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Ranking triangular interval-valued fuzzy numbers based on the relative preference relation

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Abstract

In this paper, we first use a fuzzy preference relation with a membership function representing preference degree for comparing two interval-valued fuzzy numbers and then utilize a relative preference relation improved from the fuzzy preference relation to rank a set of interval-valued fuzzy numbers. Since the fuzzy preference relation is a total ordering relation that satisfies reciprocal and transitive laws on interval-valued fuzzy numbers, the relative preference relation is also a total ordering relation. Practically, the fuzzy preference relation is more reasonable on ranking interval-valued fuzzy numbers than defuzzification because defuzzification does not present preference degree between fuzzy numbers and loses messages. However, fuzzy pair-wise comparison for the fuzzy preference relation is more complex and difficult than defuzzification. To resolve fuzzy pair-wise comparison tie, the relative preference relation takes the strengths of defuzzification and the fuzzy preference relation into consideration. The relative preference relation expresses preference degrees of interval-valued fuzzy numbers over average as the fuzzy preference relation does, and ranks fuzzy numbers by relative crisp values as defuzzification does. In fact, the application of relative preference relation was shown in traditional fuzzy numbers, such as triangular and trapezoidal fuzzy numbers, for previous approaches. In this paper, we extend and utilize the relative preference relation on interval-valued fuzzy numbers, especially for triangular interval-valued fuzzy numbers. Obviously, interval-valued fuzzy numbers based on the relative preference relation are easily and quickly ranked, and able to reserve fuzzy information.

Keywords: Fuzzy preference relation, Interval-valued fuzzy numbers, Ranking, Relative preference relation, Triangular.

1 Introduction

Since Jain, Dubois and Prade [9, 14, 15] proposed related fuzzy numbers concept, ranking fuzzy numbers had been one of important issues on fuzzy aspect [36]. In the past, many researchers presented high interesting in ranking fuzzy numbers. For instance, Murakami, Maeda and Imamura [25] used fuzzy decision analysis on the development of centralized regional energy control system, Bortolan and Degani [1] viewed some fuzzy ranking methods for ranking fuzzy numbers, Lee and Li [22] proposed fuzzy numbers comparison based on the probability measure of fuzzy events, Kauffman and Gupta [16] introduced theory and application of fuzzy Arithmetic, Choobineh and Li [6] yielded an index for ordering fuzzy numbers, and Dias [8] used fuzzy numbers to rank alternatives, Lee, Cho and Lee-Kwang [21] utilized satisfaction function to rank fuzzy values, Requena, Delgado and Verdagay [27] automatically ranked fuzzy numbers through an artificial neural network, Fortemps and Roubens [11] presented ranking and defuzzification methods through area compensation, Cheng [5] ranked fuzzy numbers by the coefficient of variance(or called CV index), and Grzegoezewski [13] proposed mertics and orders in space of fuzzy numbers, Raj and Kumar [26] used maximizing and minimizing sets for ranking fuzzy alternatives with fuzzy weights, and Li [23] proposed fuzzy method in group decision making, Chen and Lu [3] based on left and right fuzzy dominance proposed an approximate approach to rank fuzzy

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numbers, Chu and Tsao [7] ranked fuzzy numbers according to an area between centroid point and original point, Chen and Lu [4] expressed the preference order of fuzzy numbers, and Tang [30] proposed the inconsistent property of Lee and Li's fuzzy ranking method. The above researches were generally useful to rank fuzzy numbers. In the past approaches, fuzzy ranking methods were often classified into two following categories. One was by defuzzification [11] and the other was by preference relation [34] (or called fuzzy pair-wise comparison). For ranking fuzzy numbers, defuzzification is easier and simpler than fuzzy pair-wise comparison, but defuzzification loses fuzzy messages. Oppositely, fuzzy pair-wise comparison, being able to reserve fuzzy messages, is complex and difficult. Yuan [34] supposed that a fuzzy ranking method should express preference relation in fuzzy terms. For comparing two fuzzy numbers A with B , it does not merely demonstrate that A is preferred or not preferred to B. Alternatively, a fuzzy ranking method must present following situations, that is, A dominates B, A is slight better than B, or A is equal to B, etc. Therefore, a fuzzy preference relation with a membership function can present preference degree between fuzzy numbers. Recently, Wang [31] proposed a relative preference relation is an application of fuzzy preference relation to rank triangular and trapezoidal fuzzy numbers. The relative preference relation being the improvement of fuzzy preference relation is useful for ranking fuzzy numbers. Practically, Wang's [31] approach was focused on general fuzzy numbers, i.e., triangular fuzzy numbers and trapezoidal fuzzy numbers, and unable to apply in interval-valued fuzzy numbers. In fact, interval-valued fuzzy numbers grasp more information than triangular and trapezoidal fuzzy numbers. Based on above, we first present a fuzzy preference relation between interval-valued fuzzy numbers in this paper. The fuzzy preference relation with a membership function indicates preference degree of two interval-valued fuzzy numbers. Then the fuzzy preference relation is improved into a relative preference relation, and the relative preference relation with a membership function denotes relative preference degrees of a set of interval-valued fuzzy numbers. The interval-valued fuzzy numbers can be ranked according to their corresponding preference degrees. For the sake of clarity, related concepts of interval-valued fuzzy numbers are expressed in Section 2. The relative preference relation of interval-valued fuzzy numbers constructed on fuzzy preference relation is proposed in Section 3. Numerical examples of ranking interval-valued fuzzy numbers based on the relative preference relation are illustrated in Section 4. Finally, conclusions are shown in Section 5.

2 Preliminaries

In this section, we review basic notions of interval-valued fuzzy numbers [12, 17].

Definition 2.1. *Based on interval-valued fuzzy sets proposed by Gorzalczany [12], an interval-valued fuzzy set A is defined on* $(-\infty,\infty)$ *given by* $A = \{x, [\mu_{A^L}(x), \mu_{A^U}(x)]\},\ x \in (-\infty,\infty),\ \mu_{A^L}, \mu_{A^U} : (-\infty,\infty) \to [0,1]$,

$$\mu_{A^L}(x) \le \mu_{A^U}(x),\ \forall x \in (-\infty,\infty),\ \mu_A(x) = [\mu_{A^L}(x), \mu_{A^U}(x)],\ x \in (-\infty,\infty),$$

where $\mu_{A^L}(x)$ *is the lower limit of degree of membership forA and* $\mu_{A^U}(x)$ *is the upper limit of degree of membership for A. Therefore, the grade of membership of an interval-valued fuzzy set A in* x^* *is presented by the interval* $[\mu_{A^L}(x^*), \mu_{A^U}(x^*)]$ *(see Figure 1), where* $\mu_{A^L}(x^*)$ *and* $\mu_{A^U}(x^*)$ *respectively denote minimum and maximum grades of membership for A.*

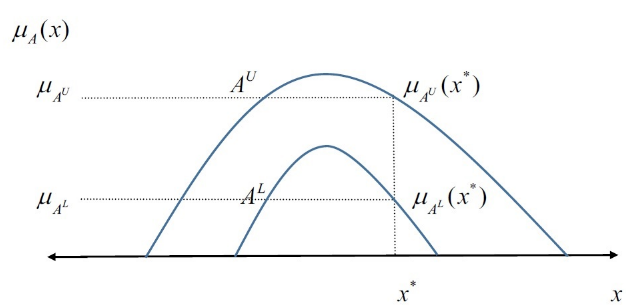

FIGURE 1. An interval-valued fuzzy set

Definition 2.2. *According to Yao and Lin's* [33] *concept, a triangular interval-valued fuzzy number A can be presented as* $A = [A^L, A^U] = [(a_l^L, a_m^L, a_r^L; w_A^L), (a_l^U, a_m^U, a_r^U; w_A^U = 1)]$ *(see Figure 2), where* A^L *and* A^U *respectively indicate the lower and upper components for A. Additionally,* $A^L \subseteq A^U$; $\mu_A(x)$ *is the membership function to denote the membership grade of x, where* $\mu_{A^L}(x)$ *and* $\mu_{A^U}(x)$ *are respectively lower and upper membership functions.*

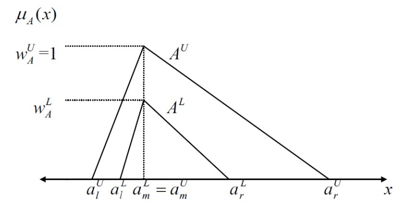

FIGURE 2. An interval-valued fuzzy number

Based on Figure 2, the related relations in an interval-valued fuzzy number are shown in the lemmas as follows.

Lemma 2.3. *If $a_l^L = a_l^U = a_m^L = a_m^U = a_r^L = a_r^U$, then A is a crisp value.*

Lemma 2.4. *If $A^L = A^U$(i.e., $a_l^L = a_l^U = a_l$, $a_m^L = a_m^U = a_m$, $a_r^L = a_r^U = a_r$) , then A is a triangular fuzzy number. The fuzzy number is indicated as a triplet (a_l, a_m, a_r).*

Lemma 2.5. *If $w_A^L = w_A^U = 1$ and $a_m^L = a_m^U$, then A is presented as $A = [A^L, A^U] = ((a_l^U, a_l^L), (a_m^L = a_m^U), (a_r^L, a_r^U))$ called a general triangular interval-valued fuzzy number (see Figure 3).*

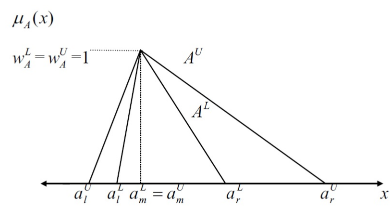

FIGURE 3. A general triangular interval-valued fuzzy number

Let $((a_l^U, a_l^L), (a_m^L = a_m^U), (a_r^L, a_r^U)) = ((a_l^U, a_l^L), (a_m), (a_r^L, a_r^U))$. In this paper, we use general triangular interval-valued fuzzy numbers above to stand for interval-valued fuzzy numbers on computation due to their popularity.

Definition 2.6. *Let $\circ$ be an operation on real numbers, such as $+, -, *, \wedge, \vee$, etc. $A = [A^L, A^U]$ and $B = [B^L, B^U]$ are assumed to be two interval-valued fuzzy numbers. By extension principle* [36]*, an extended operation $\circ$ on interval-valued fuzzy numbers is defined by $\mu_{A^L \circ B^L}(z) = sup_{x,y:z=x\circ y}\{\mu_{A^L}(x) \wedge \mu_{B^L}(y)\}$ and $\mu_{A^U \circ B^U}(z) = sup_{x,y:z=x\circ y}\{\mu_{A^U}(x) \wedge \mu_{B^U}(y)\}$.*

Definition 2.7. *Let $A = [A^L, A^U]$ be an interval-valued fuzzy number. Then $(A^L)_\alpha^\Gamma$, $(A^L)_\alpha^\Psi$, $(A^U)_\alpha^\Gamma$, and $(A^U)_\alpha^\Psi$ are respectively defined as $(A^L)_\alpha^\Gamma = inf_{\mu_{A^L}(z) \geq \alpha}(z)$, $(A^L)_\alpha^\Psi = sup_{\mu_{A^L}(z) \geq \alpha}(z)$, $(A^U)_\alpha^\Gamma = inf_{\mu_{A^U}(z) \geq \alpha}(z)$, and $(A^U)_\alpha^\Psi = sup_{\mu_{A^U}(z) \geq \alpha}(z)$.*

Definition 2.8. *A fuzzy preference relation R* [20] *is a fuzzy subset of $\Re \times \Re$ with a membership function $\mu_R(A, B)$ representing preference degree of fuzzy numbers A over B. (i) R is reciprocal iff $\mu_R(A, B) = 1 - \mu_R(B, A)$ for all interval-valued fuzzy numbers A and B. (ii) R is transitive iff $\mu_R(A, B) \geq \frac{1}{2}$ and $\mu_R(B, C) \geq \frac{1}{2} \Rightarrow \mu_R(A, C) \geq \frac{1}{2}$ for all interval-valued fuzzy numbers A, B, and C. (iii) R is a fuzzy total ordering iff R is both reciprocal and transitive* [10, 19, 20]*. Comparing A with B by the fuzzy preference relation R , A is preferred to B iff $\mu_R(A, B) > \frac{1}{2}$, and A is equal to B iff $\mu_R(A, B) = \frac{1}{2}$.*

Definition 2.9. *Let $\succ$ be a binary relation on fuzzy numbers defined by $A \succ B$ iff A is preferred to B (i.e., $\mu_R(A, B) > \frac{1}{2}$).*

Definition 2.10. *Let A and B be two general fuzzy numbers. Based on Lee's extended fuzzy preference relation* [20]*, the preference degree of A over B is defined as $\int_0^1((A - B)_\alpha^\Gamma + (A - B)_\alpha^\Psi)d\alpha$.*

Definition 2.11. *For two interval-valued fuzzy numbers $A = [A^L, A^U]$ and $B = [B^L, B^U]$, the preference degree of A over B in this paper is defined as*

$$p \int_0^1 ((A^L - B^L)_\alpha^\Gamma + (A^L - B^L)_\alpha^\Psi) d\alpha + (1 - p) \int_0^1 ((A^U - B^U)_\alpha^\Gamma + (A^U - B^U)_\alpha^\Psi) d\alpha$$

where $0 \leq p \leq 1$. In the preference relation above, the coefficient p represents the weight of lower interval-valued fuzzy numbers and the coefficient $1 - p$ stands for the weight of upper interval-valued fuzzy numbers.

According to Definition 2.11, related definitions of fuzzy preference relation between two fuzzy numbers are presented as follows.

Definition 2.12. *Let $A = [A^L, A^U]$ and $B = [B^L, B^U]$ be two interval-valued fuzzy numbers, where A is in an interval $[(a_l^U, a_l^L), (a_r^L, a_r^U)]$ and B is in an interval $[(b_l^U, b_l^L), (b_r^L, b_r^U)]$. Let $T^+ = [T^{L+}, T^{U+}]$ be in an interval $[(t_l^{U+}, t_l^{L+}), (t_r^{L+}, t_r^{U+})] = [(max\{a_l^U, b_l^U\}, max\{a_l^L, b_l^L\}), (max\{a_r^L, b_r^L\}, max\{a_r^U, b_r^U\})]$ to indicate the maximum of A and B, and $T^- = [T^{L-}, T^{U-}]$ be in an interval*

$$[(t_l^{U-}, t_l^{L-}), (t_r^{L-}, t_r^{U-})] = [(min\{a_l^U, b_l^U\}, min\{a_l^L, b_l^L\}), (min\{a_r^L, b_r^L\}, min\{a_r^U, b_r^U\})]$$

to denote the minimum of A and B. Then an extending difference $\|(A^L, B^L)\|$ and $\|(A^U, B^U)\|$ between A and B is defined as

$$\|(A^L, B^L)\| = \begin{cases} \int_0^1((T^{L+} - T^{L-})_\alpha^\Gamma + (T^{L+} - T^{L-})_\alpha^\Psi)d\alpha & if t_l^{L+} \geq t_r^{L-} \\ \int_0^1((T^{L+} - T^{L-})_\alpha^\Gamma + (T^{L+} - T^{L-})_\alpha^\Psi + 2(t_r^{L-} - t_l^{L+}))d\alpha & if t_l^{L+} < t_r^{L-} \end{cases} \quad (1)$$

and

$$\|(A^U, B^U)\| = \begin{cases} \int_0^1((T^{U+} - T^{U-})_\alpha^\Gamma + (T^{U+} - T^{U-})_\alpha^\Psi)d\alpha & if t_l^{U+} \geq t_r^{U-} \\ \int_0^1((T^{U+} - T^{U-})_\alpha^\Gamma + (T^{U+} - T^{U-})_\alpha^\Psi + 2(t_r^{U-} - t_l^{U+}))d\alpha & if t_l^{U+} < t_r^{U-} \end{cases}. \quad (2)$$

Therefore, $\|(A^L, B^L)\| = \|(B^L, A^L)\|$ and $\|(A^U, B^U)\| = \|(B^U, A^U)\|$. The extending difference is a value between two varied interval-valued fuzzy numbers to assume that the intersection of the two fuzzy numbers is ϕ or a point. It is an important concept for fuzzy preference relation to ensure that it is in the interval [0,1]. Additionally, the extending difference is also used in three fuzzy numbers or more. Furthermore, the extending difference can be recognized as the range for a set of fuzzy numbers as the intersection of the minimum and the maximum fuzzy numbers is ϕ or a point. Otherwise, the two interval-valued fuzzy numbers(i.e., the minimum and the maximum) must be adjusted to make that the intersection of the two fuzzy numbers adjusted is ϕ or a point. In fact, the similar computation by Wang [31] was shown in triangular and trapezoidal fuzzy numbers. In this paper, we improve Wang's computation on interval-valued fuzzy numbers to yield fuzzy preference relation between two interval-valued fuzzy numbers.

Definition 2.13. *Let $A = [A^L, A^U]$ and $B = [B^L, B^U]$ be two interval-valued fuzzy numbers, where A is in an interval $[(a_l^U, a_l^L), (a_r^L, a_r^U)]$ and B is in an interval $[(b_l^U, b_l^L), (b_r^L, b_r^U)]$. A fuzzy preference relation P is a fuzzy subset of $\Re \times \Re$ with a membership function $\mu_P(A, B)$ representing preference degree of A over B . For $0 \leq p \leq 1$, define*

$$\mu_P(A, B) = \frac{1}{2}(p \times \frac{\int_0^1((A^L - B^L)_\alpha^\Gamma + (A^L - B^L)_\alpha^\Psi)d\alpha}{\|(A^L, B^L)\|} + (1-p) \times \frac{\int_0^1((A^U - B^U)_\alpha^\Gamma + (A^U - B^U)_\alpha^\Psi)d\alpha}{\|(A^U, B^U)\|} + 1) \quad (3)$$

Obviously, $\mu_P(A, B) \geq \frac{1}{2}$ iff $\int_0^1((A^L - B^L)_\alpha^\Gamma + (A^L - B^L)_\alpha^\Psi)d\alpha \geq 0$ and $\int_0^1((A^U - B^U)_\alpha^\Gamma + (A^U - B^U)_\alpha^\Psi)d\alpha \geq 0$. Moreover, $\mu_P(A, B) = \frac{1}{2}$ as $A = B$.

Lemma 2.14. *The fuzzy preference relation P is reciprocal iff $\mu_P(A, B) = 1 - \mu_P(B, A)$.*

Proof. $\mu_P(A, B) = \frac{1}{2}(p \times \frac{\int_0^1((A^L - B^L)_\alpha^\Gamma + (A^L - B^L)_\alpha^\Psi)d\alpha}{\|(A^L, B^L)\|} + (1-p) \times \frac{\int_0^1((A^U - B^U)_\alpha^\Gamma + (A^U - B^U)_\alpha^\Psi)d\alpha}{\|(A^U, B^U)\|} + 1)$

$$= \frac{1}{2}(p \times \frac{-\int_0^1((B^L - A^L)_\alpha^\Gamma + (B^L - A^L)_\alpha^\Psi)d\alpha}{\|(A^L, B^L)\|} + (1-p) \times \frac{-\int_0^1((B^U - A^U)_\alpha^\Gamma + (B^U - A^U)_\alpha^\Psi)d\alpha}{\|(A^U, B^U)\|} + 1)$$

$$= \frac{1}{2}(p \times \frac{-\int_0^1((B^L - A^L)_\alpha^\Gamma + (B^L - A^L)_\alpha^\Psi)d\alpha}{\|(B^L, A^L)\|} + (1-p) \times \frac{-\int_0^1((B^U - A^U)_\alpha^\Gamma + (B^U - A^U)_\alpha^\Psi)d\alpha}{\|(B^U, A^U)\|} + 1)$$

$$= 1 - \frac{1}{2}(p \times \frac{\int_0^1((B^L - A^L)_\alpha^\Gamma + (B^L - A^L)_\alpha^\Psi)d\alpha}{\|(B^L, A^L)\|} + (1-p) \times \frac{\int_0^1((B^U - A^U)_\alpha^\Gamma + (B^U - A^U)_\alpha^\Psi)d\alpha}{\|(B^U, A^U)\|} + 1) = 1 - \mu_P(B, A).$$

□

Lemma 2.15. *The fuzzy preference relation P is transitive iff $\mu_P(A, B) \geq \frac{1}{2}$ and $\mu_P(B, C) \geq \frac{1}{2} \Longrightarrow \mu_P(A, C) \geq \frac{1}{2}$.*

Proof. Let A, B, and C be three interval-valued fuzzy numbers, where $A = [A^L, A^U]$, $B = [B^L, B^U]$, and $C = [C^L, C^U]$. Since $\mu_P(A, B) \geq \frac{1}{2}$ and $\mu_P(B, C) \geq \frac{1}{2}$,

$$\int_0^1 ((A^L - B^L)_\alpha^\Gamma + (A^L - B^L)_\alpha^\Psi)d\alpha \geq 0, \int_0^1 ((A^U - B^U)_\alpha^\Gamma + (A^U - B^U)_\alpha^\Psi)d\alpha \geq 0,$$

$$\int_0^1 ((B^L - C^L)_\alpha^\Gamma + (B^L - C^L)_\alpha^\Psi)d\alpha \geq 0, \text{ and } \int_0^1 ((B^U - C^U)_\alpha^\Gamma + (B^U - C^U)_\alpha^\Psi)d\alpha \geq 0.$$

Due to $\int_0^1 ((A^L - B^L)_\alpha^\Gamma + (A^L - B^L)_\alpha^\Psi)d\alpha + \int_0^1 ((B^L - C^L)_\alpha^\Gamma + (B^L - C^L)_\alpha^\Psi)d\alpha \geq 0$ and thus

$$\int_0^1 ((A^L - B^L)_\alpha^\Gamma + (A^L - B^L)_\alpha^\Psi)d\alpha + \int_0^1 ((B^L - C^L)_\alpha^\Gamma + (B^L - C^L)_\alpha^\Psi)d\alpha = \int_0^1 ((A^L - B^L)_\alpha^\Gamma + (A^L - B^L)_\alpha^\Psi + (B^L - C^L)_\alpha^\Gamma + (B^L - C^L)_\alpha^\Psi)d\alpha$$

$$= \int_0^1 ((A^L)_\alpha^\Gamma - (B^L)_\alpha^\Psi + (B^L)_\alpha^\Gamma - (C^L)_\alpha^\Psi + (A^L)_\alpha^\Psi - (B^L)_\alpha^\Gamma + (B^L)_\alpha^\Psi - (C^L)_\alpha^\Gamma)d\alpha = \int_0^1 ((A^L)_\alpha^\Gamma - (C^L)_\alpha^\Psi + (A^L)_\alpha^\Psi - (C^L)_\alpha^\Gamma)d\alpha$$

$= \int_0^1 ((A^L - C^L)_\alpha^\Gamma + (A^L - C^L)_\alpha^\Psi)d\alpha \geq 0.$

Similarly, $\int_0^1 ((A^U - C^U)_\alpha^\Gamma + (A^U - C^U)_\alpha^\Psi)d\alpha \geq 0$, due to $\int_0^1 ((A^U - B^U)_\alpha^\Gamma + (A^U - B^U)_\alpha^\Psi)d\alpha + \int_0^1 ((B^U - C^U)_\alpha^\Gamma + (B^U - C^U)_\alpha^\Psi)d\alpha \geq 0$. Based on $\int_0^1 ((A^L - C^L)_\alpha^\Gamma + (A^L - C^L)_\alpha^\Psi)d\alpha \geq 0$ and $\int_0^1 ((A^U - C^U)_\alpha^\Gamma + (A^U - C^U)_\alpha^\Psi)d\alpha \geq 0$, we have $\mu_P(A, C) \geq \frac{1}{2}$. □

According to Lemma 2.14 and Lemma 2.15 , the fuzzy preference relation P is a total ordering relation [10, 19].

Lemma 2.16. *Let A and B be two interval-valued fuzzy numbers. By the fuzzy preference relation P, A is preferred to B iff $\mu_P(A, B) > \frac{1}{2}$.*

Lemma 2.17. *$A \succ B$ iff $\mu_P(A, B) > \frac{1}{2}$, where $\succ$ is a binary relation on interval-valued fuzzy numbers presented by Definition 2.9.*

Lemma 2.18. *A and B are two triangular interval-valued fuzzy numbers, where $A = ((a_l^U, a_l^L), (a_m), (a_r^L, a_r^U))$ and $B = ((b_l^U, b_l^L), (b_m), (b_r^L, b_r^U))$. Let*

$$T^+ = [T^{L+}, T^{U+}] = [(t_l^{U+}, t_l^{L+}), (t_r^{L+}, t_r^{U+})] = ((max\{a_l^U, b_l^U\}, max\{a_l^L, b_l^L\}), max\{a_m, b_m\}, (max\{a_r^L, b_r^L\}, max\{a_r^U, b_r^U\})) \quad (4)$$

$$T^- = [T^{L-}, T^{U-}] = [(t_l^{U-}, t_l^{L-}), (t_r^{L-}, t_r^{U-})] = ((min\{a_l^U, b_l^U\}, min\{a_l^L, b_l^L\}), min\{a_m, b_m\}, (min\{a_r^L, b_r^L\}, min\{a_r^U, b_r^U\})). \quad (5)$$

We define extending difference values $\|(A^L, B^L)\|$ and $\|(A^U, B^U)\|$ between A and B to be

$$\|(A^L, B^L)\| = \begin{cases} \dfrac{(t_l^{L+} - t_r^{L-}) + 2(t_m^+ - t_m^-) + (t_r^{L+} - t_l^{L-})}{2} & if t_l^{L+} \geq t_r^{L-} \\ \dfrac{(t_l^{L+} - t_r^{L-}) + 2(t_m^+ - t_m^-) + (t_r^{L+} - t_l^{L-})}{2} + 2(t_r^{L-} - t_l^{L+}) & if t_l^{L+} < t_r^{L-} \end{cases} \quad (6)$$

$$\text{and } \|(A^U, B^U)\| = \begin{cases} \dfrac{(t_l^{U+} - t_r^{U-}) + 2(t_m^+ - t_m^-) + (t_r^{U+} - t_l^{U-})}{2} & if t_l^{U+} \geq t_r^{U-} \\ \dfrac{(t_l^{U+} - t_r^{U-}) + 2(t_m^+ - t_m^-) + (t_r^{U+} - t_l^{U-})}{2} + 2(t_r^{U-} - t_l^{U+}) & if t_l^{U+} < t_r^{U-} \end{cases}. \quad (7)$$

Proof. As $T^+ = [(t_l^{U+}, t_l^{L+}), (t_r^{L+}, t_r^{U+})]$, $T^- = [(t_l^{U-}, t_l^{L-}), (t_r^{L-}, t_r^{U-})]$, and $t_l^{L+} \geq t_r^{L-}$. Then $T^{L+} - T^{L-} = (t_l^{L+} - t_r^{L-}, t_m^+ - t_m^-, t_r^{L+} - t_l^{L-})$. Thus

$$\int_0^1 (T^{L+} - T^{L-})_\alpha^\Gamma d\alpha = \int_0^1 (((t_m^+ - t_m^-) - (t_l^{L+} - t_r^{L-}))\alpha + (t_l^{L+} - t_r^{L-}))d\alpha = \frac{(t_m^+ - t_m^-) + (t_l^{L+} - t_r^{L-})}{2}$$

and

$$\int_0^1 (T^{L+}-T^{L-})^{\Psi}_{\alpha} d\alpha = \int_0^1 (((t_m^+ - t_m^-) - (t_r^{L+} - t_l^{L-}))\alpha + (t_r^{L+} - t_l^{L-}))d\alpha = \frac{(t_m^+ - t_m^-) + (t_r^{L+} - t_l^{L-})}{2}.$$

Based on above, if $t_l^{L+} \geq t_r^{L-}$, then

$$\|(A^L, B^L)\| = \frac{(t_m^+ - t_m^-) + (t_l^{L+} - t_r^{L-})}{2} + \frac{(t_m^+ - t_m^-) + (t_r^{L+} - t_l^{L-})}{2} = \frac{(t_l^{L+} - t_r^{L-}) + 2(t_m^+ - t_m^-) + (t_r^{L+} - t_l^{L-})}{2}.$$

Likewise, $T^{U+} - T^{U-} = (t_l^{U+} - t_r^{U-}, t_m^+ - t_m^-, t_r^{U+} - t_l^{U-})$ and $\|(A^U, B^U)\| = \dfrac{(t_l^{U+} - t_r^{U-}) + 2(t_m^+ - t_m^-) + (t_r^{U+} - t_l^{U-})}{2}$ if $t_l^{U+} \geq t_r^{U-}$.

The other conditions can be proved as similar as above, so their proofs are omitted. □

Based on the proof above, we obtain

$$\int_0^1 ((A^L - B^L)^{\Gamma}_{\alpha} + (A^L - B^L)^{\Psi}_{\alpha})d\alpha = \frac{(a_l^L - b_r^L) + 2(a_m - b_m) + (a_r^L - b_l^L)}{2} \tag{8}$$

and

$$\int_0^1 ((A^U - B^U)^{\Gamma}_{\alpha} + (A^U - B^U)^{\Psi}_{\alpha})d\alpha = \frac{(a_l^U - b_r^U) + 2(a_m - b_m) + (a_r^U - b_l^U)}{2}. \tag{9}$$

The extending difference values $\|(A^L, B^L)\|$ and $\|(A^U, B^U)\|$ are yielded by fuzzy preference relation to ensure that the fuzzy preference relation must be in the interval [0,1]. We illustrate an example to demonstrate the extending difference values for triangular interval-valued fuzzy numbers as the intersection of triangular interval-valued fuzzy numbers is not ϕ or a point. The demonstration of utilizing triangular interval-valued fuzzy numbers is that they can be presented by certain figures, whereas fuzzy numbers of Definition 2.1 are only expressed in intervals. For instance, the membership functions of upper triangular interval-valued fuzzy numbers A^U and B^U are present as Figure 4 .

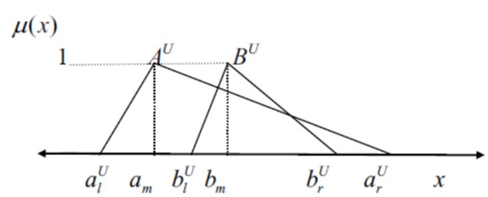

Figure 4. The membership functions for upper components of A and B

Based on A^U and B^U, the membership functions of T^{U-} and T^{U+} are derived and shown in Figure 5.

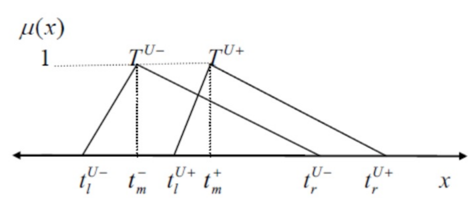

Figure 5. The membership functions of minimum and maximum for upper components of A and B

Since $t_l^{U+} < t_r^{U-}$, the extending difference relationship of T^{U-} and T^{U+} are expressed in Figure 6. In the figure, T^{U+} is moved $(t_r^{U-} - t_l^{U+})$ units into T^{U+*} by right. Then the difference between T^{U-} and T^{U+*} will be the extending difference between A^U and B^U . Thus the extending difference $\|(A^U, B^U)\|$ will be $\dfrac{(t_l^{L+} - t_r^{L-}) + 2(t_m^+ - t_m^-) + (t_r^{L+} - t_l^{L-})}{2} + 2(t_r^{U-} - t_l^{U+})$. Moreover, $t_l^{U+} + (t_r^{U-} - t_l^{U+}) = t_r^{U-}$, so the left boundary of T^{U+*} is t_r^{U-}.

Obviously, yielding the extending difference for A^U and B^U on $t_l^{U+} \geq t_r^{U-}$ is as similar as that of Figure 6 due to the intersection of A^U and B^U being ϕ or a point, and thus its computation is omitted. Likewise, the extending difference computation for A^L and B^L can be also presented above. In other words, Lemma 2.18 is clearly demonstrated.

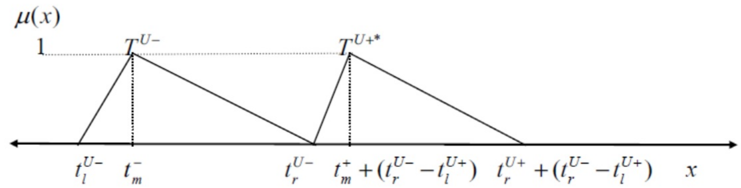

FIGURE 6. The membership functions of minimum and adjusting maximum for upper components of A and B

Lemma 2.19. *A and B are two triangular interval-valued fuzzy numbers, where* $A = ((a_l^U, a_l^L), (a_m), (a_r^L, a_r^U))$ *and* $B = ((b_l^U, b_l^L), (b_m), (b_r^L, b_r^U))$. *Based on Definition 2.13, for* $0 \le p \le 1$,

$$\mu_P(A, B) = \frac{1}{2}(p \times \frac{(a_l^L - b_r^L) + 2(a_m - b_m) + (a_r^L - b_l^L)}{2 \times \|(A^L, B^L)\|} + (1-p) \times \frac{\int_0^1((A^U - B^U)_\alpha^\Gamma + (A^U - B^U)_\alpha^\Psi)d\alpha}{2 \times \|(A^U, B^U)\|} + 1) \quad (10)$$

By the fuzzy preference relation P on pair-wise comparison, preference degree of two interval-valued fuzzy numbers can be derived.

3 The Relative Preference Relation Constructed on The Fuzzy Preference Relation

The fuzzy preference relation P is useful to rank interval-valued fuzzy numbers. However, time complexity on fuzzy operation is $O(n^2)$ for ranking interval-valued fuzzy numbers due to pair-wise comparison [24] of the fuzzy preference relation, whereas time complexity on fuzzy operation is only $O(n)$ for ranking n interval-valued fuzzy numbers by defuzzification [11]. Therefore, we modify the fuzzy preference relation P into a relative preference relation P^* for ranking n interval-valued fuzzy numbers. By the relative preference relation P^*, time complexity on fuzzy operation for ranking n interval-valued fuzzy numbers is $O(n)$. In addition, the notions of the relative preference relation are presented as follows. Let $S = \{x_1, x_2, ..., x_n\}$ denote a set composed of n interval-valued fuzzy numbers. An interval-valued fuzzy number $X_i = [X_i^L, X_i^U] = [(x_{il}^L, x_{il}^U), (x_{ir}^L, x_{ir}^U)]$ belongs to the set S, where $i = 1, 2, ..., n$. $\overline{X} = [\overline{X}^L, \overline{X}^U]$ yielded by extension principle is average of the n interval-valued fuzzy numbers in S. A relative preference relation P^* with a membership function $\mu_{P^*}(X_i, \overline{X})$ represents preference degree of X over $\overline{X}$ in S. Therefore,

$$\mu_{P^*}(X_i, \overline{X}) = \frac{1}{2}(p \times \frac{\int_0^1((X_i^L - \overline{X}^L)_\alpha^\Gamma + (X_i^L - \overline{X}^L)_\alpha^\Psi)d\alpha}{\|(T_S^{L+}, T_S^{L-})\|} + (1-p) \times \frac{\int_0^1((X_i^U - \overline{X}^U)_\alpha^\Gamma + (X_i^U - \overline{X}^U)_\alpha^\Psi)d\alpha}{\|(T_S^{U+}, T_S^{U-})\|} + 1) \quad (11)$$

for $0 \le p \le 1$, where

$$\|(T_S^{L+}, T_S^{L-})\| = \begin{cases} \int_0^1((T_S^{L+} - T_S^{L-})_\alpha^\Gamma + (T_S^{L+} - T_S^{L-})_\alpha^\Psi)d\alpha & \text{if } t_{sl}^{L+} \ge t_{sr}^{L-} \\ \int_0^1((T_S^{L+} - T_S^{L-})_\alpha^\Gamma + (T_S^{L+} - T_S^{L-})_\alpha^\Psi + 2(t_{sr}^{L-} - t_{sl}^{L+}))d\alpha & \text{if } t_{sl}^{L+} < t_{sr}^{L-} \end{cases},$$

$$\|(T_S^{U+}, T_S^{U-})\| = \begin{cases} \int_0^1((T_S^{U+} - T_S^{U-})_\alpha^\Gamma + (T_S^{U+} - T_S^{U-})_\alpha^\Psi)d\alpha & \text{if } t_{sl}^{U+} \ge t_{sr}^{U-} \\ \int_0^1((T_S^{U+} - T_S^{U-})_\alpha^\Gamma + (T_S^{U+} - T_S^{U-})_\alpha^\Psi + 2(t_{sr}^{U-} - t_{sl}^{U+}))d\alpha & \text{if } t_{sl}^{U+} < t_{sr}^{U-} \end{cases},$$

T_S^{L+} is in an interval $[t_{sl}^{L+}, t_{sr}^{L+}]$, T_S^{L-} is in an interval $[t_{sl}^{L-}, t_{sr}^{L-}]$, T_S^{U+} is in an interval $[t_{sl}^{U+}, t_{sr}^{U+}]$, T_S^{U-} is in an interval $[t_{sl}^{U-}, t_{sr}^{U-}]$, $t_{sl}^{L+} = max_i\{x_{il}^L\}$, $t_{sr}^{L+} = max_i\{x_{ir}^L\}$, $t_{sl}^{L-} = min_i\{x_{il}^L\}$, $t_{sr}^{L-} = min_i\{x_{ir}^L\}$, $t_{sl}^{U+} = max_i\{x_{il}^U\}$, $t_{sr}^{U+} = max_i\{x_{ir}^U\}$, $t_{sl}^{U-} = min_i\{x_{il}^U\}$, $t_{sr}^{U-} = min_i\{x_{ir}^U\}$, $i = 1, 2, ..., n$. Obviously, $0 < \mu_{P^*}(X_i, \overline{X}) < 1$, $i = 1, 2, ..., n$. $\mu_{P^*}(X_i, \overline{X}) < \frac{1}{2}$ denotes that $\overline{X}$ is preferred to X_i, whereas $\mu_{P^*}(X_i, \overline{X}) > \frac{1}{2}$ indicates that X_i is preferred to $\overline{X}$.

Based on the above lemmas of Section 2, the relative preference relation P^* is a total ordering relation. Additionally, X_i is preferred to X_j iff $\mu_{P^*}(X_i, \overline{X}) > \mu_{P^*}(X_j, \overline{X})$ as X_i and X_j are two interval-valued fuzzy numbers in S. Then n interval-valued fuzzy numbers $X_1, X_2, ..., X_n$ can be ranked according to $\mu_{P^*}(X_1, \overline{X})$, $\mu_{P^*}(X_2, \overline{X})$,..., $\mu_{P^*}(X_n, \overline{X})$, i.e., relative preference degrees of the interval-valued fuzzy numbers over average. Therefore, time complexity on fuzzy operation is $O(n)$ for ranking n interval-valued fuzzy numbers. The main difference between P^* and P is the comparing basis of fuzzy numbers. The range of P^* includes a set of interval-valued fuzzy numbers, whereas P is only used in two interval-valued fuzzy numbers. Furthermore, $X_i \succ X_j$ iff $\mu_{P^*}(X_i,\overline{X}) > \mu_{P^*}(X_j,\overline{X})$, where $\succ$ is a binary relation on interval-valued fuzzy numbers presented on Definition 2.9. Besides, We assume $S=\{X_1,X_2,...,X_n\}$ to indicate a set consisting of n triangular interval-valued fuzzy numbers, where $X_i=((x_{il}^U,x_{il}^L),x_m,(x_{ir}^L,x_{ir}^U))$, $i=1,2,...,n$. Let $\overline{X}=((\overline{x}_l^U,\overline{x}_l^L),\overline{x}_m,(\overline{x}_r^L,\overline{x}_r^U))$ be average of the interval-valued fuzzy numbers, where $\overline{x}_l^L=\frac{\sum_{i=1}^n x_{il}^L}{n}$, $\overline{x}_m=\frac{\sum_{i=1}^n x_{im}}{n}$, $\overline{x}_r^L=\frac{\sum_{i=1}^n x_{ir}^L}{n}$, $\overline{x}_l^U=\frac{\sum_{i=1}^n x_{il}^U}{n}$, and $\overline{x}_r^U=\frac{\sum_{i=1}^n x_{ir}^U}{n}$. The relative preference relation P^* with a membership function $\mu_{P^*}(X_i,\overline{X})$ represents preference degree X_i of over $\overline{X}$ in S, where $i=1,2,...,n$. Therefore,

$$\mu_{P^*}(X_i,\overline{X})=\frac{1}{2}(p\times\frac{(x_{il}^L-\overline{x}_r^L)+2(x_{im}-\overline{x}_m)+(x_{ir}^L-\overline{x}_l^L)}{2\left\|(T_S^{L+},T_S^{L-})\right\|}+(1-p)\times\frac{(x_{il}^U-\overline{x}_r^U)+2(x_{im}-\overline{x}_m)+(x_{ir}^U-\overline{x}_l^U)}{2\left\|(T_S^{U+},T_S^{U-})\right\|}+1) \tag{12}$$

for $0\le p\le 1$, where

$$\left\|(T_S^{L+},T_S^{L-})\right\|=\begin{cases}\frac{(t_{sl}^{L+}-t_{sr}^{L-})+2(t_{sm}^{+}-t_{sm}^{-})+(t_{sr}^{L+}-t_{sl}^{L-})}{2} & \text{if } t_{sl}^{L+}\ge t_{sr}^{L-}\\ \frac{(t_{sl}^{L+}-t_{sr}^{L-})+2(t_{sm}^{+}-t_{sm}^{-})+(t_{sr}^{L+}-t_{sl}^{L-})}{2}+2(t_{sr}^{L-}-t_{sl}^{L+}) & \text{if } t_{sl}^{L+}< t_{sr}^{L-}\end{cases},$$

$$\left\|(T_S^{U+},T_S^{U-})\right\|=\begin{cases}\frac{(t_{sl}^{U+}-t_{sr}^{U-})+2(t_{sm}^{+}-t_{sm}^{-})+(t_{sr}^{U+}-t_{sl}^{U-})}{2} & \text{if } t_{sl}^{U+}\ge t_{sr}^{U-}\\ \frac{(t_{sl}^{U+}-t_{sr}^{U-})+2(t_{sm}^{+}-t_{sm}^{-})+(t_{sr}^{U+}-t_{sl}^{U-})}{2}+2(t_{sr}^{U-}-t_{sl}^{U+}) & \text{if } t_{sl}^{U+}< t_{sr}^{U-}\end{cases},$$

$T_S^{L+}=(t_{sl}^{L+},t_{sm}^{+},t_{sr}^{L+})$, $T_S^{L-}=(t_{sl}^{L-},t_{sm}^{-},t_{sr}^{L-})$, $T_S^{U+}=(t_{sl}^{U+},t_{sm}^{+},t_{sr}^{U+})$, $T_S^{U-}=(t_{sl}^{U-},t_{sm}^{-},t_{sr}^{U-})$, $t_{sl}^{L+}=max_i\{x_{il}^L\}$, $t_{sm}^{+}=max_i\{x_{im}\}$, $t_{sr}^{L+}=max_i\{x_{ir}^L\}$, $t_{sl}^{L-}=min_i\{x_{il}^L\}$, $t_{sm}^{-}=min_i\{x_{im}\}$, $t_{sr}^{L-}=min_i\{x_{ir}^L\}$, $t_{sl}^{U+}=max_i\{x_{il}^U\}$, $t_{sr}^{U+}=max_i\{x_{ir}^U\}$, $t_{sl}^{U-}=min_i\{x_{il}^U\}$, $t_{sr}^{U-}=min_i\{x_{ir}^U\}$, $i=1,2,...,n$.

Practically, the relative preference relation P^* has the advantage of geometry based on the characteristics of itself. For $X_i=((x_{il}^U,x_{il}^L),x_m,(x_{ir}^L,x_{ir}^U))$, $i=1,2,...,n$ and $\overline{X}=((\overline{x}_l^U,\overline{x}_l^L),\overline{x}_m,(\overline{x}_r^L,\overline{x}_r^U))$, the addition $\oplus$ of the two interval-valued fuzzy numbers from Kuo and Liang's [17] approach is defined as $X_i\oplus\overline{X}=((x_{il}^U+\overline{x}_l^U,x_{il}^L+\overline{x}_l^L),x_m+\overline{x}_m,(x_{ir}^L+\overline{x}_r^L,x_{ir}^U+\overline{x}_r^U))$. Thus the subtraction $\ominus$ of the two interval-valued fuzzy numbers through extension principle [36] is reasonably recognized as $X_i\ominus\overline{X}=((x_{il}^U-\overline{x}_r^U,x_{il}^L-\overline{x}_r^L),x_m-\overline{x}_m,(x_{ir}^L-\overline{x}_l^L,x_{ir}^U-\overline{x}_l^U))$. The interval-valued fuzzy number $((x_{il}^U-\overline{x}_r^U,x_{il}^L-\overline{x}_r^L),x_m-\overline{x}_m,(x_{ir}^L-\overline{x}_l^L,x_{ir}^U-\overline{x}_l^U))$ can be classified into $(x_{il}^L-\overline{x}_r^L,x_m-\overline{x}_m,x_{ir}^L-\overline{x}_l^L)$ in the lower membership function and $(x_{il}^U-\overline{x}_r^U,x_m-\overline{x}_m,x_{ir}^U-\overline{x}_l^U)$ in the upper membership function for the value of x. The above results are consistent with the difference of two triangular fuzzy numbers. Obviously, the relative preference relation P^* in Eq.(12) displays the difference in fractions because it combines $\frac{(x_{il}^L-\overline{x}_r^L)+2(x_{im}-\overline{x}_m)+(x_{ir}^L-\overline{x}_l^L)}{2}$ with $\frac{(x_{il}^U-\overline{x}_r^U)+2(x_{im}-\overline{x}_m)+(x_{ir}^U-\overline{x}_l^U)}{2}$. Furthermore, the relative preference relation P^* is from $\int_0^1((X_i^L-\overline{X}^L)_\alpha^\Gamma+(X_i^L-\overline{X}^L)_\alpha^\Psi)d\alpha$ and $\int_0^1((X_i^U-\overline{X}^U)_\alpha^\Gamma+(X_i^U-\overline{X}^U)_\alpha^\Psi)d\alpha$ to stand for considering both geometry characteristic and computation convenience due to integral and $\alpha-cut$. Besides, the relative preference relation P^* in denominators also expresses the difference between maximum and minimum interval-valued fuzzy numbers. Since the relative preference relation P^* only emphasize the difference between interval-valued fuzzy numbers, interval-valued fuzzy numbers no matter positive, negative, or mixed are derived in the relation. Thus the y axis in original point for all figures is omitted.

4 Numerical Examples

To demonstrate the relative preference relation P^*, we illustrate three numerical examples to present fuzzy numbers ranked by P^*.

In the first example, $A_1=((5,6,),7,(8,9))$, $A_2=((-2,-1),0,(1,2))$, and $A_3=((-9,-8),-7,(-6,-5))$ are three triangular interval-valued fuzzy numbers which are respectively positive, mixed, and negative. Intuitively, A_1 is preferred to A_2 and A_2 is preferred to A_3. Through extension principle, average $\overline{A}$ of the three triangular interval-valued fuzzy numbers A_1, A_2, and A_3 is derived as $((-2,-1),0,(1,2))$. By the relative preference relation P^*, $\left\|(T_S^{L+},T_S^{L-})\right\|$ of A_1, A_2, and A_3 is yielded as $\frac{(6-(-6))+2(7-(-7))+(8-(-8))}{2}=28$, where $t_{sl}^{L+}=max\{6,-1,-8\}=6$, $t_{sm}^{+}=max\{7,0,-7\}=7$, $t_{sr}^{L+}=max\{8,1,-6\}=8$, $t_{sl}^{L-}=min\{6,-1,-8\}=-8$, $t_{sm}^{-}=min\{7,0,-7\}=-7$, $t_{sr}^{L-}=min\{8,1,-6\}=-6$, and $t_{sl}^{L+}>t_{sr}^{L-}$. Similarly, $\left\|(T_S^{U+},T_S^{U-})\right\|$ of A_1, A_2, and A_3 by the relative preference relation P^* is yielded as $\frac{(5-(-5))+2(7-(-7))+(9-(-9))}{2}=28$, where $t_{sl}^{U+}=max\{5,-2,-9\}=5$, $t_{sm}^{+}=max\{7,0,-7\}=7$,

$t_{sr}^{U+} = max\{9, 2, -5\} = 9$, $t_{sl}^{U-} = min\{5, -2, -9\} = -9$, $t_{sm}^{-} = min\{7, 0, -7\} = -7$, $t_{sr}^{U-} = min\{9, 2, -5\} = -5$, and $t_{sl}^{U+} > t_{sr}^{U-}$. As $p = 0.5$,

$$\mu_{P^*}(A_1, \overline{A}) = \frac{1}{2}[0.5 \times \frac{(6-1)+2(7-0)+(8-(-1))}{2 \times 28} + (1-0.5) \times \frac{(5-2)+2(7-0)+(9-(-2))}{2 \times 28} + 1] = 0.75,$$

$$\mu_{P^*}(A_2, \overline{A}) = \frac{1}{2}[0.5 \times \frac{((-1)-1)+2(0-0)+(1-(-1))}{2 \times 28} + (1-0.5) \times \frac{((-2)-2)+2(0-0)+(2-(-2))}{2 \times 28} + 1] = 0.5,$$

$$\mu_{P^*}(A_3, \overline{A}) = \frac{1}{2}[0.5 \times \frac{((-8)-1)+2(-7))+((-6)-(-1))}{2 \times 28} + (1-0.5) \times \frac{((-9)-2)+2(-7)+((-5)-(-2))}{2 \times 28} + 1] = 0.25.$$

Based on $\mu_{P^*}(A_1, \overline{A}) = 0.75$, $\mu_{P^*}(A_2, \overline{A}) = 0.5$, and $\mu_{P^*}(A_3, \overline{A}) = 0.25$, we know that $A_1 \succ A_2 \succ A_3$ for $p = 0.5$. Obviously, the ranking result is consistent with people's intuition. Furthermore, p in Eq.(11) and Eq.(12) is not only equal to 0.5. Table 1 expresses the values of $\mu_{P^*}(A_i, \overline{A})$ in varied p's values from 0 to 1, where $i = 1, 2, 3$.

p	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$\mu_{P^*}(A_1, \overline{A})$	0.75	0.75	0.75	0.75	0.75	0.75	0.75	0.75	0.75	0.75	0.75
$\mu_{P^*}(A_2, \overline{A})$	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
$\mu_{P^*}(A_3, \overline{A})$	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25

Table 1: The relative preference relation values after considering varied weights in the first example

According to the varied values of p in Table 1, A_1 is always preferred to A_2, and A_2 is always preferred to A_3 . The is due to the locations of the three interval-valued fuzzy numbers.

In the second example,

$$B_1 = ((0.19, 0.22), 0.3, (0.51, 0.63)), \quad B_2 = ((0.13, 0.16), 0.32, (0.58, 0.65)), \quad \text{and} \quad B_3 = ((0.22, 0.25), 0.4, (0.71, 0.76))$$

are three triangular interval-valued fuzzy numbers. Through extension principle, average $\overline{B}$ of the three triangular interval-valued fuzzy numbers B_1, B_2, and B_3 is yielded as ((0.18,0.21),0.34,(0.6,0.68)). By the relative preference relation P^*, $\|(T_S^{L+}, T_S^{L-})\|$ of B_1, B_2, and B_3 is derived as

$$\frac{(0.25-0.51)+2(0.4-0.3)+(0.71-0.16)}{2} + 2(0.51-0.25) = 0.765,$$

where $t_{sl}^{L+} = max\{0.22, 0.16, 0.25\} = 0.25$, $t_{sm}^{+} = max\{0.3, 0.32, 0.4\} = 0.4$, $t_{sr}^{L+} = max\{0.51, 0.58, 0.71\} = 0.71$, $t_{sl}^{L-} = min\{0.22, 0.16, 0.25\} = 0.16$, $t_{sm}^{-} = min\{0.3, 0.32, 0.4\} = 0.3$, $t_{sr}^{L-} = min\{0.51, 0.58, 0.71\} = 0.51$, and $t_{sl}^{L+} < t_{sr}^{L-}$. Similarly, $\|(T_S^{U+}, T_S^{U-})\|$ of B_1, B_2, and B_3 by the relative preference relation P^* is derived as

$$\frac{(0.22-0.63)+2(0.4-0.3)+(0.76-0.13)}{2} + 2(0.63-0.22) = 1.03,$$

where $t_{sl}^{U+} = max\{0.19, 0.13, 0.22\} = 0.22$, $t_{sm}^{+} = max\{0.3, 0.32, 0.4\} = 0.4$, $t_{sr}^{U+} = max\{0.63, 0.65, 0.76\} = 0.76$, $t_{sl}^{U-} = min\{0.19, 0.13, 0.22\} = 0.13$, $t_{sm}^{-} = min\{0.3, 0.32, 0.4\} = 0.3$, $t_{sr}^{U-} = min\{0.63, 0.65, 0.76\} = 0.63$, and $t_{sl}^{U+} < t_{sr}^{U-}$. Let $p = 0.5$. Then

$$\mu_{P^*}(B_1, \overline{B}) = \frac{1}{2}[0.5 \times \frac{(0.22-0.6)+2(0.3-0.34)+(0.51-0.21)}{2 \times 0.765} + (1-0.5) \times \frac{(0.19-0.68)+2(0.3-0.34)+(0.63-0.18)}{2 \times 1.03} + 1 = 0.465,$$

$$\mu_{P^*}(B_2, \overline{B}) = \frac{1}{2}[0.5 \times \frac{(0.16-0.6)+2(0.32-0.34)+(0.58-0.21)}{2 \times 0.765} + (1-0.5) \times \frac{(0.13-0.68)+2(0.32-0.34)+(0.65-0.18)}{2 \times 1.03} + 1] = 0.474,$$

and

$$\mu_{P^*}(B_3,\overline{B})=\frac{1}{2}[0.5\times\frac{(0.25-0.6)+2(0.4-0.34)+(0.71-0.21)}{2\times 0.765}]$$

$$+(1-0.5)\times\frac{(0.22-0.68)+2(0.4-0.34)+(0.76-0.18)}{2\times 1.03}+1=0.579.$$

Based on $\mu_{P^*}(B_1,\overline{B})=0.465$, $\mu_{P^*}(B_2,\overline{B})=0.474$, and $\mu_{P^*}(B_3,\overline{B})=0.579$, we know that $B_3\succ B_2\succ B_1$ as $p=0.5$. Additionally, Table 2 presents the values of $\mu_{P^*}(B_i,\overline{B})$ in varied p's values from 0 to 1, where $i=1,2,3$.

p	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$\mu_{P^*}(B_1,\overline{B})$	0.483	0.479	0.476	0.472	0.469	0.465	0.462	0.458	0.455	0.451	0.448
$\mu_{P^*}(B_2,\overline{B})$	0.474	0.474	0.474	0.474	0.474	0.474	0.474	0.474	0.474	0.474	0.475
$\mu_{P^*}(B_3,\overline{B})$	0.579	0.579	0.579	0.579	0.579	0.579	0.579	0.579	0.579	0.579	0.579

Table 2: The relative preference relation values after considering varied weights in the second example

According to the varied values of p in Table 2, $B_3\succ B_1\succ B_2$ as $p\leq 0.2$, but $B_3\succ B_2\succ B_1$ as $p\geq 0.3$. The is due to the consideration of different weights for the lower membership function over upper membership function.

Practically, ranking interval-valued fuzzy numbers by the relative preference relation P^* on fuzzy operation is as similar as defuzzification [11]. Generally, the number of fuzzy pair-wise comparison on fuzzy operation is $C_2^n=\frac{n(n-1)}{2}$ as n interval-valued fuzzy numbers are ranked. The work is easy for n being small, but fuzzy pair-wise comparison is complicated when n is large. For instance, fuzzy pair-wise comparison has 45 (i.e., $C_2^n=\frac{10(10-1)}{2}$) times for ten interval-valued fuzzy numbers being ranked by fuzzy pair-wise comparison, whereas fuzzy operation number is only 11 (i.e., 10+1) for ranking ten interval-valued fuzzy numbers by P^*. To describe the strength of the relative preference relation P^*, an example of ranking ten interval-valued fuzzy numbers is displayed in the following. The situation expresses lots of interval-valued fuzzy numbers ranked.

In the third example,

$$C_1=((0.596,0.637),0.752,(0.916,0.939)),\quad C_2=((0.491,0.519),0.704,(0.822,0.865)),$$

$$C_3=((0.304,0.314),0.573,(0.657,0.694)),\quad C_4=((0.401,0.413),0.567,(0.675,0.713)),$$

$$C_5=((0.328,0.358),0.591,(0.764,0.801)),\quad C_6=((0.219,0.236),0.768,(0.811,0.836)),$$

$$C_7=((0.298,0.335),0.416,(0.559,0.598)),\quad C_8=((0.147,0.163),0.431,(0.805,0.867)),$$

$$C_9=((0.355,0.362),0.535,(0.711,0.773)),\quad \text{and}\quad C_{10}=((0.443,0.461),0.719,(0.923,0.966))$$

are ten triangular interval-valued fuzzy numbers. Based on extension principle, average $\overline{C}$ of $C_1,C_2,...,C_{10}$ is yielded as $((0.3582,0.3798),0.6056,(0.7643,0.8052))$. By P^*, $\left\|(T_S^{L+},T_S^{L-})\right\|$ of $C_1,C_2,...,C_{10}$ is

$$\frac{(0.637-0.559)+2(0.768-0.416)+(0.923-0.163)}{2}=0.771,$$

where $t_{sl}^{L+}=0.637$, $t_{sm}^{+}=0.768$, and $t_{sr}^{L+}=0.923$, $t_{sl}^{L-}=0.163$, $t_{sm}^{-}=0.416$, $t_{sr}^{L-}=0.559$, $t_{sl}^{L+}>t_{sr}^{L-}$. Likewise, $\left\|(T_S^{U+},T_S^{U-})\right\|$ of $C_1,C_2,...,C_{10}$ by P^* is

$$\frac{(0.596-0.598)+2(0.768-0.416)+(0.966-0.147)}{2}+2(0.598-0.596)=0.7645,$$

where $t_{sl}^{U+}=0.596$, $t_{sm}^{+}=0.768$, and $t_{sr}^{U+}=0.966$, $t_{sl}^{U-}=0.147$, $t_{sm}^{-}=0.416$, $t_{sr}^{U-}=0.598$, $t_{sl}^{U+}<t_{sr}^{U-}$. Let $p=0.5$. Then

$$\mu_{P^*}(C_1,\overline{C})=\frac{1}{2}[0.5\times\frac{(0.637-0.7643)+2(0.752-0.6056)+(0.916-0.3798)}{2\times 0.771}$$

$$+(1-0.5)\times\frac{(0.596-0.8052)+2(0.752-0.6056)+(0.939-0.3582)}{2\times 0.7645}+1]=0.7256,$$

$$\mu_{P^*}(C_2,\overline{C})=\frac{1}{2}[0.5\times\frac{(0.519-0.7643)+2(0.704-0.6056)+(0.822-0.3798)}{2\times 0.771}$$

$$+(1-0.5)\times\frac{(0.491-0.8052)+2(0.704-0.6056)+(0.865-0.3582)}{2\times 0.7645}+1]=0.6037,$$

$$\mu_{P^*}(C_3,\overline{C})=\frac{1}{2}[0.5\times\frac{(0.314-0.7643)+2(0.573-0.6056)+(0.657-0.3798)}{2\times 0.771}$$

$$+(1-0.5)\times\frac{(0.304-0.8052)+2(0.573-0.6056)+(0.694-0.3582)}{2\times 0.7645}+1]=0.4268,$$

$$\mu_{P^*}(C_4,\overline{C})=\frac{1}{2}[0.5\times\frac{(0.413-0.7643)+2(0.567-0.6056)+(0.675-0.3798)}{2\times 0.771}$$

$$+(1-0.5)\times\frac{(0.401-0.8052)+2(0.567-0.6056)+(0.713-0.3582)}{2\times 0.7645}+1]=0.4608,$$

$$\mu_{P^*}(C_5,\overline{C})=\frac{1}{2}[0.5\times\frac{(0.358-0.7643)+2(0.591-0.6056)+(0.764-0.3798)}{2\times 0.771}$$

$$+(1-0.5)\times\frac{(0.328-0.8052)+2(0.591-0.6056)+(0.801-0.3582)}{2\times 0.7645}+1]=0.4844,$$

$$\mu_{P^*}(C_6,\overline{C})=\frac{1}{2}[0.5\times\frac{(0.236-0.7643)+2(0.768-0.6056)+(0.811-0.3798)}{2\times 0.771}$$

$$+(1-0.5)\times\frac{(0.219-0.8052)+2(0.768-0.6056)+(0.836-0.3582)}{2\times 0.7645}+1]=0.5755,$$

$$\mu_{P^*}(C_7,\overline{C})=\frac{1}{2}[0.5\times\frac{(0.335-0.7643)+2(0.416-0.6056)+(0.559-0.3798)}{2\times 0.771}$$

$$+(1-0.5)\times\frac{(0.298-0.8052)+2(0.416-0.6056)+(0.598-0.3582)}{2\times 0.7645}+1]=0.2954,$$

$$\mu_{P^*}(C_8,\overline{C})=\frac{1}{2}[0.5\times\frac{(0.163-0.7643)+2(0.431-0.6056)+(0.805-0.3798)}{2\times 0.771}$$

$$+(1-0.5)\times\frac{(0.147-0.8052)+2(0.431-0.6056)+(0.867-0.3582)}{2\times 0.7645}+1]=0.3365,$$

$$\mu_{P^*}(C_9,\overline{C})=\frac{1}{2}[0.5\times\frac{(0.362-0.7643)+2(0.535-0.6056)+(0.711-0.3798)}{2\times 0.771}$$

$$+(1-0.5)\times\frac{(0.355-0.8052)+2(0.535-0.6056)+(0.773-0.3582)}{2\times 0.7645}+1]=0.4399,$$

$$\mu_{P^*}(C_1,\overline{C})=\frac{1}{2}[0.5\times\frac{(0.461-0.7643)+2(0.719-0.6056)+(0.923-0.3798)}{2\times 0.771}$$

$$+(1-0.5)\times\frac{(0.443-0.8052)+2(0.719-0.6056)+(0.966-0.3582)}{2\times 0.7645}+1]=0.6561.$$

Based on above, the ranking order of the ten triangular interval-valued fuzzy numbers is $C_1 \succ C_{10} \succ C_2 \succ C_6 \succ C_5 \succ C_4 \succ C_9 \succ C_3 \succ C_8 \succ C_7$ as $p = 0.5$. By the relative preference relation P^*, it is obvious that time complexity on fuzzy operation is $O(n)$ (i.e., the fuzzy operation number is $n+1$) to rank n interval-valued fuzzy numbers. On the other hand, time complexity on fuzzy operation is $O(n^2)$ (i.e., the fuzzy operation number is $C_2^n = \frac{n(n-1)}{2}$) as n interval-valued fuzzy numbers are ranked by fuzzy pair-wise comparison including the fuzzy preference relation P. For fuzzy operation on ranking ten interval-valued fuzzy numbers, computation number (i.e., eleven) by P^* is smaller than calculation number (i.e., forty-five) by those pair-wise comparison methods. This is the strength of the relative preference relation P^* on fuzzy operation. Additionally, the previous examples are expressed through triangular interval-valued fuzzy numbers because the kind of fuzzy numbers are widely applied in many decision-making problems.

Besides, we compare the proposed method with others concerning the ranking of interval-valued fuzzy numbers. In 2016, Lee, Chung, Lee, Gan, and Chou [18] transferred triangular interval-valued fuzzy numbers into triangular fuzzy numbers and then they yielded Lee's [19] strength and weakness matrices to rank feasible alternatives composed of interval-valued fuzzy ones. Based on above, Lee et al.'s [18] method was used in triangular fuzzy numbers. Moreover, computing the strength and weakness matrices had to compare fuzzy numbers on pair-wise. In addition, Kuo and Linag [17] associated VIKOR [29] with other techniques on interval-valued fuzzy numbers to rank alternatives. These techniques included normalized Euclidean distance [2], hierarchical structure [28], etc. except for VIKOR. The association of different techniques made Kuo and Linag's method complicated for operation. Strictly speaking, Lee et al.'s or Kuo and Linag's are indirectly utilized in ranking interval-valued fuzzy numbers. In fact, related approaches of directly ranking interval-valued fuzzy numbers may be few. From 2008 to now, there were two approaches of ranking methods for interval-valued intuitionistic fuzzy numbers proposed by Yue [35], Wu and Chiclana [32], but they did not directly rank interval-valued fuzzy numbers, either. On the other hand, our proposed ranking method can be recognized as the one of related approaches for directly and mainly ranking interval-valued fuzzy numbers. Furthermore, the comparison of these related ranking methods of interval-valued fuzzy numbers is shown in Table 3.

Methods	Contributions
The proposed method	Directly and mainly ranking interval-valued fuzzy numbers
Lee et al.'s [18]	Transferring interval-valued fuzzy numbers into triangular fuzzy numbers and utilizing in decision-making
Kuo and Linag's [17]	Associating VIKOR [29], normalized Euclidean distance [2], with hierarchical structure [28], and utilizing in decision-making
Yue's [35]	Ranking interval-valued intuitionistic fuzzy numbers and utilizing in group decision-making
Wu and Chiclana's [32]	Ranking interval-valued intuitionistic fuzzy numbers

Table 3: The comparison of related ranking methods for interval-valued fuzzy numbers

To sum up, the strengths of our proposed method from the previous descriptions have easy and reasonable computation on geometry, used in many kinds of interval-valued fuzzy numbers including positive, mixed, and negative ones, the feasibility and universality on weights by considering the varied values of p.

5 Conclusions

In this paper, we first use the fuzzy preference relation P to rank interval-valued fuzzy numbers. The fuzzy preference relation P, satisfying reciprocity and transitivity, is a total ordering relation on interval-valued fuzzy numbers. However, the time complexity of fuzzy operation by P is $O(n^2)$ for ranking n interval-valued fuzzy numbers because belongs to fuzzy pair-wise comparison. In this paper, we propose the relative preference relation P^* revised from P to resolve the tie of fuzzy pair-wise comparison because the time complexity of fuzzy operation by P^* is $O(n)$ for ranking n interval-valued fuzzy numbers. Since P^* with a membership function represents preference degrees of n interval-valued fuzzy numbers over average(i.e., a specify comparison basis), the interval-valued fuzzy numbers are easily ranked by P^*. Practically, the main differences between P^* and the fuzzy pair-wise comparison methods including P are the number of comparing triangular interval-valued fuzzy numbers and their comparison basis. Obviously, P^* has the strength that the fuzzy pair-wise comparison methods have, but no weakness of the pair-wise comparison ones. Therefore, ranking a set of fuzzy numbers is easy and fast by the relative preference relation P^*. Besides, P^* has the feasibility and universality on weights by considering the varied values of p.

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