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Multiple attribute group decision making with linguistic variables and complete unknown weight information

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Abstract

Interval type-2 fuzzy sets, each of which is characterized by the footprint of uncertainty, are a very useful means to depict the linguistic information in the process of decision making. In this article, we investigate the group decision making problems in which all the linguistic information provided by the decision makers is expressed as interval type-2 fuzzy decision matrices where each of the elements is characterized by interval type-2 fuzzy set, and the information about attribute weights is completely unknown. We first introduce the average centroid matrix of the interval type-2 fuzzy decision matrix, and then utilize the interval type-2 fuzzy averaging operator to aggregate all individual interval type-2 fuzzy decision matrices into a collective interval type-2 fuzzy decision matrix. Based on the average centroid matrix of the collective interval type-2 fuzzy decision matrix and information theory, we develop an optimization model by which a straightforward formula for deriving attribute weights can be obtained. Furthermore, based on the interval type-2 fuzzy averaging operator, we utilize the average centroid measure to give an approach to ranking the given alternatives and then selecting the most desirable one(s). Finally, we give an illustrative example.

Keywords: Multi-attribute group decision making, the average centroid matrix, interval type-2 fuzzy decision matrix, interval type-2 fuzzy weighted arithmetic averaging operator.

1 Introduction

Multi-attribute group decision making (MAGDM) problems are widespread in real life decision situations, and it is to find the most desirable alternative(s) from a set of feasible alternatives according to the decision information about attribute weights and attribute values provided by a group of decision makers. The estimation of the attribute values plays an important role in MAGDM. In the process of decision making, the information about attribute values is usually uncertain or fuzzy due to the increasing complexity of the socio-economic environment and the vagueness of inherent subjective nature of human thinking. Therefore, a better approach could be the use of linguistic information.

Many representations of the linguistic information have been presented for assessing the qualitative aspects of the real world. For example, Zadeh [52] introduced linguistic information as a tool for representing imprecise expressions such as low, high and very high by using fuzzy sets [51]. Herrera and Martinez [11] expressed linguistic information by means of 2-tuples and developed a computational technique for computing with words (CW) without any loss of information. Wang and Hao [35] provided a new 2-tuple fuzzy linguistic representation model for CW, which is based on the concept of symbolic proportion. J.M. Merigó and Gil-Lafuente [29] studied induced 2-tuple linguistic generalized aggregation operators and their application in decision-making. Liu et al. [19] proposed some 2-tuple linguistic Heronian aggregation operators and applied them to MAGDM problem with 2-tuple linguistic information. In [43, 44, 27, 28], some linguistic weighted aggregation operators were developed and different practical methods for group decision making with linguistic preference relations were proposed. Xu [45] proposed two uncertain linguistic aggregation operators and developed an approach to multiple attribute group decision making with uncertain linguistic

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information. Herrera et al. [12] introduced a representation model for unbalanced linguistic information and developed an unbalanced linguistic computational model that uses the 2-tuple fuzzy linguistic computational model to accomplish processes of CW with unbalanced term sets in a precise way and without loss of information. Xu [46] investigated the MAGDM problems having multiple sources of uncertain linguistic information assessed in different linguistic label sets. Wang et al. [36] introduced the conversion between linguistic variables and clouds, and developed a linguistic multi-criteria group decision-making method based on the cloud weighted aggregation operators. In [9, 37], the concentration and dilation of intuitionistic fuzzy sets were used for dealing with various linguistic hedges. Morente-Molinera et al. [32] provided insights about the evolution of multi-granular fuzzy linguistic modeling approaches during the last years, and presented some possible approaches to improve the current multi-granular linguistic methodologies. Liao et al. [18] investigated several different types of (weighted) correlation coefficients for the hesitant fuzzy linguistic term set. Xue et al. [47] an integrated model based on hesitant 2-tuple linguistic term sets and an extended QUALIFLEX approach for handling robot selection problems with incomplete weight information. Cabrerizo et al. [4] developed a method based on PSO and granular computing of linguistic information to solve group decision making problems defined in heterogeneous contexts. Yan et al. [49] dealt with linguistic MADM with multiple priorities in terms of strict priority hierarchy or weak priority hierarchy. In [50, 30], an overview of fuzzy research, especially, the current research status on linguistic decision making were analyzed through different visualization method.

In many MAGDM situations, the decision makers may provide their preferences for alternatives to interval type-2 fuzzy linguistic terms [6][7]. In such cases, the preference information provided by decision makers is very suitable to be expressed in interval type-2 fuzzy sets (IT2 FSs) [24], which provide more exibility to present fuzziness and uncertainty than traditional type-1 fuzzy sets. In recent years, some authors have applied the IT2 FS theory to the field of decision making. Wu and Mendel [39][40] presented a method using the linguistic weighted average and IT2 FSs for handling fuzzy multiple criteria hierarchical group decision-making problems, Wu and Mendel's fuzzy multiple criteria hierarchical group decision making method is to make decisions by means of aggregating the opinions of decision-makers. Chen and Lee [8] present a new method for handling fuzzy multiple criteria hierarchical group decision-making problems based on arithmetic operations and fuzzy preference relations of IT2 FSs. Chen and Lee [6][16] present an interval type-2 fuzzy TOPSIS method to handle fuzzy multiple attributes group decision-making problems based on IT2 FSs, and use some examples to illustrate the fuzzy multiple attributes group decision-making process of the proposed method. Chen and Lee [7][17] present a new method to handle fuzzy MAGDM problems based on the ranking values and the arithmetic operations of IT2 FSs. In the situations where the information about attribute weights provided by the DMs is usually incomplete because the increasing complexity of the socio-economic environment makes it less and less possible for the DMs to consider all relevant aspects of a problem [15][14][33]. Wang and Liu [38] investigated the IT2 fuzzy group decision making problems with partial weight information, in which nonlinear optimization model with inequality constraints were established to determine the weights of attributes. Ma et al. [20] utilized some interval type-2 fuzzy dependent and interval type-2 power aggregation operators to develop approaches for MAGDM with interval type-2 fuzzy numbers, in which begin with natural linguistic evaluation. Moharrer et al. [31] proposed a novel two-phase methodology based on IT2 FSs to model the human perceptions of the linguistic terms used to describe the online services satisfaction. Wu and Liu [42] incorporated clustering analysis and information aggregation operator into the problems of large-scale multiple-criteria group decision-making with IT2 FSs. Abdullah and Najib [1] proposed a new fuzzy analytic hierarchy process characterized by IT2 FS for linguistic variables. Abdullah and Zulkifli [2] proposed the integration of fuzzy AHP and IT2 fuzzy DEMATEL and applied it to human resources management.

Due to time pressure or because of the complexity and uncertainty of the problem, the information about attribute weights may be completely unknown and the attribute weights can only be derived from the given linguistic decision-making information. As presented above, many approaches have been developed to dealing with the decision making problems based on IT2 FS theory, however, all of them are not suitable to handle this case. In this paper, we shall focus our attention on the issue of MAGDM under linguistic environment where all the information provided by the decision makers is characterized by IT2 FSs and the information about the attribute weights is completely unknown.

To do that, the remainder of this paper is organized as follows. Section 2 give a review of basic concepts and operations related to IT2 FS. Section 3 presents the considered problem and constructs the average centroid matrix of the interval type-2 fuzzy decision matrix. Based on the collective average centroid matrix and information theory, we establish a deviation-based optimization model to determine the weights of attributes and develop an approach to MAGDM with linguistic information. Section 4 provides a practical example to illustrate the developed approach. Finally, we conclude the paper in Section 5.

2 Preliminaries

The type-2 fuzzy set $\tilde{A}$ also can be represented as follows:

$$\tilde{A} = \int_{x\in X}\int_{u\in[0,1]} \mu_{\tilde{A}}(x,u)/(x,u) = \int_{x\in X}\left[\int_{u\in[0,1]} \mu_{\tilde{A}}(x,u)/u\right]\Big/ x \tag{1}$$

where x is the primary variable of $\tilde{A}$, $J_x = \{u \in [0,1], \mu_{\tilde{A}}(x,u) > 0\}$ is the primary membership of x, u is the secondary variables of $\tilde{A}$, and $\int_{u\in[0,1]} \mu_{\tilde{A}}(x,u)/u$ is the secondary membership function (MF) at x. $\int\int$ denotes the union over all admissible x and u. For discrete universes of discourse, $\int$ is replaced by $\sum$.

Definition 2.1. [23, 26] *Let $\tilde{A}$ be a type-2 fuzzy set in the universe of discourse X represented by the type-2 membership function $\mu_{\tilde{A}}$. If all $\mu_{\tilde{A}}(x,u) = 1$, then $\tilde{A}$ is called an IT2 FS. An IT2 FS $\tilde{A}$ can be regarded as a special case of a type-2 fuzzy set, shown as follows:*

$$\tilde{A} = \int_{x\in X}\int_{u\in[0,1]} 1/(x,u) = \int_{x\in X}\left[\int_{u\in[0,1]} 1/u\right]\Big/ x \tag{2}$$

where x and u are the primary variable and primary membership of $\tilde{A}$, respectively, $J_x = \{u \in [0,1], \mu_{\tilde{A}}(x,u) = 1\}$ is the primary membership of x, u is the secondary variables of $\tilde{A}$, and $\int_{u\in[0,1]} 1/u$ is the secondary membership function (MF) at x.

Uncertainty about $\tilde{A}$ is conveyed by the union of all primary memberships, called the footprint of uncertainty of $\tilde{A}$ (FOU($\tilde{A}$)), i.e.,

$$\mathrm{FOU}(\tilde{A}) = \bigcup_{x\in X} J_x \tag{3}$$

Depending on the subsets J_x, there are more special cases of IT2 FS. Suppose X be connected, then IT2 FS are the closed IT2 FS (CIT2 FS) if J_x is a closed interval for every $x \in X$[26][34][22].

In this paper, we take the CIT2 FS with trapezoidal FOU. The trapezoidal IT2 FS $\tilde{A}_i$ is shown in Fig.1, where the twelve points to determine an FOU. $(a_{i1}^U, a_{i2}^U, a_{i3}^U, a_{i4}^U, H_1(\tilde{A}_i^U), H_2(\tilde{A}_i^U))$ determines a trapezoidal UMF $\tilde{A}_i^U$ with the heights $H_1(\tilde{A}_i^U)$ and $H_2(\tilde{A}_i^U)$, and $(a_{i1}^L, a_{i2}^L, a_{i3}^L, a_{i4}^L, H_1(\tilde{A}_i^L), H_2(\tilde{A}_i^L))$ determines a trapezoidal LMF $\tilde{A}_i^L$ with the heights $H_1(\tilde{A}_i^L)$ and $H_2(\tilde{A}_i^L)$. The FOU is shown as the shaded region, and it is bounded by an upper MF (UMF) and a lower MF (LMF), both of which are T1 FSs.

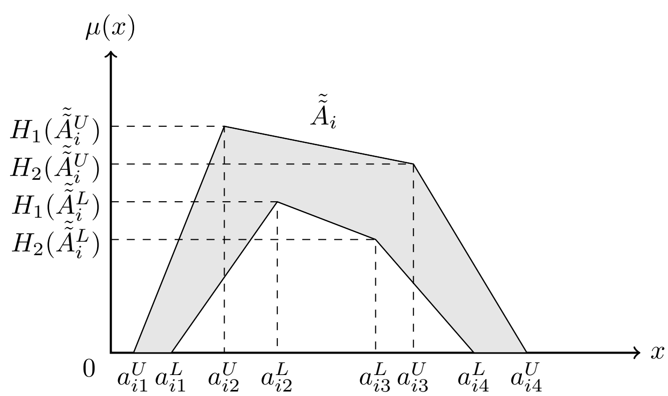

Figure 1: The FOU of the trapezoidal IT2 FS $\tilde{A}_i$.

In this paper, we present a method to use IT2 FSs for handling fuzzy multi-attribute group decision making problems, where the reference points and the heights of the upper and the lower membership functions of IT2 FSs are used to characterize IT2 FSs.

Definition 2.2. [17] *The addition operation on the trapezoidal IT2 FSs* $\tilde{A}_1 = (\tilde{A}_1^U,\ \tilde{A}_1^L) = ((a_{11}^U, a_{12}^U,\ a_{13}^U, a_{14}^U;\ H_1(\tilde{A}_1^U), H_2(\tilde{A}_1^U)),\ (a_{11}^L, a_{12}^L,\ a_{13}^L, a_{14}^L;\ H_1(\tilde{A}_1^L), H_2(\tilde{A}_1^L)))$ *and* $\tilde{A}_2 = (\tilde{A}_2^U, \tilde{A}_2^L) = ((a_{21}^U, a_{22}^U,\ a_{23}^U, a_{24}^U;\ H_1(\tilde{A}_2^U),\ H_2(\tilde{A}_2^U)),\ (a_{21}^L,\ a_{22}^L,\ a_{23}^L, a_{24}^L;\ H_1(\tilde{A}_2^L), H_2(\tilde{A}_2^L)))$ *is defined as follows:*

$$\begin{aligned}\tilde{A}_1 \oplus \tilde{A}_2 =&(\tilde{A}_1^U, \tilde{A}_1^L) \oplus (\tilde{A}_2^U, \tilde{A}_2^L)\\ =&((a_{11}^U + a_{21}^U, a_{12}^U + a_{22}^U, a_{13}^U + a_{23}^U, a_{14}^U + a_{24}^U; \min\{H_1(\tilde{A}_1^U), H_1(\tilde{A}_2^U)\},\\ &\min\{H_2(\tilde{A}_1^U), H_2(\tilde{A}_2^U)\}), (a_{11}^L + a_{21}^L, a_{12}^L + a_{22}^L, a_{13}^L + a_{23}^L, a_{14}^L + a_{24}^L;\\ &\min\{H_1(\tilde{A}_1^L), H_1(\tilde{A}_2^L)\}, \min\{H_2(\tilde{A}_1^L), H_2(\tilde{A}_2^L)\}))\end{aligned}$$

Definition 2.3. [17] *The multiplication operation on the trapezoidal IT2 FS*

$$\begin{aligned}\tilde{A}_1 &= (\tilde{A}_1^U, \tilde{A}_1^L)\\ &= ((a_{11}^U, a_{12}^U, a_{13}^U, a_{14}^U; H_1(\tilde{A}_1^U), H_2(\tilde{A}_1^U)), (a_{11}^L, a_{12}^L, a_{13}^L, a_{14}^L; H_1(\tilde{A}_1^L), H_2(\tilde{A}_1^L)))\end{aligned}$$

and the crisp value k *is defined as*

$$\begin{aligned}k\tilde{A}_1 = ((ka_{11}^U, ka_{12}^U, ka_{13}^U, ka_{14}^U; H_1(\tilde{A}_1^U),&H_2(\tilde{A}_1^U)),\\ &(ka_{11}^L, ka_{12}^L, ka_{13}^L, ka_{14}^L; H_1(\tilde{A}_1^L), H_2(\tilde{A}_1^L)))\end{aligned}$$

Based on the algebraic operations on IT2 FSs, we introduce an IT2 fuzzy weighted averaging (IT2FWA) operator as

Definition 2.4. *Let* $\tilde{A}_j = (\tilde{A}_j^U, \tilde{A}_j^L) = ((a_{j1}^U,\ a_{j2}^U, a_{j3}^U, a_{j4}^U;\ H_1(\tilde{A}_j^U), H_2(\tilde{A}_j^U)),\ (a_{j1}^L, a_{j2}^L,\ a_{j3}^L, a_{j4}^L;\ H_1(\tilde{A}_j^L), H_2(\tilde{A}_j^L)))$ $(j = 1, 2, \cdots, n)$ *be a collection of trapezoidal IT2 FSs,* $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T$ *be the weighting vector of* α_j $(j = 1, 2, \cdots, n)$ *with* $0 \leqslant \omega_j \leqslant 1$ *and* $\sum_{j=1}^n \omega_j = 1$*, then the IT2FWA operator is expressed as*

$$\begin{aligned}&IT2FWA\left(\tilde{A}_j, \tilde{A}_2, \cdots, \tilde{A}_n\right) = \sum_{j=1}^{n} (\omega_j \tilde{A}_j) =\\ &\left(\left(\sum_{j=1}^{k} \omega_j a_{j1}^U, \sum_{j=1}^{k} \omega_j a_{12}^U, \sum_{j=1}^{k} \omega_j a_{13}^U, \sum_{j=1}^{k} \omega_j a_{14}^U; \min_j \left\{H_1(\tilde{A}_j^U)\right\}, \min_j \left\{H_2(\tilde{A}_j^U)\right\}\right),\right.\\ &\left.\left(\sum_{j=1}^{k} \omega_j a_{11}^L, \sum_{j=1}^{k} \omega_j a_{12}^L, \sum_{j=1}^{k} \omega_j a_{13}^L, \sum_{j=1}^{k} \omega_j a_{14}^L; \min_j \left\{H_1(\tilde{A}_j^L)\right\}, \min_j \left\{H_2(\tilde{A}_j^L)\right\}\right)\right).\end{aligned} \tag{4}$$

The centroid of an IT2 FS provides a measure of the uncertainty of such a FS, the centroid of IT2 FS $\tilde{A}$, $c(\tilde{A})$, is defined as follows.

Definition 2.5. [13, 21] *The centroid* $C(\tilde{A})$ *of an IT2 FS* $\tilde{A}$ *is the union of the centroids of all its embedded T1 FSs ,* $c(A_e)$*, that is,*

$$C(\tilde{A}) \equiv \bigcup_{\forall A_e} c(A_e) = [c_l(\tilde{A}), c_u(\tilde{A})], \tag{5}$$

where $\bigcup$ *is the union operation, and*

$$c_l(\tilde{A}) = \min_{\forall A_e} c(A_e)$$

$$c_u(\tilde{A}) = \max_{\forall A_e} c(A_e)$$

$$c(A_e) = \frac{\sum_{i=1}^N x_i \mu_{A_e}(x_i)}{\sum_{i=1}^N \mu_{A_e}(x_i)}$$

Karnik and Mendel [13, 25] have developed iterative algorithms - KM Algorithms - for computing $c_l(\tilde{A})$ and $c_u(\tilde{A})$ in the following structure:

$$c_l(\tilde{A}) = \frac{\sum_{i=1}^L x_i \overline{\mu}_{\tilde{A}}(x_i) + \sum_{i=L+1}^N x_i \underline{\mu}_{\tilde{A}}(x_i)}{\sum_{i=1}^L \overline{\mu}_{\tilde{A}}(x_i) + \sum_{i=L+1}^N \underline{\mu}_{\tilde{A}}(x_i)}$$

$$c_u(\tilde{A}) = \frac{\sum_{i=1}^{R} x_i \underline{\mu}_{\tilde{A}}(x_i) + \sum_{i=R+1}^{N} x_i \overline{\mu}_{\tilde{A}}(x_i)}{\sum_{i=1}^{R} \underline{\mu}_{\tilde{A}}(x_i) + \sum_{i=R+1}^{N} \overline{\mu}_{\tilde{A}}(x_i)}$$

where L and R are called switch points, and these switch points are determined by the KM algorithms [21, 13, 41].

Centroid-based ranking method [24]: First compute the average centroid $\bar{c}(\tilde{A}_i)$ for each IT2 FS $\tilde{A}_i$

$$c(\tilde{A}_i) = \frac{c_l(\tilde{A}_i) + c_u(\tilde{A}_i)}{2}, i = 1, 2, \ldots, n \tag{6}$$

and then sort $c(\tilde{A}_i)$ to obtain the rank of $\tilde{A}_i$. The larger the average centroid $\bar{c}(\tilde{A}_i)$, the greater the IT2 FS $\tilde{A}_i$.

Note that the ranking method can be viewed as a generalization of Yager's first ranking method for T1 FSs to IT2 FSs [48].

3 The maximizing deviation method for group multiple attribute decision making under linguistic environment

For a MAGDM problem, let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set of alternatives, $D = \{d_1, d_2, \ldots, d_l\}$ be the set of decision makers, and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)^T$ be the weight vector of decision makers, such as $\lambda_k \geqslant 0, k = 1, 2, \ldots, l$, and $\sum_{k=1}^{l} \lambda_k = 1$. Let $G = \{g_1, g_2, \ldots, g_m\}$ be a finite set of attributes, and $\omega = (\omega_1, \omega_2, \ldots, \omega_m)^T$ be the weight vector of attributes, such as $\omega_j \geqslant 0$, $\sum_{j=1}^{m} \omega_j = 1$. Let $\tilde{A}^{(k)} = (\tilde{A}_{ij}^{(k)})_{n\times m}$ be an interval type-2 fuzzy decision matrix, where $\tilde{A}_{ij}^{(k)}$ is an IT2 FS, provided by the decision maker $d_k \in D$ for the alternative $x_i \in X$ with respect to the attribute $g_j \in G$.

In general, attributes can be classified into two types: benefit attributes and cost attributes. In other words, the attribute set G can be divided into two subsets: G_1 and G_2, which are the subsets of benefit attributes and cost attributes, respectively. Furthermore, $G_1 \cup G_2 = G$ and $G_1 \cap G_2 = \emptyset$, where $\emptyset$ is empty set. The decision matrices $\tilde{A}^{(k)} = (\tilde{A}_{ij}^{(k)})_{n\times m}$ to be normalized unless all the attributes G are of the benefit attributes. In this paper we choose the following normalization formula to derive the normalized decision matrices $\tilde{B}^{(k)} = (\tilde{B}_{ij}^{(k)})_{n\times m}$:

$$\tilde{B}_{ij}^{(k)} = \begin{cases} \tilde{A}_{ij}^{(k)} & j \in G_1 \\ (\tilde{A}_{ij}^{(k)})^c & j \in G_2 \end{cases} \tag{7}$$

where $(\tilde{A}_{ij}^{(k)})^c$ is the complement of $\tilde{A}_{ij}^{(k)}$. Hence, we obtain the normalized decision matrices $\tilde{B}^{(k)} (k = 1, 2, \ldots, l)$.

In the process of group decision making, we need to fuse all the individual decision opinion into a group opinion so as to make a final decision. To do that, we use the IT2FWA (4) operator to aggregate all individual normalized decision matrices $\tilde{B}^{(k)} = (\tilde{B}_{ij}^{(k)})_{n\times m} (k = 1, 2, \ldots, l)$ into the collective normalized decision matrix $\tilde{B} = (\tilde{B}_{ij})_{n\times m}$, where

$$\tilde{B}_{ij} = \sum_{k=1}^{l} \lambda_k \tilde{B}_{ij}^{(k)}. \tag{8}$$

In the situations where the information about attribute weights is completely known, that is, the weight vector $\omega = (\omega_1, \omega_2, \ldots, \omega_m)^T$ of the attributes $g_j (j = 1, 2, \ldots, m)$ can be completely determined in advance, then based on the collective normalized decision matrix $\tilde{B} = (\tilde{B}_{ij})_{n\times m}$, we can utilize the the IT2FWA (4) operator

$$\tilde{B}_i = \sum_{j=1}^{m} \omega_j \tilde{B}_{ij} \tag{9}$$

to get the overall value $\tilde{B}_i$ of the alternative $x_i, (i = 1, 2, \ldots, n)$. The greater the value of $\bar{c}(\tilde{B}_i)$, the better the alternative x_i will be.

If the information about the attribute weights is completely unknown in the considered problem, then we need to determine the attribute weights in advance. How to utilize the interval type-2 fuzzy decision matrix to find the most desirable alternative(s) is an interesting and important issue, which is worth paying attention to.

As interpreted in Section 2, The centroid of an IT2 FS provides a measure of the uncertainty of such a FS, as a result, we utilize the average centroid function c to define the following concept:

Definition 3.1. *Let $\tilde{B} = (\tilde{B}_{ij})_{n\times m}$ be the interval type-2 fuzzy decision matrix, then we call $C = (c_{ij})_{n\times m}$ the average centroid matrix of $\tilde{B}$, where*

$$c_{ij} \triangleq c(\tilde{B}_{ij}) = \frac{c_l(\tilde{B}_{ij}) + c_u(\tilde{B}_{ij})}{2}, i = 1, 2, \ldots, n; j = 1, 2, \ldots, m.$$

According to the information theory, if all alternatives have similar attribute values with respect to an attribute, then a small weight should be assigned to the attribute, this is due to that such attribute does not help in differentiating alternatives [53]. Based on the collective average centroid matrix C, we introduce the deviation $d_{ij}(\omega)$ between the alternative x_i and the other alternatives with respect to the attribute g_j:

$$d_{ij}(\omega) = \sum_{k\neq i} |c_{ij} - c_{kj}|\omega_j, \quad 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m$$

and let

$$d_j(\omega) = \sum_{i=1}^{n} \sum_{k\neq i} |c_{ij} - c_{kj}|\omega_j, \quad 1 \leqslant j \leqslant m$$

denote the sum of all the deviations $d_{ij}(\omega), 1 \leqslant i \leqslant n$, with respect to the attribute g_j.

Then we construct the deviation function with respect to all the attributes:

$$d(\omega) = \sum_{j=1}^{m} d_j(\omega) = \sum_{j=1}^{m} \sum_{i=1}^{n} \sum_{k\neq i} |c_{ij} - c_{kj}|\omega_j \tag{10}$$

Obviously, a reasonable vector of attribute weights $\omega = (\omega_1, \omega_2, \ldots, \omega_m)^T$ should be determined so as to maximize $d(\omega)$. and thus, we establish the following optimization model:

$$(\mathbf{M-1})\ \text{Maximize } d(\omega) = \sum_{j=1}^{m} \sum_{i=1}^{n} \sum_{k\neq i} |c_{ij} - c_{kj}|\omega_j$$
$$s.t. \quad \sum_{j=1}^{m} \omega_j^2 = 1,$$
$$\omega_j \geqslant 0, j = 1, 2, \ldots, m.$$

Lagrange multipliers are used in multivariable calculus to find maxima and minima of a function subject to constraints, and it is an alternative to the method of substitution and works particularly well for non-linear constraints [10].

Lemma 3.2. (The method of Lagrange multipliers) *[10] Assume $f(x)$ and $g(x)$ are smooth functions of n variables and let x^* be a local maximum (resp. minimum) point of $f(x)$ subject to the constraint $g(x)$. Then there exists $\lambda^* \in R$ such that, if L denotes the function*

$$L(x, \lambda) = f(x) + \lambda g(x),$$

then

$$\frac{\partial L(x^*, \lambda^*)}{\partial x_j} = 0, \quad j = 1, 2, \cdots, n$$

and

$$\frac{\partial L(x^*, \lambda^*)}{\partial \lambda} = 0.$$

where L is called Lagrange function and the numbers λ Lagrange multipliers.

Based the Lemma 3.2, we have

Theorem 3.3. *The optimal solution of* **(M-1)** *is $\omega^* = (\omega_1^*, \omega_2^*, \ldots, \omega_m^*)^T$ where*

$$\omega_j^* = \frac{\sum_{i=1}^{n} \sum_{k\neq i} |r_{ij} - r_{kj}|}{\sqrt{\sum_{j=1}^{m} \left(\sum_{i=1}^{n} \sum_{k\neq i} |r_{ij} - r_{kj}|\right)^2}}, \quad 1 \leqslant j \leqslant m$$

(11)

Proof. Based on the model ((**M** − **1**)) and Lemma 3.2, we construct the Lagrange function:

$$L(\omega,\lambda)=\sum_{j=1}^{m}\sum_{i=1}^{n}\sum_{k\neq i}|c_{ij}-c_{kj}|\omega_j+\frac{1}{2}\lambda\left(\sum_{j=1}^{m}\omega_j^2-1\right) \tag{12}$$

where λ is the Lagrange multiplier.

Differentiating (12) with respect to ω_j $(1\leqslant j\leqslant m)$ and λ, and setting these partial derivatives equal to zero, the following set of equations is obtained:

$$\begin{cases}\frac{\partial L(\omega,\lambda)}{\partial\omega_j}=\sum_{i=1}^{n}\sum_{k\neq i}|c_{ij}-c_{kj}|+\lambda\omega_j=0, \quad 1\leqslant j\leqslant m,\\ \frac{\partial L(\omega,\lambda)}{\lambda}=\sum_{j=1}^{m}\omega_j^2-1=0.\end{cases}$$

we can solve the first m equations for ω_j $(1\leqslant j\leqslant m)$ respectively. This gives,

$$\omega_j=-\frac{\sum_{i=1}^{n}\sum_{k\neq i}|c_{ij}-c_{kj}|}{\lambda}, \quad 1\leqslant j\leqslant m. \tag{13}$$

Plugging these into the constraint gives,

$$\frac{\sum_{j=1}^{m}\left(\sum_{i=1}^{n}\sum_{k\neq i}|c_{ij}-c_{kj}|\right)^2}{\lambda^2}=1.$$

We can solve this for λ.

$$\lambda=-\sqrt{\sum_{j=1}^{m}\left(\sum_{i=1}^{n}\sum_{k\neq i}|c_{ij}-c_{kj}|\right)^2} \tag{14}$$

which can satisfy $\omega_j\geqslant 0$.

Plugging Eq.(13) into Eq.(14), then we get the optimal solution $\omega^*=(\omega_1^*,\omega_2^*,\ldots,\omega_m^*)^T$ where

$$\omega_j^*=\frac{\sum_{i=1}^{n}\sum_{k\neq i}|c_{ij}-c_{kj}|}{\sqrt{\sum_{j=1}^{m}\left(\sum_{i=1}^{n}\sum_{k\neq i}|c_{ij}-c_{kj}|\right)^2}}, \quad 1\leqslant j\leqslant m$$

Obviously, $\omega_j^*\geqslant 0$, for all j. □

Normalizing (11), we get the normalized attribute weights:

$$\omega_j=\frac{\sum_{i=1}^{n}\sum_{k\neq i}|c_{ij}-c_{kj}|}{\sum_{j=1}^{m}\sum_{i=1}^{n}\sum_{k\neq i}|c_{ij}-c_{kj}|}, \quad 1\leqslant j\leqslant m \tag{15}$$

In such case, we have $0\leqslant\omega_j\leqslant 1$ and $\sum_{j=1}^{m}\omega_j=1$.

Based on the analysis above, in the following we develop an approach to IT2 fuzzy MAGDM with complete unknown weight information.

Step 1 Normalize the IT2 fuzzy decision matrices $\tilde{A}^{(k)}=(\tilde{A}_{ij}^{(k)})_{n\times m}(k=1,2,\ldots,l)$ provided by the DMs $d_k(k=1,2,\ldots,l)$ using the transformation (7).

Step 2 Utilize (8) to aggregate all individual normalized interval type-2 fuzzy decision matrices $\tilde{B}^{(k)}=(\tilde{B}_{ij}^{(k)})_{n\times m}(k=1,2,\ldots,l)$ into a collective normalized interval type-2 fuzzy decision matrix $\tilde{B}=(\tilde{B}_{ij})_{n\times m}$.

Step 3 Based on Definition 3.1, we calculate the average centroid matrix $C=(c_{ij})_{n\times m}$ of the collective normalized IT2 fuzzy decision matrix $\tilde{B}=(\tilde{B}_{ij})_{n\times m}$.

Step 4 Establish the optimization model $((M-1))$ and utilize (15) to derive the weight vector of attributes $\omega=(\omega_1,\omega_2,\ldots,\omega_m)^T$.

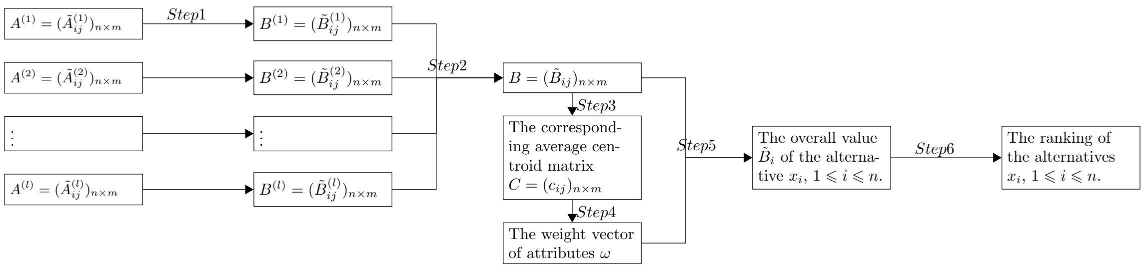

Figure 2: Decision-making Process.

Step 5 Based on the collective normalized interval type-2 fuzzy decision matrix $\tilde{B} = (\tilde{B}_{ij})_{n\times m}$, we utilize the weighted arithmetic average operator (9) to get the overall value $\tilde{B}_i$ of the alternative $x_i, (i = 1, 2, \ldots, n)$.

Step 6 Calculate the average centroid $c(\tilde{B}_i)$ of the overall value $\tilde{B}_i$, $1 \leqslant i \leqslant n$, and rank the alternatives x_i based on the average centroid $c(\tilde{B}_i)$, $1 \leqslant i \leqslant n$. The greater value the average centroid $c(\tilde{B}_i)$, the better the alternative x_i will be.

Time complexity depends mainly on the number of attributes, m, the number of experts, l, and the number of alternatives, n, in the group decision. In the normalization step, time complexity is considered as $O(lmn)$. The time complexity of computing the collective normalized interval type-2 fuzzy decision matrix in Step 3 is $O(lmn)$. In Step 3, the time complexity of calculating the average centroid matrix of the collective normalized IT2 fuzzy decision matrix $O(mn)$. The time complexity of solving the nonlinear optimization model to derive the weight vector of attributes is $O(mn(n-1))$. In Step 5, the time complexity of deriving the overall value of the alternative is $O(mn)$. The time complexity of ranking the alternatives is $O(n)$. So the total time complexity of the proposed approach is the largest one of all steps.

4 Illustrative example

In this section, we use an example to illustrate the proposed method.

Table 1 shows the linguistic terms "Very Low" (VL), "Low" (L), "Medium Low" (ML), "Medium" (M), "Medium High" (MH), "High" (H), "Very High" (VH) and their corresponding IT2 FSs, respectively. In addition there are complementary relations about the linguistic terms (or IT2 FSs) in Table 2.

Linguistic terms	IT2 FSs($\tilde{A}$)
Very Low (VL)	$((0, 0, 0, 0.1; 1, 1), (0, 0, 0, 0.05; 0.9, 0.9))$
Low (L)	$((0, 0.1, 0.1, 0.3; 1, 1), (0.05, 0.1, 0.1, 0.2; 0.9, 0.9))$
Medium Low (ML)	$((0.1, 0.3, 0.3, 0.5; 1, 1), (0.2, 0.3, 0.3, 0.4; 0.9, 0.9))$
Medium (M)	$((0.3, 0.5, 0.5, 0.7; 1, 1), (0.4, 0.5, 0.5, 0.6; 0.9, 0.9))$
Medium High (MH)	$((0.5, 0.7, 0.7, 0.9; 1, 1), (0.6, 0.7, 0.7, 0.8; 0.9, 0.9))$
High (H)	$((0.7, 0.9, 0.9, 1; 1, 1), (0.8, 0.9, 0.9, 0.95; 0.9, 0.9))$
Very High (VH)	$((0.9, 1, 1, 1; 1, 1), (0.95, 1, 1, 1; 0.9, 0.9))$

Table 1: Linguistic terms and their corresponding IT2 FSs.

$\tilde{A}$	VL	L	ML	M	MH	H	VH
$\tilde{A}^c$	VH	H	MH	M	ML	L	VL

Table 2: The complementary relations.

Assume that the problem discussed here is concerned with a manufacturing company, searching the best global supplier for one of its most critical parts used in assembling process (adapted from [5]). There are three potential global suppliers $x_i(i = 1, 2, 3)$ to be evaluated with four attributes: (1) g_1 : Quality of the product, (2) g_2 : Risk factor, (3) g_3 : Service performance of supplier and (4) g_4 : Supplier's profile. An expert group is formed which consists of three experts $d_k(k = 1, 2, 3)$ (whose weight vector is $\lambda = (0.3, 0.45, 0.25)^T$) from each strategic decision area.

The experts $d_k(k = 1, 2, 3)$ use the linguistic terms shown in Table 1 to represent the characteristics of the potential global suppliers $x_i(i = 1, 2, 3)$ with respect to different attributes $g_i(i = 1, 2, 3, 4)$, respectively, listed in Table 3.

	$\tilde{A}^{(1)}$				$\tilde{A}^{(2)}$				$\tilde{A}^{(3)}$			
	g_1	g_2	g_3	g_4	g_1	g_2	g_3	g_4	g_1	g_2	g_3	g_4
x_1	MH	H	VH	VH	H	VH	H	H	MH	H	H	H
x_2	H	MH	H	H	MH	H	VH	VH	H	VH	VH	H
x_3	VH	VH	M	H	H	VH	MH	VH	MH	H	MH	VH

Table 3: The decision matrices $\tilde{A}^{(k)}(k = 1, 2, 3)$.

If the information about attribute weights is completely unknown, then we use the developed approach to handle the above problem, which needs the following steps:

Step 1 Considering that the attributes are the benefit attributes except to the attribute Risk factor $((g_2))$, then based on the formula (7) and Table 4, the decision matrices $\tilde{A}^{(k)}(k = 1, 2, 3)$ can be updated to the following normalized matrices respectively, listed in Table 4.

	$\tilde{B}^{(1)}$				$\tilde{B}^{(2)}$				$\tilde{B}^{(3)}$			
	g_1	g_2	g_3	g_4	g_1	g_2	g_3	g_4	g_1	g_2	g_3	g_4
x_1	MH	L	VH	VH	H	VL	H	H	MH	L	H	H
x_2	H	ML	H	H	MH	L	VH	VH	H	VL	VH	H
x_3	VH	VL	M	H	H	VL	MH	VH	MH	L	MH	VH

Table 4: The normalized decision matrices $\tilde{B}^{(k)}(k = 1, 2, 3)$.

Step 2 Based on Table 4, we utilize the weighted averaging operator (8) to aggregate all individual normalized IT2 fuzzy decision matrices $\tilde{B^{(k)}} = (\tilde{B}_{ij}^{(k)})_{3\times 4}(k = 1, 2, 3)$ into a collective IT2 fuzzy decision matrix $\tilde{B} = (\tilde{B}_{ij})_{3\times 4}$ shown as follows:

$$\tilde{B} = \begin{pmatrix} \tilde{B}_{11} & \tilde{B}_{12} & \tilde{B}_{13} & \tilde{B}_{14} \\ \tilde{B}_{21} & \tilde{B}_{22} & \tilde{B}_{23} & \tilde{B}_{24} \\ \tilde{B}_{31} & \tilde{B}_{32} & \tilde{B}_{33} & \tilde{B}_{34} \end{pmatrix}$$

where $\tilde{B}_{ij} = \bigoplus_{k=1}^{3} (\lambda_k \tilde{B}_{ij}^{(k)})$, $\tilde{B}_{ij}^{(k)}(k = 1, 2, 3)$is an IT2 FS, $1 \leqslant i \leqslant 3, 1 \leqslant j \leqslant 4$, $1 \leqslant k \leqslant 3$, and k denotes the number of DMs,

$\tilde{B}_{11} = ((0.59, 0.79, 0.79, 0.945; 1, 1), (0.69, 0.79, 0.79, 0.8675; 0.9, 0.9))$,
$\tilde{B}_{12} = ((0, 0.055, 0.055, 0.21; 1, 1), (0.0275, 0.055, 0.055, 0.1325; 0.9, 0.9))$,
$\tilde{B}_{13} = ((0.76, 0.93, 0.93, 1; 1, 1), (0.845, 0.93, 0.93, 0.965; 0.9, 0.9))$,
$\tilde{B}_{14} = \tilde{B}_{13}$,
$\tilde{B}_{21} = ((0.61, 0.81, 0.81, 0.955; 1, 1), (0.71, 0.81, 0.81, 0.8825; 0.9, 0.9))$,
$\tilde{B}_{22} = ((0.03, 0.135, 0.135, 0.31; 1, 1), (0.0825, 0.135, 0.135, 0.2225; 0.9, 0.9))$,
$\tilde{B}_{23} = ((0.84, 0.97, 0.97, 1; 1, 1), (0.905, 0.97, 0.97, 0.985; 0.9, 0.9))$,
$\tilde{B}_{24} = ((0.79, 0.945, 0.945, 1; 1, 1), (0.8675, 0.945, 0.945, 0.9725; 0.9, 0.9))$,
$\tilde{B}_{31} = ((0.71, 0.88, 0.88, 0.975; 1, 1), (0.795, 0.88, 0.88, 0.9275; 0.9, 0.9))$,
$\tilde{B}_{32} = ((0, 0.025, 0.025, 0.15; 1, 1), (0.0125, 0.025, 0.025, 0.0875; 0.9, 0.9))$,
$\tilde{B}_{33} = ((0.44, 0.64, 0.64, 0.84; 1, 1), (0.54, 0.64, 0.64, 0.74; 0.9, 0.9))$,
$\tilde{B}_{34} = \tilde{B}_{23}$.

Step 3 Based on Definition 3.1, we calculate the average centroid matrix $C = (c_{ij})_{n\times m}$ of the collective normalized IT2 fuzzy decision matrix $\tilde{B} = (\tilde{B}_{ij})_{n\times m}$.

$$C = \begin{pmatrix} 0.7784 & 0.0808 & 0.9042 & 0.9042 \\ 0.7958 & 0.1530 & 0.9441 & 0.9192 \\ 0.8607 & 0.0509 & 0.6400 & 0.9441 \end{pmatrix}$$

Step 4 Establish the optimization model $((M-1))$ and utilize (15) to derive the weight vector of attributes $\omega = (0.1558, 0.1932, 0.5755, 0.0755)^T$.

Step 5 Based on the collective IT2F decision matrix $\tilde{B} = (\tilde{B}_{ij})_{3\times 4}$, we utilize the weighted averaging operator (9) to get the overall value $\tilde{B}_i = \bigoplus_{j=1}^{4}(\omega_j \tilde{B}_{ij})$ of the alternative x_i, i=1,2,3: $\tilde{B}_1 = ((0.5867, 0.7391, 0.7391, 0.8388; 1, 1), (0.6629, 0.7391, 0.7391, 0.7890; 0.9, 0.9))$, $\tilde{B}_2 = ((0.6439, 0.7819, 0.7819, 0.8597; 1, 1), (0.7129, 0.7819, 0.7819, 0.8208; 0.9, 0.9))$, $\tilde{B}_3 = ((0.4273, 0.5835, 0.5835, 0.7398; 1, 1), (0.5054, 0.5835, 0.5835, 0.6616; 0.9, 0.9))$.

Step 6 Calculate the average centroid $c(\tilde{B}_i)$ of the overall value $\tilde{B}_i, 1 \leqslant i \leqslant 3$
$c(\tilde{B}_1) = 0.7255$, $c(\tilde{B}_2) = 0.7664$, $c(\tilde{B}_3) = 0.5835$. Because $c(\tilde{B}_2) > c(\tilde{B}_1) > c(\tilde{B}_3)$, the preference order of the alternatives x_1, x_2 and x_3 is $x_2 \succ x_1 \succ x_3$. That is, the most desirable global supplier among x_1, x_2 and x_3 is x_2.

5 Conclusions

We have investigated the MAGDM problems under linguistic environment, and developed an approach to handling the situations where the attribute values are characterized by IT2 FSs, and the information about attribute weights is completely unknown. The approach utilizes the interval type-2 fuzzy weighted arithmetic averaging operator to aggregate all individual interval type-2 fuzzy decision matrices into a collective interval type-2 fuzzy decision matrix, and then we have introduced the average centroid matrix of the collective interval type-2 fuzzy decision matrix. According to the information theory, a small weight should be assigned to the attribute with similar attribute values corresponding to an alternative, we have established an optimization model which integrates all the given ranking values. By solving the model, a straightforward formula has been obtained for determining the attribute weights. We have utilized the average centroid measure to find the ranking of the alternatives and then to select the most desirable one. All these procedures have been illustrated by a numerical example concerning with a manufacturing company, searching the best global supplier for one of its most critical parts used in assembling process [3].

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