ON PROJECTIVE L- MODULES

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ABSTRACT**.** The concepts of free modules, projective modules, injective modules and the like form an important area in module theory. The notion of free fuzzy modules was introduced by Muganda as an extension of free modules in the fuzzy context. Zahedi and Ameri introduced the concept of projective and injective L-modules. In this paper we give an alternate definition for projective L-modules. We prove that every free L-module is a projective L-module. Also we prove that if $\mu \in L(P)$ is a projective L-module, and if $0 \to \eta \xrightarrow{f} \nu \xrightarrow{g} \mu \to 0$ is a short exact sequence of L-modules then $\eta \oplus \mu > \nu$. Further it is proved that if $\mu \in L(P)$ is a projective L-module then μ is a fuzzy direct summand of a free L-module.

1. **Introduction**

Though the notion of a fuzzy set was introduced by L.A. Zadeh [11] in 1965, its application to algebraic concepts started only in 1971 when A. Rosenfeld [10] introduced fuzzy subgroups of a group. Tremendous and rapid growth of fuzzy algebraic concepts resulted in a vast literature. The book of Mordeson and Malik [7] gives an account of all these up to 1998. The notion of L-modules as an extension of classical module theory is available in this book. However many concepts are yet to be “fuzzified”.

The notion of free fuzzy modules was introduced by Muganda [8] in 1993 as an extension of free modules in the fuzzy context. The concept of a free L-module is available in [7]. In 1995 Zahedi and Ameri [12] introduced the concepts of fuzzy projective and injective modules. In section 2 of this paper, we give the essential preliminaries and in section 3 we give an alternate definition for a projective L-module and prove some results using this definition.

2. **Preliminaries**

Throughout this paper unless otherwise stated, $L(\vee, \wedge, 1, 0)$ represents a complete Brouwerian lattice with maximal element ‘1’ and minimal element ‘0’; R a ring with unity ‘1’ and M a left module over R. ‘$\vee$’ denotes the supremum and ‘$\wedge$’ the infimum in L. ‘$\subseteq$’ denotes the inclusion and ‘$\subset$’ the proper inclusion. The set of all L- subsets of M is denoted by L^M. We call L a regular lattice if $a \wedge b > 0$ $\forall\, a, b > 0$ in L.

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For $x \in M$, $a \in L$ the L-subset which takes the value a at x and 0 elsewhere is denoted by $a_{\{x\}}$:

$$a_{\{x\}}(y) = \begin{cases} a & if \;\; y = x \\ 0 & if \;\; y \neq x \end{cases}$$

In this section we review some definitions and results which will be used in the sequel. For details we refer to Mordeson and Malik [7]. For knowledge regarding lattices we refer to Birkhoff [1] and for modules to Goodearl, Warfield [2] and Hungerford [3] .

Definition 2.1. [7] For $\mu, \nu \in L^M$, we define $\mu + \nu$ and $-\mu$ as follows.
For $x \in M$, $\quad (\mu + \nu)(x) = \vee \{\mu(y) \wedge \nu(z) : y, z \in M,\ y + z = x\}$
and $\quad -\mu(x) = \mu(-x)$

Definition 2.2. [7] Let $\mu_i \in L^M$; $i \in I$ be a family of L-subsets of M. Then we define

$$\sum_{i\in I} \mu_i(x) = \vee \left\{ \wedge_{i\in I} \mu_i(x_i) : x_i \in M, i \in I, \sum_{i\in I} x_i = x \right\},$$

where in the expression $x = \sum_{i\in I} x_i$, at most finitely many x_i's are $\neq 0$.

Definition 2.3. [7] For $\mu \in L^M$ and $a \in L$ we set:
(i) $\mu^* = \{x \in M : \mu(x) > 0\}$; called the support of μ .
(ii) $\mu_a = \{x \in M : \mu(x) \geq a\}$; called the a- cut or a-level subset of μ and
$\mu_a^{>} = \{x \in M : \mu(x) > a\}$; called the strict a-cut or strict a-level subset of μ .

Definition 2.4. [7] Let f be a mapping from X into Y, and let $\mu \in L^X$ and $\nu \in L^Y$. The L-subsets $f(\mu) \in L^Y$ and $f^{-1}(\nu) \in L^X$, defined by $\forall y \in Y$,

$$f(\mu)(y) = \begin{cases} \vee\{\mu(x) : x \in X, f(x) = y\} & \text{if } f^{-1}(y) \neq \phi \\ 0 & \text{otherwise} \end{cases}$$

and $\forall x \in X$,

$$f^{-1}(\nu)(x) = \nu(f(x)$$

,

are called, respectively, the image of μ under f and the pre-image of ν under f.

Definition 2.5. [7] Let $\mu \in L^M$. Then μ is said to be an L- submodule of M if
(i) $\mu(0) = 1$
(ii) $\mu(x + y) \geq \mu(x) \wedge \mu(y)$ $\forall\, x, y \in M$
(iii) $\mu(rx) \geq \mu(x)$ $\forall\, r \in R$, $\forall\, x \in M$
By saying μ is a left L-module we mean μ is an L-submodule of some left module M over a ring R. The set of all L-submodules of M is denoted by $L(M)$.

Remark 2.6. We note from [7] that if $\mu, \eta \in L(M)$, then $\mu + \eta \in L(M)$. Also if $\mu_i \in L(M)$, $(i \in I)$, then $\sum_{i \in I} \mu_i \in L(M)$. From [4] we see that $\mu \in L(M)$ if and only if μ_a is a submodule of M $\forall\, a \in L$.

Definition 2.7. [7] Let M and N be R-modules and let $\mu \in L(M)$ and $\nu \in L(N)$.
A homomorphism f of M onto N is called a weak homomorphism of μ into ν if $f(\mu) \subseteq \nu$. If f is a weak homomorphism of μ into ν, then we say that μ is weakly homomorphic to ν and we write $\mu \sim \nu$.
A homomorphism f of M onto N is called a homomorphism of μ onto ν if $f(\mu) = \nu$. If f is a homomorphism of μ onto ν, then we say that μ is homomorphic to ν and we write $\mu \approx \nu$.
An isomorphism f of M onto N is called a weak isomorphism of μ into ν if $f(\mu) \subseteq \nu$. If f is a weak isomorphism of μ into ν, then we say that μ is weakly isomorphic to ν and we write $\mu > \nu$.
An isomorphism f of M onto N is called an isomorphism of μ onto ν if $f(\mu) = \nu$. If f is an isomorphism of μ onto ν, then we say that μ is isomorphic to ν and we write $\mu \cong \nu$.

Definition 2.8. [6] Let A_i; $i \in Z$ be a family of R-modules and let $\mu_i \in L(A_i)$. Suppose that $\ldots \xrightarrow{f_{i-1}} A_{i-1} \xrightarrow{f_i} A_i \xrightarrow{f_{i+1}} A_{i+1} \xrightarrow{f_{i+2}} \ldots$ is an exact sequence of R-modules. Then the sequence $\ldots \xrightarrow{f_{i-1}} \mu_{i-1} \xrightarrow{f_i} \mu_i \xrightarrow{f_{i+1}} \mu_{i+1} \xrightarrow{f_{i+2}} \ldots$ of L-modules is said to be exact if, for all $i \in Z$,
(i) $f_{i+1}(\mu_i) \subseteq \mu_{i+1}$ & (ii) $f_i(\mu_{i-1})(x) > 0$ if $x \in \operatorname{Ker} f_{i+1}$
and $f_i(\mu_{i-1})(x) = 0$ if $x \notin \operatorname{Ker} f_{i+1}$

Definition 2.9. [6] Let $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ be a short exact sequence of R-modules. Let $\mu \in L(A)$, $\eta \in L(B)$, and $\nu \in L(C)$. Then an exact sequence of L-modules of the form $0 \longrightarrow \mu \xrightarrow{f} \eta \xrightarrow{g} \nu \longrightarrow 0$ is called a short exact sequence of L-modules.

Definition 2.10. [3] Two short exact sequences of R-modules are said to be isomorphic if there is a commutative diagram of module homomorphisms

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow \varphi & & \downarrow \psi & & \downarrow \xi & & \\ 0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \longrightarrow & 0 \end{array}$$

such that ϕ, ψ, ξ are isomorphisms.
Definition 2.11. [6] Let

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow \varphi & & \downarrow \psi & & \downarrow \xi & & \\ 0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \longrightarrow & 0 \end{array}$$

be two isomorphic short exact sequences of R-modules with the given isomorphisms. Let $\mu \in L(A)$, $\nu \in L(B)$, $\eta \in L(C)$, $\mu' \in L(A')$, $\nu' \in L(B')$ and $\eta' \in L(C')$ be such that

$$0 \longrightarrow \mu \xrightarrow{f} \nu \xrightarrow{g} \eta \longrightarrow 0 \qquad \ldots (1)$$

and

$$0 \longrightarrow \mu' \xrightarrow{f'} \nu' \xrightarrow{g'} \eta' \longrightarrow 0 \qquad \ldots (2)$$

are short exact sequences of L-modules. Then the sequence (1) is said to be weakly isomorphic to the sequence (2) if $\varphi(\mu) \subseteq \mu'$, $\psi(\nu) \subseteq \nu'$, and $\xi(\eta) \subseteq \eta'$.
The sequence (1) is said to be isomorphic to the sequence (2) if $\varphi(\mu) = \mu'$, $\psi(\nu) = \nu'$, and $\xi(\eta) = \eta'$.

Definition 2.12. [7] Let $\mu, \eta, \nu \in L(M)$. Then μ is said to be the direct sum of η and ν if
(i) $\mu = \eta + \nu$,
(ii) $\eta \cap \nu = 1_{\{0\}}$.
In this case we write $\mu = \eta \oplus \nu$

Definition 2.13. [6] Let A and B be two R-modules; $\mu \in L(A)$, $\eta \in L(B)$. Consider the direct sum $A \oplus B$. We extend μ and η on $A \oplus B$ by taking $\mu(x) = 0$ *if* $x \notin A$ and $\eta(x) = 0$ *if* $x \notin B$, i.e. $\mu(a,b) = 0$ *if* $b \neq 0$ and $\eta(a,b) = 0$ *if* $a \neq 0$ for $(a,b) \in A \oplus B$.
Then $\mu, \eta \in L(A \oplus B)$. Moreover,

$$(\mu \cap \eta)(x) = \mu(x) \wedge \eta(x) = \begin{cases} 1 & if\ x = 0 \\ 0 & if\ x \neq 0 \end{cases}$$

Therefore $\mu + \eta$ is a direct sum and we denote it by $\mu \oplus \eta$.

Remark 2.14. Note that

$$\begin{aligned}(\mu + \eta)(a, b) &= \vee \{\mu(a_1, b_1) \wedge \eta(a_2, b_2) : (a_1, b_1), (a_2, b_2) \in A \oplus B; \\ &\qquad (a_1, b_1) + (a_2, b_2) = (a, b)\} \\ &= \mu(a, 0) \wedge \eta(0, b) \\ &= \mu(a) \wedge \eta(b).\end{aligned}$$

Definition 2.15. [7] Let $\mu_i \in L(M)$, $\forall i \in I$. Then we say that μ is the direct sum of $\{\mu_i : i \in I\}$, denoted by $\bigoplus_{i \in I} \mu_i$, if

(i) $\mu = \sum_{i \in I} \mu_i$

(ii) $\mu_j \cap \sum_{i \in I - \{j\}} \mu_i = 1_{\{0\}}$

3. Free *L*-modules and Projective *L*-modules

The notion of free fuzzy modules was introduced by Muganda [8] in 1993 and later generalized to that of free L-modules [7]. Zahedi and Ameri [12] introduced the concepts of fuzzy projective and injective modules in 1995. In this paper we give an alternate definition for projective L-modules and prove that every free L-module is a projective L-module. Also we prove that if $\mu \in L(P)$ is a projective L-module, and if $0 \to \eta \xrightarrow{f} \nu \xrightarrow{g} \mu \to 0$ is a short exact sequence of L-modules then $\eta \oplus \mu > \nu$. Further it is proved that if $\mu \in L(P)$ is a projective L-module then μ is a fuzzy direct summand of a free L-module.

Definition 3.1. [7] Let F be a free module over R on the set X with respect to the function i: $X \to F$. Let β be an L-subset of X. Let $\mu \in L(F)$. Then μ is said to be free with respect to β if $i(\beta) = \mu$ on $i(X)$ and for every module A over R, and $\eta \in L(A)$ with k: $X \to A$ and $k(\beta) = \eta$ on $k(X)$, there exists a unique homomorphism h: $F \to A$ such that $k = hi$ and $h(\mu) \subseteq \eta$.

Definition 3. 2. Let $\mu \in L^S$. Then μ is said to have the supremum property if for each subset $A \subseteq S$, there exists a y in A such that $\vee\{\mu(x) : x \in A\} = \mu(y)$.

Definition 3.3. Let P be a projective R-module and let $\mu \in L(P)$. Then μ is said to be a projective L-submodule of P if for every epimorphism of R-modules g: $A \to B$, $\eta \in L(A)$ with supremum property, $\nu \in L(B)$ with $g(\eta) = \nu$ on $g(A)$, and for every R-module homomorphism f: $P \to B$ with $f(\mu) = \nu$ on $f(P)$, there exists an R-module homomorphism h: $P \to A$ such that $gh = f$ and $h(\mu) \subseteq \eta$.

It is well known from module theory that every free R-module is a projective R-module. The same is also true in the case of L-modules as we see in the following theorem.

Theorem 3.4. *Every free L-module is a projective L-module.*

Proof. Suppose F is free on X with respect to the function i: $X \to F$ and let $\mu \in L(F)$ be free with respect to $\beta \in L^X$. We have to show that μ is a projective L-module. Consider the epimorphism of R-modules g: $A \to B$. Let $\eta \in L(A)$ satisfy the supremum property, $\nu \in L(B)$ and $g(\eta) = \nu$ on $g(A)$. Also let f: $F \to B$ be a homomorphism such that $f(\mu) = \nu$ on $f(F)$. We show that there exists an R-module homomorphism h: $F \to A$ such that $gh = f$ and $h(\mu) \subseteq \eta$.

Now for $x \in X$, $i(x) \in F$, $f(i(x)) \in B$. Since g is onto, there exists $a \in A$ such that $g(a) = f(i(x))$. Since $\eta \in L(A)$ satisfies the supremum property, there exists a_x in A such that $g(a_x) = f(i(x))$ and $\eta(a_x) = \vee\{\eta(a): a\in A,\ g(a) = f(i(x))\}$ and if $f(i(x)) = f(i(y))$, we choose $a_x=a_y$. Now consider the map $k: X\to A$ defined by $k(x)= a_x$. Since F is free on X, this extends to an R-module homomorphism $h: F\to A$ such that $hi=k$. Thus we have $(hi)(x) = k(x) = a_x\ \forall\ x\in X$ which implies $(ghi)(x) = g(a_x) = f(i(x))\ \forall\ x\in X$. Since F is free on X it follows that $gh = f$.

It remains to prove that $h(\mu)\subseteq \eta$. Since μ is a free L-module with respect to β, we have $i(\beta) = \mu$ on $i(X)$. Also we have $f(\mu) = \nu$ on $f(F)$ and $g(\eta) = \nu$ on $g(A)$. If we show that $k(\beta) = \eta$ on $k(X)$, then by the definition of a free L-module it follows that $h(\mu)\subseteq \eta$. Therefore we need only prove that $k(\beta) = \eta$ on $k(X)$.

Consider the restriction of the map $g: A\to B$ to the subset $k(X)$ of A. For convenience we denote this restriction by g itself. Then g is a one-to-one mapping from $k(X)$ onto $f(i(X))$. Therefore we can consider $g^{-1}: f(i(X))\to k(X)$. Then obviously $k = g^{-1}f\,i$. Since $g(\eta) = \nu$ on $g(A)$ we get $g(\eta) = \nu$ on $g(k(X)) = f(i(X))$. i.e. for $a_x\in k(X)$ we get

$$\nu(g(a_x)) = g(\eta)(g(a_x)) = \vee\{\eta(a): g(a)=g(a_x),\ a\in A\} = \eta(a_x)$$

Thus we have $g^{-1}(\nu)(a_x) = \eta(a_x)\ \ \forall\ a_x\in k(X)$. i.e. on $k(X)$,

$$\eta = g^{-1}(\nu) = g^{-1}(f(\mu))=g^{-1}(f(\,i(\beta)) = (g^{-1} f\ i)(\beta)$$

i.e. $\eta = k(\beta)$ on $k(X)$. This completes the proof of the theorem. □

Now we quote a theorem from our earlier work which is used to prove the subsequent theorem.

Theorem 3.5. *[6] Let* $0\to A_1 \xrightarrow{f} B \xrightarrow{g} A_2 \to 0$ *be a short exact sequence of R-modules and let* $\mu_1\in L(A_1)$, $\mu_2\in L(A_2)$, $\eta\in L(B)$ *be such that* $0\to \mu_1 \xrightarrow{f} \eta \xrightarrow{g} \mu_2 \to 0$ *is a short exact sequence of L-modules. If there is an R-module homomorphism* $h: A_2\to B$ *with* $gh= I_{A_2}$ *such that* $h(\mu_2)\subseteq\eta$, *then the sequence* $0\to\mu_1 \xrightarrow{i} \mu_1\oplus\mu_2 \xrightarrow{\pi} \mu_2\to 0$ *is weakly isomorphic to the given sequence* $0\to \mu_1 \xrightarrow{f} \eta \xrightarrow{g} \mu_2 \to 0$. *In particular* $\mu_1\oplus\mu_2 > \eta$. □

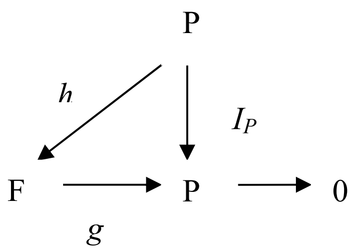
It is well known from module theory that an R-module P is projective if and only if every short exact sequence $0\to A \xrightarrow{f} B \xrightarrow{g} P \to 0$ splits so that $B\cong A\oplus P$. Also we know that P is projective if and only if there exists a free R-module F and an R-module K such that $F\cong K\oplus P$. Analogous to these, in the case of L-modules we have the following theorems.

Theorem 3.6. *Let P be a projective module and* $\mu\in L(P)$ *be a projective L-module. If* $0\to A \xrightarrow{f} B \xrightarrow{g} P \to 0$ *is a short exact sequence of R-modules and* $\eta\in L(A)$, $\nu\in L(B)$ *are such that* $0\to\eta \xrightarrow{f} \nu \xrightarrow{g} \mu \to 0$ *is a short exact sequence of L-modules then* $\eta\oplus\mu$ *is weakly isomorphic to* ν. *i.e.* $\eta\oplus\mu > \nu$.

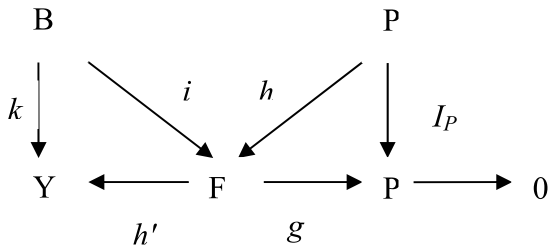
Proof. Since P is projective we see that $B \cong A \oplus P$ and the sequence $0 \to A \xrightarrow{i} A \oplus P \xrightarrow{\pi} P \to 0$ is isomorphic to $0 \to A \xrightarrow{f} B \xrightarrow{g} P \to 0$. Since $\mu \in L(P)$ is a projective L-module there exists a homomorphism h: $P \to B$ such that $gh = I_P$ and $h(\mu) \subseteq \nu$. Then by the above theorem the result follows. □

Theorem 3.7. *Let P be a projective module and $\mu \in L(P)$ be a projective L-module. Then there exists a free R-module F and a free L-module $\xi \in L(F)$ such that $\xi = \sigma \oplus \mu$ for some L-module σ.*

Proof. Since P is projective, there exists a free R-module F and an epimorphism g:$F \to P$ such that $0 \to Ker\ g \xrightarrow{\subseteq} F \xrightarrow{g} P \to 0$ is split exact so that $F \cong Ker\ g \oplus P$. Suppose F is free on B. Now consider the diagram:



Since P is projective there exists an R-module homomorphism h: $P \to F$ such that $gh = I_P$. Define $\xi \in L(F)$ by $\xi = g^{-1}(\mu)$. We will show that ξ is a free L-submodule of F. Obviously $g(\xi)(x) = \mu(x)$ $\forall$ $x \in$ $g(F) = P$; i.e. $g(\xi) = \mu$ on $g(F) = P$. Thus we have $I_P(\mu) = \mu$ on $I_P(P) = P$ and $g(\xi) = \mu$ on $g(F) = P$. Therefore, since $\mu \in L(P)$ is projective, h: $P \to F$ is such that $gh = I_P$ and $h(\mu) \subseteq \xi = g^{-1}(\mu)$. Take $\beta = i^{-1}(\xi)$ so that $i(\beta) = \xi$ on $i(B)$. Let Y be any R-module and $\eta \in L(Y)$, and let k: $B \to Y$ be a given map. Since F is free on B, there exists an R-module homomorphism h': $F \to Y$ such that $h'i = k$. If $\eta \in L(Y)$ is such that $k(\beta) = \eta$ on $k(B)$, then we have to show that $h'(\xi) \subseteq \eta$. But obviously $\eta = k(\beta) = k(i^{-1}(\xi)) = (h'i)\,(i^{-1}(\xi)) = h'(\xi)$. Therefore ξ is a free L-submodule of F.



Now it remains to show that $\xi = \sigma \oplus \mu$ for some L-module σ. We have $F \cong Ker\ g \oplus P$. Define $\sigma \in L(F)$ by

$$\sigma(x) = \begin{cases} \xi(x) & \text{if } x \in Ker\ g \\ 0 & \text{if } x \notin Ker\ g \end{cases}$$

Also, we can extend the $\mu \in L(P)$ to $\mu \in L(F)$ by defining $\mu(x)=0$, if $x \notin P$. Then, for all $x \in F$, we have:

$$\begin{aligned}(\sigma + \mu)(x) &= \vee \{\sigma(y) \wedge \mu(z) : y, z \in F,\ y + z = x\} \\ &= \vee \{\xi(y) \wedge \xi(z) : y \in Ker\, g,\ z \in P;\ y + z = x\} \\ &= \xi(y+z),\quad y \in Ker\, g,\ z \in P;\ y + z = x \\ &= \xi(x) \qquad [\square\ \xi \text{ is an } L\text{-module}]\end{aligned}$$

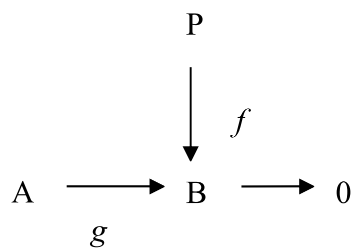
Also

$$(\sigma \cap \mu)(x) = \sigma(x) \wedge \mu(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$$

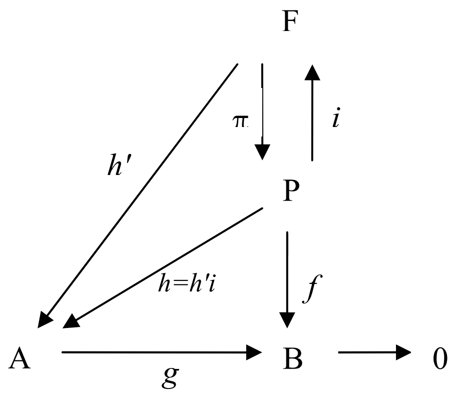
Therefore $\sigma + \mu$ is a direct sum. Thus we get $\xi = \sigma \oplus \mu$. □

Theorem 3.8. *Let P be an R-module and $\mu \in L(P)$. Let F be a free R-module and K be an R-module such that $F = K \oplus P$. If $\xi \in L(F)$ is a free L-module such that $\xi = \sigma \oplus \mu$ for some $\sigma \in L(K)$, then μ is a projective L-module.*

Proof. Consider the diagram



Let $\eta \in L(A)$ satisfies the supremum property, $\nu \in L(B)$, $g(\eta) = \nu$ on $g(A)$, $f(\mu) = \nu$ on $f(P)$. Since $F \cong K \oplus P$, we have the canonical maps $i: P \to F \cong K \oplus P$ (injection) and $\pi : F \cong K \oplus P \to P$ (projection).



Since F is free it is projective and therefore there exists an R-module homomorphism $h':F\to A$ such that $gh'=f\pi$. Consider $h=h'i$: $P\to A$. Then $gh=gh'i=(f\pi)i=f(\pi i)=f\ I_P=f$. Therefore P is projective. Now since $\xi\in L(F)$ is free it is projective. Since $\eta\in L(A)$ satisfies the supremum property, $\nu\in L(B)$, $g(\eta)=\nu$ on $g(A)$, $f(\mu)=\nu$ on $f(P)$ and since $\xi=\sigma\oplus\mu$; $F=K\oplus P$ we get $(f\pi)(\xi)=\nu$ on $(f\pi)(F)=f(P)$.
For, given $b\in(f\pi)(F)=f(P)$, we have

$$\begin{aligned}(f\pi)(\xi)(b) &= \vee\ \{\xi(y) : y\in F;\ (f\pi)(y)=b\}\\ &= \vee\ \{(\sigma\oplus\mu)(k+p):\ k\in K, p\in P;\ (f\pi)(k+p)=b\}\\ &= \vee\ \{(\sigma(k)\wedge\mu(p)\ :\ k\in K,\ p\in P;\ f(p)=b\}\\ &= \vee\ \{(\sigma(0)\wedge\mu(p) : p\in P;\ f(p)=b\}\\ &= \vee\ \{\mu(p) :\ p\in P;\ f(p)=b\}\\ &= f(\mu)(b)\\ &= \nu(b)\end{aligned}$$

Thus $(f\pi)(\xi)(b)=\nu(b)\ \forall\ b\in(f\pi)(F)=f(P)$. Therefore, since $\xi\in L(F)$ is projective, we get $h'(\xi)\subseteq\eta$ where $gh'=f\pi$. Now to prove that $\mu\in L(P)$ is projective we need only prove that $h(\mu)\subseteq\eta$. Now

$$\begin{aligned}h(\mu)(a) &= \vee\ \{\mu(x) :\ x\in P;\ h(x)=a\}\\ &= \vee\ \{\mu(x) :\ x\in P;\ (h'i)(x)=a\}\\ &= \vee\ \{\xi(i(x)) :\ x\in P;\ h'(i(x))=a\}\\ &\le \vee\ \{\xi(y) :\ y\in F;\ h'(y)=a\}\\ &= h'(\xi)(a)\\ &\le \eta(a)\end{aligned}$$

Thus we have $h(\mu)(a)\le\eta(a)\ \ \forall\ a\in A$. Therefore $\mu\in L(P)$ is projective. □

Corollary 3.9. *Let P_i $(i \in I)$ be a projective R-module and $\mu_i \in L(P_i)$ $\forall i \in I$. Then $\bigoplus_{i\in I} \mu_i$ is projective only if μ_i is projective $\forall i \in I$.*

Proof. In the above proof we used only the fact that ξ is a projective L-module and therefore the result follows by replacing ξ with $\bigoplus_{i\in I} \mu_i$, σ with $\sum_{i\in I} \mu_i$ and μ with μ_j .

□

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