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On characterizations of the fully rational fuzzy choice functions

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Abstract

In the present paper, we introduce the fuzzy Nehring axiom, fuzzy Sen axiom and weaker form of the weak fuzzy congruence axiom. We establish interrelations between these axioms and their relation with fuzzy Chernoff axiom. We express full rationality of a fuzzy choice function using these axioms along with the fuzzy Chernoff axiom.

Keywords: Full rationality, New weak fuzzy congruence axiom, fuzzy Nehring axiom, fuzzy Sen axiom, fuzzy Chernoff axiom.

1 Introduction

Rational choice theory is a mathematical approach used by social scientists to understand the human behaviour. To study the rationality of consumers Samuelson [28] introduced the theory of the revealed preference in terms of some preference relations associated with a demand function. The foundation of this theory is built on the weak axiom of consumer behavior [28] and the strong axiom of consumer behavior [20]. Georgescu-Roegen [17], Uzawa [34] and Arrow [2] have transformed this revealed preference theory to the more general framework of a choice function. In their work, they assumed that the domain of the choice function contains all non-empty finite sets of alternatives. Sen [29, 30] continued the work of Uzawa and Arrow and noticed that it is sufficient that the domain of the choice function contains all two-element and three-element sets. Without any restriction on the domain of choice function Richter [26], Hansson [19], Suzumura [32, 33] and many others studied the rationality of choice functions by introducing the revealed preference axioms, the congruence axioms, the consistency conditions etc. Following [31], a choice function on a choice space $(X, \mathcal{B})$, where X is a non-empty universe of alternatives and $\mathcal{B}$ a family of non-empty subsets of X, is defined as a function $C : \mathcal{B} \longrightarrow \mathcal{P}(X)$ such that $\phi \neq C(S) \subseteq S$, for all $S \in \mathcal{B}$. $C(S)$ is called the choice set of S. For $S \in \mathcal{B}$ and a relation $Q \subseteq X \times X$, the set of Q-greatest elements in S is $G(S, Q) = \{x \in S : (x, y) \in Q \text{ for all } y \in S\}$. The choice function C is said to be *G-rational* if there exists a preference relation Q on X such that $C(S) = G(S, Q)$ for all $S \in \mathcal{B}$. In this case, Q is called a *G-rationalization* of C. If Q is reflexive, complete and transitive then C is called full rational. In [31] Somdeb has expressed a full rationality of the choice function through new weak congruence axiom, Nehring's axiom and Sen's axiom. The aim of this paper is to obtain an extension of Somdeb results in the context of fuzzy choice functions.

In the real world most preferences and choices of a consumer are vague. To analyze a decision based on such preference and choices, the classical choice theory will have limitations. We need a different tool to analyze such situations. Thus, the fuzzy set theory is an appropriate tool to study them. Precisely, fuzzy relations are used to study vague preferences and fuzzy choice functions for vague choices. The case of vague preferences and exact choices was studied by De Baets and Fodor [9], Fodor [12] and Kulshreshtha and Shekar [21]. Barrett et al. [4, 5, 6] have studied the crisp choice functions generated by the fuzzy preference relation. Roubens [27] introduced choice functions based on t-norms of valued binary relations and studied properties of choice functions and rationality conditions. In [3] Banerjee has studied the revealed preference theory in the context of the fuzzy sets. In this approach the domain

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of a choice function consists of crisp sets of alternatives and the range is made of fuzzy sets of alternatives. Later, Georgescu [13, 14] introduced the fuzzy choice functions whose domain and range both are fuzzy sets of alternatives and studied the rationality of fuzzy choice functions and various fuzzy revealed preference axioms, fuzzy congruence axioms and fuzzy consistency conditions. Also, she proved Arrow-Sen theorem in the context of fuzzy choice function [16]. Later, Martinetti et al [23] generalised the results obtained by Georgescu in [16]. Also, they studied fuzzy and probabilistic choice functions and introduced new set of rationality conditions [24]. In [15] Georgescu has proved that the full rationality of a fuzzy choice function is equivalent to i) Weak fuzzy congruence axiom, ii) Fuzzy Arrow axiom, iii) strong fuzzy congruence axiom (see Theorem 4.7 [15]). Recently, in [8, 10, 11] we studied full rationality, quasi-transitive rationality and acyclic rationality of a fuzzy choice function with various conditions on the domain of a fuzzy choice function and Martinetti et al [22] studied role of acyclicity in the framework of rationality of fuzzy choice functions. In [1] Alcantud has introduced sequential fuzzy choice and studied rationality of sequential fuzzy choice function.

As a result of our study, the results of Somdeb are extended in the context of fuzzy choice functions defined on a non-empty family of non-zero fuzzy subsets of alternatives. The rest of the paper is divided into 5 sections. In section 2, we recall preliminary results related to the fuzzy implications defined on the real interval [0, 1] and also few basic definitions of the fuzzy relation. Section 3 contains basic definitions and results on the fuzzy choice functions. In section 4 we introduce fuzzy forms of the new weak congruence axiom, Nehring's axiom and Sen's axiom and establish relations between them. In section 5 we have shown that new weak fuzzy congruence axiom does not imply the full rationality of the fuzzy choice functions. We show that the full rationality of fuzzy choice function is equivalent to i) new weak fuzzy congruence axiom and fuzzy Chernoff axiom, ii) fuzzy Nehring axiom and the fuzzy Chernoff axiom and iii) fuzzy Sen axiom and the fuzzy Chernoff axiom.

2 Preliminaries

In this section, we shall recall some preliminary concepts with respect to the residuated structure of [0, 1] and some basic notions on fuzzy relations. The basic references for these matters are [7, 18, 35].

Let [0, 1] be the unit interval. For any $a, b, \in [0,1]$, we denote $a \vee b = max(a, b)$ and $a \wedge b = min(a, b)$. More generally, for any $\{a_i\}_{i\in I}$ we denote $\bigvee_{i\in I} a_i = sup\{a_i : i \in I\}$ and $\bigwedge_{i\in I} a_i = inf\{a_i : i \in I\}$. Clearly, $([0,1], \wedge, \vee, 0, 1)$ becomes a bounded distributive lattice. The binary operation $\longrightarrow$ called implication or residuation on [0,1] is defined as $a \longrightarrow b = \begin{cases} 1 & \text{if } a \leq b \\ b & \text{if } a > b \end{cases}$ The biresiduation operation $\longleftrightarrow$ on [0,1] is defined by $a \longleftrightarrow b = (a \longrightarrow b) \wedge (b \longrightarrow a)$. The negation operation $\neg$ associated with the Gödel t-norm is defined by $\neg a = \begin{cases} 1 & \text{if } a = 0 \\ 0 & \text{if } a > 0 \end{cases}$. The following lemma depicts some properties of the residuation

Lemma 2.1. [7, 16, 18] *For any $a, b, c \in [0,1]$ the following properties hold:*
(i) $a \wedge b \leq c \Longleftrightarrow a \leq b \longrightarrow c$
(ii) $a \wedge (a \longrightarrow b) = a \wedge b$
(iii) $1 \longrightarrow a = a$

Let X be a non-empty set of alternatives. A fuzzy subset A of X is a function $A : X \longrightarrow [0,1]$. If $x \in A$ then $A(x)$ is called the degree of membership of x in A. We denote by $\mathcal{P}(X)$ the family of crisp subsets of X and by $\mathcal{F}(\mathcal{X})$ the family of fuzzy subsets of X. Then $\mathcal{P}(\mathcal{X}) \subseteq \mathcal{F}(\mathcal{X})$. A fuzzy subset A of X is called *normal*, if $A(x) = 1$ for some $x \in X$ and is called non-zero if $A(x) > 0$ for some $x \in X$. For any $x_1, x_2, ..., x_n \in X$ we shall denote the *characteristic function* of $\{x_1, x_2, ..., x_n\}$ by $[x_1, x_2, ..., x_n]$. Thus $[x_1, x_2, ..., x_n](y) = \begin{cases} 1 & \text{if } y \in \{x_1, x_2, ..., x_n\} \\ 0 & \text{otherwise} \end{cases}$. For $A, B \in \mathcal{F}(\mathcal{X})$ let us denote $I(A, B) = \bigwedge_{z\in X} [A(z) \longrightarrow B(z)]$ and $E(A, B) = \bigwedge_{z\in X} [A(z) \longleftrightarrow B(z)]$, $I(A, B)$ is called the subsethood degree of A in B and $E(A, B)$ the degree of equality of A and B. It is clear that $A \subseteq B$ if and only if $I(A, B) = 1$ and $A = B$ if and only if $E(A, B) = 1$. For any $x \in X$ we have $I(A, B) \leq A(x) \longrightarrow B(x)$ and $E(A, B) \leq A(x) \longleftrightarrow B(x)$. A fuzzy preference relation Q on X is a fuzzy set $Q : X^2 \longrightarrow [0,1]$. For any $x, y \in X$ the real number $Q(x, y)$ shows the degree to which x is preferred to y. A fuzzy preference relation Q on X is said to be i) reflexive If $Q(x, x) = 1$ for all $x \in X$; ii) complete if $Q(x, y) = 1$ or $Q(y, x) = 1$ for all distinct $x, y \in X$ and iii) transitive if $Q(x, y) \wedge Q(y, z) \leq Q(x, z)$ for all

$x, y, z \in X$. The transitive closure of Q is denoted $\overline{Q}$ and is given by

$$\overline{Q}(x,y) = Q(x,y) \vee \left\{ \bigvee_{k\in N} \bigvee_{z_1,z_2,\ldots z_k \in X} [Q(x,z_1) \wedge Q(z_1,z_2) \wedge \cdots \wedge Q(z_k,y)] \right\}.$$

3 Fuzzy Choice Function

For self-completion of the paper, in this section, we recall the definition of the fuzzy choice function, fuzzy revealed preference relations, fuzzy congruence axioms, fuzzy Arrow axiom and few results from [16]. Let X be a non-empty set of alternatives and $\mathcal{B}$ a non-empty family of non-zero fuzzy subsets of X. The pair $(X, \mathcal{B})$ is called a fuzzy choice space. A *fuzzy choice function* (or fuzzy consumer) on $(X, \mathcal{B})$ is a function $C : \mathcal{B} \longrightarrow \mathcal{F}(\mathcal{X})$ such that for each $S \in \mathcal{B}$, $C(S)$ is non-zero fuzzy subset of X and $C(S) \subseteq S$. In the language of fuzzy consumers, X is called the set of *commodity bundles* and $\mathcal{B}$ is the family of *fuzzy budgets*. If x is a bundle and S a fuzzy budget then the real numbers $S(x)$ and $C(S)(x)$ denotes the availability degree of the bundle x in the fuzzy budget S and the degree to which the bundle x is chosen from the fuzzy budget S respectively. The results proved by Georgescu [16] are under the assumption that the fuzzy choice function satisfies the following two hypotheses:
(H1) Every $S \in \mathcal{B}$ and $C(S)$ are normal fuzzy subsets of X;
(H2) $\mathcal{B}$ includes the fuzzy sets $[x_1, x_2, ..., x_n]$, for any $n \geq 1$ and $x_1, x_2, ..., x_n \in X$.
The hypothesis (H1) asserts that every choice set has at least one unambiguous choice (i.e. S and $C(S)$ are non-empty) whereas (H2) asserts that $\mathcal{B}$ includes all non-empty finite subsets of X. Throughout this paper we assume the above two hypotheses.

Let $(X, \mathcal{B})$ be a fuzzy choice space and Q a fuzzy preference relation on X. For any $S \in \mathcal{B}$, the fuzzy subset $G(S, Q)$ of X is given by, $G(S,Q)(x) = S(x) \wedge \bigwedge_{z\in X} [S(z) \longrightarrow Q(x,z)]$, for any $x \in X$. Note that the real number $G(S,Q)(x)$ represents the greatness of the alternative x in the fuzzy set S. In general $G(S,Q)$ is not a fuzzy choice function. If Q is a reflexive and transitive fuzzy preference relation on a finite set X then $G(S,Q)$ is a fuzzy choice function on (X, B).

A fuzzy choice function $C : \mathcal{B} \longrightarrow \mathcal{F}(\mathcal{X})$ is said to be *G-rational*, if there exists a fuzzy preference relation Q on X such that $C(S) = G(S,Q)$ for all $S \in \mathcal{B}$. In this case, Q is called the *G-rationalization* of C. A fuzzy choice function C is called *full rational*, if there exists a fuzzy preference relation Q on X which is reflexive, complete and transitive such that $C(S) = G(S,Q)$ for all $S \in \mathcal{B}$.

Let $C : \mathcal{B} \longrightarrow \mathcal{F}(\mathcal{X})$ be a fuzzy choice function on $(X, \mathcal{B})$. One can define the fuzzy preference relation R on X for all $x, y \in X$ as, $R(x,y) = \bigvee_{S\in\mathcal{B}} [C(S)(x) \wedge S(y)]$. If a fuzzy choice function C is G-rational with rationalization R then C is called normal fuzzy choice function [15].

Lemma 3.1. [15, 16] *If C is a G-rational fuzzy choice function on $(X, \mathcal{B})$ with rationalization Q, then $R \subseteq Q$.*

If C verifies (H1) and (H2) then another fuzzy preference relation $\bar{R}$ on X is given by, $\bar{R}(x,y) = C([x,y])(x)$, for all $x, y \in X$.

Lemma 3.2. [15, 16] *Let C be a fuzzy choice function on $(X, \mathcal{B})$. If C satisfies (H1) and (H2), then*
(i) $\bar{R} \subseteq R$
(ii) *R and $\bar{R}$ are reflexive and complete.*

Definition 3.3. [15, 16] *Let W be the transitive closure of the fuzzy revealed preference relation R. A fuzzy choice function $C : \mathcal{B} \longrightarrow \mathcal{F}(\mathcal{X})$ defined on $(X, \mathcal{B})$ is said to satisfy*
(i) ***Weak Fuzzy Congruence Axiom (WFCA)*** *if for any $S \in \mathcal{B}$ and $x, y \in X$, we have $R(x,y) \wedge C(S)(y) \wedge S(x) \leq C(S)(x)$.*
(ii) ***Strong Fuzzy Congruence Axiom (SFCA)*** *if for any $S \in \mathcal{B}$ and $x, y \in X$, we have $W(x,y) \wedge C(S)(y) \wedge S(x) \leq C(S)(x)$.*
(iii) ***Fuzzy Arrow Axiom (FAA)*** *if for any $S_1, S_2 \in \mathcal{B}$ and $x \in X$, we have $I(S_1, S_2) \wedge S_1(x) \wedge C(S_2)(x) \leq E(S_1 \cap C(S_2), C(S_1))$.*
(iv) ***Fuzzy Chernoff Axiom (FCA)*** *if for any $S_1, S_2 \in \mathcal{B}$ and $x \in X$, we have $I(S_1, S_2) \wedge S_1(x) \wedge C(S_2)(x) \leq I(S_1 \cap C(S_2), C(S_1))$.*

In [15] Georgescu has shown the equivalence between the full rationality of a fuzzy choice function and the above axioms:

Theorem 3.4. [15] *If $C : \mathcal{B} \longrightarrow \mathcal{F}(\mathcal{X})$ is a fuzzy choice function defined on $(X, \mathcal{B})$, then the following are equivalent:*
(i) *C is full rational*
(ii) *C satisfies FAA*
(iii) *C is normal and R is reflexive, complete and transitive*
(iv) *C satisfies WFCA*
(v) *C satisfies SFCA.*

4 Axioms and Relationships

In this section, we introduce *new weak fuzzy congruence axiom, fuzzy forms of Nehring's* and *Sen's axioms.* We discuss relations between them as well as their links with the weak fuzzy congruence axiom and the fuzzy Chernoff axiom.We begin with the definitions of new weak fuzzy congruence axiom, fuzzy Nehring's axiom and fuzzy Sen's axiom.

Definition 4.1. *A fuzzy choice function C on $(X, \mathcal{B})$ is said to satisfy*
(i) ***New Weak Fuzzy Congruence Axiom (NWFCA)****, if for any $S \in \mathcal{B}$ and $x, y \in X$, we have $\bar{R}(x,y) \wedge C(S)(y) \wedge S(x) \leq C(S)(x)$.*
(ii) ***Fuzzy Nehring's Axiom (FNA)****, if for any $S \in \mathcal{B}$ and $x, y \in X$, we have $\bar{R}(x,y) \wedge \bar{R}(y,x) \wedge S(x) \wedge S(y) \leq C(S)(x) \longleftrightarrow C(S)(y)$.*
(iii) ***Fuzzy Sen's Axiom (FSA)****, if for all $S, T \in \mathcal{B}$ and $x \in X$, we have $I(S,T) \wedge C(S)(x) \wedge C(T)(x) \leq I(C(S), C(T))$.*

We notice that FNA and FSA are fuzzy versions of the Nehring's axiom (see [25]) and Sen's axiom (see [30]) respectively. Since $\bar{R} \subseteq R$, C satisfying WFCA must satisfy NWFCA. However, the converse may fail to be true.

Example 4.2. *Let $X = \{x, y, z, w\}$. Consider $\mathcal{B} = \mathcal{P}(X) \setminus \{\phi\}$ i.e the set of characteristic functions of non-empty subsets X. Define a fuzzy choice function C on $\mathcal{B}$ as $C([a])(a) = 1$, for all $a \in X$; $C([x,y])(x) = 1$; $C([x,y])(y) = 0.7$; $C([x,z])(x) = 1$; $C([x,z])(z) = 0.7$; $C([y,z])(y) = 1$; $C([y,z])(z) = 0.7$; $C([y,w])(y) = 0.7$; $C([y,w])(w) = 1$; $C([z,w])(z) = 0.7$; $C([z,w])(w) = 1$; $C([x,w])(x) = 1$; $C([x,w])(w) = 0.8$; $C([x,y,z])(x) = 1$; $C([x,y,z])(y) = 1$; $C([x,y,z])(z) = 0.8$; $C([x,y,w])(x) = 1$; $C([x,y,w])(y) = 0.7$; $C([x,y,w])(w) = 0.8$; $C([y,z,w])(y) = 0.7$; $C([y,z,w])(z) = 0.7$; $C([y,z,w])(w) = 1$; $C([x,z,w])(x) = 1$; $C([x,z,w])(z) = 0.8$; $C([x,z,w])(w) = 0.8$; $C(X)(x) = 1$; $C(X)(y) = 0.8$; $C(X)(z) = 0.7$ and $C(X)(w) = 0.8$. Then the fuzzy revealed preference relation $\bar{R}$ is given by*

$$\bar{R} = \begin{array}{c} \\ x \\ y \\ z \\ w \end{array} \begin{array}{c} \begin{array}{cccc} x & y & z & w \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0.7 & 1 & 1 & 0.7 \\ 0.7 & 0.7 & 1 & 0.7 \\ 0.8 & 1 & 1 & 1 \end{pmatrix} \end{array}$$

Clearly, for every $S \in \mathcal{B}$ and $a, b \in X$, we have $\bar{R}(a,b) \wedge C(S)(b) \wedge S(a) \leq C(S)(a)$. Therefore, C satisfies the NWFCA. Now, the fuzzy revealed preference relation R is given by,

$$R = \begin{array}{c} \\ x \\ y \\ z \\ w \end{array} \begin{array}{c} \begin{array}{cccc} x & y & z & w \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0.8 \\ 0.9 & 0.9 & 1 & 0.9 \\ 0.8 & 1 & 1 & 1 \end{pmatrix} \end{array}$$

For $x, y \in X$, we have $R(y,x) \wedge C(X)(x) \wedge X(x) = 1$ and $C(X)(y) = 0.8$, this shows that C does not satisfy the WFCA.

Now we establish the links between the NWFCA, FSA, FNA and FCA. The following lemma gives the relation between NWFCA and FNA.

Lemma 4.3. *If a fuzzy choice function C satisfies the NWFCA, then C satisfies the FNA.*

Proof. For any $x, y \in X$ and $S \in \mathcal{B}$, we have $\bar{R}(x,y) \wedge \bar{R}(y,x) \wedge S(x) \wedge S(y) \wedge C(S)(x) \leq \bar{R}(y,x) \wedge S(y) \wedge C(S)(x) \leq C(S)(y)$
By Lemma 2.1-(i), we get

$$\bar{R}(x,y) \wedge \bar{R}(y,x) \wedge S(x) \wedge S(y) \leq C(S)(x) \longrightarrow C(S)(y) \tag{1}$$

For any $x, y \in X$ and $S \in \mathcal{B}$, we have $\bar{R}(x,y) \wedge \bar{R}(y,x) \wedge S(x) \wedge S(y) \wedge C(S)(y) \leq \bar{R}(x,y) \wedge S(x) \wedge C(S)(y) \leq C(S)(x)$
By Lemma 2.1-(i)

$$\bar{R}(x,y) \wedge \bar{R}(y,x) \wedge S(x) \wedge S(y) \leq C(S)(y) \longrightarrow C(S)(x) \tag{2}$$

Thus, $\bar{R}(x,y) \wedge \bar{R}(y,x) \wedge S(x) \wedge S(y) \leq (C(S)(x) \longrightarrow C(S)(y)) \wedge (C(S)(y) \longrightarrow C(S)(x)) = C(S)(x) \longleftrightarrow C(S)(y)$. The inequalities 1 and 2 holds for any $x, y \in X$ and $S \in \mathcal{B}$. Thus, by idempotent property of $\wedge$, C satisfies FNA. □

The following example illustrates that the converse of the above lemma is not true in general.

Example 4.4. *Let $X = \{x, y, z\}$. Consider $\mathcal{B} = \mathcal{P}(X) \setminus \{\phi\}$ i.e. the set of all characteristic functions of non-empty subsets of X. Now, define a fuzzy choice function C on $\mathcal{B}$ as $C([a])(a) = 1$, for all $a \in X$; $C([x,y])(x) = 1$; $C([x,y])(y) = 0.8$; $C([x,z])(x) = 0.8$; $C([x,z])(z) = 1$; $C([y,z])(y) = 1$; $C([y,z])(z) = 0.9$; $C(X)(x) = 0.9$; $C(X)(y) = 1$ and $C(X)(z) = 1$. Then the fuzzy revealed preference relation $\bar{R}$ is given by*

$$\bar{R} = \begin{array}{c} \\ x \\ y \\ z \end{array} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{pmatrix} 1 & 1 & 0.8 \\ 0.8 & 1 & 1 \\ 1 & 0.9 & 1 \end{pmatrix} \end{array}$$

Clearly, C satisfies the FNA. But, for $x, y \in X$ and $X \in \mathcal{B}$, we have $\bar{R}(x,y) \wedge C(X)(y) \wedge X(x) = 1$ and $C(X)(x) = 0.9$. This shows that C does not satisfy the NWFCA.

But the following lemma shows that a fuzzy choice function satisfying FNA and FCA will also satisfy NWFCA.

Lemma 4.5. *If a fuzzy choice function C satisfies the FNA and FCA, then C satisfies the NWFCA.*

Proof. Suppose that C satisfies FNA and FCA. Let $x, y \in X$ and $S \in \mathcal{B}$. Since $\bar{R}$ is complete, we have $\bar{R}(x,y) = 1$ or $\bar{R}(y,x) = 1$. First suppose that $\bar{R}(y,x) = 1$. Then by the FNA it follows that

$$\bar{R}(x,y) \wedge C(S)(y) \wedge S(x) = \bar{R}(x,y) \wedge \bar{R}(y,x) \wedge C(S)(y) \wedge S(x) = \bar{R}(x,y) \wedge \bar{R}(y,x) \wedge C(S)(y) \wedge S(y) \wedge S(x) \le C(S)(x)$$

Now, consider $\bar{R}(x,y) = 1$. Then by FCA, we get

$$\begin{aligned} \bar{R}(x,y) \wedge S(x) \wedge C(S)(y) &= S(x) \wedge S(y) \wedge C(S)(y) = I\left([x,y], S\right) \wedge [x,y](y) \wedge C(S)(y) \le I([x,y] \cap C(S), C([x,y])) \\ &\le C(S)(y) \longrightarrow C([x,y])(y) \end{aligned}$$

By Lemma 2.1-(i) and idempotent property of $\wedge$, we get $\bar{R}(x,y) \wedge S(x) \wedge C(S)(y) \le C([x,y])(y) = \bar{R}(y,x)$. By idempotent property of $\wedge$, FNA and 2.1-2.1 we get

$$\begin{aligned} \bar{R}(x,y) \wedge S(x) \wedge C(S)(y) &= \bar{R}(x,y) \wedge \bar{R}(y,x) \wedge S(x) \wedge S(y) \wedge C(S)(y) \le C(S)(y) \wedge [C(S)(y) \longrightarrow C(S)(x)] \\ &= C(S)(x) \wedge C(S)(y) \le C(S)(x) \end{aligned}$$

This proves NWFCA. □

The following examples show that the FNA and FCA are independent.

Example 4.6. *In Example 4.4 for $[x,z], X \in \mathcal{B}$, we have $I([x,z], X) \wedge [x,z](x) \wedge C(X)(x) = 0.9$ and $I([x,z] \cap C(X), C([x,z])) = 0.8$. This shows that C does not satisfy the FCA.*

Example 4.7. *Let $X = \{x, y, z\}$. Define a fuzzy choice function C on $\mathcal{B} = \mathcal{P}(\mathcal{X}) \setminus \{\phi\}$ as, $C([a])(a) = 1$, for all $a \in X$; $C([x,y])(x) = 1$; $C([x,y])(y) = 1$; $C([x,z])(x) = 1$; $C([x,z])(z) = 0.9$; $C([y,z])(y) = 1$; $C([y,z])(z) = 0.8$; $C(X)(x) = 1$; $C(X)(y) = 0.6$ and $C(X)(z) = 0.8$. Then the relation $\bar{R}$ is given by*

$$\bar{R} = \begin{array}{c} \\ x \\ y \\ z \end{array} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0.9 & 0.8 & 1 \end{pmatrix} \end{array}$$

Obviously, C satisfies the fuzzy Chernoff axiom. For $x, y \in X$ and $X \in \mathcal{B}$, we have $\bar{R}(x,y) \wedge \bar{R}(y,x) \wedge X(x) \wedge X(y) = 1$ and $C(X)(x) \longleftrightarrow C(X)(y) = 0.6$. This shows that the FCA does not imply FNA.

The following examples show that the NWFCA and FCA are independent.

Example 4.8. *In Example 4.7 for $x, y \in X$ and $X \in \mathcal{B}$, we have $\bar{R}(y,x) \wedge C(X)(x) \wedge X(y) = 1$ and $C(X)(y) = 0.6$. This shows FCA does not imply NWFCA.*

Example 4.9. *In Example 4.2, C satisfies NWFCA. For $[x,y]$ and $X \in \mathcal{B}$, we have $I\left([x,y], X\right) \wedge [x,y](x) \wedge C(X)(x) = 1$ and $I\left([x,y] \cap C(X), C([x,y])\right) = 0.7$. This shows that C does not satisfy FCA.*

In the following lemma we show a fuzzy choice function satisfying the FNA and FCA will also satisfy the FSA

Lemma 4.10. *If a fuzzy choice function C satisfies the FNA and FCA, then C satisfies the FSA.*

Proof. Let $S,T \in \mathcal{B}$ and $x \in X$. Then for any $z \in X$, we have

$$\begin{aligned} I(S,T) \wedge C(S)(x) \wedge C(T)(x) \wedge C(S)(z) &\leq C(S)(x) \wedge C(S)(z) = C(S)(x) \wedge S(z) \wedge S(x) \wedge C(S)(z) \\ &= I([x,z],S) \wedge C(S)(x) \wedge C(S)(z) = I([x,z],S) \wedge [x,z](x) \wedge C(S)(x) \wedge C(S)(z) \\ &\leq C(S)(z) \wedge I([x,z] \cap C(S), C([x,z])), \text{ by FCA} \\ &\leq C(S)(z) \wedge (C(S)(z) \longrightarrow C([x,z])(z)) = C(S)(z) \wedge C([x,z])(z) \text{ by Lemma 2.1-(i)} \\ &\leq C([x,z])(z) \end{aligned}$$

Similarly, $I(S,T) \wedge C(S)(x) \wedge C(T)(x) \wedge C(S)(z) \leq C([x,z])(x)$. For any $z \in X$, by Lemma 2.1-(i) it follows that

$$\begin{aligned} &I(S,T) \wedge C(S)(x) \wedge C(T)(x) \wedge C(S)(z) \leq C(S)(z) \wedge I(S,T) \leq S(z) \wedge I(S,T) \leq S(z) \wedge (S(z) \longrightarrow T(z)) \\ &= S(z) \wedge T(z) \leq T(z) \end{aligned}$$

By all the inequalities above and the idempotent property of $\wedge$, we have

$$I(S,T) \wedge C(S)(x) \wedge C(T)(x) \wedge C(S)(z) \leq C([x,z])(x) \wedge C([x,z])(z) \wedge T(z)$$

Now by above inequality and FNA, we get

$$\begin{aligned} &I(S,T) \wedge C(S)(x) \wedge C(T)(x) \wedge C(S)(z) = I(S,T) \wedge C(S)(x) \wedge C(T)(x) \wedge C(S)(z) \wedge T(x) \\ &\leq C([x,z])(x) \wedge C([x,z])(z) \wedge T(z) \wedge T(x) = \bar{R}(x,z) \wedge \bar{R}(z,x) \wedge T(x) \wedge T(z) \leq C(T)(x) \longleftrightarrow C(T)(z) \\ &\leq C(T)(x) \longrightarrow C(T)(z) \end{aligned}$$

By Lemma 2.1-(i), we have $I(S,T) \wedge C(S)(x) \wedge C(T)(x) \wedge C(S)(z) \leq C(T)(z)$. Again by Lemma 2.1-(i), we have $I(S,T) \wedge C(S)(x) \wedge C(T)(x) \leq C(S)(z) \longrightarrow C(T)(z)$. The above inequality is true for all $z \in X$. Therefore, $I(S,T) \wedge C(S)(x) \wedge C(T)(x) \leq I(C(S), C(T))$. This shows that C satisfies the FSA. □

The following example shows that the FSA neither implies FNA nor FCA

Example 4.11. *Let $X = \{x,y,z,w\}$. Define a fuzzy choice function C on $\mathcal{B} = \mathcal{P}(\mathcal{X}) \setminus \{\phi\}$ as, $C([a])(a) = 1$, for all $a \in X$; $C[x,y](x) = 0.8$, $C[x,y](y) = 1$, $C[x,z](x) = 0.7$, $C[x,z](z) = 1$, $C[x,w](x) = 1$, $C[x,w](w) = 0.9$, $C[y,z](y) = 1$, $C[y,z](z) = 0.8$, $C[y,w](y) = 1$, $C[y,w](w) = 0.7$, $C[z,w](z) = 0.7$, $C[z,w](w) = 1$, $C[x,y,z](x) = 0.8$, $C[x,y,z](y) = 1$, $C[x,y,z](z) = 1$, $C[x,y,w](x) = 0.9$, $C[x,y,w](y) = 0.8$, $C[x,y,w](w) = 1$, $C[x,z,w](x) = 1$, $C[x,z,w](z) = 1$, $C[x,z,w](w) = 1$, $C[y,z,w](y) = 1$, $C[y,z,w](z) = 1$, $C[y,z,w](w) = 1$, $C(X)(x) = 1$, $C(X)(y) = 1$, $C(X)(z) = 1$, $C(X)(w) = 1$. Then the relation $\bar{R}$ is given by*

$$\bar{R} = \begin{array}{c} \\ x \\ y \\ z \\ w \end{array} \begin{array}{c} \begin{array}{cccc} x & y & z & w \end{array} \\ \begin{pmatrix} 1 & 0.8 & 0.7 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0.8 & 1 & 0.7 \\ 1 & 0.7 & 1 & 1 \end{pmatrix} \end{array}$$

Clearly, C satisfies FSA. But for $x,w \in [x,y,w]$ and $[x,y,w] \in \mathcal{B}$, we have $\bar{R}(x,w) \wedge \bar{R}(w,x) \wedge [x,y,w](x) \wedge [x,y,w](w) = 1$ and $C([x,y,w])(x) \longleftrightarrow C([x,y,w])(w) = 0.9$. This shows that C does not satisfy FNA.

Example 4.12. *In Example 4.4, for every $S,T \in \mathcal{B}$ and $a \in X$, we have $I(S,T) \wedge C(S)(x) \wedge C(T)(x) \leq I(C(S), C(T))$. This shows that C satisfies the FSA. For $[x,z], X \in \mathcal{B}$ we have $I([x,z],X) \wedge [x,z](z) \wedge C(X)(z) = 1$ and $I([x,z] \cap C(X), C([x,z])) = 0.8$. This shows that C does not satisfy the FCA.*

The following lemma shows that the FCA and FSA together imply the NWFCA.

Lemma 4.13. *If a fuzzy choice function C satisfies the FCA and FSA, then C satisfies the NWFCA.*

Proof. For any $x, y \in X$ and $S \in \mathcal{B}$ by Lemma 2.1-(i) it follows that

$$\bar{R}(x,y) \wedge C(S)(y) \wedge S(x) = \bar{R}(x,y) \wedge C(S)(y) \wedge S(y) \wedge S(x) \wedge C(S)(y) \wedge S(x)$$
$$= \bar{R}(x,y) \wedge I([x,y],S) \wedge C(S)(y) \wedge [x,y](y) \wedge C(S)(y) \wedge S(x) \leq \bar{R}(x,y) \wedge I([x,y] \cap C(S), C([x,y])) \wedge C(S)(y) \wedge S(x) \text{ by FCA}$$
$$\leq \bar{R}(x,y) \wedge [C(S)(y) \longrightarrow C([x,y])(y)] \wedge C(S)(y) \wedge S(x) = \bar{R}(x,y) \wedge C(S)(y) \wedge C([x,y])(y) \wedge S(x)$$
$$= C([x,y])(x) \wedge C(S)(y) \wedge C([x,y])(y) \wedge S(x) = C([x,y])(x) \wedge C(S)(y) \wedge S(y) \wedge S(x) \wedge C([x,y])(y)$$
$$= C([x,y])(x) \wedge I([x,y],S) \wedge C([x,y])(y) \wedge C(S)(y) \leq C([x,y])(x) \wedge I(C([x,y]), C(S)) \text{ by FSA}$$
$$\leq C([x,y])(x) \wedge (C([x,y])(x) \longrightarrow C(S)(x)) = C([x,y])(x) \wedge C(S)(x) \leq C(S)(x)$$

This shows that C satisfies the NWFCA. □

The following example shows that the FCA does not imply the FSA.

Example 4.14. *Let $X = \{x, y, z\}$. Define a fuzzy choice function C on $\mathcal{B} = \mathcal{P}(X) \setminus \{\phi\}$ as, $C([a])(a) = 1$, for all $a \in X$, $C([x,y])(x) = 1$, $C([x,y])(y) = 1$, $C([x,z])(x) = 1$, $C([x,z])(z) = 1$, $C([y,z])(y) = 1$, $C([y,z])(z) = 1$, $C(X)(x) = 1$, $C(X)(y) = 0.8$, $C(X)(z) = 0.9$. Here C satisfies FCA, but $I([x,y],X) \wedge C([x,y])(x) \wedge C(X)(x) = 1$ and $I(C([x,y]), C(X)) = 0.8$. This shows that C does not satisfy the FSA.*

Remark 4.15. *Examples 4.12 and 4.14 show that the FCA and FSA are independent.*

In the following example we aim to prove that the FSA alone does not imply the NWFCA.

Example 4.16. *In Example 4.11 for $x, w \in [x,y,w]$ and $[x,y,w] \in \mathcal{B}$, we have $\bar{R}(x,w) \wedge C([x,y,w])(w) \wedge [x,y,w](x) = 1$ and $C([x,y,w])(x) = 0.9$. Hence, C does not satisfy the NWFCA.*

The following example shows that the NWFCA does not imply the FSA.

Example 4.17. *In Example 4.2, we have $I([x,y,z],X) \wedge C(X)(x) \wedge C([x,y,z])(x) = 1$ and $I(C([x,y,z]), C(X)) = 0.7$. This shows that the NWFCA does not imply the FSA.*

Remark 4.18. *Examples 4.16 and 4.17 together show that the NWFCA and the FSA are independent.*

5 Characterizations

Recall that axioms NWFCA and FCA are mutually independent and NWFCA alone does not imply the full rationality and hence the Fuzzy Arrow axiom. This gives us an opportunity to combine these axioms for characterization of full rationality of fuzzy choice functions.

Theorem 5.1. *A fuzzy choice function C is full rational if and only if C satisfies the FCA and NWFCA.*

Proof. Let C be a full rational fuzzy choice function. Then by Theorem 3.4 C satisfies WFCA and hence by Lemma 2.1-(i) C satisfies the NWFCA. Let Q be a reflexive, complete and transitive fuzzy preference relation on X such that $C(S) = G(S,Q)$, for all $S \in \mathcal{B}$. Let $S, T \in \mathcal{B}$ and $x \in X$. Then for any $z, t \in X$, we have

$$I(S,T) \wedge S(x) \wedge C(T)(x) \wedge S(z) \wedge C(T)(z) \wedge S(t) \leq S(t) \wedge I(S,T) \wedge S(x) \wedge C(T)(z)$$
$$\leq S(t) \wedge (S(t) \longrightarrow T(t)) \wedge S(x) \wedge C(T)(z) = S(t) \wedge T(t) \wedge S(x) \wedge C(T)(z), \text{ by Lemma 2.1-(i)}$$
$$\leq T(t) \wedge C(T)(z) = T(t) \wedge G(T,Q)(z) \leq T(t) \wedge (T(t) \longrightarrow Q(z,t)) = T(t) \wedge Q(z,t), \text{ by Lemma 2.1-(i)}$$
$$\leq Q(z,t)$$

Thus, $I(S,T) \wedge S(x) \wedge C(T)(x) \wedge S(z) \wedge C(T)(z) \leq S(t) \longrightarrow Q(z,t)$. The above inequality is true for all $t \in X$. Therefore

$$I(S,T) \wedge S(x) \wedge C(T)(x) \wedge S(z) \wedge C(T)(z) \leq S(z) \wedge \bigwedge_{t \in X} [S(t) \longrightarrow Q(z,t)] = G(S,Q)(z) = C(S)(z)$$

Again by Lemma 2.1-(i), we get $I(S,T) \wedge S(x) \wedge C(T)(x) \leq S(z) \wedge C(T)(z) \longrightarrow C(S)(z)$. The above inequality is true for all $z \in X$. Thus, $I(S,T) \wedge S(x) \wedge C(T)(x) \leq I(S \cap C(T), C(S))$. This shows that C satisfies the FCA.
Conversely, suppose C satisfies the FCA and NWFCA. By Lemma 2.1-(i) $\bar{R}$ is reflexive and complete. First we shall prove

that $\bar{R}$ is transitive. Let $x, y, z \in X$ and denote $S = [x, y, z]$. Since C is normal, $C([x, y, z])(x) = 1$ or $C([x, y, z])(y) = 1$ or $C([x, y, z])(z) = 1$. If $C([x, y, z])(x) = 1$, then by Lemma 2.1-(i), it follows that

$$\begin{aligned}\bar{R}(x, y) \wedge \bar{R}(y, z) &= \bar{R}(x, y) \wedge \bar{R}(y, z) \wedge C([x, y, z])(x) \wedge [x, z](x) \leq C([x, y, z])(x) \wedge [x, z](x)\\ &= I([x, z], [x, y, z]) \wedge C([x, y, z])(x) \wedge [x, z](x) \leq I([x, z] \cap C([x, y, z]), C([x, z])), \text{ by FCA}\\ &\leq [x, z](x) \wedge C([x, y, z])(x) \longrightarrow C([x, z])(x) = C([x, z])(x) = \bar{R}(x, z)\end{aligned}$$

If $C([x, y, z])(y) = 1$, then, by NWFCA

$$\bar{R}(x, y) \wedge \bar{R}(y, z) = \bar{R}(x, y) \wedge C([x, y, z])(y) \wedge [x, y, z](x) \wedge \bar{R}(y, z) \leq C([x, y, z])(x) \wedge \bar{R}(y, z) \quad \leq C([x, y, z])(x) \quad (3)$$

Now,

$$\begin{aligned}C([x, y, z])(x) &= I([x, z], [x, y, z]) \wedge C([x, y, z])(x) \wedge [x, z](x) \leq I([x, z] \cap C([x, y, z]), C([x, z]))\\ &\leq [x, z](x) \wedge C([x, y, z])(x) \longrightarrow C([x, z])(x) = C([x, y, z])(x) \longrightarrow C([x, z])(x)\end{aligned}$$

By Lemma 2.1-(i), we have $C([x, y, z])(x) \leq C([x, z])(x)$. Therefore, Equation (3) gives $\bar{R}(x, y) \wedge \bar{R}(y, z) \leq C([x, z])(x) = \bar{R}(x, z)$. Similarly, we can prove the inequality $\bar{R}(x, y) \wedge \bar{R}(y, z) \leq \bar{R}(x, z)$ for $C([x, y, z])(z) = 1$. Hence, $\bar{R}$ is transitive. Now, it remains to prove $C(S) = G(S, \bar{R})$, for all $S \in \mathcal{B}$. Let $S \in \mathcal{B}$ and $x \in X$. For any $y \in X$, we have $I([x, y], S] = S(x) \wedge S(y)$. Then

$$\begin{aligned}C(S)(x) \wedge S(y) &= S(x) \wedge S(y) \wedge C(S)(x) \wedge [x, y](x) \wedge C(S)(x) = I([x, y], S) \wedge C(S)(x) \wedge [x, y](x) \wedge C(S)(x)\\ &\leq I([x, y] \cap C(S), C([x, y])) \wedge C(S)(x) \leq C(S)(x) \wedge ([x, y](x) \wedge C(S)(x) \longrightarrow C([x, y])(x))\\ &= C(S)(x) \wedge C([x, y])(x) \leq C([x, y])(x) = \bar{R}(x, y)\end{aligned}$$

By Lemma 2.1-(i) the above inequality reduces to $C(S)(x) \leq S(y) \longrightarrow \bar{R}(x, y)$, for all $y \in X$. Thus, $C(S)(x) \leq S(x) \wedge \bigwedge_{y \in X} [S(y) \longrightarrow \bar{R}(x, y)] = G(S, \bar{R})(x)$. Next, since C is normal for every $S \in \mathcal{B}$, there exists $z \in X$ such that $C(S)(z) = 1$. Then for any $x \in X$, by Lemma 2.1-(i), it follows that

$$G(S, \bar{R})(x) \leq S(x) \wedge (S(z) \longrightarrow \bar{R}(x, z)) = S(x) \wedge \bar{R}(x, z) = \bar{R}(x, z) \wedge C(S)(z) \wedge S(x) \leq C(S)(x), \text{ by NWFCA}$$

Thus, $G(S, \bar{R}) = C(S)$. Hence, C is full rational with rationalization $\bar{R}$ □

Theorem 5.2. *Let C be a normal fuzzy choice function on $(X, \mathcal{B})$. Then C is full rational if and only if C satisfies the NWFCA.*

Proof. Let C be a full rational fuzzy choice function. Then by Theorem 3.4 C satisfies WFCA and hence by Lemma 2.1-(i) C satisfies the NWFCA.
Conversely, suppose that C satisfies the NWFCA. For any $x, y \in X$, we have

$$\begin{aligned}\bar{R}(x, y) = C([x, y])(x) &= [x, y](x) \wedge \bigwedge_{z \in X} [[x, y](z) \longrightarrow R(x, z)] = [[x, y](x) \longrightarrow R(x, x)] \wedge [[x, y](y) \longrightarrow R(x, y)]\\ &= R(x, x) \wedge R(x, y) = R(x, y)\end{aligned}$$

Thus, $R = \bar{R}$. This implies C holds WFCA. Hence by Theorem 3.4, C is full rational. □

The following are few more characterizations for the property of the full rationality of a fuzzy choice function.

Theorem 5.3. *A fuzzy choice function C is full rational if and only if it satisfies the FNA and FCA.*

Proof. First suppose C is full rational. Then by Theorem 5.1 and Lemma 4.3, C satisfies FNA and FCA.
The converse follows by Theorem 5.1 and Lemma 4.5 □

Theorem 5.4. *A fuzzy choice function C is full rational if and only if it satisfies the FSA and FCA.*

Proof. Suppose C is full rational. Then by the above theorem C satisfies the FNA and FCA. Hence by Lemma 4.10 C satisfies the FSA. Conversely, suppose C satisfies FSA and FCA. Then by Theorem 5.1 and Lemma 4.13, C is full rational. □

Summarizing Theorems 3.4, 5.1, 5.2, 5.3 and 5.4 we get the following extended characterization theorem for the full rationality of the fuzzy choice functions

Theorem 5.5. *Let $C : \mathcal{B} \longrightarrow \mathcal{F}(X)$ be a fuzzy choice function on $(X, \mathcal{B})$. The following are equivalent:*
(i) *C is full rational*
(ii) *C satisfies the NWFCA and FCA*
(iii) *C is normal and satisfies NWFCA*
(iv) *C satisfies the FAA*
(v) *C satisfies the FNA and FCA*
(vi) *C satisfies the FSA and FCA*
(vii) *C satisfies the WFCA*
$(viii)$ *C satisfies the SFCA*
(ix) *C is normal and R is reflexive, complete and transitive.*

6 Conclusions

In this paper, we have defined the fuzzy choice functions on the domains that contain a non-empty family of non-zero fuzzy subsets of the universal set. We have introduced fuzzy forms of Sen's axiom and Nehring's axiom. We weakened weak fuzzy congruence axiom and called it new weak fuzzy congruence axiom. We established links between these axioms. In [15], Georgescu has shown that the weak fuzzy congruence axiom is a necessary and sufficient condition for a fuzzy choice function to be full rational. In the case of the new weak fuzzy congruence axiom this is not true. Therefore we combine independent axioms new weak fuzzy congruence axiom and fuzzy Chernoff axiom to characterize the full rationality of a fuzzy choice function. Also, we have shown that the full rationality of the fuzzy choice function is equivalent to the (i) fuzzy Nehring axiom and the fuzzy Chernoff axiom and (iii) fuzzy Sen axiom and the fuzzy Chernoff axiom. Combining our results with the results of Georgescu [15] we get Theorem 5.5. The results of this paper are obtained in the framework of a fuzzy set theory based on the Gödel t-norm, particularly the idempotent property. Therefore an open problem is to verify the results for an arbitrary continuous t-norm.

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