

## Migrativity equations and Mayor's aggregation operators

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### Abstract

There has been a growing interest in the study of the notion of  $\alpha$ -migrativity and generalizations in recent years, and it has been investigated for families of certain operators such as t-norms, t-conorms, uninorms, nullnorms. This paper is mainly devoted to investigating the migrativity equations between semi-t-operators or semi-uninorms, and Mayor's aggregation operators. The results that we obtain are complete and different from the known ones concerning migrativity for t-norms, t-conorms, uninorms and nullnorms.

*Keywords:* Migrativity, Mayor's aggregation operators, semi-t-operators, semi-uninorms.

## 1 Introduction

Migrativity of binary operators is an important property. In recent years, there has been a growing interest in studying the notion of  $\alpha$ -migrativity and generalizations [2, 3, 9, 10, 17, 18, 19, 20, 24, 27, 28, 29, 30, 31]. The interest of this property comes from its applications, such as image processing [26], the nature of the image itself does not change when a part of the image shrunk in proportion; decision making [23], it has nothing to do with the sequence of information selection when repeated and partial information is aggregated into a whole conclusion. As is pointed out by Mesiar et al.[22], it is important to ensure in some applications that variations in the value of some functions caused by considering just a given fraction of one of the input variables is independent of the actual choice of variable. The notion of  $\alpha$ -migrative t-norms was introduced [7] when Durante and Sarkoci investigated the convex combination of continuous t-norms and the drastic product  $T_D$ , and further studied by Fodor and Rudas [8]. Afterwards a lot of work on  $\alpha$ -migrativity for various operators have been done by many authors.

Nullnorms and t-operators as aggregation operators were introduced in [4, 14], respectively, which are generalizations of the concepts of t-norms and t-conorms. As is pointed out in [16], nullnorms and t-operators are equivalent since they have the same block structures in  $[0, 1]^2$ . Moreover, uninorms as aggregation operators were introduced to generalize and unify the concepts of t-norms and t-conorms. Our consideration was motivated by intention of getting algebraic structures which have weaker assumptions than t-operators or uninorms. A characterization of such binary operators is interesting not only from a theoretical point of view, but also for their applications, since they have been proved to be useful in several fields like fuzzy logic framework [13], expert system [15], neural networks [15] or fuzzy quantifiers [13].

This paper is mainly devoted to investigating the migrativity equations between semi-t-operators or semi-uninorms, and Mayor's aggregation operators. This paper is organized as follows. In Section 2, we review the definitions and structures of semi-uninorms, semi-t-operators and Mayor's aggregation operators, which will be used throughout this paper. In Section 3, we explore the migrative property for Mayor's aggregation operators and semi-t-operators. In Section 4, we characterize the migrativity equations for Mayor's aggregation operators and semi-uninorms. Section 5 is conclusion and further work.

## 2 Preliminaries

We assume that the readers are familiar with the basic theory of t-norms, t-conorms and uninorms. The deninitions, notions and results on them can be found in [1, 4, 12]. We will just recall some basic facts about Mayor's aggregation operators, semi-uninorms and semi-t-operators to be used later in this paper.

**Definition 2.1.** ([5]) A function  $A^{(n)} : [0, 1]^n \rightarrow [0, 1]$  is called an aggregation operator if it is non-decreasing in each variable and fulfills the boundary conditions  $A^{(n)}(0, \dots, 0) = 0$  and  $A^{(n)}(1, \dots, 1) = 1$ .

**Definition 2.2.** ([21]) A binary operator  $F : [0, 1]^2 \rightarrow [0, 1]$  is called a GM aggregation operator if it is commutative, non-decreasing in each variable and satisfies the boundary conditions for all  $x \in [0, 1]$ :  $F(0, x) = F(0, 1)x$  and  $F(1, x) = (1 - F(0, 1))x + F(0, 1)$ .

It should be noted that GM aggregation operators mean (Gaspar) Mayor's aggregation operators in this paper. Let GM denote the family of all GM aggregation operators. The following properties of GM aggregation operators are essential for their further characterization.

**Theorem 2.3.** ([21]) Let  $F$  be a GM aggregation operator. Then, the following results hold:

- (i)  $F$  is associative if and only if  $F$  is a t-norm or a t-conorm.
- (ii)  $F = \min$  or  $F = \max$  if and only if  $F(0, 1) = 0$  or  $F(0, 1) = 1$ , and  $F(x, x) = x$  for all  $x \in [0, 1]$ .
- (iii)  $F$  is idempotent if and only if  $\min \leq F \leq \max$ .

**Definition 2.4.** ([6]) A binary operator  $T : [0, 1]^2 \rightarrow [0, 1]$  is called a semi-t-norm if it is non-decreasing in each variable and has the neutral element 1. Furthermore, a semi-t-norm  $T$  is called a t-norm if it is associative and commutative.

In the dual context, a binary operator  $S : [0, 1]^2 \rightarrow [0, 1]$  is called a semi-t-conorm if it is non-decreasing in each variable and has the neutral element 0. Furthermore, a semi-t-conorm  $S$  is called a t-conorm if it is associative and commutative.

**Example 2.5.** ([25])

(i) Let

$$T_a(x, y) = \begin{cases} \min(x, y) & \text{if } x + y > 1, \\ 0 & \text{if } 0 \leq x, y \leq \frac{1}{2}, \\ \frac{1}{2}x & \text{if } 0 \leq x \leq \frac{1}{2}, \frac{1}{2} \leq y \leq 1 - x, \\ \frac{1}{3}y & \text{if } \frac{1}{2} \leq x \leq 1, 0 \leq y \leq 1 - x. \end{cases}$$

Then  $T_a$  is a non-associative and non-commutative semi-t-norm.

(ii) Let

$$T_b(x, y) = \begin{cases} \min(x, y) & \text{if } x + y > 1, \\ xy & \text{if } x + y \leq 1. \end{cases}$$

Then  $T_b$  is a commutative but not associative semi-t-norm.

(iii) Let

$$T_c(x, y) = \begin{cases} 0 & \text{if } (x, y) \in [0, 0.5] \times [0, 1), \\ \min(x, y) & \text{otherwise.} \end{cases}$$

Then  $T_c$  is an associative but not commutative semi-t-norm.

(iv) Let

$$S_a(x, y) = \begin{cases} \max(x, y) & \text{if } x + y \leq 1, \\ 1 & \text{if } \frac{1}{2} \leq x, y \leq 1, \\ \min(1.01y, 1) & \text{if } 0 \leq x \leq \frac{1}{2}, 1 - x \leq y \leq 1, \\ \min(1.03x, 1) & \text{if } \frac{1}{2} \leq x \leq 1, 1 - x \leq y \leq \frac{1}{2}. \end{cases}$$

Then  $S_a$  is a non-associative and non-commutative semi-t-conorm.

(v) Let

$$S_b(x, y) = \begin{cases} \max(x, y) & \text{if } x + y \leq 1, \\ x + y - xy & \text{if } x + y > 1. \end{cases}$$

Then  $S_b$  is a commutative but not associative semi-t-conorm.

(vi) Let

$$S_c(x, y) = \begin{cases} 1 & \text{if } (x, y) \in [0.5, 1] \times (0, 1], \\ \max(x, y) & \text{otherwise.} \end{cases}$$

Then  $S_c$  is an associative but not commutative semi-t-conorm.

**Definition 2.6.** ([14]) A binary operation  $F : [0, 1]^2 \rightarrow [0, 1]$  is called a *t-operator* if it is commutative, associative, non-decreasing in each variable, fulfills  $F(0, 0) = 0$ ,  $F(1, 1) = 1$  and such that the functions  $F_0$  and  $F_1$  are continuous, where  $F_0(x) = F(0, x)$  and  $F_1(x) = F(1, x)$ .

**Definition 2.7.** ([6]) A binary operation  $F : [0, 1]^2 \rightarrow [0, 1]$  is called a *semi-t-operator* if it is non-decreasing in each variable, fulfills  $F(0, 0) = 0$ ,  $F(1, 1) = 1$  and such that the functions  $F_0, F_1, F^0, F^1$  are continuous, where  $F_0(x) = F(0, x)$ ,  $F_1(x) = F(1, x)$ ,  $F^0(x) = F(x, 0)$  and  $F^1(x) = F(x, 1)$ . A commutative and associative semi-t-operators is called a *t-operator*.

Let  $\mathcal{F}_{a,b}$  denote the family of all semi-t-operators such that  $F(0, 1) = a$  and  $F(1, 0) = b$ .

**Theorem 2.8.** ([6]) Let  $F : [0, 1]^2 \rightarrow [0, 1]$ ,  $F(0, 1) = a$  and  $F(1, 0) = b$ . Operation  $F \in \mathcal{F}_{a,b}$  if and only if there exists a semi-t-norm  $T_F$  and a semi-t-conorm  $S_F$  such that

$$F(x, y) = \begin{cases} aS_F\left(\frac{x}{a}, \frac{y}{a}\right) & \text{if } (x, y) \in [0, a]^2, \\ b + (1 - b)T_F\left(\frac{x-b}{1-b}, \frac{y-b}{1-b}\right) & \text{if } (x, y) \in [b, 1]^2, \\ a & \text{if } (x, y) \in [0, a] \times [a, 1], \\ b & \text{if } (x, y) \in [b, 1] \times [0, b], \\ x & \text{otherwise,} \end{cases} \quad (1)$$

for  $a \leq b$  and

$$F(x, y) = \begin{cases} bS_F\left(\frac{x}{b}, \frac{y}{b}\right) & \text{if } (x, y) \in [0, b]^2, \\ a + (1 - a)T_F\left(\frac{x-a}{1-a}, \frac{y-a}{1-a}\right) & \text{if } (x, y) \in [a, 1]^2, \\ a & \text{if } (x, y) \in [0, a] \times [a, 1], \\ b & \text{if } (x, y) \in [b, 1] \times [0, b], \\ y & \text{otherwise,} \end{cases} \quad (2)$$

for  $a \geq b$ .

**Definition 2.9.** ([11]) A binary operator  $U : [0, 1]^2 \rightarrow [0, 1]$  is called a *semi-uninorm* if it is non-decreasing in each variable, and for which there exists a neutral element  $e \in [0, 1]$  such that  $U(e, x) = U(x, e) = x$  for all  $x \in [0, 1]$ . A commutative and associative semi-uninorm  $U$  is called a *uninorm*.

We denote the class of all semi-uninorms with neutral element  $e$  by  $U_e$ . Evidently, semi-t-norms and semi-t-conorms are two special classes of semi-uninorms with neutral  $e = 1$  and  $e = 0$ , respectively. A semi-uninorm  $U$  is conjunctive if  $U(0, 1) = U(1, 0) = 0$  and disjunctive when  $U(0, 1) = U(1, 0) = 1$ .

**Theorem 2.10.** ([11]) Let  $U \in U_e$  with  $e \in (0, 1)$ . Then,

(i)  $U(1, x) = U(x, 1) = x$  for all  $x \in [0, e]$  if and only if  $U$  has the following form:

$$U(x, y) = \begin{cases} eT_U(\frac{x}{e}, \frac{y}{e}) & \text{if } (x, y) \in [0, e]^2, \\ e + (1 - e)S_U(\frac{x-e}{1-e}, \frac{y-e}{1-e}) & \text{if } (x, y) \in [e, 1]^2, \\ \min(x, y) & \text{otherwise.} \end{cases} \quad (3)$$

(ii)  $U(0, x) = U(x, 0) = x$  for all  $x \in (e, 1]$  if and only if  $U$  has the following form:

$$U(x, y) = \begin{cases} eT_U(\frac{x}{e}, \frac{y}{e}) & \text{if } (x, y) \in [0, e]^2, \\ e + (1 - e)S_U(\frac{x-e}{1-e}, \frac{y-e}{1-e}) & \text{if } (x, y) \in [e, 1]^2, \\ \max(x, y) & \text{otherwise,} \end{cases} \quad (4)$$

where  $T_U$  is a semi-t-norm and  $S_U$  is a semi-t-conorm.

The family of all semi-uninorms in case (i) will be denoted by  $U_e^{\min}$  and the family of all semi-uninorms in case (ii) by  $U_e^{\max}$ .

### 3 Migrativity for Mayor's aggregation operators and semi-t-operators

In this section, we investigate the migrativity property for Mayor's aggregation operators and semi-t-operators. Let us suppose that  $G \in \mathbb{GM}$  is a GM aggregation operator and  $F \in F_{a,b}$  is a semi-t-operator. Depending on the inequality between  $a$  and  $b$  of the operator  $F$ , there are two cases: migrativity for  $G$  and  $F$  with  $a \geq b$ , and migrativity for  $G$  and  $F$  with  $a < b$ .

**Definition 3.1.** Let  $\alpha \in [0, 1]$ ,  $G \in \mathbb{GM}$  and  $F \in F_{a,b}$ . A GM aggregation operator  $G$  is said to be  $\alpha$ -migrative over a semi-t-operator  $F$  or  $(\alpha, F)$ -migrative if

$$G(F(\alpha, x), y) = G(x, F(\alpha, y)) \quad (5)$$

for all  $x, y \in [0, 1]$ .

**Definition 3.2.** Let  $\alpha \in [0, 1]$ ,  $G \in \mathbb{GM}$  and  $F \in F_{a,b}$ . A semi-t-operator  $F$  is said to be  $\alpha$ -migrative over a GM aggregation operator  $G$  or  $(\alpha, G)$ -migrative if

$$F(G(\alpha, x), y) = F(x, G(\alpha, y)) \quad (6)$$

for all  $x, y \in [0, 1]$ .

#### 3.1 Migrativity for $G \in \mathbb{GM}$ and $F \in F_{a,b}$ with $a \geq b$

**Theorem 3.3.** Let  $\alpha \in [0, b]$ ,  $G \in \mathbb{GM}$  be a GM aggregation operator and  $F \in F_{a,b}$  be a semi-t-operator with  $a \geq b$ . Then  $G$  is  $(\alpha, F)$ -migrative if and only if one of the following statements holds:

(i)  $\alpha = 0$  and there exist two commutative semi-t-norms  $T_1$  and  $T_2$  such that

$$G(x, y) = \begin{cases} aT_1(\frac{x}{a}, \frac{y}{a}) & \text{if } (x, y) \in [0, a]^2, \\ a + (1 - a)T_2(\frac{x-a}{1-a}, \frac{y-a}{1-a}) & \text{if } (x, y) \in [a, 1]^2, \\ \min(x, y) & \text{otherwise.} \end{cases} \quad (7)$$

(ii) the structure of  $F$  is

$$F(x, y) = \begin{cases} bS_F(\frac{x}{b}, \frac{y}{b}) & \text{if } (x, y) \in [0, b]^2, \\ b & \text{if } (x, y) \in [b, 1] \times [0, b], \\ y & \text{otherwise.} \end{cases} \quad (8)$$

And  $G$  is a commutative semi-t-conorm such that  $G(x, \alpha) = F(\alpha, x)$  for  $x \in [0, b]$ ,  $G(G(\alpha, x), y) = G(x, G(\alpha, y))$  for any  $x \in [0, b]$  and  $y \in [0, 1]$ , and  $G(x, y) = \max(x, y)$  for any  $(x, y) \in [0, \alpha] \times [b, 1] \cup [b, 1] \times [0, \alpha]$ . Moreover, if the underlying semi-t-conorm  $S_F$  of  $F$  is continuous and  $F(\alpha, \alpha) = \alpha$ . Then  $G$  is  $(\alpha, F)$ -migrative if and only if  $G(\alpha, x) = F(\alpha, x) = \max(\alpha, x)$  for all  $x \in [0, 1]$ .

(iii)  $\alpha = 0$  and  $F$  has the form (8).

*Proof.* Let  $G(0, 1) = k$ . Firstly, taking  $x = 0$  and  $y = 1$  in Eq.(5), then we have  $(1 - k)\alpha + k = G(\alpha, 1) = G(F(\alpha, 0), 1) = G(0, F(\alpha, 1)) = G(0, a) = ka$ , which means  $(1 - k)\alpha = k(a - 1) = 0$ . Thus we obtain three possibilities:  $\alpha = k = 0$ ;  $a = k = 1$ ;  $a = 1$  and  $\alpha = 0$ .

(i) When  $\alpha = k = 0$ , then it is obvious that  $G$  is a commutative semi-t-norm because  $G(x, 1) = G(1, x) = x$  for any  $x \in [0, 1]$ . Now let  $\alpha = 0$  and  $x = 1$  in Eq.(5), we have

$$G(a, y) = G(F(0, 1), y) = G(1, F(0, y)) = F(0, y) = \begin{cases} y & \text{if } 0 \leq y \leq a, \\ a & \text{if } a \leq y \leq 1. \end{cases} \quad (9)$$

So we can obtain the following:

For  $x \in [0, a]$  and  $y \in [a, 1]$ , we have  $x = G(x, a) \leq G(x, y) \leq G(x, 1) = x$ . That is,  $G(x, y) = x = \min(x, y)$ .

For  $x \in [a, 1]$  and  $y \in [0, a]$ , we have  $G(x, y) = y = \min(x, y)$  from the commutativity of  $G$ .

For  $x, y \in [0, a]$ , we have  $0 = G(0, 0) \leq G(x, y) \leq G(a, a) = a$ .

For  $x, y \in [a, 1]$ , we have  $a = G(a, a) \leq G(x, y) \leq G(1, 1) = 1$ .

To summarize,  $G$  has the form (7).

Conversely, let  $\alpha = 0$  and  $G$  be given by formula (7). In order to verify that  $G$  is  $(\alpha, F)$ -migrative, the following cases need to be considered.

If  $(x, y) \in [0, a]^2$ , then  $G(F(0, x), y) = G(x, y) = G(x, F(0, y))$ .

If  $(x, y) \in [a, 1]^2$ , then  $G(F(0, x), y) = G(a, y) = a \wedge y = a = x \wedge a = G(x, a) = G(x, F(0, y))$ .

If  $(x, y) \in [0, a] \times [a, 1]$ , then  $G(F(0, x), y) = G(x, y) = x \wedge y = x = x \wedge a = G(x, a) = G(x, F(0, y))$ .

If  $(x, y) \in [a, 1] \times [0, a]$ , then  $G(F(0, x), y) = G(a, y) = a \wedge y = y = x \wedge y = G(x, y) = G(x, F(0, y))$ .

(ii) When  $a = k = 1$ , then it is obvious that  $F$  has the form (8) and  $G$  is a commutative semi-t-conorm because  $G(0, x) = G(x, 0) = x$ . For any  $x \in [0, 1]$ , we have  $F(\alpha, x) = G(F(\alpha, x), 0) = G(x, F(\alpha, 0)) = G(x, \alpha)$ . Specially,  $G(\alpha, x) = G(x, \alpha) = F(\alpha, x) = x$  for  $x \in [b, 1]$ . So we have  $y = G(0, y) \leq G(x, y) \leq G(\alpha, y) = y$  for  $(x, y) \in [0, \alpha] \times [b, 1]$ , that is,  $G(x, y) = y = \max(x, y)$  for  $(x, y) \in [0, \alpha] \times [b, 1]$ . Therefore,  $G(x, y) = \max(x, y)$  for  $(x, y) \in [0, \alpha] \times [b, 1] \cup [b, 1] \times [0, \alpha]$  because of the commutativity of  $G$ . For any  $x \in [0, b]$  and  $y \in [0, 1]$ , we have  $G(G(\alpha, x), y) = G(F(\alpha, x), y) = G(x, F(\alpha, y)) = G(x, G(\alpha, y))$ .

Conversely, in order to verify that  $G$  is  $(\alpha, F)$ -migrative, the following cases need to be considered.

If  $(x, y) \in [0, b]^2$ , then  $G(F(\alpha, x), y) = G(G(x, \alpha), y) = G(x, G(\alpha, y)) = G(x, F(\alpha, y))$ .

If  $(x, y) \in [0, b] \times [b, 1]$ , then  $G(F(\alpha, x), y) = G(G(x, \alpha), y) = G(x, G(\alpha, y)) = G(x, y) = G(x, F(\alpha, y))$ .

If  $(x, y) \in [b, 1] \times [0, b]$ , then  $G(F(\alpha, x), y) = G(x, y) = G(G(x, \alpha), y) = G(x, G(\alpha, y)) = G(x, F(\alpha, y))$ .

If  $(x, y) \in [b, 1]^2$ , then  $G(F(\alpha, x), y) = G(x, y) = G(x, F(\alpha, y))$ .

Moreover, if the underlying semi-t-conorm  $S_F$  of  $F$  is continuous and  $F(\alpha, \alpha) = \alpha$ , then it is obvious that  $G(\alpha, y) = F(\alpha, y) = \max(\alpha, y)$  for all  $y \in [0, b]$ . From the structure of  $F$ , it follows that  $F(\alpha, y) = y$  for  $y \in [b, 1]$ . Thus we have  $G(\alpha, y) = F(\alpha, y) = \max(\alpha, y)$  for all  $y \in [0, 1]$ . Conversely, it is easy to verify that  $G$  is  $(\alpha, F)$ -migrative.

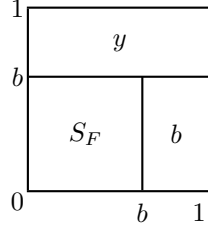
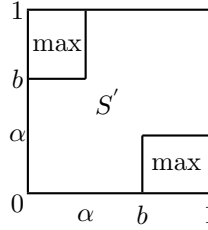
(iii) When  $a = 1$  and  $\alpha = 0$ , then  $F$  has the form (8). Conversely, it is obvious that  $G(F(0, x), y) = G(x, y) = G(x, F(0, y))$  for all  $x, y \in [0, 1]$ .  $\square$

**Theorem 3.4.** Let  $\alpha \in (b, a)$ ,  $G \in \mathbb{GM}$  be a GM aggregation operator and  $F \in F_{a,b}$  be a semi-t-operator with  $a \geq b$ . Then  $G$  is  $(\alpha, F)$ -migrative if and only if one of the following statements holds:

(i)  $F(x, y) = y$  for all  $x, y \in [0, 1]$ .

(ii)  $F$  has the form (8) and there exist two commutative semi-t-conorms  $S_1$  and  $S_2$  such that

$$G(x, y) = \begin{cases} bS_1\left(\frac{x}{b}, \frac{y}{b}\right) & \text{if } (x, y) \in [0, b]^2, \\ b + (1 - b)S_2\left(\frac{x-b}{1-b}, \frac{y-b}{1-b}\right) & \text{if } (x, y) \in [b, 1]^2, \\ \max(x, y) & \text{otherwise.} \end{cases} \quad (10)$$

Figure 1:  $F$  in Theorem 3.3(ii)Figure 2:  $G$  in Theorem 3.3(ii)

(iii)  $G$  has the form (7) and the structure of  $F$  is

$$F(x, y) = \begin{cases} a + (1-a)T_F\left(\frac{x-a}{1-a}, \frac{y-a}{1-a}\right) & \text{if } (x, y) \in [a, 1]^2, \\ a & \text{if } (x, y) \in [0, a] \times [a, 1], \\ y & \text{otherwise.} \end{cases} \quad (11)$$

*Proof.* Let  $G(0, 1) = k$ . Firstly, taking  $x = 0$  and  $y = 1$  in Eq.(5), then we have  $(1-k)b + k = G(b, 1) = G(F(\alpha, 0), 1) = G(0, F(\alpha, 1)) = G(0, a) = ka$ , which means  $(1-k)b = k(a-1) = 0$ . Thus we obtain three possibilities:  $b = k = 0$ ;  $a = k = 1$ ;  $a = 1$  and  $b = 0$ .

(i) When  $a = 1$  and  $b = 0$ , then it is obvious that  $F(x, y) = y$  for all  $x, y \in [0, 1]$ . Conversely, we have  $G(F(\alpha, x), y) = G(x, F(\alpha, y))$  for any  $(x, y) \in [0, 1]^2$ .

(ii) When  $a = k = 1$ , then it is obvious that  $F$  has the form (8) and  $G$  is a commutative semi-t-conorm from  $G(x, 0) = G(0, x) = x$ . Now let  $x = 0$  in Eq.(5), we have

$$G(b, y) = G(F(\alpha, 0), y) = G(0, F(\alpha, y)) = F(\alpha, y) = \begin{cases} b & \text{if } 0 \leq y \leq b, \\ y & \text{if } b \leq y \leq 1. \end{cases} \quad (12)$$

So we can obtain the following:

For  $x \in [0, b]$  and  $y \in [b, 1]$ , we have  $y = G(0, y) \leq G(x, y) \leq G(b, y) = y$ . That is,  $G(x, y) = y = \max(x, y)$ .

For  $x \in [b, 1]$  and  $y \in [0, b]$ , we have  $G(x, y) = x = \max(x, y)$  from the commutativity of  $G$ .

For  $x, y \in [0, b]$ , we have  $0 = G(0, 0) \leq G(x, y) \leq G(b, b) = b$ .

For  $x, y \in [b, 1]$ , we have  $b = G(b, b) \leq G(x, y) \leq G(1, 1) = 1$ .

To summarize,  $G$  has the form (10).

Conversely, let  $F$  be given by formula (8) and  $G$  be given by formula (10). In order to verify that  $G$  is  $(\alpha, F)$ -migrative, the following cases need to be considered.

If  $(x, y) \in [0, b]^2$ , then  $G(F(\alpha, x), y) = G(b, y) = b \vee y = b = x \vee b = G(x, b) = G(x, F(\alpha, y))$ .

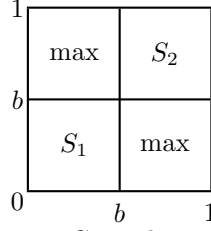
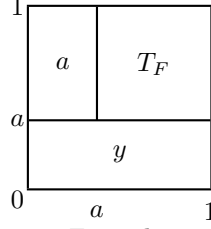
If  $(x, y) \in [b, 1]^2$ , then  $G(F(\alpha, x), y) = G(x, y) = G(x, F(\alpha, y))$ .

If  $(x, y) \in [0, b] \times [b, 1]$ , then  $G(F(\alpha, x), y) = G(b, y) = b \vee y = y = x \vee y = G(x, y) = G(x, F(\alpha, y))$ .

If  $(x, y) \in [b, 1] \times [0, b]$ , then  $G(F(\alpha, x), y) = G(x, y) = x \vee y = x = x \vee b = G(x, b) = G(x, F(\alpha, y))$ .

(iii) When  $b = k = 0$ , then it is obvious that  $F$  has the form (11) and  $G$  is a commutative semi-t-norm from  $G(x, 1) = G(1, x) = x$ . Now let  $x = 1$  in Eq.(5), we have

$$G(a, y) = G(F(\alpha, 1), y) = G(1, F(\alpha, y)) = F(\alpha, y) = \begin{cases} y & \text{if } 0 \leq y \leq a, \\ a & \text{if } a \leq y \leq 1. \end{cases}$$


 Figure 3:  $G$  in Theorem 3.4(ii)

 Figure 4:  $F$  in Theorem 3.4(iii)

(13)

So we can obtain the following:

For  $x \in [0, a]$  and  $y \in [a, 1]$ , we have  $x = G(x, a) \leq G(x, y) \leq G(x, 1) = x$ . That is,  $G(x, y) = x = \min(x, y)$ .

For  $x \in [a, 1]$  and  $y \in [0, a]$ , we have  $G(x, y) = y = \min(x, y)$  from the commutativity of  $G$ .

For  $x, y \in [0, a]$ , we have  $0 = G(0, 0) \leq G(x, y) \leq G(a, a) = a$ .

For  $x, y \in [a, 1]$ , we have  $a = G(a, a) \leq G(x, y) \leq G(1, 1) = 1$ .

To summarize,  $G$  has the form (7).

Conversely, let  $F$  be given by formula (11) and  $G$  be given by formula (7). In order to verify that  $G$  is  $(\alpha, F)$ -migrative, the following cases need to be considered.

If  $(x, y) \in [0, a]^2$ , then  $G(F(\alpha, x), y) = G(x, y) = G(x, F(\alpha, y))$ .

If  $(x, y) \in [a, 1]^2$ , then  $G(F(\alpha, x), y) = G(a, y) = a \wedge y = a = x \wedge a = G(x, a) = G(x, F(\alpha, y))$ .

If  $(x, y) \in [0, a] \times [a, 1]$ , then  $G(F(\alpha, x), y) = G(x, y) = x \wedge y = x = x \wedge a = G(x, a) = G(x, F(\alpha, y))$ .

If  $(x, y) \in [a, 1] \times [0, a]$ , then  $G(F(\alpha, x), y) = G(a, y) = a \wedge y = y = x \wedge y = G(x, y) = G(x, F(\alpha, y))$ .  $\square$

**Theorem 3.5.** Let  $\alpha \in [a, 1]$ ,  $G \in \mathbb{GM}$  be a GM aggregation operator and  $F \in F_{a,b}$  be a semi- $t$ -operator with  $a \geq b$ . Then  $G$  is  $(\alpha, F)$ -migrative if and only if one of the following statements holds:

(i)  $\alpha = 1$  and  $G$  has the form (10).

(ii)  $F$  has the form (11) and  $G$  is a commutative semi- $t$ -norm such that  $G(x, \alpha) = F(\alpha, x)$  for  $x \in [a, 1]$ ,  $G(G(\alpha, x), y) = G(x, G(\alpha, y))$  for  $x \in [a, 1]$  and  $y \in [0, 1]$ , and  $G(x, y) = \min(x, y)$  for  $(x, y) \in [a, 1] \times [0, a] \cup [0, a] \times [a, 1]$ . Moreover, if the underlying semi- $t$ -norm  $T_F$  of  $F$  is continuous and  $F(\alpha, \alpha) = \alpha$ . Then  $G$  is  $(\alpha, F)$ -migrative if and only if  $G(\alpha, x) = F(\alpha, x) = \min(\alpha, x)$  for all  $x \in [0, 1]$ .

(iii)  $\alpha = 1$  and  $F$  has the form (11).

*Proof.* The proof is similar to the one of Theorem 3.3.  $\square$

**Theorem 3.6.** Let  $\alpha \in [0, 1]$ ,  $G \in \mathbb{GM}$  be a GM aggregation operator with  $G(0, 1) = k$ , and  $F \in F_{a,b}$  be a semi- $t$ -operator with  $a \geq b$ , where the underlying semi- $t$ -norm  $T_F$  and semi- $t$ -conorm  $S_F$  are associative. Then  $F$  is  $(\alpha, G)$ -migrative if and only if the following statements hold.

(i)  $G(\alpha, y) = F(k\alpha, y) = F(y, k\alpha)$  for  $y \in [0, b]$ .

(ii)  $G(\alpha, y) = y$  for  $y \in [b, a]$ .

(iii)  $G(\alpha, y) = F((1 - k)\alpha + k, y) = F(y, (1 - k)\alpha + k)$  for  $y \in [a, 1]$ .

*Proof.* Firstly, let us prove that  $G(\alpha, b) = b$  and  $G(\alpha, a) = a$ . Taking  $x = 1$  and  $y = b$  in Eq.(6), then we have  $b = F(G(\alpha, 1), b) = F(1, G(\alpha, b))$ , which implies that  $G(\alpha, b) \leq b$  because of the structure of  $F$ . Now let  $x = 0$  and  $y = b$  in Eq.(6), then we have  $b = F(G(\alpha, 0), b) = F(0, G(\alpha, b)) = G(\alpha, b)$ . Similarly, taking  $x = 0$  and  $y = a$  in Eq.(6), then we have  $a = F(G(\alpha, 0), a) = F(0, G(\alpha, a))$ , which implies that  $G(\alpha, a) \geq a$  because of the structure of  $F$ . Now let  $x = 1$  and  $y = a$  in Eq.(6), then we have  $a = F(G(\alpha, 1), a) = F(1, G(\alpha, a)) = G(\alpha, a)$ .

For any  $y \in [0, b]$ , we have  $G(\alpha, y) \leq G(\alpha, b) = b$ . Thus, it follows that  $G(\alpha, y) = F(G(\alpha, y), 0) = F(y, G(\alpha, 0)) = F(y, k\alpha)$  and  $F(k\alpha, y) = F(G(\alpha, 0), y) = F(0, G(\alpha, y)) = G(\alpha, y)$  from the structure of  $F$ . That is,  $G(\alpha, y) = F(k\alpha, y) = F(y, k\alpha)$  for any  $y \in [0, b]$ . For any  $y \in [a, b]$ , we have  $b = G(\alpha, b) \leq G(\alpha, y) \leq G(\alpha, a) = a$ . So from the structure of  $F$ , it follows that  $y = F(G(\alpha, x), y) = F(x, G(\alpha, y)) = G(\alpha, y)$  for any  $y \in [b, a]$  and  $x \in [0, 1]$ . For any  $y \in [a, 1]$ , we have  $G(\alpha, y) \geq G(\alpha, a) = a$ . Thus, it follows that  $G(\alpha, y) = F(G(\alpha, y), 1) = F(y, G(\alpha, 1)) = F(y, (1-k)\alpha + k)$  and  $F((1-k)\alpha + k, y) = F(G(\alpha, 1), y) = F(1, G(\alpha, y)) = G(\alpha, y)$  from the structure of  $F$ . That is,  $G(\alpha, y) = F((1-k)\alpha + k, y) = F(y, (1-k)\alpha + k)$  for any  $y \in [a, 1]$ .

Conversely, in order to verify that  $F$  is  $(\alpha, G)$ -migrative, the following cases need to be considered.

If  $(x, y) \in [0, b]^2$ , then  $F(G(\alpha, x), y) = F(F(x, k\alpha), y) = F(x, F(k\alpha, y)) = F(x, G(\alpha, y))$ .

If  $(x, y) \in [b, 1] \times [0, b]$ , then  $F(G(\alpha, x), y) = b = F(x, G(\alpha, y))$ .

If  $(x, y) \in [0, 1] \times [b, a]$ , then  $F(G(\alpha, x), y) = y = F(x, y) = F(x, G(\alpha, y))$ .

If  $(x, y) \in [0, a] \times [a, 1]$ , then  $F(G(\alpha, x), y) = a = F(x, G(\alpha, y))$ .

If  $(x, y) \in [a, 1]^2$ , then  $F(G(\alpha, x), y) = F(F(x, (1-k)\alpha + k), y) = F(x, F((1-k)\alpha + k, y)) = F(x, G(\alpha, y))$ .  $\square$

### 3.2 Migrativity for $G \in \mathbb{GM}$ and $F \in F_{a,b}$ with $a < b$

**Theorem 3.7.** Let  $\alpha \in [0, a]$ ,  $G \in \mathbb{GM}$  be a GM aggregation operator and  $F \in F_{a,b}$  be a semi-t-operator with  $a < b$ . Then  $G$  is  $(\alpha, F)$ -migrative if and only if  $\alpha = 0$  and  $G$  has the form (7).

*Proof.* The proof is similar to the one of Theorem 3.3(i).  $\square$

**Theorem 3.8.** Let  $\alpha \in (a, b)$ ,  $G \in \mathbb{GM}$  be a GM aggregation operator and  $F \in F_{a,b}$  be a semi-t-operator with  $a < b$ . Then  $G$  is not  $(\alpha, F)$ -migrative.

*Proof.* Let  $G(0, 1) = k$ . Taking  $x = 0$  and  $y = 1$  in Eq.(5), then we have  $(1-k)\alpha + k = G(\alpha, 1) = G(F(\alpha, 0), 1) = G(0, F(\alpha, 1)) = G(0, \alpha) = k\alpha$ . So we obtain that  $(1-k)\alpha = k(\alpha - 1) = 0$ , which means  $\alpha = 0$  or  $\alpha = 1$ . Note that, both  $\alpha = 0$  and  $\alpha = 1$  contradict the assumption that  $a < \alpha < b$ .  $\square$

**Theorem 3.9.** Let  $\alpha \in [b, 1]$ ,  $G \in \mathbb{GM}$  be a GM aggregation operator and  $F \in F_{a,b}$  be a semi-t-operator with  $a < b$ . Then  $G$  is  $(\alpha, F)$ -migrative if and only if  $\alpha = 1$  and  $G$  has the form (10).

*Proof.* The proof is similar to the one of Theorem 3.4(ii).  $\square$

**Example 3.10.** (i) Take  $a = 0.5$  and  $b = 1$ , let  $T_1 = T_2 = T_b$  and  $S_F = S_a$ . That is, the structures of  $G$  and  $F$  are:

$$G(x, y) = \begin{cases} 0.5T_b(\frac{x}{0.5}, \frac{y}{0.5}) & \text{if } (x, y) \in [0, 0.5]^2, \\ 0.5 + 0.5T_b(\frac{x-0.5}{0.5}, \frac{y-0.5}{0.5}) & \text{if } (x, y) \in [0.5, 1]^2, \\ \min(x, y) & \text{otherwise.} \end{cases}$$

$$F(x, y) = \begin{cases} 0.5S_a(\frac{x}{0.5}, \frac{y}{0.5}) & \text{if } (x, y) \in [0, 0.5]^2, \\ x & \text{if } (x, y) \in [0.5, 1] \times [0, 1], \\ a & \text{otherwise.} \end{cases}$$

Then it follows from Theorem 3.7 that  $G$  is  $(0, F)$ -migrative.

(ii) Take  $a = 0$  and  $b = 1$ , that is,  $F(x, y) = x$  for all  $x, y \in [0, 1]$ . Let  $\alpha \in (0, 1)$  and  $G \in \mathbb{GM}$  is an arbitrary GM aggregation operator. Then it follows from Theorem 3.8 that  $G$  is not  $(\alpha, F)$ -migrative.

(iii) Take  $a = 0.5$  and  $b = 0.8$ , let  $S_1 = S_2 = S_b$ ,  $S_F = S_a$  and  $T_F = \min$ . That is, the structures of  $G$  and  $F$  are:

$$G(x, y) = \begin{cases} 0.8S_b(\frac{x}{0.8}, \frac{y}{0.8}) & \text{if } (x, y) \in [0, 0.8]^2, \\ 0.8 + 0.8S_b(\frac{x-0.8}{0.2}, \frac{y-0.8}{0.2}) & \text{if } (x, y) \in [0.8, 1]^2, \\ \max(x, y) & \text{otherwise.} \end{cases}$$



$$F(x, y) = \begin{cases} 0.5S_a\left(\frac{x}{0.5}, \frac{y}{0.5}\right) & \text{if } (x, y) \in [0, 0.5]^2, \\ x & \text{if } (x, y) \in [0.5, 0.8] \times [0, 1], \\ 0.5 & \text{if } (x, y) \in [0, 0.5] \times [0.5, 1], \\ 0.8 & \text{if } (x, y) \in [0.8, 1] \times [0, 0.8], \\ \min(x, y) & \text{otherwise.} \end{cases}$$

Then it follows from Theorem 3.9 that  $G$  is  $(1, F)$ -migrative.

**Theorem 3.11.** Let  $\alpha \in [0, 1]$ ,  $G \in \mathbb{GM}$  be a GM aggregation operator with  $G(0, 1) = k$ , and  $F \in F_{a,b}$  be a semi-t-operator with  $a < b$ , where the underlying semi-t-norm  $T_F$  and semi-t-conorm  $S_F$  are associative. Then  $F$  is  $(\alpha, G)$ -migrative if and only if the following statements hold.

- (i)  $G(\alpha, y) = F(k\alpha, y) = F(y, k\alpha)$  for  $y \in [0, a]$ .
- (ii)  $G(\alpha, y) = y$  for  $y \in [a, b]$ .
- (iii)  $G(\alpha, y) = F((1-k)\alpha + k, y) = F(y, (1-k)\alpha + k)$  for  $y \in [b, 1]$ .

*Proof.* The proof is similar to the one of Theorem 3.6. □

## 4 Migrativity for Mayor's aggregation operators and semi-uninorms

In this section, we investigate the migrativity property for Mayor's aggregation operators and semi-uninorms. Let us suppose that  $F \in \mathbb{GM}$  is a GM aggregation operator and  $U$  is a semi-uninorm form  $U_e^{\min} \cup U_e^{\max}$ . Again, there are two cases: migrativity for  $F$  and  $U \in U_e^{\min}$ , and migrativity for  $F$  and  $U \in U_e^{\max}$ .

**Definition 4.1.** Let  $\alpha \in [0, 1]$ ,  $F \in \mathbb{GM}$  and  $U \in U_e$ . A GM aggregation operator  $F$  is said to be  $\alpha$ -migrative over a semi-uninorm  $U$  or  $(\alpha, U)$ -migrative if

$$F(U(\alpha, x), y) = F(x, U(\alpha, y)) \tag{14}$$

for all  $x, y \in [0, 1]$ .

**Definition 4.2.** Let  $\alpha \in [0, 1]$ ,  $F \in \mathbb{GM}$  and  $U \in U_e$ . A semi-uninorm  $U$  is said to be  $\alpha$ -migrative over a GM aggregation operator  $F$  or  $(\alpha, F)$ -migrative if

$$U(F(\alpha, x), y) = U(x, F(\alpha, y)) \tag{15}$$

for all  $x, y \in [0, 1]$ .

**Lemma 4.3.** Let  $F \in \mathbb{GM}$  be a GM aggregation operator and  $U \in U_e$  be a semi-uninorm, then  $F$  is always  $(e, U)$ -migrative.

*Proof.* It is obvious that  $F(U(e, x), y) = F(x, y) = F(x, U(e, y))$  for all  $x, y \in [0, 1]$ . □

### 4.1 Migrativity for $F \in \mathbb{GM}$ and $U \in U_e^{\min}$

In this subsection, we will respectively discuss the migrative equation for  $F$  over  $U \in U_e^{\min}$  and  $U \in U_e^{\max}$  over  $F$  in detail.

**Theorem 4.4.** Let  $\alpha \in [0, e]$ ,  $F \in \mathbb{GM}$  be a GM aggregation operator and  $U \in U_e^{\min}$  be a semi-uninorm. Then  $F$  is  $(\alpha, U)$ -migrative if and only if  $F$  is a commutative semi-t-norm such that  $F(y, \alpha) = U(\alpha, y)$  for  $y \in [0, e]$ ,  $F(\alpha, y) = \alpha$  for  $y \in [e, 1]$ , and  $F(F(\alpha, x), y) = F(x, F(\alpha, y))$  for any  $(x, y) \in [0, e] \times [0, 1]$ . Moreover, if the underlying semi-t-norm  $T_U$  of  $U$  is continuous and  $U(\alpha, \alpha) = \alpha$ . Then  $F$  is  $(\alpha, U)$ -migrative if and only if  $F(\alpha, x) = U(\alpha, x) = \min(\alpha, x)$  for all  $x \in [0, 1]$ .

*Proof.* First, let us prove  $k = 0$ . Letting  $x = 0$  and  $y = 1$  in Eq.(14), then  $k = F(0, 1) = F(U(\alpha, 0), 1) = F(0, U(\alpha, 1)) = F(0, \alpha) = k\alpha$ , which means  $k(1 - \alpha) = 0$ . That is,  $k = 0$  or  $\alpha = 1$ . Note that  $\alpha = 1$  contradicts the assumption  $\alpha < e$ . Therefore, the only possibility is  $k = 0$ , which means  $F$  is a commutative semi-t-norm because of  $F(x, 1) = F(1, x) = x$  for all  $x \in [0, 1]$ .

For any  $y \in [0, 1]$ , we have  $F(\alpha, y) = F(U(\alpha, 1), y) = F(1, U(\alpha, y)) = U(\alpha, y)$ . Specially,  $F(\alpha, y) = \alpha$  for  $y \in [e, 1]$ . Then it is easy for us to obtain  $F(F(\alpha, x), y) = F(U(\alpha, x), y) = F(x, U(\alpha, y)) = F(x, F(\alpha, y))$  for any  $(x, y) \in [0, e] \times [0, 1]$ .

Conversely, let us verify that  $F$  is  $(\alpha, U)$ -migrative.

If  $(x, y) \in [0, e]^2$ , then  $F(U(\alpha, x), y) = F(F(x, \alpha), y) = F(x, F(\alpha, y)) = F(x, U(\alpha, y))$ .

If  $(x, y) \in [e, 1]^2$ , then  $F(U(\alpha, x), y) = F(\alpha, y) = \alpha = F(x, \alpha) = F(x, U(\alpha, y))$ .

If  $(x, y) \in [0, e] \times [e, 1]$ , then  $F(U(\alpha, x), y) = F(F(x, \alpha), y) = F(x, F(\alpha, y)) = F(x, \alpha) = F(x, U(\alpha, y))$ .

If  $(x, y) \in [e, 1] \times [0, e]$ , then  $F(U(\alpha, x), y) = F(\alpha, y) = F(F(\alpha, x), y) = F(x, F(\alpha, y)) = F(x, U(\alpha, y))$ .

Moreover, if the underlying semi-t-norm  $T_U$  of  $U$  is continuous and  $U(\alpha, \alpha) = \alpha$ , then it is obvious that  $F(\alpha, x) = U(\alpha, x) = \min(\alpha, x)$  for  $x \in [0, e]$ . And from the structure of  $U$ , it follows that  $U(\alpha, x) = \min(\alpha, x)$  for  $x \in [e, 1]$ . Thus we have  $F(\alpha, x) = U(\alpha, x) = \min(\alpha, x)$  for all  $x \in [0, 1]$ . Conversely, it is easy to verify that  $F$  is  $(\alpha, U)$ -migrative.  $\square$

**Theorem 4.5.** *Let  $\alpha \in (e, 1]$ ,  $F \in \mathbb{GM}$  be a GM aggregation operator and  $U \in U_e^{\min}$  be a semi-uninorm. Then  $F$  is not  $(\alpha, U)$ -migrative.*

*Proof.* Suppose that  $F$  is  $(\alpha, U)$ -migrative. Taking  $x = e$  and  $y = 0$  in Eq.(14), we have  $k\alpha = F(\alpha, 0) = F(U(\alpha, e), 0) = F(e, U(\alpha, 0)) = F(e, 0) = ke$ , which means  $k = 0$  because of  $\alpha > e$ . While taking  $x = e$  and  $y = 1$  in Eq.(14), we have  $(1 - k)\alpha + k = F(\alpha, 1) = F(U(\alpha, e), 1) = F(e, U(\alpha, 1)) = F(e, 1) = (1 - k)e + k$ , which means  $k = 1$  because of  $\alpha > e$ . It is obvious that  $k = 1$  contradicts the result  $k = 0$ .  $\square$

**Theorem 4.6.** *Let  $\alpha \in [0, e]$ ,  $F \in \mathbb{GM}$  be a GM aggregation operator with  $F(0, 1) = k$ , and  $U \in U_e^{\min}$  be a semi-uninorm, where the underlying semi-t-norm  $T_U$  of  $U$  is associative. Then  $U$  is  $(\alpha, F)$ -migrative if and only if one of the following statements hold.*

(i)  *$F$  is a commutative semi-t-norm such that  $F(\alpha, y) = U(\alpha, y) = U(y, \alpha)$  for  $y \in [0, e]$  and  $F(\alpha, y) = \alpha$  for  $y \in [e, 1]$ . Moreover, if the underlying semi-t-norm  $T_U$  of  $U$  is continuous and  $U(\alpha, \alpha) = \alpha$ , then  $U$  is  $(\alpha, F)$ -migrative if and only if  $F(\alpha, y) = U(\alpha, y) = \min(\alpha, y)$  for  $y \in [0, 1]$ .*

(ii)  *$\alpha = 0$  and one of the following statements holds:*

- *$F$  is a commutative semi-t-norm.*
- *$e = 1$  and  $U(k, y) = U(y, k) = ky$  for all  $y \in [0, 1]$ .*

*Proof.* Let  $x = 0$  and  $y = e$  in Eq.(15), then  $k\alpha = F(\alpha, 0) = U(F(\alpha, 0), e) = U(0, F(\alpha, e)) = 0$ , which means  $k = 0$  or  $\alpha = 0$ .

(i) Assume  $k = 0$ . We know that  $F$  is a commutative semi-t-norm because of  $F(x, 1) = F(1, x) = x$ . Now let us prove that  $F(\alpha, e) = \alpha$ . It follows that  $F(\alpha, e) = U(F(\alpha, e), 1) = U(e, F(\alpha, 1)) = F(\alpha, 1) = \alpha$  from  $F(\alpha, e) \leq F(1, e) = e$ . So we obtain  $F(\alpha, y) = \alpha$  for any  $y \in [e, 1]$  because  $F(\alpha, 1) = \alpha$ . For any  $y \in [0, e]$ , then we have  $U(\alpha, y) = U(F(\alpha, e), y) = U(e, F(\alpha, y)) = F(\alpha, y)$  and  $F(\alpha, y) = U(F(\alpha, y), e) = U(y, F(\alpha, e)) = U(y, \alpha)$ , that is,  $F(\alpha, y) = U(\alpha, y) = U(y, \alpha)$  for  $y \in [0, e]$ . Conversely, in order to verify that  $U$  is  $(\alpha, F)$ -migrative, we need consider the following cases:

If  $(x, y) \in [0, e]^2$ , then  $U(F(\alpha, x), y) = U(U(x, \alpha), y) = U(x, U(\alpha, y)) = U(x, F(\alpha, y))$ .

If  $(x, y) \in [e, 1]^2$ , then  $U(F(\alpha, x), y) = U(\alpha, y) = \alpha = U(x, \alpha) = U(x, F(\alpha, y))$ .

If  $(x, y) \in [0, e] \times [e, 1]$ , then  $U(F(\alpha, x), y) = F(\alpha, x) \wedge y = F(\alpha, x) = U(x, \alpha) = U(x, F(\alpha, y))$ .

If  $(x, y) \in [e, 1] \times [0, e]$ , then  $U(F(\alpha, x), y) = U(\alpha, y) = F(\alpha, y) = x \wedge F(\alpha, y) = U(x, F(\alpha, y))$ .

Moreover, if the underlying semi-t-norm  $T_U$  of  $U$  is continuous and  $U(\alpha, \alpha) = \alpha$ , then we can easily obtain that  $F(\alpha, y) = U(\alpha, y) = \min(\alpha, y)$  for any  $y \in [0, e]$ . And from the structure of  $U$ , it follows that  $U(\alpha, y) = \alpha = \min(\alpha, y)$  for  $y \in [e, 1]$ . Thus we have  $F(\alpha, y) = U(\alpha, y) = \min(\alpha, y)$  for all  $x, y \in [0, 1]$ . Conversely, it is easy to verify that  $U$  is  $(\alpha, F)$ -migrative.

(ii) Assume  $\alpha = 0$ . Taking  $x = 1$  and  $y = e$  in Eq.(15), then we have  $k = F(0, 1) = U(F(0, 1), e) = U(1, F(0, e)) = U(1, ke) = ke$ , which means  $k = 0$  or  $e = 1$ . When  $k = 0$ , that is,  $F$  is a commutative semi-t-norm. And we can easily obtain the converse case  $U(F(0, x), y) = U(0, y) = 0 = U(x, 0) = U(x, F(0, y))$  for all  $(x, y) \in [0, 1]^2$ . When  $e = 1$ , that is,  $U$  is a associative semi-t-norm. Let  $x = 1$  in Eq.(15), then  $U(k, y) = U(F(0, 1), y) = U(1, F(0, y)) =$

$F(0, y) = ky$  and  $ky = F(0, y) = U(F(0, y), 1) = U(y, F(0, 1)) = U(y, k)$  for any  $y \in [0, 1]$ . Conversely, it is obvious that  $U(F(0, x), y) = U(kx, y) = U(U(x, k), y) = U(x, U(k, y)) = U(x, ky) = U(x, F(0, y))$  for all  $(x, y) \in [0, 1]^2$ .  $\square$

**Theorem 4.7.** *Let  $\alpha \in (e, 1]$ ,  $F \in \mathbb{GM}$  be a GM aggregation operator and  $U \in U_e^{\min}$  be a semi-uninorm. Then  $U$  is  $(\alpha, F)$ -migrative if and only if  $\alpha = 1$  and  $F$  is a commutative semi-t-norm.*

*Proof.* Let  $x = 0$  and  $y = e$  in Eq.(15), then  $k\alpha = F(\alpha, 0) = U(F(\alpha, 0), e) = U(0, F(\alpha, e)) = 0$ , which means  $k = 0$  or  $\alpha = 0$ . Note that  $\alpha = 0$  contradicts the assumption  $\alpha > e$ . So the only one possibility is  $k = 0$ , which means  $F$  is a commutative semi-t-norm. Now suppose  $\alpha \in (e, 1)$ , then similar to the proof of Theorem 4.6, we have  $F(\alpha, y) = U(\alpha, y)$  for all  $y \in [0, 1]$ . While from the structure of  $U$  and  $F$ , it follows that  $\alpha < y \leq U(\alpha, y) = F(\alpha, y) \leq \alpha$  for any  $y \in (\alpha, 1)$ . This is a contradiction. So we obtain  $\alpha = 1$ . Conversely, it is obvious that  $U(F(1, x), y) = U(x, y) = U(x, F(1, y))$  for all  $x, y \in [0, 1]$ .  $\square$

## 4.2 Migrativity for $F \in \mathbb{GM}$ and $U \in U_e^{\max}$

In this subsection, we will only list the results of migrativity for  $F \in \mathbb{GM}$  and  $U \in U_e^{\max}$ , because they can be derived in the manner that be used in subsection 4.1.

**Theorem 4.8.** *Let  $\alpha \in [0, e)$ ,  $F \in \mathbb{GM}$  be a GM aggregation operator and  $U \in U_e^{\max}$  be a semi-uninorm. Then  $F$  is not  $(\alpha, U)$ -migrative.*

**Theorem 4.9.** *Let  $\alpha \in (e, 1]$ ,  $F \in \mathbb{GM}$  be a GM aggregation operator and  $U \in U_e^{\max}$  be a semi-uninorm. Then  $F$  is  $(\alpha, U)$ -migrative if and only if  $F$  is a commutative semi-t-conorm such that  $F(y, \alpha) = U(\alpha, y)$  for  $y \in [e, 1]$ ,  $F(\alpha, y) = \alpha$  for  $y \in [0, e]$ , and  $F(F(\alpha, x), y) = F(x, F(\alpha, y))$  for any  $(x, y) \in [e, 1] \times [0, 1]$ . Moreover, if the underlying semi-t-conorm  $S_U$  of  $U$  is continuous and  $U(\alpha, \alpha) = \alpha$ . Then  $F$  is  $(\alpha, U)$ -migrative if and only if  $F(\alpha, x) = U(\alpha, x) = \max(\alpha, x)$  for all  $x \in [0, 1]$ .*

**Theorem 4.10.** *Let  $\alpha \in [0, e)$ ,  $F \in \mathbb{GM}$  be a GM aggregation operator and  $U \in U_e^{\max}$  be a semi-uninorm. Then  $U$  is  $(\alpha, F)$ -migrative if and only if  $\alpha = 0$  and  $F$  is a commutative semi-t-conorm.*

**Example 4.11.** *Take  $e = 0.6$ , let  $F = S_b$ ,  $T_U = T_a$  and  $S_U = \max$ . That is,  $F$  is a GM aggregation operator and the structures of  $U$  is:*

$$U(x, y) = \begin{cases} 0.6T_a(\frac{x}{0.6}, \frac{y}{0.6}) & \text{if } (x, y) \in [0, 0.6]^2, \\ \max(x, y) & \text{otherwise.} \end{cases}$$

*Then it follows from Theorem 4.8 that  $F$  is not  $(0, U)$ -migrative because  $F(U(0, 0.6), 0) = F(0, 0) = 0 \neq 0.6 = F(0.6, 0) = F(0.6, U(0, 0))$ . While we know that  $U$  is  $(0, F)$ -migrative from Theorem 4.10.*

**Theorem 4.12.** *Let  $\alpha \in [e, 1]$ ,  $F \in \mathbb{GM}$  be a GM aggregation operator with  $F(0, 1) = k$ , and  $U \in U_e^{\max}$  be a semi-uninorm, where the underlying semi-t-norm  $T_U$  and semi-t-conorm  $S_U$  are associative. Then  $U$  is  $(\alpha, F)$ -migrative if and only if one of the following statements hold.*

(i)  *$F$  is a commutative semi-t-conorm such that  $F(\alpha, y) = U(\alpha, y) = U(y, \alpha)$  for  $y \in [e, 1]$  and  $F(\alpha, y) = \alpha$  for  $y \in [0, e]$ . Moreover, if the underlying semi-t-conorm  $S_U$  of  $U$  is continuous and  $U(\alpha, \alpha) = \alpha$ , then  $U$  is  $(\alpha, F)$ -migrative if and only if  $F(\alpha, y) = U(\alpha, y) = \max(\alpha, y)$  for  $y \in [0, 1]$ .*

(ii)  *$\alpha = 1$  and one of the following statements holds:*

- *$F$  is a commutative semi-t-norm.*
- *$F$  is a commutative semi-t-conorm.*
- *$e = 0$  and  $U(k, y) = U(y, k) = (1 - k)y + k$  for all  $y \in [0, 1]$ .*

## 5 Conclusions

In this paper, we investigated the migrativity equations between semi-t-operators or semi-uninorms, and Mayor's aggregation operators. The results that we obtained are complete and different from the known ones concerning migrativity for t-norms, t-conorms, uninorms and nullnorms, so we can provide a reference value for the operator selection and failure rate reduction in some aggregation process. Moreover, we listed several examples in order to illustrate our results. In future work, we will concentrate on migrativity for other operators.

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## References

- [1] C. Alsina, M. J. Frank, B. Schweizer, *Associative Functions: Triangular Norms and Copuias*, World Scientific, New Jersey, 2006.
- [2] H. Bustince, B. De Baets, J. Fernandez, R. Mesiar, J. Montero, *A generalization of the migrativity property of aggregation functions*, *Information Sciences*, **191** (2012), 76-85.
- [3] H. Bustince, J. Montero, R. Mesiar, *Migrativity of aggregation functions*, *Fuzzy Sets and Systems*, **160** (2009), 766-777.
- [4] T. Calvo, B. De Baets, J. C. Fodor, *The functional equations of Frank and Alsina for uninorms and nullnorms*, *Fuzzy Sets and Systems*, **120** (2001), 385-394.
- [5] T. Calvo, G. Mayor, R. Mesiar (Eds.), *Aggregation Operators*, New Trends and Applications, Physica-Verlag, Heidelberg, 2002.
- [6] P. Drygaś, *Distributivity between semi-t-operators and semi-nullnorms*, *Fuzzy Sets and Systems*, **264** (2015), 100-109.
- [7] F. Durante, P. Sarkoci, *A note on the convex combinations of triangular norms*, *Fuzzy Sets and Systems*, **159** (2008), 77-80.
- [8] J. Fodor, I. J. Rudas, *On continuous triangular norms that are migrative*, *Fuzzy Sets and Systems*, **158** (2007), 1692-1697.
- [9] J. Fodor, I. J. Rudas, *Migrative t-norms with respect to continuous ordinal sums*, *Information Sciences*, **181** (2011), 4860-4866.
- [10] J. Fodor, I. J. Rudas, *An extension of the migrative property for triangular norms*, *Fuzzy Sets and Systems*, **168** (2011), 70-80.
- [11] D. Jočić, I. Štajner-Papuga, *Distributivity equations and Mayor's aggregation operators*, *Knowledge Based Systems*, **52** (2013), 194-200.
- [12] E. P. Klement, R. Mesiar, E. Pap, *Triangular Norms*, Kluwer Acad. Publ., Dordrecht, 2000.
- [13] G. J. Klir, B. Yuan, *Fuzzy Sets and Fuzzy Logic, Theory and Application*, Prentice Hall PTR, Upper Saddle River, New Jersey, 1995.
- [14] M. Mas, G. Mayor, J. Torrens, *T-operators*, *Int. J. Uncertain. Fuzziness Knowl.-Based Systems*, **7** (1999), 31-50.
- [15] M. Mas, G. Mayor, J. Torrens, *The modularity condition for uninorms and t-operators*, *Fuzzy Sets and Systems*, **126** (2002), 207-218.
- [16] M. Mas, G. Mayor, J. Torrens, *The distributivity condition for uninorms and t-operators*, *Fuzzy Sets and Systems*, **128** (2002), 209-225.
- [17] M. Mas, M. Monserrat, D. Ruiz-Aguilera, J. Torrens, *Migrativity of uninorms over t-norms and t-conorms*, *Advanced Intelligent Systems and Computing*, **228** (2012), 155-166.
- [18] M. Mas, M. Monserrat, D. Ruiz-Aguilera, J. Torrens, *On migrative t-conorms and uninorms*, *Information Sciences*, **299** (2012), 286-295.
- [19] M. Mas, M. Monserrat, D. Ruiz-Aguilera, J. Torrens, *An extension of the migrative property for uninorms*, *Information Sciences*, **246** (2013), 191-198.

- [20] M. Mas, M. Monserrat, D. Ruiz-Aguilera, J. Torrens, *Migrative uninorms and nullnorms over  $t$ -norms and  $t$ -conorms*, Fuzzy Sets and Systems, **261** (2015), 20-32.
- [21] G. Mayor, *Contribució a l'estudi dels models matemàtics per a la lògica de la vaguetat*, Ph.D. Thesis, Universitat de les Illes Balears, 1984.
- [22] R. Mesiar, H. Bustince, J. Fernandez, *On the  $\alpha$ -migrativity of semicopulas, quasi-copulas, and copulas*, Information Sciences, **180** (2010), 1967-1976.
- [23] J. Montero, V. López, D. Gómez, *The role of fuzziness in decision making*, Fuzzy Logic, **215** (2007), 337-349.
- [24] F. Qin, D. Ruiz-Aguilera, *On the  $\alpha$ -migrativity of idempotent uninorms*, International Journal Uncertain Fuzziness Knowledge-Based Systems, **23** (2015), 105-115.
- [25] F. Qin, Y. M. Wang, *Distributivity between semi- $t$ -operators and Mayor's aggregation operators*, Information Sciences, **346-347** (2016), 6-16.
- [26] B. Roy, *Decision sciences or decision aid sciences*, European Journal of Operational Research, **66** (1993), 184-203.
- [27] Y. Su, W. Zong, H. W. Liu, *A note on an extension of the migrative property for uninorms*, Information Sciences, **281** (2014), 334-337.
- [28] Y. Su, W. Zong, H. W. Liu, F. X. Zhang, *On migrativity property for uninorms*, Information Sciences, **300** (2015), 114-123.
- [29] Y. Su, W. Zong, H. W. Liu, *Migrativity property for uninorms*, Fuzzy Sets and Systems, **287** (2016), 213-226.
- [30] Y. Su, W. Zong, H. W. Liu, P. J. Xue, *Migrativity property for uninorms and semi- $t$ -operators*, Information Sciences, **325** (2015), 455-465.
- [31] W. Zong, Y. Su, H. W. Liu, *Migrativity property for nullnorms*, International Journal Uncertain Fuzziness Knowledge-Based Systems, **22** (2014), 749-759.