. For example, ramps in a highway system may be modeled as a fuzzy incidence graph so that unexpected flow between cities and highways can be effectively studied and controlled. Like node and edge connectivity in graphs, node connectivity and arc connectivity in fuzzy incidence graphs are introduced in this article. Their relationships with fuzzy connectivity parameters are discussed and results similar to Whitney's theorems are obtained. Also, the incidence is used to model flows in human trafficking networks.

Keywords: Fuzzy incidence graph, Incidence cut, Cutpair, Complete fuzzy incidence, Human trafficking.

1 Introduction

The study of human trafficking presents a distinctive challenge. There is no accurate data concerning the amount of trafficking from country to country. The only attempt to quantify the flow from country to country appears in [23]. Here countries are given three designations, source, transit, and destination countries. The amount of trafficking for these designations are linguistic. The amount is given by the terms very low, low, medium, high, and very high. This presents a problem in using traditional mathematics to model the situation in terms of network flow. However, it presents an interesting way to use fuzzy mathematics to create such a model. The authors are involved in a major research project to place the results in [23] into a fuzzy network model. Several approaches are underway [16, 17]. This paper is a first step to use the new concept of fuzzy incidence in a fuzzy graph. In order to accomplish this, we introduce the concepts of an incidence cutpair and $s-t$ incidence connectivity reducing set of pairs. The removal of such pairs reduces the flow in the network. This gives information concerning where to attack the fuzzy graph in order to reduce the flow. We conclude the paper by showing the development at this point of our use of the concept of incidence pair to reduce the flow in a fuzzy network by using real world data concerning the vulnerability of a country to human trafficking and a country's government response to reduce human trafficking. Graph theory provides tools to model different types of real world networks. When there are external agencies influencing the real flow in a network, we need to consider more relations, specifically the relationship of links with corresponding nodes, usually referred as incidences. Connectivity concepts of fuzzy incidence graphs are developed in this article, using which the structure and properties of fuzzy networks like human trafficking network can be studied. Fuzzy logic is an essential tool for modeling, machine logic and several other natural phenomena involving 'fuzziness'. After Zadeh's magical invention of fuzzy sets[25] in 1965, Rosenfeld[19] and Yeh and Bang [24] simultaneously introduced fuzzy graphs in 1975. Today fuzzy graph theory is emerging as one of the major areas of research in mathematics. Many other authors, including Mordeson [13, 14], Bhutani[5, 4], Sunitha [21, 22], Akram [1, 2] and Mathew [11, 10] have also contributed to the growth of fuzzy graph theory significantly. So many variants of fuzzy graphs also were introduced during this period. Bipolar fuzzy graphs [2] and intuitionistic fuzzy graphs [1] are some of the examples. Recently Dinesh [6, 7] has introduced a new concept, namely fuzzy incidence graph (FIG). Mathew and Mordeson studied some of their properties and applied

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Received: November 2017; Revised: January 2018; Accepted: July 2018.

them in problems related to human trafficking[16, 17, 20]. FIG is an excellent model for interconnection systems having ramps or minor links between node-edge pairs. A fuzzy graph is a fuzzy incidence graph where all the ramps in the graph have zero capacities. Basic definitions of fuzzy graphs used in this paper are from [13, 9]. Also, connectivity parameters of fuzzy incidence graphs can be seen in [12] and fuzzy endnodes in [15]. Different types of incidence edges are discussed in [8]. Related human trafficking data and fuzzy models are available in [16, 17, 3, 23].

2 Preliminaries

Some of the preliminary definitions and results about incidence fuzzy graphs are given below for a better understanding. Most of them are taken from [12]. Let (V, E) be a graph. Then, $G = (V, E, I)$ is called an incidence graph, where $I \subseteq V \times E$. If (u, vw) is in I, then (u, vw) is called an incidence pair or simply pair [12]. Two edges uv and vw are said to be adjacent if all the four pairs $(u, uv), (v, uv), (v, vw)$ and (w, vx) are in I. Throughout this paper $\wedge$ denotes minimum and $\vee$ denotes the maximum. Let $G = (V, E)$ be a graph and σ be a fuzzy subset of V and μ, a fuzzy subset of $V \times E$. Let Ψ be a fuzzy subset of E. If $\Psi(v, e) \leq \sigma(v) \wedge \mu(e)$ for all $v \in V$ and $e \in E$, then Ψ is called a **fuzzy incidence** of G. If Ψ is a fuzzy incidence, then $\widetilde{G} = (\sigma, \mu, \Psi)$ is called a **fuzzy incidence graph (FIG)** of G [6]. As usual, the support of a fuzzy set ξ defined over S, denoted by ξ^* or Supp(ξ) is the set $\{x \in S : \xi(x) > 0\}$. Let $xy \in$ Supp(μ). Then xy is an edge of the fuzzy incidence graph $\widetilde{G} = (\sigma, \mu, \Psi)$ and if $(x, xy), (y, xy) \in$ Supp(Ψ), then (x, xy) and (y, xy) are called **pairs**. Two nodes v_i and v_j joined by an incidence path in a fuzzy incidence graph are said to be **connected**. The **incidence strength** of a fuzzy incidence graph $\widetilde{G} = (\sigma, \mu, \Psi)$ is defined to be $\wedge\{\Psi(v, e)|(v, e) \in$ Supp$(\Psi)\}$ [7].

The fuzzy incidence graph $\widetilde{G} = (\sigma, \mu, \Psi)$ is a **cycle** if (Supp(σ), Supp(μ), Supp(Ψ)) is a cycle. The fuzzy incidence graph $\widetilde{G} = (\sigma, \mu, \Psi)$ is a **fuzzy cycle** if (Supp(σ), Supp(μ), Supp(Ψ)) is a cycle and there exists no unique $xy \in$ Supp(μ) such that $\mu(xy) = \wedge\{\mu(uv)|uv \in$ Supp$(\mu)\}$. The fuzzy incidence graph $\widetilde{G} = (\sigma, \mu, \Psi)$ is a **fuzzy incidence cycle** if it is a fuzzy cycle and there exists no unique $(x, yz) \in$ Supp(Ψ) such that $\Psi(x, yz) = \wedge\{\Psi(u, vw)|(u, vw) \in$ Supp$(\Psi)\}$ [12]. The fuzzy incidence graph $\widetilde{G} = (\sigma, \mu, \Psi)$ is a **tree** if (Supp(σ), Supp(μ), Supp(Ψ)) is a tree and is a **forest** if (Supp(σ), Supp(μ), Supp(Ψ)) is a forest. Let $\widetilde{G} = (\sigma, \mu, \Psi)$ be a fuzzy incidence graph. Then $\widetilde{H} = (\tau, \nu, \Omega)$ is a **fuzzy incidence subgraph** of $\widetilde{G}$ if $\tau \subseteq \sigma, \nu \subseteq \mu$, and $\Omega \subseteq \Psi$. A fuzzy incidence subgraph $\widetilde{H}$ of $\widetilde{G}$ is a **fuzzy incidence spanning subgraph** of $\widetilde{G}$ if $\tau = \sigma$ [7]. Also define $\Psi^\infty(s, t)$ to be the incidence strength of a path from s to t of greatest incidence strength, where $s, t \in \sigma^* \cup \mu^*$. We shall also use the notation $ICONN_G(s, t)$ to denote $\Psi^\infty(s, t)$ [8].

Consider an incidence path $u_0, (u_0, e_1), e_1, (u_1, e_1), u_1, ..., (u_n, e_{n+1}), u_n$. Since $\Psi(u_{i-1}, e_i) \leq \sigma(u_{i-1}) \wedge \mu(e_i)$, the strength of the incidence path is $\Psi(u_0, e_1) \wedge ... \wedge \Psi(u_n, e_{n+1})$. An edge $xy \in E$ is called a **bridge** if there exists $u, v \in E\backslash\{xy\}$ such that $\mu'^\infty(u, v) < \mu^\infty(u, v)$, where $\mu' = \mu$ restricted to $E\backslash\{xy\}$. If $w \in V$ and E' denotes the set difference of E and the set of edges with w as an end node. Then w is called a **cutnode** if $\mu'^\infty(u, v) < \mu^\infty(u, v)$ for some $u, v \in \sigma^*$ such that $u \neq w \neq v$, where $\mu' = \mu$ restricted to E'. w is called a **fuzzy incidence cutnode**, or simply an **incidence cutnode** if $\Psi'^\infty(u, uv) < \Psi^\infty(u, uv)$ for some $(u, uv) \in V \times E'$ such that $u \neq w \neq v$, where $\Psi' = \Psi$ restricted to $V \times E'$. A pair (x, xy) is called a **fuzzy incidence cutpair** or simply an **incidence cutpair** if $\Psi'^\infty(u, uv) < \Psi^\infty(u, uv)$ for some pair (u, uv) in $\widetilde{G}$, where $\Psi' = \Psi$ restricted to $(V \times E)\backslash\{(x, xy)\}$ [12]. $\widetilde{G}$ is a **fuzzy incidence forest** if $\widetilde{G}$ has a fuzzy incidence spanning subgraph $\widetilde{F} = (\sigma, \nu, \Omega)$ which is also a forest such that $\forall(u, vw) \in$ Supp$(\Psi)\backslash$Supp$(\Omega), \Psi(u, vw) < \Omega^\infty(u, vw)$. A connected fuzzy incidence forest is called a fuzzy incidence tree [7]. It is established in [6] that $\widetilde{F}$ in the definition of a fuzzy incidence forest is unique. $\widetilde{G}$ is said to **fuzzy incidence complete** if for all $(u, vw) \in V \times E, \Psi(u, vw) = \sigma(u) \wedge \mu(vw)$. Note that if $\widetilde{G}$ is fuzzy incidence complete, then $\Psi(u, uv) = \sigma(v) \wedge \mu(uv) = \mu(uv) = \sigma(v) \wedge \mu(uv) = \Psi(v, uv)$. Also, we state an important result from fuzzy incidence graphs without proof.

Theorem 2.1. *If $\widetilde{G}$ is a fuzzy incidence forest, with unique spanning forest $\widetilde{F}$, then the node edge pairs of $\widetilde{F}$ (as in the definition of fuzzy incidence forest) are exactly the incidence cutpairs of $\widetilde{G}$.*

Even though the incidence pairs are generally of the form (u, vw), we consider only pairs of the form (u, uv) in this article.

3 Incidence Degree and Ramp Degree of Fuzzy Incidence Graphs

Apart from the human trafficking application, there are several possible applications for fuzzy incidence graphs. One of them is in the design of an efficient highway system. When we design a network of roads connecting a number of

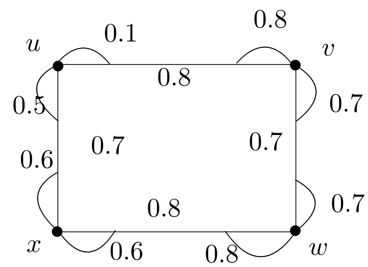

Figure 1: FIG with 8 Incidence Pairs

cities, there are several branching parameters to be considered. For example, the number and capacity of ramps at nodes will usually be different. Towards this purpose, we define incidence degrees of nodes and edges as in the following definitions.

Definition 3.1. *Let $\widetilde{G} = (\sigma, \mu, \Psi)$ be a fuzzy incidence graph. The incidence degree of a node $u \in \sigma^*$ is defined as $d_i(u) = \sum_{uv \in \mu^*} \psi(u, uv)$. Also the strong incidence degree of a node $u \in \sigma^*$ is defined as $d_{si}(u) = \sum_{uv \in \mu^*} \psi(u, uv)$, where (u, uv) is a strong pair of $\tilde{G}$.*

Definition 3.2. *The incidence degree of an edge $uv \in \mu^*$ of an FIG $\widetilde{G}$ is defined as $d_i(uv) = \psi(u, uv) + \psi(v, uv)$. The strong incidence degree of an edge uv is defined as the sum of ψ values of strong pairs incident on it. If there are no strong pairs incident on the edge uv, then define $d_{si}(uv) = 0$. We may also refer $d_i(uv)$ and $d_{si}(uv)$ as ramp degree and strong ramp degree of the edge uv, respectively.*

As in graph theory, minimum and maximum of node degrees can be obtained. We denote them as $\delta_i(\widetilde{G}), \Delta_i(\widetilde{G}), \delta_{si}(\widetilde{G})$, $\Delta_{si}(\widetilde{G})$ respectively. Also, we call the minimum and maximum of incidence degrees of edges as minimum ramp degree and maximum ramp degree of the incidence graph and denote them as $\delta_R(\widetilde{G})$ and $\Delta_R(\widetilde{G})$ respectively. We explain these definitions in the following example.

Example 3.3. *Consider the incidence fuzzy graph given in Figure 1 having 4 nodes, 4 edges and 8 ramps or incidence edges. For simplicity assume that $\sigma(u) = 1$ for every node $u \in \sigma^*$.*

All edges in this FIG are strong, but the pair (u, uv) is not strong. All other pairs are strong. Hence, $d_i(u) = 0.6$, $d_i(v) = 1.5$, $d_{si}(u) = 0.5, d_i(uv) = 0.9$ and $d_{si}(uv) = 0.8$. $\delta_i(\widetilde{G}) = 0.6, \Delta_i(\widetilde{G}) = 1.5, \delta_{si}(\widetilde{G}) = 0.5, \Delta_{si}(\widetilde{G}) = 1.5, \delta_R(\widetilde{G}) = 0.9$ and $\Delta_R(\widetilde{G}) = 1.4$.

Next we have two basic propositions whose proof are immediate.

Proposition 3.4. In a non trivial connected FIG $\widetilde{G} = (\sigma, \mu, \Psi), 0 \leq d_i(v) \leq d(v)$ and $0 \leq d_{si}(v) \leq d_i(v)$ for every node $v \in \sigma^*$.

Proof. Let $\widetilde{G} = (\sigma, \mu, \Psi)$ be a nontrivial FIG. Let $v \in \sigma^*$. Since $\psi(v, vz) \leq \mu(vz)$ for every $z \in \sigma^*, d_i(v) \leq d(v)$. Also if all pairs at v are strong, then both $d_i(v)$ and $d_{si}(v)$ are equal. Otherwise it is obvious that $d_{si}(v) < d_i(v)$ for $v \in \sigma^*$. □

Next we have a result analogous to the handshaking lemma in graph theory.

Proposition 3.5. In a non trivial connected FIG $\widetilde{G} = (\sigma, \mu, \Psi)$, the sum of incidence degrees of nodes $u \in \sigma^*$ is less than or equal to twice the membership values of edges $e \in \mu^*$. That is, $\sum_{u \in \sigma^*}(d_i(u)) \leq 2 \sum_{e \in \mu^*} \mu(e)$.

The proof of Proposition 3.5 follows from the fact that any incidence pair in ψ^* contribute a maximum of 2 towards the incidence degree sum of the FIG. As seen before, it is not necessary that $\mu(uv) = \wedge\{\sigma(u), \sigma(v)\}$ in a fuzzy complete incidence. Hence, we have the following definition.

Definition 3.6. *Let $\widetilde{G} = (\sigma, \mu, \Psi)$ be a fuzzy incidence graph. Then, $\widetilde{G}$ is said to be a complete fuzzy incidence graph(CFIG) if it is a fuzzy complete incidence and $G = (\sigma, \mu)$ is a complete fuzzy graph where G is the underlying fuzzy graph.*

Example 3.7. *Let $\widetilde{G}_1 = (\sigma, \mu, \Psi)$ be the FIG in Figure 2(a) and $\widetilde{G}_2 = (\sigma', \mu', \Psi')$ in Figure 2(b). In both FIGs $\sigma(a) = 0.6, \sigma(b) = 0.7, \sigma(c) = 0.8$.*

The FIG in Figure $2(a)$ is not a complete fuzzy incidence graph, where as that in Figure $2(b)$ is. But Figure $2(a)$ is a fuzzy complete incidence. Also note that any complete fuzzy incidence is a fuzzy complete incidence. Next two propositions deal with ramp degree of complete fuzzy incidence graphs.

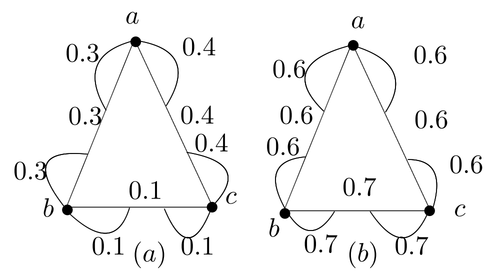

Figure 2: (a) A non CFIG (b) A CFIG

Proposition 3.8. Let $\widetilde{G} = (\sigma, \mu, \Psi)$ be a CFIG. Let $\sigma^* = \{v_0, v_1, ..., v_n\}$ and $\mu^* = \{e_1, e_2, ..., e_m\}$. Let $e_j, e_k \in \mu^*$ be such that $\mu(e_j) \leq \mu(e_l)$ for every $l \neq j$ and $\mu(e_k) \geq \mu(e_l)$ for every $l \neq k$. Also let $e_j = (v_0, v_j)$ and $e_k = (v_k, v_n)$. Then $\delta_R(\widetilde{G}) = 2\psi(v_0, v_0v_j)$ and $\Delta_R(\widetilde{G}) = 2\psi(v_k, v_kv_n)$.

Proof. Since $\widetilde{G}$ is a CFIG, it is a fuzzy complete incidence and hence by Proposition 9 of [8], every pair of $\widetilde{G}$ strong. Hence $d_i(xy) = d_{si}(xy)$ for every edge $xy \in \mu^*$ and since $\mu(xy) \leq \sigma(x) \wedge \sigma(y)$, it follows that minimum ramp degree of the FIG is is $2\psi(v_0, v_0v_j)$ and maximum is $2\psi(v_k, v_kv_n)$. □

In the next proposition, we compare the incidence degrees of a CFIG $\widetilde{G}$ and the strong degree of the underlying fuzzy graph G.

Proposition 3.9. Let $\widetilde{G} = (\sigma, \mu, \Psi)$ be a CFIG. Let $\sigma^* = \{v_1, v_2, ..., v_n\}$ be such that $d(v_1) \leq d(v_2) \leq ... \leq d(v_n)$. Then $d_i(v_i) = d_{si}(v_i) = d_s(v_i) = d(v_i)$ for $i = 1, 2, ..., n$ and $\delta_i(\widetilde{G}) = \delta(G) = \sum_{l \neq 1} \psi(v_1, v_1v_l)$ and $\Delta_i(\widetilde{G}) = \Delta(G) = \sum_{j \neq n} \psi(v_n, v_nv_j)$ where $G(\sigma, \mu)$ is the underlying fuzzy graph of $\widetilde{G}$.

Proof of Proposition 3.9 easily follows from the fact that a CFIG has no δ pairs. Hence a formal proof is omitted. The underling fuzzy graph G of a FIG $\widetilde{G}$ can be obtained by removing all incidence pairs from $\widetilde{G}$.

Definition 3.10. *Let $\widetilde{G} = (\sigma, \mu, \Psi)$ be a CFIG with $\sigma^* = \{v_1, v_2, ..., v_n\}$. The incidence degree sequence of $\widetilde{G}$ is defined to be $(q_1, q_2, ..., q_n), q_i \in \boldsymbol{R}$ such that $q_1 \leq q_2 \leq ... \leq q_n$, where $q_i = d_i(v_i)$.*

In Figure 1, the incidence degree sequence is $(0.6, 1.2, 1.5, 1.5)$. For convenience we may write it as $(0.6, 1.2, 1.5^2)$. One can obtain minimum and maximum incidence degree nodes of complete fuzzy incidence graphs using incidence degree sequences. As mentioned before, a fuzzy complete incidence need not be a CFIG, but in any fuzzy complete incidence graph we have the following result.

Theorem 3.11. Let $\widetilde{G} = (\sigma, \mu, \Psi)$ be a fuzzy complete incidence. Then there exists at least one $u \in \sigma^*$ such that $d_i(u) = d_{si}(u)$.

Proof. Let $\widetilde{G} = (\sigma, \mu, \Psi)$ be a fuzzy complete incidence. By Proposition 9 of [8], $\widetilde{G}$ has no δ-incidence pairs. Thus all incidence pairs in $\widetilde{G}$ are strong and the theorem follows. □

4 Fuzzy Incidence Cuts (FIC) and Fuzzy Incidence Cuts of Pairs(FICP)

In this section, we generalize the concepts of fuzzy incidence cutnodes and fuzzy incidence cutpairs in a FIG, similar to the cutset concept in graphs. A fuzzy incidence subgraph $\widetilde{H} = (\tau, \nu, \Omega)$ of $\widetilde{G} = (\sigma, \mu, \psi)$ is said to be a subgraph of $\widetilde{G}$ if $\tau(u) = \sigma(u)$ for every $u \in \tau^*$, $\nu(uv) = \mu(uv)$ for every $uv \in \nu^*$ and $\Omega(u, vw) = \psi(u, vw)$ for every pair $(u, vw) \in \Omega^*$[8]. In a FIG, both the elements of σ^* and μ^* may treated as the nodes of a fuzzy graph. Motivated by this we generalize the definitions of incidence cutsets and connectivity in a different way as follows.

Definition 4.1. *A disconnection of a fuzzy incidence graph $\widetilde{G} = (\sigma, \mu, \psi)$ is a set $D \subseteq \sigma^* \cup \mu^*$ whose removal disconnects $\widetilde{G}$. The weight of D is defined to be $\sum_{v \in D} min\{\mu(v, u), u \in \sigma^*\} + \sum_{e \in D} \mu(e)$.*

Definition 4.2. *The node connectivity of a fuzzy incidence graph $\widetilde{G}$, denoted by $\Omega(\widetilde{G})$, is defined to be the minimum weight of a disconnection in $\widetilde{G}$.*

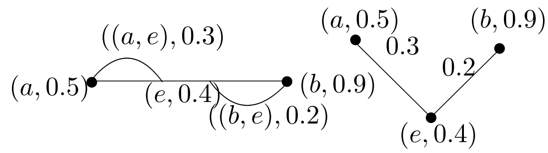

Figure 3: An Incidence Edge

Example 4.3. *Consider the example of a single incidence edge (Figure 3). As noted there should be two supporting incidence pairs for any incidence edge.*

Since it is a unique incidence path from a to b, both incidence pairs are strong. Now we can visualize this FIG as a fuzzy graph, given in the second picture. This visualization will be difficult for FIGs having more number of nodes, edges and pairs. From this fuzzy incidence graph, it can be seen that there is only one disconnection for $\widetilde{G}$, namely the edge e and its weight is $\mu(e) = 0.4$. Hence node connectivity $\Omega(\widetilde{G}) = 0.4$. Now we generalize this to a fuzzy incidence cut of $\widetilde{G}$ as follows.

Definition 4.4. *Let $\widetilde{G} = (\sigma, \mu, \psi)$ be a connected FIG. A set $X = \{s_1, s_2, ..., s_m\} \subset \sigma^* \cup \mu^*$ is said to be a fuzzy incidence cut (FIC) of $\widetilde{G}$ if, $ICONN_{G-X}(s,t) < ICONN_G(s,t)$ for some pair of elements $s, t \in \sigma^* \cup \mu^* - X$.*

Note that if X consists of a single node $s \in \sigma^*$, then it is a fuzzy incidence cutnode and when $s \in \mu^*$, is an edge, it is an incidence bridge.

Definition 4.5. *Let X be a fuzzy incidence cut in $\widetilde{G}$. The strong weight of X, denoted by $s(X)$ is defined as $s(X) = \sum_{s,xy \in X, s \in \sigma^*, xy \in \mu^*} \{\psi(s, su) + \wedge\{\psi(x, xy), \psi(y, xy)\}$, where (s, su) is a strong pair of minimum weight at s, and $\wedge$ is taken over all strong pairs incident at xy.*

Definition 4.6. *The incidence connectivity of a connected fuzzy incidence graph $\widetilde{G}$ is defined as the minimum strong weights of fuzzy incidence cuts of $\widetilde{G}$. It is denoted by $\kappa_i(\widetilde{G})$.*

Next we illustrate these definitions. Consider the following example of a FIG having 4 nodes, 5 edges and 10 incidence pairs.

Example 4.7. *Let $\widetilde{G} = (\sigma, \mu, \psi)$ be the FIG given in Figure 4. Its fuzzy graph visualization will have 9 nodes and 10 edges. For simplicity we take $\sigma(x) = 1$ for every $x \in \sigma^*$.*

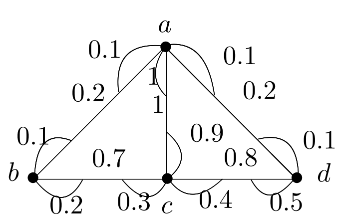

Figure 4: FIG with an Incidence Cutnode

In Figure 4, $\widetilde{G}$, c is a fuzzy incidence cutnode. So if we let $X = \{c\}$, then it is a fuzzy incidence cut and whose strong weight is 0.3, which is the minimum of ψ values of pairs at c. But incidence connectivity of $\widetilde{G}$ is the minimum of such values. There are several fuzzy incidence bridges and incidence cuts in $\widetilde{G}$, but none of them give a lesser value than 0.2. Thus $\kappa_i(\widetilde{G}) = 0.2$ Note that a fuzzy incidence cut can contain even edges which are not strong. But in an FIC of minimum strong weight, all edges present will be strong as the removal of a non strong edge will not reduce the incidence flow between any pair $s, t \in \sigma^* \cup \mu^*$.

Next we define fuzzy incidence cut of pairs and fuzzy incidence pair connectivity as follows.

Definition 4.8. *Let $\widetilde{G} = (\sigma, \mu, \psi)$ be a connected FIG. A set of of strong pairs $S = \{t_1, t_2, ..., t_m\} \subset \psi*$ is said to be a fuzzy incidence cut of pairs (FICP) of $\widetilde{G}$ if, $ICONN_{\widetilde{G}-S}(s,t) < ICONN_{\widetilde{G}}(s,t)$ for some pair of elements $s, t \in \sigma^* \cup \mu^* - X$.*

Note that a FICP of one element is a cutpair of $\widetilde{G}$.

Definition 4.9. *Let S be a fuzzy incidence cut of pairs in $\widetilde{G}$. The weight of S, denoted by $w(X)$ is defined as* $w(X) = \sum_{(u,uv)\in S} \psi(u, uv)$.

Definition 4.10. *The incidence connectivity of pairs of a connected fuzzy incidence graph $\widetilde{G}$ is defined as the minimum weight of all fuzzy incidence cuts of pairs of $\widetilde{G}$. It is denoted by $\kappa_i^{'}(\widetilde{G})$.*

Example 4.11. *Consider the following example of an incidence graph with $\mid \sigma^* \mid = 6$. Suppose that all the pairs in this FIG except (a, ab) and (b, ab) have the same ψ value, say 0.3.*

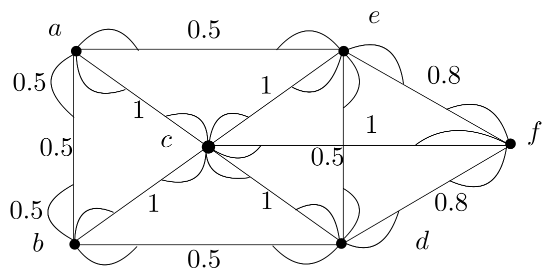

Figure 5: FIG with Equal Pair Values

In this FIG (Figure 5), there are several incidence cuts. But edge ab is a unique incidence bridge of $\tilde{G}$ since $ICONN_{\widetilde{G}-ab}(a, b) = 0.3 < 0.5 = ICONN_{\widetilde{G}}(a, b)$. Let $S = \{ab\}$. S is an incidence cut. So the incidence connectivity $\kappa_i(\widetilde{G}) = \wedge\{0.5, 0.5\} = 0.5$. Also, there are several cuts of pairs in this FIG. But clearly the pair (a, ab), which is a cut pair is of less weight. Therefore, $\kappa_i^{'}(\widetilde{G}) = 0.5$. This example shows that a fuzzy incidence bridge need not be a fuzzy bridge in the underlying fuzzy graph. Here ab is a fuzzy incidence bridge of $\widetilde{G}$, but not a fuzzy bridge in the corresponding underlying fuzzy graph. Also note that the end nodes of an incidence bridge need not be an incidence cutnode. Here, neither a nor b are incidence cutnodes. But as noted in [6], the adjacent pairs of an incidence bridge are cutpairs. Here both (a, ab) and (b, ab) are cutpairs. In Figure 5, the removal of the incidence bridge ab reduces the incidence connectivity between its end nodes only. So we shall have the following definition of some other types of incidence bridges.

Definition 4.12. *Let $\widetilde{G} = (\sigma, \mu, \Psi)$ be fuzzy incidence graph. An edge $uv \in \mu^*$ is said to be a fuzzy incidence bond if $ICONN_{\widetilde{G}-uv}(s, t) < ICONN_{\widetilde{G}}(s, t)$ for some $s, t \in \sigma^* \cup \mu^*$ such that at least one of s, t is different from u and v. uv is called a fuzzy incidence cutbond if $ICONN_{\widetilde{G}-uv}(s, t) < ICONN_{\widetilde{G}}(s, t)$ for some $s, t \in \sigma^* \cup \mu^*$, $s, t \neq u, v$.*

It is obvious that any fuzzy incidence cutbond is a fuzzy incidence bond, which is also a fuzzy incidence bridge. Consider the following example.

Example 4.13. *Let $\widetilde{G} = (\sigma, \mu, \Psi)$ be the FIG given Figure 6, with $\sigma(x) = 1$ for every $x \in \sigma^*$.*

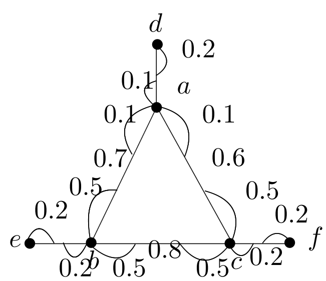

Figure 6: FIG with an Incidence Cutbond

For the FIG in Figure 6, any pendent edge is an incidence bond, since their removal reduces the incidence connectivity from its endnode to any other node in the FIG to 0, from a non zero value. Also the edge bc is an incidence cutbond. since $ICONN_{\widetilde{G}-bc}(e, f) = 0.1 < 0.2 = ICONN_{\widetilde{G}}(e, f)$. Also, we can see that removal of a, b or c will reduce the incidence connectivity between nodes in $\widetilde{G}$. Motivated by this, we have the following result.

Proposition 4.14. At least one of the end nodes of a fuzzy incidence bond is a fuzzy incidence cutnode.

Proof. Let $\widetilde{G} = (\sigma, \mu, \Psi)$ be a FIG and $e = (u, v)$ be a fuzzy incidence bond in $\widetilde{G}$. The deletion of e from $\widetilde{G}$ reduces the fuzzy incidence strength between s and t in $\sigma^* \cup \mu^*$ with at least one of s and t different from u and v. If both s and t are different from u and v, then u as well as v are incidence cutnodes. If one of s or t coincides with u or v, then u or v which is neither s nor t will be a fuzzy incidence cutnode. □

From the proof, it follows that both end nodes of a fuzzy incidence cutbond are fuzzy incidence cutverties. In Figure 6, b and c are fuzzy incidence cutnodes. A fuzzy incidence graph $\widetilde{G} = (\sigma, \mu, \psi)$ is said to be a fuzzy incidence tree if there exists a spanning subgraph $\widetilde{F} = (\tau, \nu, \Omega)$ of $\widetilde{G}$, which is a tree, such that for every pair $(u, vw) \in \psi^* - \Omega^*, \psi(u, vw) < ICONN_{\widetilde{F}}(u, vw)$ [12]. In other words, there exists an incidence path in $\widetilde{F}$ from u to vw so that each of its pairs has greater ψ value than $\psi(u, vw)$. Note that the existence of an incidence path between any two elements of $\sigma^* \cup \mu^*$ is automatically guaranteed in this definition. Consider the following example.

Example 4.15. *Consider a fuzzy incidence tree with* $\mid \sigma^* \mid = 5$. *Assume for simplicity that* $\sigma(u) = 1$ *for every* $u \in \sigma^*$.

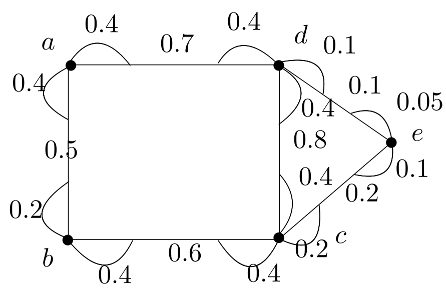

Figure 7: A Fuzzy Incidence Tree

In the FIT, all pairs are strong except (b, ab) and (e, ed). So the unique $\widetilde{F}$ is the FIG in Figure 7, without pairs (b, ab) and (e, ed). It is comparatively easy to study the connectivity parameters of fuzzy incidence trees. Both incidence connectivity and incidence connectivity of pairs of an incidence tree are equal to the minimum of the ψ values of strong pairs of $\widetilde{G}$ as seen from the following theorem.

Theorem 4.16. Let $\widetilde{G} = (\sigma, \mu, \psi)$ be a fuzzy incidence tree. Then, $\kappa_i(\widetilde{G}) = \kappa_i'(\widetilde{G}) = \wedge\{\psi(u, uv) : (u, uv)$ is a strong pair in $\widetilde{G}\}$.

Proof. Let $\widetilde{G} = (\sigma, \mu, \psi)$ be a fuzzy incidence tree. Consider the fuzzy incidence graph $\widetilde{F}$ in the definition of $\widetilde{G}$. A pair (x, xy) in $\widetilde{G}$ is an incidence cutpair if and only if (x, xy) is a pair of $F = (\sigma, \mu, \Omega)$ (Theorem 1.1). Also strong pairs of $\widetilde{G}$ are essentially the strong pairs of $\widetilde{F}$ and all pairs in $\widetilde{F}$ are strong pairs. Thus each strong pair in $\widetilde{F}$ is a fuzzy incidence cutpair of $\widetilde{G}$. Clearly the strong weight of each such FIC of pairs is $\psi(x, xy)$. Hence fuzzy incidence connectivity of pairs $\kappa_i'(\widetilde{G})$ of $\widetilde{G}$ is the minimum ψ value of all pairs in $\widetilde{F}$ and hence the minimum ψ value of all strong pairs in $\widetilde{G}$.

Now every internal node of $\widetilde{F}$ is an incidence cutnode of $\widetilde{G}$ and hence are fuzzy incidence node cuts of G. Hence fuzzy incidence connectivity $\kappa_i(\widetilde{G})$ of $\widetilde{G}$ is the minimum ψ value of all pairs in $\widetilde{F}$ and hence the minimum ψ value of all strong pairs in $\widetilde{G}$. Hence the proof is complete. □

Generally the following inequality holds in any fuzzy incidence graph.

Theorem 4.17. In a connected fuzzy incidence graph $\widetilde{G} = (\sigma, \mu, \psi)$, $\kappa_i(\widetilde{G}) \leq \kappa_i'(\widetilde{G}) \leq \delta_{si}(\widetilde{G})$.

Proof. First we prove the second inequality. Let $\widetilde{G} = (\sigma, \mu, \psi)$ be a connected fuzzy incidence graph. Let $v \in \sigma^*$ be such that $d_{si}(v) = \delta_{si}(\widetilde{G})$. Let P be the set of strong pairs incident at v. If these are the only pairs incident at v, then $\widetilde{G} - P$ is disconnected. If not, let (v, vu) be a pair which is not strong at v. Then vu is an edge different from the edges of the pairs in P. By definition of a strong pair, $\psi(v, vu) < ICONN_{\widetilde{G}}(v, vu)$, which implies that there exists a strongest $v - vu$ incidence path say Q in $\widetilde{G}$ which should definitely pass through one of the strong pairs at v. Thus the removal of P from $\widetilde{G}$ will reduce the incidence strength between v and u. Thus in both cases, P is a fuzzy incidence cut of pairs. The strong weight of this FICP is $\delta_{si}(\widetilde{G})$. Hence it follows that $\kappa_i'(\widetilde{G}) \leq \delta_{si}(\widetilde{G})$.
Next to prove $\kappa_i(\widetilde{G}) \leq \kappa_i'(\widetilde{G})$. Let P be a FICP with weight $\kappa_i'(\widetilde{G})$. We have the following cases.
Case 1. Every pair in P has a common node v(say).

In this case, let $P = \{p_i = (v, vv_i), i = 1, 2, ..., n\}$.
Let $S = \{vv_1, vv_2, ..., vv_n\}$. Then clearly P is a fuzzy incidence cut. Now, $min_{u\in\sigma_*}\psi(v_i, uv_i) \le \psi(v_i, vv_i)$.
Therefore, $\sum_i (min_{u\in\sigma_*}\psi(v_i, uv_i)) \le \psi(v, vv_1) + \psi(v, vv_2) + ... + \psi(v, vv_n)$.

That is, $\kappa_i(\widetilde{G}) \le \kappa'_i(\widetilde{G})$.
Case 2. Not all pairs in P have a node in common.
Let $P = \{p_i = (u_i, u_iv_i), i = 1, 2, ..., n\}$ for some n. Let $X_1 = \{u_1, u_2, ..., u_n\}$ and $X_2 = \{u_1v_1, u_2v_2, ..., u_nv_n\}$. By assumption, $ICONN_{\widetilde{G}-P}(s, t) < ICONN_{\widetilde{G}}(s, t)$ for some $s, t \in \sigma^* \cup \mu^*$.
Sub Case I. s and t are not members of $X_1 \cup X_2$.
In this case, take $X = X_1$ or $X = X_2$. Then clearly X is a fuzzy incidence cut since its deletion from $\widetilde{G}$ reduces the incidence strength between s and t and, $\kappa_i(\widetilde{G}) \le$ *weight* of $X \le$ *weight* of $P = \kappa'_i(\widetilde{G})$.
Sub Case 2. Either s or t is in $X_1 \cup X_2$.
Without loss of generality suppose that s is in $X_1 \cup X_2$. Suppose $s \in X_1$. Then take $X = X_2$. Clearly X is a fuzzy incidence cut, for; the deletion of X from $\widetilde{G}$ will reduce the incidence strength between s and t. Thus, $\kappa_i(\widetilde{G}) \le$ *weight* of $X \le$ *weight* of $P = \kappa'_i(\widetilde{G})$.
Sub Case 3. Both s and t are in $X_1 \cup X_2$. Clearly $s \in \sigma^*$ and $t \in \mu^*$. Hence $t = rv$ where $r, v \in \sigma^*$. The removal of P reduces the incidence strength between s and v and the conclusion follows.
Thus in all cases, $\kappa_i(\widetilde{G}) \le \kappa'_i(\widetilde{G}) \le \delta_{si}(\widetilde{G})$. □

As in the case of incidence fuzzy trees, we can easily determine the incidence connectivity parameters of a complete fuzzy incidence graph, using the following result.

Theorem 4.18. In a complete fuzzy incidence graph, $\widetilde{G} = (\sigma, \mu, \psi)$, $\kappa_i(\widetilde{G}) = \kappa_{si}(\widetilde{G}) = \kappa'_{si}(\widetilde{G}) = \kappa'_i(\widetilde{G}) = \delta_{si}(\widetilde{G}) = \delta_i(\widetilde{G})$.

Proof. Let $\widetilde{G} = (\sigma, \mu, \psi)$, be a CFIG such that $|\sigma^*| = n$. Since ψ is a complete fuzzy incidence, the deletion of any set P of at most $n-2$ pairs from $\widetilde{G}$ will not reduce the incidence strength between any pair of elements of $\sigma^* \cup \mu^*$. But a set of $n-1$ pairs incident at a node $u \in \sigma^*$ is a FICP with weight $d_{si}(u) = d_i(u)$. Let $v \in \sigma^*$ be such that $d_i(v) = \delta_i(G)$. Clearly the set of edges incident at v is a FICP with minimum weight. Therefore, $\kappa'_i(\widetilde{G}) = d_i(v) = \delta_i(\widetilde{G})$. Now we prove that $\kappa_i(\widetilde{G}) = \delta_i(\widetilde{G})$. If possible suppose that $\kappa_i(\widetilde{G}) \ne \delta_i(\widetilde{G})$. By Theorem 3.17, $\kappa_i(\widetilde{G}) \le \kappa'_i(\widetilde{G}) \le \delta_{si}(\widetilde{G})$. Hence $\kappa_i(\widetilde{G}) < \delta_{si}(\widetilde{G})$ Note that any incidence cut of elements from σ^* will have cardinality $n-1$. Among such incidence cuts, the one which does not contain v such that $d_{si}(v) = \delta_{si}(G)$, say S_1 will have the minimum weight since the set of pairs adjacent with elements in S_1 with one end at v are the pairs with minimum ψ value among elements of S_1. Thus, $\kappa_i(\widetilde{G}) = s(S_1) < \delta_{si}(\widetilde{G})$.

Now let E_1 be the set of pairs incident with v. Then E_1 is a FICP such that, $s'(E_1) = s(S_1) < \delta_{si}(\widetilde{G})$, which contradicts the fact that $\kappa'_i(\widetilde{G}) = \delta_{si}(\widetilde{G})$. Hence, $\kappa_i(\widetilde{G}) = \kappa'_i(\widetilde{G}) = \delta_{si}(\widetilde{G})$. The conclusion of the theorem follows from the fact that all pairs of a complete fuzzy incidence are strong. □

In the next section, we shall discuss incidence connectivity locally between fixed elements and present results similar to Menger's theorem.

5 $s-t$ Incidence Connectivity Reducing Sets

Now we focuss on a particular 2-element subset $\{s, t\}$ of $\sigma^* \cup \mu^*$ and collections of incidence pairs whose removal reduces the incidence connectivity $ICONN_{\widetilde{G}}(s, t)$ between s and t. Consider the following definitions of $s-t$ incidence connectivity reducing sets.

Definition 5.1. *A set $S \subset \sigma^* \cup \mu^* - \{s, t\}$ is said to be an $s-t$ incidence connectivity reducing set if $ICONN_{\widetilde{G}-S}(s, t) < ICONN_{\widetilde{G}}(s, t)$, where $\widetilde{G} - S$ is the fuzzy incidence subgraph of $\widetilde{G}$ obtained by removing all elements in S and the associated pairs.*

Definition 5.2. *A set $P \subset \psi^*$ is said to be an $s-t$ incidence connectivity reducing set of pairs if $ICONN_{\widetilde{G}-P}(s, t) < ICONN_{\widetilde{G}}(s, t)$, where $\widetilde{G} - P$ is the fuzzy incidence subgraph of $\widetilde{G}$ obtained by removing all pairs in P.*

Definition 5.3. *An $s-t$ incidence connectivity reducing set with n elements of $\sigma^* \cup \mu^*$ is said to be a minimum $s-t$ incidence connectivity reducing set if there exist no $s-t$ incidence connectivity reducing set with less than n elements. A minimum $s-t$ incidence connectivity reducing set is denoted by $I_{\widetilde{G}}(s, t)$. Minimum $s-t$ incidence connectivity reducing set of pairs can be similarly defined and is denoted by $P_{\widetilde{G}}(s, t)$.*

Being an incidence cut and an incidence cut of pairs, the weight of these sets can be calculated using definitions 4.5 and 4.9.

Example 5.4. *Consider the following fuzzy incidence graph given in Figure 8.*

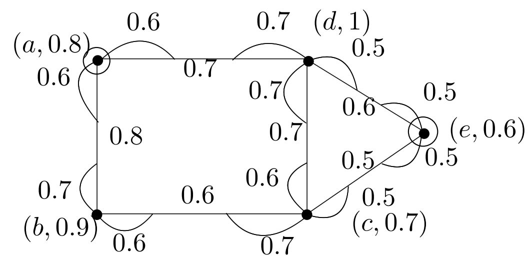

Figure 8: Incidence Connectivity Reducing Sets

Consider a and e. $S_1 = \{d, c\}, S_2 = \{d, ce\}$ are two $a - b$ incidence connectivity reducing sets with size two and weight one. Also $P = \{(d, de), (c, ce)\}$ is a incidence connectivity reducing set of pairs with size 2 and weight one. Next we characterize $s - t$ incidence connectivity reducing sets in the following theorems.

Theorem 5.5. Let $\widetilde{G} = (\sigma, \mu, \psi)$ be a connected fuzzy incidence graph and $s, t \in \sigma^* \cup \mu^*$. Then, a set $S \subset \sigma^* \cup \mu^*$ is an $s - t$ incidence connectivity reducing set if and only if every strongest incidence path from s to t contains at least one element of S.

Proof. Suppose that S is an $s - t$ incidence connectivity reducing set in $\widetilde{G}$ and P a strongest $s - t$ incidence path in $\widetilde{G}$. If P contains no node of S, the removal of S keep P intact and hence $G - S$ contains P. Thus $ICONN_{\widetilde{G}-S}(s, t) = ICONN_{\widetilde{G}}(s, t)$, which contradicts the fact that S is an $s - t$ incidence connectivity reducing set. Thus P must contains at least one member of S.

Conversely, suppose that every strongest incidence path from s to t contains at least one element of S, where $S \subset \sigma^* \cup \mu^*$ and s, t not in S. Then the removal of S destroys all strongest $s - t$ incidence paths in $\widetilde{G}$ and hence $ICONN_{\widetilde{G}-S}(s, t) < ICONN_{\widetilde{G}}(s, t)$. Thus it follows that S is an $s - t$ incidence connectivity reducing set in $\widetilde{G}$. □

We have a similar characterization for $s - t$ incidence connectivity reducing set of pairs.

Theorem 5.6. Let $\widetilde{G} = (\sigma, \mu, \psi)$ be a connected fuzzy incidence graph and s, t any two elements in $\sigma^* \cup \mu^*$. Then, a set K of pairs in $\widetilde{G}$ is an $s - t$ incidence connectivity reducing set of pairs if and only if every strongest incidence path from s to t contains at least one pair from K.

Proof. Let K be an $s - t$ incidence connectivity reducing set of pairs in $\widetilde{G}$ and Q a strongest $s - t$ incidence path in $\widetilde{G}$. If Q contains no incidence pair of K, the removal of K keep Q intact and hence $G - K$ contains Q. Thus $ICONN_{\widetilde{G}-K}(s, t) = ICONN_{\widetilde{G}}(s, t)$, which contradicts the fact that K is an $s - t$ incidence connectivity reducing set of pairs. Thus Q must contains at least one pair from K.

Conversely, suppose that every strongest incidence path from s to t contains at least one pair of K. Then the removal of K removes all strongest $s - t$ incidence paths in $\widetilde{G}$ and hence $ICONN_{\widetilde{G}-K}(s, t) < ICONN_{\widetilde{G}}(s, t)$. Thus it follows that K is an $s - t$ incidence connectivity reducing set of pairs in $\widetilde{G}$. □

In all connectivity problems in graph theory, internally disjoint paths are very important. In the next theorem internally disjoint paths in an FIG and minimal incidence connectivity reducing sets are related.

Theorem 5.7. Let $\widetilde{G} = (\sigma, \mu, \psi)$ be a fuzzy incidence graph. For any two elements $s, t \in \sigma^* \cup \mu^*$ such that (s, t) not a strong incidence pair, the maximum number of internally disjoint strongest $s - t$ incidence paths in $\widetilde{G}$ is equal to the number of elements in a minimal $s - t$ incidence connectivity reducing set.

Proof. Let $\widetilde{G} = (\sigma, \mu, \psi)$ be a fuzzy incidence graph and let $s, t \in \sigma^* \cup \mu^*$. There are different cases.
Case 1. $s, t \in \sigma^*$ or μ^*.

If all the elements in a minimal $s - t$ incidence connectivity reducing set S are in σ^* or in μ^*. Then they are elements of a minimal $s - t$ strength reducing sets of nodes in the corresponding fuzzy graph $\widetilde{G}'$ obtained from $\widetilde{G}$ by identifying all elements of $\sigma^* \cup \mu^*$ as nodes and elements of ψ^* as edges. By Menger's theorem for fuzzy graphs, the number of internally disjoint strongest paths in $\widetilde{G}'$ is equal to the number of elements in S. Each such path in G' can be expanded as a strongest incidence path in $\widetilde{G}$ by shrinking edges to nodes and joining node edge pairs by incidence edges. We can easily observe that these are all the strongest $s-t$ incidence paths in $\widetilde{G}$ and the theorem follows. The case where both $s,t \in \mu^*$ can be proved similarly.

Case 2. One of $s,t \in \sigma^*$ and other in μ^*.

Let $s \in \sigma^*$ and $t \in \mu^*$. By assumption, (s,t) is not a strong pair. Hence, $ICONN_{\widetilde{G}}(s,t) > \psi(s,t)$. Thus, (s,t) is not part of any strongest incidence path from s to t. So the edge st in the fuzzy graph $\widetilde{G}'$ will be δ. By Menger's theorem for fuzzy graphs, the number of internally disjoint $s-t$ paths in $\widetilde{G}'$ is equal to number of nodes in an $s-t$ strength reducing set S of $\widetilde{G}'$. As did in Case I, the maximum number of internally disjoint strongest $s-t$ incidence paths in $\widetilde{G}$ will be the number of elements in the $s-t$ incidence connectivity reducing set obtained by modifying S.

□

Actually when (s,t) is a strong pair, their do not exists an $s-t$ incidence connectivity reducing set in $\widetilde{G}$ as seen from the next example.

Example 5.8. *Consider the fuzzy incidence graph in Figure 9. For convenience, assume that* $\sigma(x) = 1$ *for all* $x \in \sigma^*$ *and* $\mu(uv) = 1$ *for all* $uv \in \mu^*$.

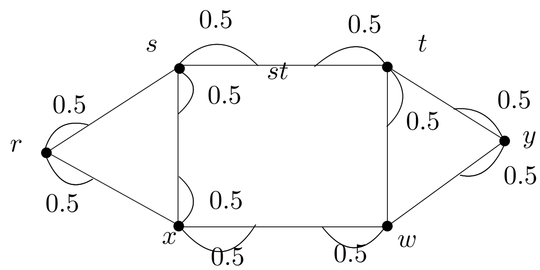

Figure 9: Fuzzy Incidence Graph in Example 5.8

For the FIG in Figure 9, there is only one $s-st$ incidence path whose incidence strength is greater than zero. It is $P: s,(s,st),st$. Incidence strength of P is 0.5. Note that there are two pairs namely (w,wt) and (w,wy) in this FIG with zero incidence values. Thus the number of internally disjoint strongest $s-t$ incidence paths is one. But there exits no $s-st$ incidence strength reducing set in this FIG and Theorem 5.7 is not valid.

In case of incidence strength reducing set of pairs this condition is not relevant. Since the proof is similar, we state the theorem without proof.

Theorem 5.9. Let $\widetilde{G} = (\sigma,\mu,\psi)$ be a fuzzy incidence graph. For any two elements $s,t \in \sigma^* \cup \mu^*$ the maximum number of internally disjoint strongest $s-t$ incidence paths in $\widetilde{G}$ is equal to the number of elements in a minimal $s-t$ incidence connectivity reducing set of pairs.

6 Application

In [18], a study of how governments are tackling modern slavery was undertaken. 161 countries were included in the assessment of government responses. Of these countries, 124 have criminalised human trafficking in line with United Nations Trafficking Protocol. 91 have National Action Plans to evaluate government responses, and 150 countries provide some sort of service for victims of modern slavery.

Vulnerability to human trafficking is affected by a complex interaction of factors related to the presence or absence of protection and respect for rights, physical safety and security, access to the necessities of life such as food, water and health care, and patterns of migration, displacement and conflict. Statistical testing grouped 24 measures of vulnerability into four dimensions covering: (1) civil and political protections, (2) social health and economic rights, (3) personal security, and (4) refugee populations and conflict. In [18], tables are provided giving measures of vulnerability to modern slavery by country with respect to these four dimensions.

Government response to human trafficking involves the following categories: (1) survivors supported, (2) criminal justice, (3) coordination and accountability, (4) addressing risk, and (5) government and business. The Walk Free Foundation included a measure on state-sanctioned forced labor to the government response rating in 2016. Government response ratings to human trafficking can be found in [18].

The data in the tables that give the measures of the four vulnerabilities of the countries can be normalized so that they fall in the interval $[0,1]$. An average or weighted average of these four measures can be taken for each country.

The countries are treated as nodes and the incidence pairs (C_i, C_iC_j) for countries C_i and C_j are assigned the above average for C_i. The government responses of C_i to the vulnerabilities would reduce the value of the (C_i, C_iC_j) and hence reduce the flow of human trafficking. In the introduction, we mention the use of linguistic terms such as very low, low, medium, high, and very high in [23] to describe the flow of human trafficking. In order to understand our approach to the study of this situation, we list five key elements.

(1) We assign the numbers $0.1, 0.3, 0.5, 0.7, 0.9$ to the terms very low, low, medium, high, very high, respectively. This places the flow problem in the realm of fuzzy logic. This is the setting that we plan to use in our research efforts to combat human trafficking. We are developing an arithmetic whose operations on these numbers keep the results in the interval $[0, 1]$. In this way, we can develop a max flow, min cut theorem in a network using these linguistic terms. The next four items involve the placement of the data given in [18] into this setting.

(2) The data in the table in [18] involving vulnerability and government response is normalized by using the idea in (1), i.e., the linguistic terms.

(3) The theory involving fuzzy incidence pairs often uses the notion of an edge as a vertex and the notion of incidence pair as an edge. This corresponds to an approach which reduces the importance or interpretation of edges. Thus in our application, the roll of an edge C_iC_j may take a back seat. One interpretation could be that an incidence pair (C_i, C_iC_j) is the measure of incidence of country C_i with the edge C_iC_j for flow from C_i to C_j.For the time being, we assume that a country C_j has no direct influence on a country C_i's government response.

(4) We are interested in the implication that an increase in a government's response decreases the country's vulnerability to human trafficking. The table in [18] assigns a large number to a country if the country is highly vulnerable and a large number to a country if its government response has a high response. However, if a country has a high government response one expects the country will have a low vulnerability. Thus in a fuzzy network involving the flow of human trafficking, a low vulnerability for a country might imply a low involvement of the country in trafficking. This is in agreement with the inequalities $\Psi(C_i, C_iC_j) \leq \mu(C_iC_j) \leq \sigma(C_i) \wedge \sigma(C_j)$ since the strength of a path depends on Ψ, i.e., the amount of flow is limited by Ψ.

(5) In order to determine if an increase in a country's government response reduces its vulnerability with respect to the data given in [18], we apply a similarity measure to the average vulnerabilities and the average government responses. Due to the way the data is presented in the tables as discussed in (4), we are most interested in the similarity of V^c, the complement of the vulnerability V, and the government response G. The complement we use is the standard complement.

Consider for example Western Europe. We have normalized the data given in the tables in [18] as mentioned in item (2). Only one country in Europe , the United Kingdom, was given a positive value for G_5. Consequently, we ignore G_5 in the following discussion.

Western Europe	V_1	V_2	V_3	V_4	Avg	G_1	G_2	G_3	G_4	G_5	Avg
Austria	0.1	0.1	0.0	0.3	0.125	0.5	0.7	0.7	0.7		0.65
Belgium	0.1	0.1	0.1	0.3	0.15	0.7	0.5	0.7	0.7		0.65
Denmark	0.0	0.1	0.0	0.1	0.05	0.5	0.7	0.5	0.7		0.6
Finland	0.1	0.1	0.1	0.1	0.1	0.4	0.7	0.5	0.6		0.55
France	0.1	0.1	0.1	0.3	0.15	0.4	0.9	0.4	0.7		0.6
Germany	0.1	0.1	0.1	0.3	0.15	0.5	0.7	0.4	0.7		0.575
Greece	0.3	0.1	0.3	0.3	0.25	0.4	0.3	0.1	0.3		0.275
Iceland	0.1	0.0	0.1	0.1	0.075	0.3	0.5	0.3	0.4		0.375
Ireland	0.0	0.1	0.1	0.3	0.125	0.5	0.8	0.1	0.5		0.475
Italy	0.1	0.1	0.1	0.3	0.15	0.3	0.7	0.3	0.7		0.5
Luxembourg	0.1	0.1	0.0	0.5	0.175	0.3	0.3	0.7	0.1		0.35
Malta											
Netherlands	0.0	0.1	0.1	0.2	0.1	0.7	0.9	0.9	0.9		0.85
Norway	0.0	0.1	0.0	0.3	0.1	0.5	0.9	0.3	0.7		0.6
Portugal	0.1	0.1	0.1	0.1	0.1	0.5	0.9	0.7	0.8		0.725
Spain	0.1	0.1	0.1	0.2	0.125	0.7	0.7	0.5	0.7		0.65
Sweden	0.1	0.1	0.0	0.3	0.125	0.7	0.7	0.7	0.7		0.7
Switzerland	0.0	0.1	0.0	0.3	0.1	0.5	0.7	0.2	0.7		0.525
United Kingdom	0.1	0.1	0.1	0.5	0.2	0.7	0.9	0.3	0.7		0.65

We next take the standard complement of the vulnerability numbers in the previous table.

Western Europe	V_1^c	V_2^c	V_3^c	V_4^c	Avg^c	G_1	G_2	G_3	G_4	G_5	Avg
Austria	0.9	0.9	1.0	0.7	0.875	0.5	0.7	0.7	0.7		0.65
Belgium	0.9	0.9	0.9	0.7	0.85	0.7	0.5	0.7	0.7		0.65
Denmark	1.0	0.9	1.0	0.9	0.95	0.5	0.7	0.5	0.7		0.6
Finland	0.9	0.9	0.9	0.9	0.9	0.4	0.7	0.5	0.6		0.55
France	0.9	0.9	0.9	0.7	0.85	0.4	0.9	0.4	0.7		0.6
Germany	0.9	0.9	0.9	0.7	0.85	0.5	0.7	0.4	0.7		0.575
Greece	0.7	0.9	0.7	0.7	0.75	0.4	0.3	0.1	0.3		0.275
Iceland	0.3	1.0	0.9	0.9	0.925	0.3	0.5	0.3	0.4		0.375
Ireland	1.0	0.9	0.9	0.7	0.875	0.5	0.8	0.1	0.5		0.475
Italy	0.9	0.9	0.9	0.7	0.85	0.3	0.7	0.3	0.7		0.5
Luxembourg	0.9	0.9	1.0	0.5	0.625	0.3	0.3	0.7	0.1		0.35
Malta											
Netherlands	1.0	0.9	0.9	0.8	0.9	0.7	0.9	0.9	0.9		0.85
Norway	1.0	0.9	1.0	0.7	0.9	0.5	0.9	0.3	0.7		0.6
Portugal	0.9	0.9	0.9	0.9	0.9	0.5	0.9	0.7	0.8		0.725
Spain	0.9	0.9	0.9	0.8	0.875	0.7	0.7	0.5	0.7		0.65
Sweden	0.9	0.9	1.0	0.7	0.875	0.7	0.7	0.7	0.7		0.7
Switzerland	1.0	0.9	1.0	0.7	0.9	0.5	0.7	0.2	0.7		0.525
United Kingdom	0.9	0.9	0.9	0.5	0.8	0.7	0.9	0.3	0.7		0.65

We next use the similarity measure defined below. The v_i denote the vulnerability of country C_i and g_i the government response of country C_i. Then $v_i^c = 1 - v_i$ and $g_i^c = 1 - g_i$.

$$S(V,G) = (\sum_{i=1}^{n} v_i \wedge g_i)/(\sum_{i=1}^{n} v_i \vee g_i), \quad S(V^c,G) = (\sum_{i=1}^{n} v_i^c \wedge g_i)/(\sum_{i=1}^{n} v_i^c \vee g_i) \text{ and } S(V,G^c) = (\sum_{i=1}^{n} v_i \wedge g_i^c)/(\sum_{i=1}^{n} v_i \vee g_i^c)$$

Using the above similarity measure, we determine the following table.

Europe	$S(V,G)$	$S(V^c,G)$	$S(V,G^c)$
Central & South Eastern Europe	0.37	0.60	0.35
Western Europe	0.23	0.66	0.31

This shows that there is a somewhat high similarity relation between V^c and G for Europe. That is, a high government response and a low vulnerability are related.

7 Future Work

The authors plan to continue this research for the other regions in the world, even using other measures of similarity and relation measures. We also plan to determine how much to decrease a v_i with respect to an increase in g_i.

8 Conclusions

Fuzzy incidence graphs have been used as good models in certain types of connectivity structures in the recent past. It represents a supporting system having extra supports, called incidence edges or ramps to the edges. The authors made an attempt to generalize the fuzzy graph connectivity concepts to fuzzy incidence graphs and presented several related results. As mentioned, the authors have collected realistic data from the documents released by the United Nations and other reliable agencies, and fitted fuzzy logic based models for studying human trafficking and illegal immigration. Even though, this incidence fuzzy graph model does not reflect all the structural properties of such a very complicated and vast network, some of the features of the network, like extra influences by third parties can be well studied.

Fuzzy graphs and its variants provide a large number of opportunities for the researchers, both in theoretical and applied aspects. For example, fuzzy incidence graphs can be used in the modeling of human loss during illegal immigration and in the design of ramp and highway systems. Also, most of the interconnection networks like internet and communication networks, do not obey any precise, crisp rule and hence fuzzy graph theory can be used for better performance. Identification and prevention of trafficking lines by different deterministic, stochastic and fuzzy methods are also under investigation by the authors and will be discussed in the forthcoming papers.

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