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A modified position value for communication situations and its fuzzification

X. H. Li[1], H. Sun[2] and G. J. Xu[3]

[1,2,3] *Department of Mathematics, Northwestern Polytechnical University, Xi'an, Shaanxi 710072, PR China*

xianghuili@mail.nwpu.edu.cn, hsun@nwpu.edu.cn, xugenjiu@nwpu.edu.cn

Abstract

Myerson treated various situations of cooperation in the field of cooperative games and proposed the communication structure. In this paper, we define and characterize an allocation rule in terms of the position value, called an average spanning tree solution, for communication situations by introducing a concept of cooperation relationship which says that two players are deemed to possess this relationship if there is a communication path through them. Considering the fact that the extent to what players participate in a coalition may be partially restricted for uncertain possibilities, we construct a graph game in multilinear extension form and continue to explore the fuzzy average spanning tree solution on a framework of communication situation with fuzzy coalition.

Keywords: Communication situation, Basic cooperation graph, Position value, Participation level, Fuzzy coalition.

1 Introduction

A cooperative game is a pair consisting of a finite set of players and a characteristic function assigning to any coalition a worth which is achieved by cooperation among the players in it. In this kind of games, it is assumed that any coalition can form ignoring such situations where the collection of possible coalitions is restricted by technical or economical structures and so on. Myerson [9] introduced the communication structure which can be described by an undirected graph with nodes as players and links as communication relationships. For the purpose to measure the influence of communication restrictions on the gains from cooperation, he defined a cooperative game, called the graph game, associated to each communication situation. The Myerson value for a communication situation is the Shapley value of its graph game, and axiomatized by Myerson [10] and Borm et al. [3]. Meessen [7] proposed another allocation rule, called the position value, for communication situations which focuses on the role of links by defining a link game. Borm et al. [3] gave a characterization of position value limited on cycle-free communication structures. The Shapley value of each link in the link game is an evaluation of its power and shared equally between the two incident nodes of this link. Later, Slikker [13] axiomatized the position value for arbitrary communication situations. In the classical cooperative game theory models, there are only two options for each player in deciding whether to take part in a coalition, that is, participate completely or not. In reality, however, agents may face the non-full cooperation situation between the non-cooperation and full cooperation due to the variability of environment or the limitation of available resource. Aubin [1, 2] proposed the concept of fuzzy coalition considering the uncertainty of players' participation levels and introduced the fuzzy cooperative game. A key issue for a fuzzy cooperative game is how to determine its characteristic function, which attracts more scholars to construct different characteristic functions by extensions of the classical case and so far three popular ones are the proportional extension [4], the extension of Chouqet integral [14], and the multilinear extension [8, 11].

Our whole research in this paper is developed by defining a concept of cooperation relationship. Any two players in a communication structure are considered to hold this relationship if they can communicate directly or indirectly, i.e., there is a path connecting them. In fact, deleting some links sometimes does not have an impact on the cooperation

Corresponding Author: H. Sun

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relationship among players. Meantime, it need spend amounts of money or human resources for maintaining the bilateral relationships and more communication links means higher expenditures. Based on the economical principle, all players are rational to remove communication channels as much as possible under the premise of ensuring the cooperation relationship among all players unchanged. This motivates us to pay attention to the minimal subgraph, called a basic cooperation graph, that keeps the cooperation relationship invariance of original communication structure. Assume each basic cooperation graph has the equal possibility to be treated as the final communication network for players. We propose and axiomatize a new allocation rule by modifying the position value, called an average spanning tree solution, which is the expected value of position values associated to all basic cooperation graphs. Further, we define a fuzzy cooperative game in multilinear extension form inspired by Meng and Zhang [8] and go on to investigate the fuzzy average spanning tree solution under a model of communication situation with fuzzy coalition [6] where agents have no full ability to join in cooperation and the bilateral communications between them are possibly restricted by some uncertain factors.

The rest of this paper is organized as follows. In Section 2, we present some basic preliminaries of cooperative games, communication situations, fuzzy cooperative games and communication situations with fuzzy coalition. In Section 3, we propose an average spanning tree solution for crisp communication situations and give its axiomatization. In Section 4, we define a graph game in multilinear extension form and study the fuzzy average spanning tree solution. Finally, in Section 5, some conclusions are given.

2 Preliminaries

2.1 Cooperative Games

A *cooperative game* can be described by a pair (N, v) where $N = \{1, 2, \ldots, n\}$ denotes the set of players and $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$ the corresponding characteristic function. Any subset $S \subseteq N$ is called a *coalition* and $v(S)$ the *worth* of S. The set of all cooperative games is denoted by G^N. A game (N, u_T) with characteristic function given by

$$u_T(S) = \begin{cases} 1, & \text{if } T \subseteq S, \\ 0, & \text{otherwise} \end{cases}$$

for any $T \subseteq N$ is called the *unanimity game*. It is easy to see that the unanimity games $\big\{(N, u_T)\big\}$ with $\emptyset \neq T \subseteq N$ form a basis of G^N. As a consequence, any game (N, v) has a unique expression as a linear combination of unanimity games whose characteristic function can be written as

$$v = \sum_{\emptyset \neq T \subseteq N} \Delta_v(T) u_T. \tag{1}$$

The coefficient $\Delta_v(T)$ is known as the Harsanyi dividend generated by coalition T in the game (N, v) with $\Delta_v(T) = \sum\limits_{S \subseteq T} (-1)^{|T|-|S|} v(S)$. Both of The Shapley value and core are famous solution concepts for cooperative games.

Definition 2.1. [12] *The Shapley value for any cooperative game (N, v) is defined by*

$$Sh_i(N, v) = \sum_{T \subseteq N; i \in T} \frac{\big(|N| - |T|\big)!\big(|T| - 1\big)!}{|N|!} \big[v(T) - v(T \backslash \{i\})\big], \quad \textit{for any } i \in N.$$

An alternative expression for this value is $Sh_i(N, v) = \sum\limits_{T \subseteq N; i \in T} \Delta_v(T) \frac{1}{|T|}$, for any $i \in N$.

Definition 2.2. *The core for any $(N, v) \in G^N$ is defined by $C(v) = \{x \in \mathbb{R}^n | \sum\limits_{i \in N} x_i = v(N), \sum\limits_{i \in S} x_i \geq v(S)$ for any $S \subseteq N\}$.*

For convenience, we assume throughout this paper that the game (N, v) is *zero-normalized*, i.e., $v\big(\{i\}\big) = 0$ for all $i \in N$.

2.2 Communication Situations

A *communication structure* [9] can be described by an undirected graph (N, L) where the nodes in N represent the set of players and edges in $L \subseteq \big\{\{i,j\} | i,j \in N, i \neq j\big\}$ represent a collection of possible communication links. A *subgraph* of (N, L) is another graph (S, A) satisfying that $S \subseteq N$ and $A \subseteq L$. A *path* in (N, L) is a sequence of nodes $(i_1, i_2, \ldots, i_k)$ such that $\{i_h, i_{h+1}\} \in L$ is different for each $h \in \{1, 2, \ldots, k-1\}$. A *cycle* in (N, L) is a sequence of nodes $(i_1, i_2, \ldots, i_{k+1})$ if $k \geq 3$, $(i_1, i_2, \ldots, i_k)$ is a path, $\{i_k, i_{k+1}\} \in L$ and $i_{k+1} = i_1$. A graph (N, L) is *cycle-free* if it does not contain any cycle. Two nodes $i, j \in N$ are *connected* in (N, L) if there exists a path from i to j. (N, L) is *connected* if any two nodes $i, j \in N$ are connected. A connected graph without cycles is named a *tree*. A *spanning tree* in a connected graph (N, L) is a tree subgraph of (N, L) which uses all its nodes and we denote the set of all spanning trees of (N, L) by $ST(N, L)$. For any $S \subseteq N$, the graph $\big(S, L(S)\big)$ is a subgraph of (N, L) with $L(S) = \big\{\{i,j\} \in L | i,j \in S\big\}$. $S \subseteq N$ is a *connected coalition* of (N, L) if the graph $\big(S, L(S)\big)$ is connected. $C \subseteq N$ is a *component* of (N, L) if $\big(C, L(C)\big)$ is maximally connected. The set of all components in subgraph $\big(S, L(S)\big)$ is denoted by $S/L(S)$. Let $d(N, L) = \big(d_1(N, L), d_2(N, L), \ldots, d_n(N, L)\big)$ be the *degree vector* of (N, L) with $d_i(N, L)$ being the number of adjacent nodes to i in (N, L). The set of incident links to i in (N, L) is denoted by L_i. A *communication situation* is a triple (N, v, L) where (N, L) is a communication structure and (N, v) a cooperative game. The set of all communication situations over N is denoted by CS^N. For any communication situation, it is often necessary to consider how to allocate the profit among players. An *allocation rule* for communication situations is a function $\psi : CS^N \to \mathbb{R}^n$. Myerson [9] introduced the Myerson value which is actually the Shapley value of a graph game (N, v^L), i.e.,

Definition 2.3. [9] *For any $(N, v, L) \in CS^N$, the Myerson value is defined to be $\mu(N, v, L) = Sh(N, v^L)$, where $v^L(S) = \sum_{C \in S/L(S)} v(C)$, $(\forall S \subseteq N)$.*

Meessen [7] proposed the position value in which two endpoints of each link divides equally the Shapley value of a link game (L, r^v), i.e.,

Definition 2.4. [7] *For any $(N, v, L) \in CS^N$, the position value is defined to be*

$$P_i(N, v, L) = \frac{1}{2} \sum_{l \in L_i} Sh_l(L, r^v), \quad \text{for any } i \in N, \tag{2}$$

where $r^v(E) = \sum_{C \in N/E} v(C)$, $(\forall E \subseteq L)$.

Since (N, v^L) also has the form $v^L = \sum_{S \subseteq N} \Delta_{v^L}(S) u_S$, where $\Delta_{v^L}(S) = \sum_{S' \subseteq S} (-1)^{|S|-|S'|} v^L(S')$ is the Harsanyi dividend of coalition S in the game (N, v^L), an alternative expression for Myerson value is $\mu_i(N, v, L) = \sum_{S \subseteq N; i \in S} \Delta_{v^L}(S) \frac{1}{|S|}$, for any $i \in N$ and for position value if (N, L) is cycle-free is $P_i(N, v, L) = \sum_{S \subseteq N; i \in S} \Delta_{v^L}(S) \frac{d_i\left(N, L(S)\right)}{\sum_{i \in N} d_i\left(N, L(S)\right)}$, for any $i \in N$.

2.3 Fuzzy Cooperative Games

Let N be a finite set of players. A *fuzzy coalition* is a fuzzy set $U \in [0,1]^N$. The i_{th} coordinate U_i of fuzzy coalition U reflects the participation level to what extent player i takes part in this coalition. We denote by $\mathcal{F}^N$ the set of fuzzy coalitions over N. The empty coalition in a fuzzy setting is denoted by $e^{\emptyset} = (0, 0, \ldots, 0)$. e^S is a crisp-like fuzzy coalition with the participation level of every player being 1 in S and 0 outside S for any $S \subseteq N$. For any $U \in \mathcal{F}^N$, the *support* of U is defined by $supp(U) = \{i \in N | U_i \neq 0\}$.

An n-person *fuzzy cooperative game* [2], is represented by a characteristic function $v^f : \mathcal{F}^N \to \mathbb{R}$ such that $v^f(e^{\emptyset}) = 0$. The set of fuzzy cooperative games is denoted by FG^N. Let $\{h_1, h_2, \ldots, h_m\}$ be the set of non-zero elements in U and $h_1 < h_2 < \cdots < h_m$, $h_0 = 0$. In 2001, Tsurumi et al. [14] proposed a class of fuzzy cooperative games.

Definition 2.5. [14] *v^f is said to be a fuzzy cooperative game with* Choquet integral form *if and only if, for any $U \in \mathcal{F}^N$,, $v^f(U) = \sum_{k=1}^{m} [h_k - h_{k-1}] \cdot v([U]_{h_k})$, where $[U]_{h_k}$ is the set of players in fuzzy coalition U with the participation level not smaller than h_k, i.e., $[U]_{h_k} = \{i | i \in N, U_i \geq h_k\}$ for each $k \in \{1, 2, \ldots, m\}$.*

In 2010, Meng and Zhang [8] proposed fuzzy cooperative games in multilinear extension form, namely,

Definition 2.6. [8] *$v^f \in FG^N$ is said to be a fuzzy cooperative game in multilinear extension form if, for any $U \in \mathcal{F}^N$, $v^f(U) = \sum_{T \subseteq supp(U)} \prod_{i \in T} U_i \prod_{i \in supp(U) \setminus T} (1 - U_i) v(T)$.*

In 2009, Yu and Zhang [17] propose the *fuzzy core* for fuzzy cooperative games.

Definition 2.7. [17] *For any fuzzy cooperative game $v^f \in FG^N$, the fuzzy core is defined by*

$$\tilde{C}^U(v^f) = \{x \in \mathbb{R}^n | \sum_{i \in N} x_i = v^f(U), \sum_{i \in (U_T)^{cr}} x_i \geq v^f(U_T) \text{ for any } T \subseteq N\}.$$

2.4 Communication Situations with Fuzzy Coalition

Here we will use the participation level as the fraction of actual ability of players when a cooperation is established. A situation where communication restriction and partial cooperation happen simultaneously can be modeled by a communication situation with fuzzy coalition.

A *communication situation with fuzzy coalition* can be described by a triple (U, v, L) where $U = (U_1, U_2, \ldots, U_n)$ is said to be a fuzzy coalition with $U_i \in [0, 1]$ for all $i \in N$ being the participation level of player i and the *associated crisp communication situation* $\big(supp(U), v, L\big) \in CS^N$. We denote the set of all communication situations with fuzzy coalition by CSF^N. An *allocation rule* over CSF^N is a function: $\Psi : CSF^N \to \mathbb{R}^n$ obtaining a payoff vector for each communication situation with fuzzy coalition.

3 An Average Spanning Tree Solution

In this section, with the proposal of a concept of cooperation relationship, we introduce an average spanning tree solution for deterministic communication situations which is defined by some modification of the position value.

For any communication situation (N, v, L), we say two players $i, j \in N$ have *cooperation relationship* if they can communicate directly or by means of intermediaries, that is, there exists a path from i to j in (N, L). A subgraph (N, L') of (N, L) is supposed to hold *cooperation relationship invariance* if there are the same cooperation relationship between any two players in (N, L') as in (N, L), i.e., i and j have cooperation relationship in (N, L') if and only if i and j have cooperation relationship in (N, L). A *basic cooperation graph* of (N, L) is a minimal subgraph (N, L') satisfying the cooperation relationship invariance. The set of all basic cooperation graphs of (N, L) is denoted by $R(N, L)$. Without loss of generality, we tacitly suppose that (N, L) referred to in this paper is connected.

Definition 3.1. *An* average spanning tree solution *for any communication situation $(N, v, L) \in CS^N$ is defined by*

$$AST(N, v, L) = \frac{1}{\big|R(N, L)\big|} \sum_{(N, L') \in R(N, L)} P(N, v, L'),$$

where $P(N, v, L')$ is given in the formula (2).

For any communication situation (N, v, L), the average spanning tree solution is the expected value of the position values for all (N, v, L') satisfying $(N, L') \in R(N, L)$. Different from the position value, from an economical view the average spanning tree solution considers that not every link is indispensable in the process of cooperation, i.e., the cooperation relationship between any two nodes may be still guaranteed through intermediaries after deleting some communication links. It is noted that only in all of the basic cooperation graphs, every of communication links is equally decisive to ensure the cooperation relationship invariance. When (N, L) is cycle-free, the average spanning tree solution coincides with the position value.

Remark 3.2. *It is apparent that any connected graph (N, L) gathers all the spanning trees as a collection of its basic cooperation graphs, i.e., $R(N, L) = ST(N, L)$. Therefore, the average spanning tree solution has an alternative form*

$$AST(N, v, L) = \frac{1}{\big|ST(N, L)\big|} \sum_{(N, L') \in ST(N, L)} P(N, v, L').$$

Example 3.3. *Four cities $N = \{1, 2, 3, 4\}$ want to build cables cooperatively in order that any two cities can communication each other. However, due to geographical reasons, it is not suitable for laying the cable between city 2 and city 4, city 3 and city 4. Given a cooperative game (N, v), where $v(\{i\}) = 0$ for any $i \in \{1, 2, 3, 4\}$, $v(\{1, 2\}) =$*

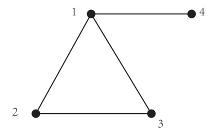

Figure 1: Graph (N, L)

$v(\{1,3\}) = v(\{1,4\}) = v(\{2,3\}) = 200$ $v(\{1,2,3\}) = 400$, $v(\{1,2,4\}) = 600$, $v(\{1,3,4\}) = 800$, $v(\{1,2,3,4\}) = 1000$ and $v(S)$ for any $S \subseteq N$ represents the surplus created by cities in coalition S. This scenario can be simply modeled by a communication situation (N, v, L) where (N, L) is depicted in Figure 1. The position value for (N, v, L) is

$$P(N, v, L) = (425, 166\frac{2}{3}, 183\frac{1}{3}, 225).$$

Considering the high expenditures to build cables, these four cities are designed to minimize the total cost of laying cables. In fact, all of the networks induced by spanning trees of a communication structure are the minimal subgraphs to keep the original cooperation relationship between players unchanged. Figure 2 shows three spanning trees of (N, L). By simple calculations, we derive the position values associated to the three spanning trees of (N, L),

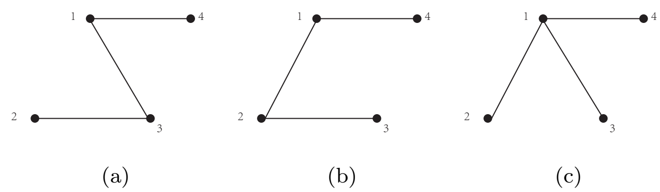

Figure 2: Three Spanning Trees of (N, L)

$$P^{(a)}(N, v, L) = (400, 100, 300, 200), \quad P^{(b)}(N, v, L) = (366\frac{2}{3}, 316\frac{2}{3}, 133\frac{1}{3}, 183\frac{1}{3}), \quad P^{(c)}(N, v, L) = (500, 116\frac{2}{3}, 166\frac{2}{3}, 216\frac{2}{3}).$$

Therefore, the final payoff vector for these four cities is

$$AST(N, v, L) = \frac{1}{3}\big[P^{(a)}(N, v, L) + P^{(b)}(N, v, L) + P^{(c)}(N, v, L)\big] = (422\frac{2}{9}, 177\frac{7}{9}, 200, 200).$$

It is noted that the link $\{1,4\}$ receives more in the position value than in the average spanning tree solution because only deleting it will definitely affect the gain of payoff 1000. However, the average spanning tree solution dilutes the powers of players 1, 4 and relatively increases the powers of players 2, 3 compared to the position value since every link in the basic cooperation graph $(N, L') \in ST(N, L)$ has the same status, i.e., the breach of any link in L' will destroy the cooperation relationship invariance.

We now state the following axioms for an allocation rule ψ over CS^N.

Axiom c1 (*Efficiency*). For any $(N, v, L) \in CS^N$, it holds that $\sum_{i \in N} \psi_i(N, v, L) = v(N)$.

Axiom c2 (*Linearity*). For all $a, b \in \mathbb{R}$ and (N, v, L), $(N, w, L) \in CS^N$, we have $\psi(N, av + bw, L) = a\psi(N, v, L) + b\psi(N, w, L)$.

Axiom c3 (*Cycle-free unanimity game degree property*). When T is a connected coalition of cycle-free (N, L), for any $(N, v, L) \in CS^N$, there exists $\alpha \in \mathbb{R}_+$ such that $\psi(N, u_T, L) = \alpha d\big(N, L(T)\big)$.

The cycle-free unanimity game degree property implies that if T is a connected coalition of cycle-free (N, L), the final allocation for (N, u_T, L) is proportional to the degree vector of the subgraph $\big(N, L(T)\big)$.

Axiom c4 [15] (*Cycle-free connectedness*). For any (N, v, L), $(N, w, L) \in CS^N$, it holds that $\psi(N, v, L) = \psi(N, w, L)$, when (N, L) is cycle-free and $v^L = w^L$.

The cycle-free connectedness property shows that the payoff for a cycle-free communication situation only depends on the worth of connected coalitions.

For any communication structure, deleting some links does not influence the cooperation relationship between players and more communication links means more maintenance costs. Therefore, all players are rational to establish communication connections with others through a communication network which is a minimal subgraph with respect to the numbers of links to ensure the cooperation relationship invariance, namely, a basic cooperation graph. This leads us to consider the following property.

Axiom c5 (*Average spanning tree property*). For any $(N,v,L)\in CS^N$, we have

$$\psi(N,v,L)=\frac{1}{\big|ST(N,L)\big|}\sum_{(N,L')\in ST(N,L)}\psi(N,v,L').$$

The average spanning tree property states that the allocation for any $(N,v,L)\in CS^N$ is exactly the expected value of all allocations for cooperative games associated with basic cooperation graph $(N,L')\in ST(N,L)$.

Remark 3.4. *Superfluous link property [3] says that for any $(N,v,L)\in CS^N$, $\psi(N,v,L)=\psi(N,v,L\backslash\{l\})$ if $r^v(E)=r^v(E\cup\{l\})$ for all $l\in L$, $E\subseteq L$. It seems that both of average spanning tree property and superfluous link property are about deletion of links. However, the superfluous link property focuses on deleting link which has no contribution on the profit of any link set, while the average spanning tree property emphasizes deleting links which are not necessary to maintain cooperation relationship invariance.*

Definition 3.5. *A graph (N,L) is an* independent cycle graph *if the intersection of nodes involved in any two cycles $(i_1,i_2,\dots,i_{k+1})$ and $(j_1,j_2,\dots,j_{m+1})$ at most has one element, i.e., $\big|\{i_1,i_2,\dots,i_{k+1}\}\cap\{j_1,j_2,\dots,j_{m+1}\}\big|\in\{0,1\}$.*

Denote by ICS^N the set of all communication situations (N,v,L) with (N,L) being an independent cycle graph.

Axiom c5' (*Recurrence deleting cycle property*). For any $(N,v,L)\in ICS^N$ and a cycle $\mathcal{C}$ in (N,L), it holds that

$$\psi(N,v,L)=\sum_{l\in\mathcal{C}}\frac{\Big|ST\big(N,L\backslash\{l\}\big)\Big|}{\big|ST(N,L)\big|}\psi\big(N,v,L\backslash\{l\}\big).$$

The recurrence deleting cycle property is a special case of average spanning tree property over ICS^N, saying that $\big|ST(N,L)\big|$ times of the allocation for (N,v,L) is equal to the sum of $\Big|ST\big(N,L\backslash\{l\}\big)\Big|$ times of the allocation for $\big(N,v,L\backslash\{l\}\big)$ over all l belonging to a cycle $\mathcal{C}$ in (N,L).

Theorem 3.6. (1) *The average spanning tree solution is the unique allocation rule over CS^N satisfying Axioms* c1-c5.
(2) *The average spanning tree solution is the unique allocation rule over ICS^N satisfying Axioms* c1-c4 *and* c5'.

Proof. (1) It is easy to verify that the average spanning tree solution satisfies all Axioms $c1$-$c5$. Therefore, it suffices to show the uniqueness. Let ψ be an allocation satisfying these five axioms, due to the linearity and average spanning tree property, in fact we only need to consider the communication situation (N,u_T,L) where (N,L) is cycle-free and (N,u_T) is an unanimity game with $|T|\geq 2$. When T is connected in cycle-free graph (N,L), it follows from the efficiency and cycle-free unanimity game degree property that $\psi_i(N,u_T,L)=\frac{d_i\big(N,L(T)\big)}{\sum_{i\in T}d_i\big(N,L(T)\big)}$ for any $i\in N$. Suppose T is not connected in cycle-free graph (N,L) and let $\mathcal{T}$ be a collection of connected subsets that contain T, we know that there exist numbers $\delta_S, S\in\mathcal{T}$ such that $(u_T)^L=\sum_{S\in\mathcal{T}}\delta_S u_S$ from Hamiache [5]. Moreover, since $(u_S)^L=u_S$ for any connected subset S in (N,L), we obtain that $(u_T)^L=(\sum_{S\in\mathcal{T}}\delta_S u_S)^L$. Along with the linearity and cycle-free connectedness, we then have that $\psi(N,u_T,L)=\sum_{S\in\mathcal{T}}\delta_S\psi(N,u_S,L)$ is uniquely determined.

(2) For any $(N,v,L)\in ICS^N$, we denote the set of all cycles in (N,L) by $\{\mathcal{C}_1,\mathcal{C}_2,\dots,\mathcal{C}_t\}$. By the recurrence deleting cycle property, we have the following calculation result

$$\begin{aligned}\psi(N,v,L)&=\sum_{l_1\in\mathcal{C}_1}\frac{\Big|ST\big(N,L\backslash\{l_1\}\big)\Big|}{\big|ST(N,L)\big|}\psi\big(N,v,L\backslash\{l_1\}\big)=\sum_{l_1\in\mathcal{C}_1}\sum_{l_2\in\mathcal{C}_2}\frac{|ST(N,L\backslash\{l_1\})|}{|ST(N,L)|}\frac{\big|ST(N,L\backslash\{l_1,l_2\})\big|}{\big|ST(N,L\backslash\{l_1\})\big|}\psi\big(N,v,L\backslash\{l_1,l_2\}\big)\\&=\sum_{l_1\in\mathcal{C}_1}\sum_{l_2\in\mathcal{C}_2}\cdots\sum_{l_t\in\mathcal{C}_t}\frac{|ST(N,L\backslash\{l_1\})|}{|ST(N,L)|}\frac{\big|ST(N,L\backslash\{l_1,l_2\})\big|}{\big|ST(N,L\backslash\{l_1\})\big|}\cdots\frac{\big|ST(N,L\backslash\{l_1,l_2,\dots,l_t\})\big|}{\big|ST(N,L\backslash\{l_1,l_2,\dots,l_{t-1}\})\big|}\psi\big(N,v,L\backslash\{l_1,l_2,\dots,l_t\}\big)\\&=\sum_{l_1\in\mathcal{C}_1}\sum_{l_2\in\mathcal{C}_2}\cdots\sum_{l_t\in\mathcal{C}_t}\frac{\Big|ST\big(N,L\backslash\{l_1,l_2,\dots,l_t\}\big)\Big|}{\big|ST(N,L)\big|}\psi\big(N,v,L\backslash\{l_1,l_2,\dots,l_t\}\big)=\frac{1}{\big|ST(N,L)\big|}\sum_{(N,L')\in ST(N,L)}\psi(N,v,L'),\end{aligned}$$

where the last "=" is true because $\big(N,v,L\backslash\{l_1,l_2,\dots,l_t\}\big)$ is a basic cooperation graph of (N,L) for any $l_1\in\mathcal{C}_1,l_2\in\mathcal{C}_2$, $\dots,l_t\in\mathcal{C}_t$. The remaining proof is omitted and refers to part (1). □

The independence proof of Axioms $c1$-$c5$ is shown as follows.

1. The allocation rule ψ^1 for any $(N,v,L) \in CS^N$, given by $\psi^1(N,v,L) = 0$ satisfies Axioms $c1$-$c5$ except for Axiom $c1$.

2. The allocation rule ψ^2 for any $(N,v,L) \in CS^N$ and $i \in N$, given by

$$\psi_i^2(N,v,L) = \begin{cases} \frac{1}{|ST(N,L)|} \sum\limits_{(N,L') \in ST(N,L)} P_i(N,v,L'), & \text{if } v = u_T \ (T \text{ is connected}), \\ \frac{1}{n} v(N), & \text{otherwise} \end{cases}$$

satisfies Axioms $c1$-$c5$ except for Axiom $c2$.

3. The allocation rule ψ^3 for any $(N,v,L) \in CS^N$ and $i \in N$, given by, $\psi_i^3(N,v,L) = \frac{1}{n} v(N)$ satisfies Axioms $c1$-$c5$ except for Axiom $c3$.

4. The allocation rule ψ^4 for any $(N,v,L) \in CS^N$, given by,

$$\psi^4(N,v,L) = \frac{1}{|ST(N,L)|} \sum_{(N,L') \in ST(N,L)} \sum_{T \subseteq N} \Delta_v(T) \frac{d\big(N, L'(T)\big)}{\sum\limits_{i \in N} d_i\big(N, L'(T)\big)}$$

satisfies Axioms $c1$-$c5$ except for Axiom $c4$.

5. The allocation rule ψ^5 for any $(N,v,L) \in CS^N$, given by, $\psi^5(N,v,L) = P(N,v,L')$, where $(N,L') \in ST(N,L)$ satisfies Axioms $c1$-$c5$ except for Axiom $c5$.

Proposition 3.7. *For any $(N,v,L) \in CS^N$, if the game (N,v) is convex,*

$$AST(N,v,L) \in C\Big(\frac{1}{|ST(N,L)|} \sum_{(N,L') \in ST(N,L)} v^{L'}\Big).$$

Proof. Because $(N,L') \in ST(N,L)$ is cycle-free and game (N,v) is convex, we know that $P(N,v,L') \in C(v^{L'})$ according to the Theorem 4 in [16]. Hence, it is easy to get that $AST(N,v,L) \in C\big(\frac{1}{|ST(N,L)|} \sum\limits_{(N,L') \in ST(N,L)} v^{L'}\big)$ from Definition 3.1. □

4 A Fuzzy Average Spanning Tree Solution

Li et al. [6] proposed the position value for communication situations with fuzzy coalition by a graph game in multilinear extension form, which is defined to be $\hat{v^L}(U) = \sum\limits_{T \subseteq supp(U)} \prod\limits_{i \in T} U_i \prod\limits_{i \in supp(U) \setminus T} (1 - U_i) v^L(T)$, for any $(U,v,L) \in CSF^N$. Let $U = e^S$, it is observed that $\hat{v^L}(U) = v^L(S)$ for any $S \subseteq N$. A probabilistic explanation of $\hat{v^L}(U)$ can be given. If each player $i \in supp(U)$ joins in a coalition $S \subseteq supp(U)$ with a probability U_i of independence, then the probability of formation of coalition S is $\prod\limits_{i \in S} U_i \prod\limits_{i \in supp(U) \setminus S} (1 - U_i)$ and therefore $\hat{v^L}(U)$ can be thought as the expected value of the worths of all subsets $S \subseteq supp(U)$ in the crisp graph game (N, v^L).

Given $(U,v,L) \in CSF^N$, in the later part of this article, for notational convenience we default to $supp(U) = N$.

Definition 4.1. [6] *A position value for any $(U,v,L) \in CSF^N$ is $P_i(U,v,L) = \frac{1}{2} \sum\limits_{l \in L_i} Sh_l(L, r^U)$, for any $i \in N$, where $r^U(E) = \hat{v^E}(U)$.*

The position value $P(U,v,L)$ can also be expressed by another form when (N,L) is cycle-free, i.e.,

$$P(U,v,L) = \sum_{S \subseteq N} \Delta_{\hat{v^L}}(U_S) \frac{d\big(N, L(S)\big)}{\sum\limits_{i \in N} d_i\big(N, L(S)\big)},$$

where U_S is a fuzzy coalition with $(U_S)_i = \begin{cases} U_i, & \text{if } i \in S, \\ 0, & \text{otherwise} \end{cases}$ and $\Delta_{\hat{v^L}}(U_S)$ for any $S \subseteq N$ is the Harsanyi dividend of fuzzy coalition U_S in the graph game $(U, \hat{v^L})$ with

$$\Delta_{\hat{v^L}}(U_S) = \sum_{S' \subseteq S} (-1)^{|S|-|S'|} \sum_{M \subseteq S'} \prod_{i \in M} U_i \prod_{i \in S' \setminus M} (1 - U_i) v^L(M).$$

In the position value, the Harsanyi dividend of fuzzy coalition U_S in $(U, \hat{v^L})$ for any $S \subseteq N$ is divided using the the degree vector $d\big(N, L(S)\big)$ as the power measure. In this section, we focus on the fuzzy version of crisp average spanning tree solution.

Definition 4.2. *A* fuzzy average spanning tree solution *for any* $(U, v, L) \in CSF^N$ *is defined by*

$$AST(U, v, L) = \frac{1}{\big|ST(N, L)\big|} \sum_{(N, L') \in ST(N, L)} P(U, v, L'),$$

where $P(U, v, L')$ *is given in the Definition 4.1.*

Now we provide several axioms for an allocation rule Ψ over CSF^N which are approximate extensions for the average spanning tree solution.
Axiom f1 (*Cycle-free efficiency*). For any $(U, v, L) \in CSF^N$ with (N, L) being cycle-free, it holds that

$$\sum_{i \in N} \Psi_i(U, v, L) = \hat{v^L}(U).$$

Axiom f2 (*Linearity*). For all $a, b \in \mathbb{R}$ and (U, v, L), $(U, w, L) \in CSF^N$, we have that

$$\Psi(U, av + bw, L) = a\Psi(U, v, L) + b\Psi(U, w, L).$$

Axiom f3 (*Cycle-free unanimity game degree property*). For any $(U, v, L) \in CSF^N$, when T is a connected coalition of cycle-free (N, L), there exists a real number $p(T) \in [0, 1]$ only related to T and U such that

$$\Psi(U, u_T, L) = \sum_{S \supseteq T} p(T) \frac{d\big(N, L(S)\big)}{\sum\limits_{i \in N} d_i\big(N, L(S)\big)},$$

Axiom f4 (*Cycle-free connectedness*). For any (U, v, L), $(U, w, L) \in CSF^N$, it holds that $\Psi(U, v, L) = \Psi(U, w, L)$ when (N, L) is cycle-free and $v^L = w^L$.
Axiom f5 (*Average spanning tree property*). For any $(U, v, L) \in CSF^N$, we have that

$$\Psi(U, v, L) = \frac{1}{\big|ST(N, L)\big|} \sum_{(N, L') \in ST(N, L)} \Psi(U, v, L').$$

Axiom f5' (*Recurrence deleting cycle property*). For any $(U, v, L) \in ICSF^N$ and a cycle $\mathcal{C}$ in (N, L), it holds that

$$\Psi(U, v, L) = \sum_{l \in \mathcal{C}} \frac{\big|ST(N, L \backslash \{l\})\big|}{\big|ST(N, L)\big|} \Psi\big(U, v, L \backslash \{l\}\big).$$

Denote by $ICSF^N$ the set of $(U, v, L) \in CSF^N$ such that the associated crisp communication situation $(N, v, L) \in ICS^N$. Next we will give the axiomatization of the fuzzy average spanning tree solution over CSF^N and $ICSF^N$ by the extended axioms above. In order to explain the main theorem clearly, we first show a lemma.

Lemma 4.3. *For any* $(U, v, L) \in CSF^N$*, if* T *is a connected coalition of* (N, L) *and* $v = u_T$*, then*

$$\Delta_{\hat{v^L}}(U_S) = \prod_{j \in S} U_j \prod_{j \in N \setminus S} (1 - U_j), \quad \textit{for any } S \supseteq T.$$

Proof. First, we show that for any $K \subseteq N\backslash T$ the equation

$$\sum_{T\subseteq M\subseteq T\cup K}\prod_{i\in M}U_i\prod_{i\in T\cup K\backslash M}(1-U_i)=\prod_{i\in T}U_i \tag{3}$$

holds by induction on $|K|$. When $|K| = 0$ or 1, the equality holds obviously. Suppose the equation (3) holds when $|K| = k'$. When $|K| = k' + 1$, let $K = K' \cup i_0$ with $|K'| = k', i_0 \in N\backslash K'$, then

$$\begin{aligned}
& \sum_{T\subseteq M\subseteq T\cup K'\cup i_0}\prod_{i\in M}U_i\prod_{i\in T\cup K\backslash M}(1-U_i)\\
= & \sum_{T\subseteq M\subseteq T\cup K'}\prod_{i\in M}U_i\prod_{i\in (T\cup K)\backslash M}(1-U_i)+\sum_{T\cup i_0\subseteq M\subseteq T\cup K'\cup i_0}\prod_{i\in M}U_i\prod_{i\in (T\cup K)\backslash M}(1-U_i)\\
= & \prod_{i\in T}U_i\prod_{i\in i_0}(1-U_i)+\sum_{T\subseteq M\backslash i_0\subseteq T\cup K'}\prod_{i\in M}U_i\prod_{i\in T\cup K\backslash M}(1-U_i)\\
= & \prod_{i\in T}U_i(1-U_{i_0})+\sum_{T\subseteq M\backslash i_0\subseteq T\cup K'}\prod_{i\in M\backslash i_0}U_i\prod_{i\in (T\cup K')\backslash (M\backslash i_0)}(1-U_i)\frac{U_{i_0}}{1-U_{i_0}}\\
= & \prod_{i\in T}U_i(1-U_{i_0})+U_{i_0}\prod_{i\in T}U_i\\
= & \prod_{i\in T}U_i.
\end{aligned}$$

Hence, the equation (3) is true.

Second, when T is a connected coalition of (N, L) and $v = u_T$, let $S = T \cup K$, $K \subseteq N\backslash T$, $S' = T \cup K'$, $|K| = k$, $|K'| = k'$, then for all $S \supseteq T$, we have

$$\begin{aligned}
& \Delta_{\hat{v^L}}(U_S)\\
= & \sum_{S'\subseteq S}(-1)^{|S|-|S'|}\sum_{M\subseteq S'}\prod_{i\in M}U_i\prod_{i\in S'\backslash M}(1-U_i)v^L(M)\\
= & \sum_{K'\subseteq K}(-1)^{k-k'}\sum_{T\subseteq M\subseteq T\cup K'}\prod_{i\in M}U_i\prod_{i\in T\cup K'\backslash M}(1-U_i)\\
= & (-1)^k\prod_{i\in T}U_i\prod_{i\in K'}(1-U_i)+\sum_{i_1\in K}(-1)^{k-1}\sum_{T\subseteq M\subseteq T\cup i_1}\prod_{i\in M}U_i\prod_{i\in (T\cup i_1)\backslash M}(1-U_i)\\
& +\sum_{\{i_1,i_2\}\subseteq K}(-1)^{k-2}\sum_{T\subseteq M\subseteq T\cup\{i_1,i_2\}}\prod_{i\in M}U_i\prod_{i\in (T\cup\{i_1,i_2\})\backslash M}(1-U_i)+\cdots\\
& +\sum_{\{i_1,i_2,\ldots,i_{k-1}\}\subseteq K}(-1)^1\sum_{T\subseteq M\subseteq T\cup\{i_1,i_2,\ldots,i_{k-1}\}}\prod_{i\in M}U_i\prod_{i\in (T\cup\{i_1,i_2,\ldots,i_{k-1}\})\backslash M}(1-U_i)\\
& +\sum_{\{i_1,i_2,\ldots,i_k\}\subseteq K}(-1)^0\sum_{T\subseteq M\subseteq T\cup\{i_1,i_2,\ldots,i_k\}}\prod_{i\in M}U_i\prod_{i\in (T\cup\{i_1,i_2,\ldots,i_k\})\backslash M}(1-U_i)\\
\overset{*}{=} & \prod_{i\in T}U_i[(-1)^k+\sum_{i_1\in K}(-1)^{k-1}+\sum_{\{i_1,i_2\}\subseteq K}(-1)^{k-2}+\cdots+\sum_{\{i_1,i_2,\ldots,i_{k-1}\}\subseteq K}(-1)^1+1]\\
= & \prod_{i\in T}U_i
\end{aligned}$$

where the "$\overset{*}{=}$" part is true by applying the equation (3).

The proof of Lemma 4.3 is completed. □

Theorem 4.4. (1) *The fuzzy average spanning tree solution is the unique allocation rule over CSF^N satisfying Axioms $f1$-$f5$.*
(2) *The fuzzy average spanning tree solution is the unique allocation rule over $ICSF^N$ satisfying Axioms $f1$-$f4$ and $f5'$.*

Proof. (1) For any $(U, v, L) \in CSF^N$, we first examine that the fuzzy average spanning solution satisfies all five axioms. Axiom $f1$. Put a crisp cooperative game (N, w^U) by $w^U(S) = \hat{v^L}(U_S)$, we have that $\Delta_{w^U}(S) = \Delta_{\hat{v^L}}(U_S)$ according to Definition 4.1. In addition, equation (1) implies $w^U(N) = \sum_{S\subseteq N}\Delta_{w^U}(S)$. So, $\hat{v^L}(U) = \sum_{S\subseteq N}\Delta_{\hat{v^L}}(U_S)$. From the Definition 4.2, we deduce that when (N, L) is cycle-free, $\sum_{i\in N} AST_i(U, v, L) = \sum_{i\in N} P_i(U, v, L) = \sum_{S\subseteq N}\Delta_{\hat{v^L}}(U_S) = \hat{v^L}(U)$.
Axiom $f2$. For any $(U, v, L), (U, w, L) \in CSF^N$, $a, b \in \mathbb{R}$, since $\Delta_{\hat{(av+bw)^L}}(U_S) = a\Delta_{\hat{v^L}}(U_S) + b\Delta_{\hat{w^L}}(U_S)$ for all $S \subseteq N$, we have that $AST(U, av + bw, L) = aAST(U, v, L) + bAST(U, w, L)$ from Definition 4.2.
Axiom $f3$. When T is a connected coalition of cycle-free (N, L), we choose $p(T)$ satisfying that $p(T) = \prod_{i\in T} U_i$. Therefore, according to the Lemma 4.3 and Definition 4.1, $AST(U, u_T, L) = P(U, u_T, L) = \sum_{S\supseteq T} p(T)\frac{d\big(N,L(S)\big)}{\sum_{i\in N} d_i\big(N,L(S)\big)}$. Axiom $f4$.

If (N,L) is cycle-free and $v^L = w^L$, we easily have $\Delta_{\hat{v^L}}(S) = \Delta_{\hat{w^L}}(S)$ for all $S \subseteq N$ and then $AST(U,v,L) = AST(U,w,L)$.

Axiom $f5$. According to Definition 4.2, we have that $AST(U,v,L') = P(U,v,L')$ for any $(N,L') \in ST(N,L)$ and then

$$AST(U,v,L) = \frac{1}{\left|ST(N,L)\right|} \sum_{(N,L')\in ST(N,L)} AST(U,v,L').$$

Now, it remains to prove the uniqueness part. Let Ψ be any allocation rule satisfying the Axioms $f1$-$f5$. By the Axioms $f2$ and $f5$, in fact we only need to prove that $\Psi(U,u_T,L)$ with $|T| \geq 2$ is unique when (N,L) is cycle-free. Now we suppose that (N,L) is cycle-free.

Case 1. When T is connected in (N,L), from Axiom $f3$ we know that there exists a $p(T)$ such that

$$\Psi(U,u_T,L) = \sum_{S\supseteq T} p(T) \frac{d\big(N,L(S)\big)}{\sum\limits_{i\in N} d_i\big(N,L(S)\big)}.$$

Then by simplification we have $\Psi(U,u_T,L) = p(T) \sum\limits_{S\supseteq T} \frac{d\left(N,L(S)\right)}{\sum\limits_{i\in N} d_i\left(N,L(S)\right)}$. Together with the Axiom $f1$, $p(T)$ is immediately determined.

Case 2. When T is not connected in (N,L), let $\mathcal{T}$ be a collection of connected coalitions in (N,L) that contain T and δ_S, $S \in \mathcal{T}$, be numbers such that $(u_T)^L = \sum\limits_{S\in\mathcal{T}} \delta_S u_S = (\sum\limits_{S\in\mathcal{T}} \delta_S u_S)^L$. From Axioms $f2$ and $f4$, we get that $\Psi(U,u_T,L) = \sum\limits_{S\in\mathcal{T}} \delta_S \Psi(U,u_S,L)$ which belongs to the case 1 and then is uniquely determined.

The part (1) is proved.

(2) The proof of this part is similar to the Theorem 3.6. So we omit it here. □

Now we illustrate that the Axioms $f1$-$f5$ are independent.

1. The allocation rule Ψ^1 defined, for every $(U,v,L) \in CSF^N$, by $\Psi^1(U,v,L) = 0$ satisfies Axioms $f1$-$f5$ except for cycle-free efficiency.

2. The allocation rule Ψ^2 defined, for every $(U,v,L) \in CSF^N$ and $i \in N$, by

$$\Psi_i^2(U,v,L) = \begin{cases} \frac{1}{\left|ST(N,L)\right|} \sum\limits_{(N,L')\in ST(N,L)} P_i(U,v,L'), & \text{if } v = u_T \ (T \text{ is connected}), \\ \frac{1}{\left|ST(N,L)\right|} \sum\limits_{(N,L')\in ST(N,L)} \frac{1}{n} \hat{v^{L'}}(U), & \text{otherwise} \end{cases}$$

   satisfies Axioms $f1$-$f5$ except for linearity.

3. The allocation rule Ψ^3 defined, for every $(U,v,L) \in CSF^N$ and $i \in N$, by

$$\Psi_i^3(U,v,L) = \frac{1}{\left|ST(N,L)\right|} \sum_{(N,L')\in ST(N,L)} \frac{1}{n} \hat{v^{L'}}(U)$$

   satisfies Axioms $f1$-$f5$ except for cycle-free unanimity game degree property.

4. The allocation rule Ψ^4 defined, for every $(U,v,L) \in CSF^N$ and $i \in N$, by

$$\Psi_i^4(U,v,L) = \begin{cases} \frac{1}{\left|ST(N,L)\right|} \sum\limits_{(N,L')\in ST(N,L)} \sum\limits_{T\subseteq N} \Delta_v(T) P_i(U,u_T,L), & T \text{ is connected} \\ \frac{1}{\left|ST(N,L)\right|} \sum\limits_{(N,L')\in ST(N,L)} \sum\limits_{T\subseteq N} \Delta_v(T) \frac{1}{n} \sum\limits_{i\in N} P_i(U,u_T,L), & T \text{ is not connected} \end{cases}$$

   satisfies Axioms $f1$-$f5$ except for cycle-free connectedness.

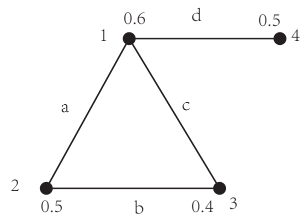

Figure 3: Graph (U, L)

5. The allocation rule Ψ^5 defined, for every $(U, v, L) \in CSF^N$, by

$$\Psi^5(U, v, L) = \begin{cases} P(U, v, L), & \text{if } (N, L) \text{ is cycle-free,} \\ 0, & \text{otherwise} \end{cases}$$

satisfies Axioms $f1$-$f5$ except for average spanning tree property.

Proposition 4.5. *For any $(U, v, L) \in CSF^N$, if game (N, v) is convex,*

$$AST(U, v, L) \in \tilde{C}(\frac{1}{|ST(N, L)|} \sum_{(N,L') \in ST(N,L)} \hat{v^{L'}}).$$

Proof. We get that (L', r^v) is convex since game (N, v) is convex and (N, L') is cycle-free from the Theorem 4 in [16]. Further, because $r^U(E) = \hat{v^E}(U) = \sum_{S \subseteq N} \prod_{i \in S} U_i \prod_{i \in N \setminus S} (1 - U_i) v^E(S)$, we easily have that (L', r^U) is convex. So,

$$\sum_{i \in N} AST_i(U, v, L) = \frac{1}{|ST(N, L)|} \sum_{(N,L') \in ST(N,L)} \hat{v^{L'}}(U),$$

and for all $S \subseteq supp(U)$,

$$\sum_{i \in S} AST_i(U, v, L) = \frac{1}{|ST(N, L)|} \sum_{(N,L') \in ST(N,L)} \sum_{i \in S} P_i(N, v, L') \geq \frac{1}{|ST(N, L)|} \sum_{(N,L') \in ST(N,L)} \hat{v^{L'}}(U_S).$$

Hence, $AST(U, v, L) \in \tilde{C}(\frac{1}{|ST(N,L)|} \sum_{(N,L') \in ST(N,L)} \hat{v^{L'}})$. □

Remark 4.6. *If players with uncertain participation levels have the Choquet behaviour, i.e., they try the best to form the biggest coalition, we can modify the graph game in multilinear extension form by graph game with Chpquet integral form, i.e., $\hat{v^L}(U) = \sum_{k=1}^{m} [h_k - h_{k-1}] v^L([U]_{h_k})$. Furthermore, it is easily observed that the fuzzy average tree solution with Chpquet integral form has the similar results with Theorem 4.4 and Proposition 4.5.*

Example 4.7. *Consider the situation in Example 3.3 with the only change: here all cities do not contribute all their resources into this cooperation. Let $U = (0.6, 0.5, 0.4, 0.5)$ be the fuzzy coalition with the coordinate being the participation level of the corresponding city. This communication situation with uncertain participation levels of players can be modeled by (U, v, L) where (U, L) is painted in Figure 3.*

We calculate the link game (L, r^U)

$$r^U(a) = 60,\ r^U(b) = 40,\ r^U(c) = 48,\ r^U(d) = 60,$$
$$r^U(a, b) = 94,\ r^U(a, c) = 102,\ r^U(b, c) = 88,\ r^U(a, d) = 150,$$
$$r^U(b, d) = 100,\ r^U(c, d) = 156,\ r^U(a, b, c) = 124,\ r^U(a, b, d) = 190,$$
$$r^U(a, c, d) = 234,\ r^U(b, c, d) = 196,\ r^U(a, b, c, d) = 250,$$

and derive the position value for (U, v, L), $P(U, v, L) = (109, 47, 46, 48)$. According to the Definition 4.1. Finally, it follows from Definition 4.2 that the payoff vector $AST(U, v, L)$ for these four companies is $AST(U, v, L) = (90\frac{1}{6}, 37\frac{1}{6}, 36\frac{1}{3}, 43)$. Since removing any one of communication links $\{1,2\}$, $\{1,3\}$ and $\{2,3\}$ will not damage the the cooperation among all cities, the importance of communication link $\{1,4\}$ in the fuzzy average spanning tree solution is weakened and then the city 4 will definitely receive less compared to the position value.

5 Conclusions

In this paper, we suppose that any two players are regarded as possessing the cooperation relationship as long as they can communicate each other by some way directly or indirectly. In fact, only in the basic cooperation graphs, every of links is absolutely necessary in keeping the cooperation relationship unchanged. We have characterized a modified position value and its fuzzification by defining a restricted game in multilinear extension form for communication situations with fuzzy coalition. Sometimes, in addition to the uncertainty of participation levels of players, the communication links between them are also limited by a probability, which may be interpreted as the possibility of bilateral exchanges. In order to deal with such issues, we need a further study.

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