

Degree of  $\mathbf{F}$ -irresolute function in  $(L, M)$ -fuzzy topological spacesA. Ghareeb<sup>1</sup>

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**Abstract**

In this paper, we present a new vision for studying  $\mathbf{F}$ -open,  $\mathbf{F}$ -continuous, and  $\mathbf{F}$ -irresolute function in  $(L, M)$ -fuzzy topological spaces based on the implication operation and  $(L, M)$ -fuzzy  $\mathbf{F}$ -open operator [3]. These kinds of functions are generalized with their elementary properties to  $(L, M)$ -fuzzy topological spaces setting based on graded concepts. Moreover, a systematic discussion of their relationship with the degree of  $\mathbf{F}$ -compactness,  $\mathbf{F}$ -connectedness,  $\mathbf{FT}_1$ , and  $\mathbf{FT}_2$  is carried out.

*Keywords:*  $(L, M)$ -fuzzy topological space,  $\mathbf{F}$ -open operator,  $\mathbf{F}$ -continuous function,  $\mathbf{F}$ -irresolute function.

**1 Introduction**

The concept of fuzzy topology was first formulated by Chang [2] as a completely new mathematical application for fuzzy sets theory [31]. Chang defined fuzzy topology as a collection of fuzzy sets which fulfills the same conditions as general topology while the topology itself still crisp. Later, Goguen [8] suggested to use a complete residuated lattice with possibly non-commutative truth function of conjunction instead the closed interval  $[0, 1]$ . Many researchers are interested in extending the Chang-Goguens approach. Accordingly, Höhle [9] and Ying [30] extended independently this approach to more general form which later known as fuzzifying topology. In 1985, Kubiak [12] and Šostak [24] defined independently a new extension to Chang-Goguens fuzzy topology. It realizes  $(L, M)$ -fuzzy topology on  $X$  as an  $M$ -fuzzy subset of  $L^X$  where  $L$  and  $M$  are appropriate lattices. Most early works mainly attempted to extend for  $(L, M)$ -fuzzy topology some basic properties and results of general topology and Chang-Goguens fuzzy topology (see [15, 11, 23, 14], for instance).

Various fuzzification of classical concepts related to topology evolved as further aspects and connections have been taken into account. For example, consider the important concept of category. A notion of fuzzy category was introduced in [27] and its study has been pursued for many years [13, 29] meaning that potential objects and morphisms have been endowed with a certain degree of the corresponding lattice. In 1999, Šostak [28] fuzzified some categories concerning to topology and algebra. The continuity degree considered as a significant example of a morphism degree in the fuzzy category. The obvious fuzzification of category led to the definition of many topological concepts endowed with degrees. Recently, Pang [17] defined the continuity, openness, and closeness degree of functions in  $L$ -fuzzifying topology. An important conception is due to Liang and Shi who approached the same degrees in  $(L, M)$ -fuzzy topology in the spirit of their pioneer paper [16] and suggested using the implication operation. Several properties in general topology generalized to  $(L, M)$ -fuzzy topology setting based on graded concepts. Recently, Shi [20] formulated a new technique for studying the semiopenness and preopenness in  $L$ -fuzzy topology by introducing  $L$ -fuzzy semiopen and  $L$ -fuzzy preopen operators based on implication operation. Also, several properties and results have been reformulated with the help of Shi's operators. Afterwards, Ghareeb [6, 3, 4, 5, 1, 7] presented other operators rely on Shi's operators and used it to discuss the related degree of continuity, connectedness, and compactness.

The main purpose of this paper is to define the degree of  $\mathbf{F}$ -openness,  $\mathbf{F}$ -continuity, and  $\mathbf{F}$ -irresoluteness for functions in  $(L, M)$ -fuzzy topology based on implication operation and  $(L, M)$ -fuzzy  $\mathbf{F}$ -open operator [2]. Their primary properties

will be introduced and discussed through graded concepts. Therefore, the function can be regard as **F**-open, **F**-continuous, and **F**-irresolute to some degree. The relationship with **F**-compactness, **F**-connectedness, **FT**<sub>1</sub> and **FT**<sub>2</sub> are discussed systematically.

## 2 Preliminaries

In the sequel, both  $L$  and  $M$  denote a completely distributive De Morgan algebras.  $X$  is a nonempty set. The smallest and greatest members in  $L$  and  $M$  are denoted by  $\perp_L, \top_L$  and  $\perp_M, \top_M$ , respectively. For each  $r, s \in L$ , the element  $r$  is wedge below  $s$  in  $L$  [18], in symbols  $r \ll s$ , if for each subset  $\mathcal{D} \subseteq L$ ,  $\bigvee \mathcal{D} \geq s$  yields to  $t \geq r$  for some  $t \in \mathcal{D}$ . We say the complete lattice  $L$  is completely distributive iff  $s = \bigvee \{r \in L \mid r \ll s\}$  for any  $s \in L$ . A member  $r \in L$  is said to be co-prime if  $r \leq s \vee t$  yields to  $r \leq s$  or  $r \leq t$ . The family of non-zero co-prime members in  $L$  is denoted by  $J(L)$ .  $L^X$  refers to the set of all  $L$ -subsets on  $X$ .  $2^{\mathcal{U}}$  denotes to the collection of all finite sub-collections of  $\mathcal{U} \subseteq L^X$ . Evidently,  $L^X$  is a completely distributive DeMorgan algebra when it inherits the structure of the lattice  $L$  in a natural way, by defining  $\bigvee, \bigwedge, \leq$  and  $'$  pointwisely. It is evidently that  $\{x_\lambda \mid \lambda \in J(L)\}$  is a collection of non-zero co-primes in  $L^X$ . For any completely distributive De Morgan algebra  $L$ , there exists an implication operation  $\mapsto: L \times L \rightarrow L$  as the right adjoint for the meet operation  $\wedge$  is defined by  $r \mapsto s = \bigvee \{t \in L \mid r \wedge t \leq s\}$ . Further, the operation  $\leftrightarrow$  is given by  $r \leftrightarrow s = (r \mapsto s) \wedge (s \mapsto r)$ . The properties of an implication operation listed in the following lemma.

**Lemma 2.1.** [10] *Let  $(L, \bigvee, \bigwedge)$  be a completely distributive lattice and  $\mapsto$  be the implication operation corresponding to  $\wedge$ . Then for all  $r, s, t \in L$ ,  $\{r_i\}_{i \in \Gamma}$ , and  $\{s_i\}_{i \in \Gamma} \subseteq L$ , we have the following statements:*

- (1)  $(r \mapsto s) \geq t \iff r \wedge t \leq s$ .
- (2)  $r \leq s \iff r \mapsto s = \top_L$ .
- (3)  $r \mapsto (s \mapsto t) = (r \wedge s) \mapsto t$ .
- (4)  $(t \mapsto r) \wedge (r \mapsto s) \leq t \mapsto s$ .
- (5)  $t \mapsto r \leq (r \mapsto s) \mapsto (t \mapsto s)$ .
- (6)  $r \mapsto \bigwedge_{i \in \Gamma} r_i = \bigwedge_{i \in \Gamma} (r \mapsto r_i)$ , hence  $r \mapsto s \leq r \mapsto t$  whenever  $s \leq t$ .
- (7)  $\bigvee_{i \in \Gamma} r_i \mapsto s = \bigwedge_{i \in \Gamma} (r_i \mapsto s)$ , hence  $r \mapsto t \geq s \mapsto t$  whenever  $r \leq s$ .

**Definition 2.2.** [10, 12, 19, 24] *For any non-empty set  $X$ , an  $(L, M)$ -fuzzy topology is defined as a function  $\tau: L^X \rightarrow M$  satisfies the following conditions:*

- (O1)  $\tau(\top_{L^X}) = \tau(\perp_{L^X}) = \top_M$ .
- (O2)  $\forall A_1, A_2 \in L^X, \tau(A_1 \wedge A_2) \geq \tau(A_1) \wedge \tau(A_2)$ .
- (O3)  $\forall \{A_i\}_{i \in \Gamma} \subseteq L^X, \tau\left(\bigvee_{i \in \Gamma} A_i\right) \geq \bigwedge_{i \in \Gamma} \tau(A_i)$ .

For any  $L$ -subset  $A \in L^X$ ,  $\tau(A)$  refers to the degree of openness of  $A$ .  $\tau^*(A) = \tau(A')$  is the degree of closeness of  $A$ . The pair  $(X, \tau)$  is said to be an  $(L, M)$ -fuzzy topological space. A function  $\varphi: (X, \tau) \rightarrow (Y, \sigma)$  is said to be continuous with respect to  $(L, M)$ -fuzzy topological spaces  $(X, \tau)$  and  $(Y, \sigma)$  iff  $\tau(\varphi^{\Leftarrow}(B)) \geq \sigma(B)$  for all  $B \in L^Y$ , where  $\varphi^{\Leftarrow}$  is given by  $\varphi^{\Leftarrow}(B)(x) = B(\varphi(x))$ .

An  $(L, M)$ -fuzzy closure operators introduced by Shi [22] as follows.

**Definition 2.3.** [22] *Let  $(X, \tau)$  be an  $(L, M)$ -fuzzy topological space. The function  $Cl^\tau: L^X \rightarrow M^{J(L^X)}$  is called an  $(L, M)$ -fuzzy closure operator induced by  $\tau$ , where  $Cl^\tau(A)(x_\lambda) = \bigwedge_{x_\lambda \preceq A_1 \geq A} (\tau(A_1'))'$ , for any  $x_\lambda \in J(L^X)$  and  $A \in L^X$ .*

**Definition 2.4.** [20] *Let  $(X, \tau)$  be an  $(L, M)$ -fuzzy topological space. The function  $\mathcal{S}: L^X \rightarrow M$  is called an  $(L, M)$ -fuzzy semiopen operator induced by  $\tau$ , where  $\mathcal{S}(A) = \bigvee_{A_1 \leq A} \left\{ \tau(A_1) \wedge \bigwedge_{x_\lambda \ll A} \bigwedge_{x_\lambda \preceq A_2 \geq A_1} (\tau(A_2'))' \right\}$ , for any  $A \in L^X$ , where  $\mathcal{S}(A)$  refers to the degree to which  $A$  is semiopen.  $\mathcal{S}^*(A) = \mathcal{S}(A')$  is the degree to which  $A$  is semiclosed.*

Based on Definition 2.3 and Definition 2.4, Shi [20] stated the next corollary.

**Corollary 2.5.** *Let  $(X, \tau)$  be an  $(L, M)$ -fuzzy topological space. Then for all  $A \in L^X$ , we have*

$$\mathcal{S}(A) = \bigvee_{A_1 \leq A} \left\{ \tau(A_1) \wedge \bigwedge_{x_\lambda \ll A} Cl^\tau(A_1)(x_\lambda) \right\}.$$

**Theorem 2.6.** [20] *Let  $(X, \tau)$  be an  $(L, M)$ -fuzzy topological space and  $\mathcal{S}$  be the corresponding  $(L, M)$ -fuzzy semiopen operator. Then  $\tau(A) \leq \mathcal{S}(A)$  for all  $A \in L^X$ .*

**Theorem 2.7.** [20] *Let  $(X, \tau)$  be an  $(L, M)$ -fuzzy topological space and  $\mathcal{S}$  be the corresponding  $(L, M)$ -fuzzy semiopen operator. Then  $\mathcal{S}(\bigvee_{i \in \Gamma} A_i) \geq \bigwedge_{i \in \Gamma} \mathcal{S}(A_i)$ , for each  $\{A_i\}_{i \in \Gamma} \subseteq L^X$ .*

**Definition 2.8.** [3] *Let  $(X, \tau)$  be an  $(L, M)$ -fuzzy topological space. For any  $A \in L^X$ , the function  $\mathcal{F} : L^X \rightarrow M$  is called an  $(L, M)$ -fuzzy **F**-open operator induced by  $\tau$  and  $\mathcal{S}$ , where*

$$\mathcal{F}(A) = \bigvee_{B \leq A} \left\{ \tau(B) \wedge \bigwedge_{x_\lambda \ll A} \bigwedge_{x_\lambda \not\leq C \geq B} (\mathcal{S}(C'))' \right\}.$$

$\mathcal{F}(A)$  refers to the degree to which  $A$  is **F**-open and  $\mathcal{F}^*(A) = \mathcal{F}(A')$  refers to the degree to which  $A$  is **F**-closed.

**Lemma 2.9.** *Let  $\mathcal{F} : L^X \rightarrow M$  be an  $(L, M)$ -fuzzy **F**-open operator induced by  $(L, M)$ -fuzzy topology  $\tau$  on  $X$ . Then  $\mathcal{F}$  satisfies the following conditions:*

- (1)  $\mathcal{F}(\perp_{L^X}) = \mathcal{F}(\top_{L^X}) = \top_M$ .
- (2)  $\mathcal{F}(\bigvee_{i \in \Gamma} A_i) \geq \bigwedge_{i \in \Gamma} \mathcal{F}(A_i)$  for any  $\{A_i\}_{i \in \Gamma} \subseteq L^X$ .

*Proof.* The proof of (1) is clear. To prove (2), suppose that  $r \in M$  and  $r \ll \bigwedge_{i \in \Gamma} \mathcal{F}(A_i)$ . Then for any  $i \in \Gamma$ , there exists  $B_i \leq A_i$  with

$$r \ll \tau(B_i) \quad \text{and} \quad r \ll \bigwedge_{x_\lambda \ll A_i} \bigwedge_{x_\lambda \not\leq C \geq B_i} (\mathcal{S}(C'))'.$$

Hence

$$r \leq \bigwedge_{i \in \Gamma} \tau(B_i) \leq \tau\left(\bigvee_{i \in \Gamma} B_i\right) \quad \text{and} \quad r \leq \bigwedge_{i \in \Gamma} \bigwedge_{x_\lambda \ll A_i} \bigwedge_{x_\lambda \not\leq C \geq B_i} (\mathcal{S}(C'))'.$$

Since  $\{x_\lambda : x_\lambda \ll \bigvee_{i \in \Gamma} A_i\} = \bigcup_{i \in \Gamma} \{x_\lambda : x_\lambda \ll A_i\}$ , we have

$$\begin{aligned} \mathcal{F}\left(\bigvee_{i \in \Gamma} A_i\right) &= \bigvee_{B \leq \bigvee_{i \in \Gamma} A_i} \left\{ \tau(B) \wedge \bigwedge_{x_\lambda \ll \bigvee_{i \in \Gamma} A_i} \bigwedge_{x_\lambda \not\leq C \geq B} (\mathcal{S}(C'))' \right\} \geq \tau\left(\bigvee_{i \in \Gamma} B_i\right) \wedge \bigwedge_{i \in \Gamma} \bigwedge_{x_\lambda \ll A_i} \bigwedge_{x_\lambda \not\leq C \geq \bigvee_{i \in \Gamma} B_i} (\mathcal{S}(C'))' \\ &\geq \tau\left(\bigvee_{i \in \Gamma} B_i\right) \wedge \bigwedge_{i \in \Gamma} \bigwedge_{x_\lambda \ll A_i} \bigwedge_{x_\lambda \not\leq C \geq B_i} (\mathcal{S}(C'))' \geq r. \end{aligned}$$

From this, we conclude that  $\mathcal{F}(\bigvee_{i \in \Gamma} A_i) \geq \bigwedge_{i \in \Gamma} \mathcal{F}(A_i)$ . □

**Definition 2.10.** [3] *Let  $\varphi : (X, \tau) \rightarrow (Y, \sigma)$  be a function between two  $(L, M)$ -fuzzy topological spaces  $(X, \tau)$  and  $(Y, \sigma)$ . Then:*

- (1)  $\varphi$  is  $(L, M)$ -fuzzy **F**-continuous function iff  $\sigma(B) \leq \mathcal{F}_1(\varphi^{\leftarrow}(B))$  holds for any  $B \in L^Y$ .
- (2)  $\varphi$  is  $(L, M)$ -fuzzy **F**-irresolute iff  $\mathcal{F}_2(B) \leq \mathcal{F}_1(\varphi^{\leftarrow}(B))$  holds for any  $B \in L^Y$ .

**Corollary 2.11.** *Let  $\varphi : (X, \tau) \rightarrow (Y, \sigma)$  be a function between two  $(L, M)$ -fuzzy topological spaces  $(X, \tau)$  and  $(Y, \sigma)$ . Then:*

- (1)  $\varphi$  is  $(L, M)$ -fuzzy **F**-continuous iff  $\sigma^*(B) \leq \mathcal{F}_1^*(\varphi^{\leftarrow}(B))$  for any  $B \in L^Y$ .

(2)  $\varphi$  is  $(L, M)$ -fuzzy  $\mathbf{F}$ -irresolute iff  $\mathcal{F}_2^*(B) \leq \mathcal{F}_1^*(\varphi^{\leftarrow}(B))$  for any  $B \in L^Y$ .

*Proof.* Clear. □

**Definition 2.12.** A function  $\varphi : (X, \tau) \rightarrow (Y, \sigma)$  between two  $(L, M)$ -fuzzy topological spaces  $(X, \tau)$  and  $(Y, \sigma)$  is called an  $(L, M)$ -fuzzy  $\mathbf{F}$ -open function iff  $\tau(A) \leq \mathcal{F}_2(\varphi^{\rightarrow}(A))$  for any  $A \in L^X$ .

**Definition 2.13.** An  $(L, M)$ -fuzzy quasi  $\mathbf{F}$ -neighborhood system on  $X$  is a collection  $\mathbf{FQ} = \{\mathbf{FQ}_{x_\lambda} | x_\lambda \in J(L^X)\}$  of functions  $\{\mathbf{FQ}_{x_\lambda} : L^X \rightarrow M\}$  which satisfying the following axioms:

- (1)  $\mathbf{FQ}_{x_\lambda}(\perp_{L^X}) = \perp_M$ .
- (2)  $\mathbf{FQ}_{x_\lambda}(A) \neq \perp_M$ , then  $x_\lambda \not\leq A'$ .
- (3)  $\mathbf{FQ}_{x_\lambda}(A_1 \wedge A_2) \leq \mathbf{FQ}_{x_\lambda}(A_1) \wedge \mathbf{FQ}_{x_\lambda}(A_2)$ .
- (4)  $\mathbf{FQ}_{x_\lambda}(A) = \bigvee_{x_\lambda \leq A_1 \geq A'} \bigwedge_{y_\mu \leq A_1'} \mathbf{FQ}_{y_\mu}(A_1)$ .

**Definition 2.14.** [4] An  $(L, M)$ -fuzzy  $\mathbf{F}$ -closure operator on  $X$  is a function  $\mathbf{FC} : L^X \rightarrow M^{J(L^X)}$  which satisfying the following axioms:

- (1)  $\mathbf{FC}(A)(x_\lambda) = \bigwedge_{\mu \ll \lambda} \mathbf{FC}(A)(x_\mu)$  for any  $x_\lambda \in J(L^X)$ .
- (2)  $\mathbf{FC}(\perp_{L^X})(x_\lambda) = \perp_M$  for any  $x_\lambda \in J(L^X)$ .
- (3)  $\mathbf{FC}(A)(x_\lambda) = \top_M$  for any  $x_\lambda \leq A$ .
- (4) for any  $r \in M_\perp$ ,  $(\mathbf{FC}(\bigvee(\mathbf{FC}(A))_{[r]}))_{[r]} \subseteq (\mathbf{FC}(A))_{[r]}$ .

$\mathbf{FC}(A)(x_\lambda)$  refers to the degree to which  $x_\lambda$  belongs to the  $\mathbf{F}$ -closure of  $A$ .

From [22] and Lemma 2.9, we can easily prove Theorems 2.15, 2.16 and 2.19.

**Theorem 2.15.** Let  $\mathcal{F}$  be the  $(L, M)$ -fuzzy  $\mathbf{F}$ -open operator on  $X$  and let  $\mathbf{FC}^{\mathcal{F}}$  be the corresponding  $(L, M)$ -fuzzy  $\mathbf{F}$ -closure operator. Then for any  $x_\lambda \in J(L^X)$  and  $A \in L^X$ , we have

$$\mathbf{FC}^{\mathcal{F}}(A)(x_\lambda) = \bigwedge_{x_\lambda \leq A_1 \geq A} (\mathcal{F}(A_1))'.$$

**Theorem 2.16.** Let  $(X, \tau)$  be an  $(L, M)$ -fuzzy topological space,  $\mathcal{F}$  be the corresponding  $(L, M)$ -fuzzy  $\mathbf{F}$ -open operator, and  $\mathbf{FQ}^{\mathcal{F}} = \{\mathbf{FQ}_{x_\lambda} | x_\lambda \in J(L^X)\}$  be the  $(L, M)$ -fuzzy quasi  $\mathbf{F}$ -neighborhood system induced by  $\mathcal{F}$ . Define the function  $\mathbf{FC}^{\mathcal{F}} : L^X \rightarrow M^{J(L^X)}$  by  $\mathbf{FC}^{\mathcal{F}}(A)(x_\lambda) = (\mathbf{FQ}_{x_\lambda}(A'))'$ . Then  $\mathbf{FC}^{\mathcal{F}}$  is an  $(L, M)$ -fuzzy  $\mathbf{F}$ -closure operator on  $X$ .

**Definition 2.17.** An  $(L, M)$ -fuzzy  $\mathbf{F}$ -neighborhood system on  $X$  is the collection  $\mathbf{FN} = \{\mathbf{FN}_{x_\lambda} | x_\lambda \in J(L^X)\}$  of functions  $\{\mathbf{FN}_{x_\lambda} : L^X \rightarrow M\}$  which satisfying the following axioms:

- (1)  $\mathbf{FN}_{x_\lambda}(\perp_{L^X}) = \perp_M$ .
- (2)  $\mathbf{FN}_{x_\lambda}(A) \neq \perp_M$ , then  $x_\lambda \not\leq A$ .
- (3)  $\mathbf{FN}_{x_\lambda}(A_1 \wedge A_2) \leq \mathbf{FN}_{x_\lambda}(A_1) \wedge \mathbf{FN}_{x_\lambda}(A_2)$ .
- (4)  $\mathbf{FN}_{x_\lambda}(A) = \bigvee_{x_\lambda \leq A_1 \geq A} \bigwedge_{y_\mu \ll A_1} \mathbf{FN}_{y_\mu}(A_1)$ .

**Definition 2.18.** An  $(L, M)$ -fuzzy  $\mathbf{F}$ -interior operator on  $X$  is a function  $\mathbf{FI} : L^X \rightarrow M^{J(L^X)}$  which satisfying the following axioms:

- (1)  $\mathbf{FI}(A)(x_\lambda) = \bigwedge_{\mu \ll \lambda} \mathbf{FI}(A)(x_\mu)$ , for any  $x_\lambda \in J(L^X)$ .
- (2)  $\mathbf{FI}(\top_{L^X})(x_\lambda) = \top_M$  for any  $x_\lambda \in J(L^X)$ .
- (3)  $\mathbf{FI}(A)(x_\lambda) = \perp_M$  for any  $x_\lambda \not\leq A$ .
- (4) For all  $r \in M_\perp$ ,  $(\mathbf{FI}(A))^{(r)} \subseteq (\mathbf{FI}(\bigvee(\mathbf{FI}(A))^{(r)}))^{(r)}$ .

$\mathbf{FI}(A)(x_\lambda)$  refers to the degree to which  $x_\lambda$  belongs to the **F**-interior of  $A$ .

**Theorem 2.19.** Let  $(X, \tau)$  be an  $(L, M)$ -fuzzy topological space and  $\mathbf{FN}^\tau = \{\mathbf{FN}_{x_\lambda}^\tau \mid x_\lambda \in J(L^X)\}$  be the corresponding  $(L, M)$ -fuzzy **F**-neighborhood system. Define the functions  $\mathbf{FI}^\tau : L^X \rightarrow M^{J(L^X)}$  by  $\mathbf{FI}^\tau(A)(x_\lambda) = \mathbf{FN}_{x_\lambda}^\tau(A)$ . Then  $\mathbf{FI}^\tau$  is an  $(L, M)$ -fuzzy **F**-interior operator on  $X$ .

**Definition 2.20.** Let  $(X, \tau)$  be an  $(L, M)$ -fuzzy topological space and  $A \in L^X$ . Define

$$\mathbf{Fcon}(A) = \bigwedge_{\substack{A_1, A_2 \in L^X \{\perp_{L^X}\}, \\ A = A_1 \vee A_2}} \left\{ \bigvee_{x_\lambda \leq A_1} \mathbf{FC}(A_2)(x_\lambda) \vee \bigvee_{y_\mu \leq A_2} \mathbf{FC}(A_1)(y_\mu) \right\}.$$

Then  $\mathbf{Fcon}(A)$  is said to be the **F**-connectedness degree of  $A$ .

**Theorem 2.21.** Let  $(X, \tau)$  be an  $(L, M)$ -fuzzy topological space and  $A \in L^X$ . Then

$$\mathbf{Fcon}(A) = \bigwedge_{\substack{A \wedge A_1 \neq \perp_{L^X}, A \wedge A_2 \neq \perp_{L^X}, \\ A \wedge A_1 \wedge A_2 \neq \perp_{L^X}, A \leq A_1 \vee A_2}} \left\{ (\mathcal{F}(A_1))' \vee (\mathcal{F}(A_2))' \right\}.$$

*Proof.* Straightforward. □

**Definition 2.22.** [25, 26] An  $L$ -fuzzy inclusion on  $X$  is a function  $\tilde{c} : L^X \times L^X \rightarrow L$  is given by the equality  $\tilde{c}(A_1, A_2) = \bigwedge_{x \in X} (A_1'(x) \vee A_2(x))$ . It is customary to denote the  $L$ -fuzzy inclusion  $\tilde{c}(A_1, A_2)$  by the symbol  $[A_1 \tilde{c} A_2]$ .

**Lemma 2.23.** [21] Let  $\varphi : X \rightarrow Y$  be a function. Then for any  $\mathcal{B} \subseteq L^Y$ , we have

$$\bigwedge_{y \in Y} \left\{ \varphi \Rightarrow (A)'(y) \vee \bigvee_{B \in \mathcal{B}} B(y) \right\} = \bigwedge_{x \in X} \left\{ B'(x) \vee \bigvee_{B \in \mathcal{B}} \varphi \Leftarrow (B)(x) \right\}.$$

In [3], Ghareeb introduced the notion of **F**-compactness in  $L$ -fuzzy topology based on  $L$ -fuzzy **F**-open operator. In the following definitions, we define the degree of **F**-compactness,  $\mathbf{FT}_1$ , and  $\mathbf{FT}_2$  in  $(L, M)$ -fuzzy topological spaces with the help of  $(L, M)$ -fuzzy **F**-open operator and the implication operation.

**Definition 2.24.** Let  $(X, \tau)$  be an  $(L, M)$ -fuzzy topological space and  $L = M$ . For any  $A \in L^X$ , let

$$\begin{aligned} \mathbf{FCM}_\tau(A) &= \bigwedge_{U \subseteq L^X} \left\{ \mathcal{F}(U) \mapsto \left( [A \tilde{c} \vee U] \mapsto \bigvee_{V \in 2^{(U)}} [A \tilde{c} \vee V] \right) \right\} \\ &= \bigwedge_{U \subseteq L^X} \left\{ \bigwedge_{A_1 \in U} \mathcal{F}(A_1) \mapsto \left\{ \bigwedge_{x \in X} \left( A' \vee \bigvee_{A_1 \in U} A_1 \right)(x) \mapsto \bigvee_{V \in 2^{(U)}} \bigwedge_{x \in X} \left( A' \vee \bigvee_{A_1 \in V} A_1 \right)(x) \right\} \right\}. \end{aligned}$$

Then  $\mathbf{FCM}_\tau(A)$  is called the **F**-compactness degree of  $A$  with respect to  $\tau$ .

**Definition 2.25.** Let  $(X, \tau)$  be an  $(L, M)$ -fuzzy topological space. Then:

- (1) The degree  $\mathbf{FT}_1(X, \tau)$  to which  $(X, \tau)$  is  $\mathbf{FT}_1$  is defined by,  $\mathbf{FT}_1(X, \tau) = \bigwedge_{a_1 \not\leq a_2} \bigvee_{a_1 \not\leq A \geq a_2} \mathcal{F}(A)$ .
- (2) The degree  $\mathbf{FT}_2(X, \tau)$  to which  $(X, \tau)$  is  $\mathbf{FT}_2$  is defined by,

$$\mathbf{FT}_2(X, \tau) = \bigwedge_{a_1 \not\leq a_2} \bigvee \{ \mathcal{F}(A_1) \wedge \mathcal{F}(A_2) \mid a_1 \not\leq A_1 \geq A_2 \geq a_2 \}.$$

### 3 Degree of **F**-openness, **F**-continuity and **F**-irresolutness for functions in $(L, M)$ -fuzzy topological spaces

In this section, we will introduce the notions of **F**-openness, **F**-continuity, and **F**-irresolutness degree for functions in  $(L, M)$ -fuzzy topological spaces. Further, we will prove that this notions can be characterized by  $(L, M)$ -fuzzy quasi **F**-neighborhood systems,  $(L, M)$ -fuzzy **F**-neighborhood systems,  $(L, M)$ -fuzzy **F**-interior operators and  $(L, M)$ -fuzzy **F**-closure operators.

**Definition 3.1.** Let  $\varphi : (X, \tau) \longrightarrow (Y, \sigma)$  be a function between two  $(L, M)$ -fuzzy topological spaces. Then:

- (1) The degree  $\mathbf{Fc}(\varphi)$  to which  $\varphi$  is  $\mathbf{F}$ -continuous is defined by,  $\mathbf{Fc}(\varphi) = \bigwedge_{B \in L^Y} \left\{ \sigma(B) \mapsto \mathcal{F}_1(\varphi^{\leftarrow}(B)) \right\}$ .
- (2) The degree  $\mathbf{Fo}(\varphi)$  to which  $\varphi$  is  $\mathbf{F}$ -open is defined by,  $\mathbf{Fo}(\varphi) = \bigwedge_{A \in L^X} \left\{ \mathcal{F}_1(A) \mapsto \mathcal{F}_2(\varphi^{\rightarrow}(A)) \right\}$ .
- (3) The degree  $\mathbf{Fi}(\varphi)$  to which  $\varphi$  is  $\mathbf{F}$ -irresolute is defined by,  $\mathbf{Fi}(\varphi) = \bigwedge_{B \in L^Y} \left\{ \mathcal{F}_2(B) \mapsto \mathcal{F}_1(\varphi^{\leftarrow}(B)) \right\}$ .

**Definition 3.2.** For a bijective function  $\varphi : (X, \tau) \longrightarrow (Y, \sigma)$  between two  $(L, M)$ -fuzzy topological spaces  $(X, \tau)$  and  $(Y, \sigma)$ , the degree  $\mathbf{F-Hom}(\varphi)$  to which  $\varphi$  is  $\mathbf{F}$ -homomorphism is given by,  $\mathbf{F-Hom}(\varphi) = \mathbf{Fi}(\varphi) \wedge \mathbf{Fo}(\varphi)$ .

**Remark 3.3.** (1) Based on Lemma 2.1 (2),  $\mathbf{Fc}(\varphi) = \top_M$  implies to  $\mathcal{F}_1(\varphi^{\leftarrow}(B)) \geq \sigma(B)$  for any  $B \in L^Y$ . This is exactly the definition of  $\mathbf{F}$ -continuous function. Similarly, for the cases  $\mathbf{Fo}(\varphi) = \top_M$  and  $\mathbf{Fi}(\varphi) = \top_M$ , then (2) and (3) in Definition 3.1 are precisely the  $\mathbf{F}$ -open and  $\mathbf{F}$ -irresolute functions definition as in the sense of Ghareeb [2].

(2) For the identity function  $i : (X, \tau) \longrightarrow (X, \tau)$ , we have  $\mathbf{Fi}(i) = \mathbf{Fo}(i) = \mathbf{F-Hom}(i) = \top_M$ .

Based on Definition 3.1 and Corollary 2.11, we have the following corollary.

**Corollary 3.4.** Let  $\varphi : (X, \tau) \longrightarrow (Y, \sigma)$  be a function between two  $(L, M)$ -fuzzy topological spaces. Then:

- (1) The degree  $\mathbf{Fc}(\varphi)$  to which  $\varphi$  is  $\mathbf{F}$ -continuous is characterized by,  $\mathbf{Fc}(\varphi) = \bigwedge_{B \in L^Y} \left\{ \sigma^*(B) \mapsto \mathcal{F}_1^*(\varphi^{\leftarrow}(B)) \right\}$ .
- (2) The degree  $\mathbf{Fi}(\varphi)$  to which  $\varphi$  is  $\mathbf{F}$ -irresolute is characterized by,  $\mathbf{Fi}(\varphi) = \bigwedge_{B \in L^Y} \left\{ \mathcal{F}_2^*(B) \mapsto \mathcal{F}_1^*(\varphi^{\leftarrow}(B)) \right\}$ .

**Definition 3.5.** For any function  $\varphi : (X, \tau) \longrightarrow (Y, \sigma)$  between two  $(L, M)$ -fuzzy topological spaces, the degree  $\mathbf{Fcl}(\varphi)$  to which  $\varphi$  is an  $\mathbf{F}$ -closed function is given by,  $\mathbf{Fcl}(\varphi) = \bigwedge_{A \in L^X} \left\{ \mathcal{F}_1^*(A) \mapsto \mathcal{F}_2^*(\varphi^{\rightarrow}(A)) \right\}$ .

**Theorem 3.6.** Let  $(X, \tau)$ ,  $(Y, \sigma)$  and  $(Z, \delta)$  be  $(L, M)$ -fuzzy topological spaces,  $\varphi : X \longrightarrow Y$  and  $\psi : Y \longrightarrow Z$  be two functions. Then:

- (1)  $\mathbf{Fi}(\varphi) \wedge \mathbf{Fi}(\psi) \leq \mathbf{Fi}(\psi \circ \varphi)$ .
- (2)  $\mathbf{Fo}(\varphi) \wedge \mathbf{Fo}(\psi) \leq \mathbf{Fo}(\psi \circ \varphi)$ .
- (3)  $\mathbf{Fcl}(\varphi) \wedge \mathbf{Fi}(\psi) \leq \mathbf{Fcl}(\psi \circ \varphi)$ .

*Proof.* Since the proof of (2) and (3) is clear, we only prove (1). By using Definition 3.1 and Lemma 2.1 (4), we obtain

$$\begin{aligned}
\mathbf{Fi}(\varphi) \wedge \mathbf{Fi}(\psi) &= \bigwedge_{B \in L^Y} \left\{ \mathcal{F}_2(B) \mapsto \mathcal{F}_1(\varphi^{\leftarrow}(B)) \right\} \wedge \bigwedge_{C \in L^Z} \left\{ \mathcal{F}_3(C) \mapsto \mathcal{F}_2(\psi^{\leftarrow}(C)) \right\} \\
&\leq \bigwedge_{C \in L^Z} \left\{ \mathcal{F}_2(\psi^{\leftarrow}(C)) \mapsto \mathcal{F}_1(\varphi^{\leftarrow}(\psi^{\leftarrow}(C))) \right\} \wedge \bigwedge_{C \in L^Z} \left\{ \mathcal{F}_3(C) \mapsto \mathcal{F}_2(\psi^{\leftarrow}(C)) \right\} \\
&= \bigwedge_{C \in L^Z} \left\{ \left( \mathcal{F}_2(\psi^{\leftarrow}(C)) \mapsto \mathcal{F}_1((\psi \circ \varphi)^{\leftarrow}(C)) \right) \wedge \left( \mathcal{F}_3(C) \mapsto \mathcal{F}_2(\psi^{\leftarrow}(C)) \right) \right\} \\
&\leq \bigwedge_{C \in L^Z} \left\{ \mathcal{F}_3(\psi^{\leftarrow}(C)) \mapsto \mathcal{F}_1((\psi \circ \varphi)^{\leftarrow}(C)) \right\} = \mathbf{Fi}(\psi \circ \varphi).
\end{aligned}$$

□

By using Definition 3.2 and Theorem 3.6, we have the following corollary.

**Corollary 3.7.** Let  $(X, \tau)$ ,  $(Y, \sigma)$  and  $(Z, \delta)$  be  $(L, M)$ -fuzzy topological spaces,  $\varphi : X \rightarrow Y$  and  $\psi : Y \rightarrow Z$  be two bijective functions. Then  $\mathbf{F}\text{-Hom}(\varphi) \wedge \mathbf{F}\text{-Hom}(\psi) \leq \mathbf{F}\text{-Hom}(\psi \circ \varphi)$ .

**Theorem 3.8.** Let  $(X, \tau)$ ,  $(Y, \sigma)$  and  $(Z, \delta)$  be  $(L, M)$ -fuzzy topological spaces and  $\psi : Y \rightarrow Z$  be a surjective function. Then:

- (1)  $\mathbf{Fo}(\psi \circ \varphi) \wedge \mathbf{Fi}(\varphi) \leq \mathbf{Fo}(\psi)$ .
- (2)  $\mathbf{Fcl}(\psi \circ \varphi) \wedge \mathbf{Fi}(\varphi) \leq \mathbf{Fcl}(\psi)$ .

*Proof.* (1) Since  $\varphi$  is a surjective function, we have  $(\psi \circ \varphi)^\Rightarrow(\varphi^\Leftarrow(B)) = \psi^\Rightarrow(B)$  for each  $B \in L^Y$ . Based on Lemma 2.1 (4), we get

$$\begin{aligned} \mathbf{Fo}(\psi \circ \varphi) \wedge \mathbf{Fi}(\varphi) &= \bigwedge_{A \in L^X} \left\{ \mathcal{F}_1(A) \mapsto \mathcal{F}_3((\psi \circ \varphi)^\Rightarrow(A)) \right\} \wedge \bigwedge_{B \in L^Y} \left\{ \mathcal{F}_2(B) \mapsto \mathcal{F}_1(\varphi^\Leftarrow(B)) \right\} \\ &\leq \bigwedge_{B \in L^Y} \left\{ \mathcal{F}_1(\varphi^\Leftarrow(B)) \mapsto \mathcal{F}_3((\psi \circ \varphi)^\Rightarrow(\varphi^\Leftarrow(B))) \right\} \wedge \bigwedge_{B \in L^Y} \left\{ \mathcal{F}_2(B) \mapsto \mathcal{F}_1(\psi^\Leftarrow(B)) \right\} \\ &= \bigwedge_{B \in L^Y} \left\{ \left( \mathcal{F}_1(\varphi^\Leftarrow(B)) \mapsto \mathcal{F}_3(\psi^\Rightarrow(B)) \right) \wedge \left( \mathcal{F}_2(B) \mapsto \mathcal{F}_1(\varphi^\Leftarrow(B)) \right) \right\} \\ &\leq \bigwedge_{B \in L^Y} \left\{ \mathcal{F}_2(B) \mapsto \mathcal{F}_3(\psi^\Rightarrow(B)) \right\} = \mathbf{Fo}(\psi). \end{aligned}$$

Analogously, we can prove (2). □

Similarly, the following theorem is true.

**Theorem 3.9.** Let  $(X, \tau)$ ,  $(Y, \sigma)$  and  $(Z, \delta)$  be  $(L, M)$ -fuzzy topological spaces,  $\varphi : X \rightarrow Y$  be an injective function and  $\psi : Y \rightarrow Z$  be any function. Then

- (1)  $\mathbf{Fo}(\psi \circ \varphi) \wedge \mathbf{Fi}(\psi) \leq \mathbf{Fo}(\varphi)$ .
- (2)  $\mathbf{Fcl}(\psi \circ \varphi) \wedge \mathbf{Fi}(\psi) \leq \mathbf{Fcl}(\varphi)$ .

**Theorem 3.10.** Let  $(X, \tau)$  and  $(Y, \sigma)$  be two  $(L, M)$ -fuzzy topological spaces and  $\varphi : X \rightarrow Y$  is a bijective function. Then

- (1)  $\mathbf{Fi}(\varphi) = \bigwedge_{A \in L^X} \left\{ \mathcal{F}_2(\varphi^\Rightarrow(A)) \mapsto \mathcal{F}_1(A) \right\}$ .
- (2)  $\mathbf{Fo}(\varphi) = \bigwedge_{B \in L^Y} \left\{ \mathcal{F}_1(\varphi^\Leftarrow(B)) \mapsto \mathcal{F}_2(B) \right\}$ .
- (3)  $\mathbf{Fi}(\varphi^{-1}) = \mathbf{Fo}(\varphi) = \mathbf{Fcl}(\varphi)$ .

*Proof.* The proof of (2) is similar to (1), we only prove (1) and (3).

(1) Since  $\varphi$  is a bijective function, we have  $\varphi^\Leftarrow(\varphi^\Rightarrow(A)) = A$  for any  $A \in L^X$ , and  $\varphi^\Rightarrow(\varphi^\Leftarrow(B)) = B$  for any  $B \in L^Y$ . It follows that

$$\begin{aligned} \bigwedge_{A \in L^X} \left\{ \mathcal{F}_2(\varphi^\Rightarrow(A)) \mapsto \mathcal{F}_1(A) \right\} &= \bigwedge_{A \in L^X} \left\{ \mathcal{F}_2(\varphi^\Rightarrow(A)) \mapsto \mathcal{F}_1(\varphi^\Leftarrow(\varphi^\Rightarrow(A))) \right\} \geq \bigwedge_{B \in L^Y} \left\{ \mathcal{F}_2(B) \mapsto \mathcal{F}_1(\varphi^\Leftarrow(B)) \right\} \\ &= \bigwedge_{B \in L^Y} \left\{ \mathcal{F}_2(\varphi^\Rightarrow(\varphi^\Leftarrow(B))) \mapsto \mathcal{F}_1(\varphi^\Leftarrow(B)) \right\} \geq \bigwedge_{A \in L^X} \left\{ \mathcal{F}_2(\varphi^\Rightarrow(A)) \mapsto \mathcal{F}_1(A) \right\}. \end{aligned}$$

Hence

$$\mathbf{Fi}(\varphi) = \bigwedge_{B \in L^Y} \left\{ \mathcal{F}_2(B) \mapsto \mathcal{F}_1(\varphi^\Leftarrow(B)) \right\} = \bigwedge_{A \in L^X} \left\{ \mathcal{F}_2(\varphi^\Rightarrow(A)) \mapsto \mathcal{F}_1(A) \right\}.$$

(3) Since  $\varphi$  is a bijective function, we get  $(\varphi^{-1})^{\leftarrow}(A) = \varphi^{\rightarrow}(A)$  and  $\varphi^{\rightarrow}(A') = \varphi^{\rightarrow}(A)'$  for any  $A \in L^X$ . Therefore

$$\mathbf{Fi}(\varphi^{-1}) = \bigwedge_{A \in L^X} \left\{ \mathcal{F}_1(A) \mapsto \mathcal{F}_2((\varphi^{-1})^{\leftarrow}(A)) \right\} = \bigwedge_{A \in L^X} \left\{ \mathcal{F}_1(A) \mapsto \mathcal{F}_2(\varphi^{\rightarrow}(A)) \right\} = \mathbf{Fo}(\varphi).$$

and

$$\mathbf{Fo}(\varphi^{-1}) = \bigwedge_{A \in L^X} \left\{ \mathcal{F}_1(A) \mapsto \mathcal{F}_2(\varphi^{\rightarrow}(A)) \right\} = \bigwedge_{A \in L^X} \left\{ \mathcal{F}_1(A') \mapsto \mathcal{F}_2(\varphi^{\rightarrow}(A')) \right\} = \bigwedge_{A \in L^X} \left\{ \mathcal{F}_1(A') \mapsto \mathcal{F}_2(\varphi^{\rightarrow}(A')) \right\} = \mathbf{Fcl}(\varphi).$$

The proof is completed.  $\square$

**Corollary 3.11.** Let  $\varphi : (X, \tau) \rightarrow (Y, \sigma)$  be a bijective function between  $(L, M)$ -fuzzy topological spaces  $(X, \tau)$  and  $(Y, \sigma)$ , then

$$(1) \mathbf{F-Hom}(\varphi) = \mathbf{Fi}(\varphi) \wedge \mathbf{Fi}(\varphi^{-1}) = \mathbf{Fi}(\varphi) \wedge \mathbf{Fcl}(\varphi).$$

$$(2) \mathbf{F-Hom}(\varphi) = \bigwedge_{A \in L^X} \left\{ \mathcal{F}_2(\varphi^{\rightarrow}(A)) \leftrightarrow \mathcal{F}_1(A) \right\}.$$

$$(3) \mathbf{F-Hom}(\varphi) = \bigwedge_{B \in L^Y} \left\{ \mathcal{F}_1(\varphi^{\leftarrow}(B)) \leftrightarrow \mathcal{F}_2(B) \right\}.$$

In the following, we characterize  $\mathbf{F}$ -irresolutness and  $\mathbf{F}$ -openness degree with the help of  $(L, M)$ -fuzzy quasi  $\mathbf{F}$ -neighborhood systems,  $(L, M)$ -fuzzy  $\mathbf{F}$ -closure operators, and  $(L, M)$ -fuzzy  $\mathbf{F}$ -interior operators.

**Corollary 3.12.** Let  $\varphi : (X, \tau) \rightarrow (Y, \sigma)$  be a function between two  $(L, M)$ -fuzzy topological spaces. Then

$$(1) \mathbf{Fi}(\varphi) = \bigwedge_{B \in L^Y} \bigwedge_{x_\lambda \in J(L^X)} \left\{ FQ_{\varphi(x)_\lambda}^{\mathcal{F}_2}(B) \mapsto FQ_{x_\lambda}^{\mathcal{F}_1}(\varphi^{\leftarrow}(B)) \right\}.$$

$$(2) \mathbf{Fi}(\varphi) = \bigwedge_{B \in L^Y} \bigwedge_{x_\lambda \in J(L^X)} \left\{ FN_{\varphi(x)_\lambda}^{\mathcal{F}_2}(B) \mapsto FN_{x_\lambda}^{\mathcal{F}_1}(\varphi^{\leftarrow}(B)) \right\}.$$

$$(3) \mathbf{Fi}(\varphi) = \bigwedge_{B \in L^Y} \bigwedge_{x_\lambda \in J(L^X)} \left\{ FI^{\mathcal{F}_2}(B)(\varphi(x)_\lambda) \mapsto FI^{\mathcal{F}_1}(\varphi^{\leftarrow}(B))(x_\lambda) \right\}.$$

$$(4) \mathbf{Fi}(\varphi) = \bigwedge_{B \in L^Y} \bigwedge_{x_\lambda \in J(L^X)} \left\{ FCl^{\mathcal{F}_2}(B)(\varphi(x)_\lambda)' \mapsto FCl^{\mathcal{F}_1}(\varphi^{\leftarrow}(B))(x_\lambda)' \right\}.$$

*Proof.* (1) Since  $FQ_{x_\lambda}^{\mathcal{F}_1}(A) = \bigvee_{x_\lambda \not\leq A'_1 > A'} \mathcal{F}(A_1)$  for any  $A \in L^X$ , and for all  $x_\lambda \in J(L^X)$  and  $B_1, B_2 \in L^Y$ , we have  $\varphi(x)_\lambda \not\leq B'_1 \geq B'_2 \Rightarrow x_\lambda \not\leq \varphi^{\leftarrow}(B_1)' \geq \varphi^{\leftarrow}(B_2)$ . Therefore, we have

$$\begin{aligned} & \bigwedge_{B_2 \in L^Y} \bigwedge_{x_\lambda \in J(L^X)} \left\{ FQ_{\varphi(x)_\lambda}^{\mathcal{F}_2}(B_2) \mapsto FQ_{x_\lambda}^{\mathcal{F}_1}(\varphi^{\leftarrow}(B_2)) \right\} = \bigwedge_{B_2 \in L^Y} \bigwedge_{x_\lambda \in J(L^X)} \left\{ \bigvee_{\varphi(x)_\lambda \not\leq B'_1 \geq B'_2} \mathcal{F}_2(B_1) \mapsto \bigvee_{x_\lambda \not\leq A'_1 \geq \varphi^{\leftarrow}(B)'} \mathcal{F}_1(A_1) \right\} \\ & \geq \bigwedge_{B_2 \in L^Y} \bigwedge_{x_\lambda \in J(L^X)} \left\{ \bigvee_{\varphi(x)_\lambda \not\leq B'_1 \geq B'_2} \mathcal{F}_2(B_1) \mapsto \bigvee_{x_\lambda \not\leq \varphi^{\leftarrow}(B_3)' \geq \varphi^{\leftarrow}(B_2)'} \mathcal{F}_1(\varphi^{\leftarrow}(B_3)) \right\} \\ & \geq \bigwedge_{B_2 \in L^Y} \bigwedge_{x_\lambda \in J(L^X)} \bigwedge_{\varphi(x)_\lambda \not\leq B'_1 \geq B'_2} \left\{ \mathcal{F}_2(B_1) \mapsto \mathcal{F}_1(\varphi^{\leftarrow}(B_1)) \right\} \geq \bigwedge_{B_2 \in L^Y} \left\{ \mathcal{F}_2(B_2) \mapsto \mathcal{F}_1(\varphi^{\leftarrow}(B_2)) \right\} = \mathbf{Fi}(\varphi). \end{aligned}$$

Since  $\mathcal{F}(A) = \bigwedge_{x_\lambda \not\leq A'} FQ_{x_\lambda}^{\mathcal{F}_1}(A)$  for any  $A \in L^X$  and  $x_\lambda \not\leq \varphi^{\leftarrow}(B)' \Rightarrow \varphi(x)_\lambda \not\leq B'$  for any  $x_\lambda \in J(L^X)$  and  $B \in L^Y$ , we have

$$\begin{aligned} \mathbf{Fi}(\varphi) &= \bigwedge_{B \in L^Y} \left\{ \mathcal{F}_2(B) \mapsto \mathcal{F}_1(\varphi^{\leftarrow}(B)) \right\} = \bigwedge_{B \in L^Y} \left\{ \bigwedge_{y_\mu \not\leq B'} FQ_{y_\mu}^{\mathcal{F}_2}(B) \mapsto \bigwedge_{x_\lambda \not\leq \varphi^{\leftarrow}(B)'} FQ_{x_\lambda}^{\mathcal{F}_1}(\varphi^{\leftarrow}(B)) \right\} \\ &\geq \bigwedge_{B \in L^Y} \left\{ \bigwedge_{\varphi(x)_\lambda \not\leq B'} FQ_{\varphi(x)_\lambda}^{\mathcal{F}_2}(B) \mapsto \bigwedge_{x_\lambda \not\leq \varphi^{\leftarrow}(B)'} FQ_{x_\lambda}^{\mathcal{F}_1}(\varphi^{\leftarrow}(B)) \right\} \geq \bigwedge_{B \in L^Y} \bigwedge_{x_\lambda \not\leq \varphi^{\leftarrow}(B)'} \left\{ FQ_{\varphi(x)_\lambda}^{\mathcal{F}_2}(B) \mapsto FQ_{x_\lambda}^{\mathcal{F}_1}(\varphi^{\leftarrow}(B)) \right\} \\ &\geq \bigwedge_{B \in L^Y} \bigwedge_{x_\lambda \in J(L^X)} \left\{ FQ_{\varphi(x)_\lambda}^{\mathcal{F}_2}(B) \mapsto FQ_{x_\lambda}^{\mathcal{F}_1}(\varphi^{\leftarrow}(B)) \right\}. \end{aligned}$$

which completes the proof of (1). Similarly, we can prove (2), (3) and (4) from Theorem 2.16 and Theorem 2.19.  $\square$



**Theorem 3.13.** Let  $\varphi : (X, \tau) \rightarrow (Y, \sigma)$  be a function between two  $(L, M)$ -fuzzy topological spaces. Then

- (1)  $\mathbf{Fo}(\varphi) = \bigwedge_{A \in L^X} \bigwedge_{x_\lambda \in J(L^X)} \left\{ FQ_{x_\lambda}^{\mathcal{F}_1}(A) \mapsto FQ_{\varphi(x)_\lambda}^{\mathcal{F}_2}(\varphi^{\Rightarrow}(A)) \right\}$ .
- (2)  $\mathbf{Fo}(\varphi) = \bigwedge_{A \in L^X} \bigwedge_{x_\lambda \in J(L^X)} \left\{ FN_{x_\lambda}^{\mathcal{F}_1}(A) \mapsto FN_{\varphi(x)_\lambda}^{\mathcal{F}_2}(\varphi^{\Rightarrow}(A)) \right\}$ .
- (3)  $\mathbf{Fo}(\varphi) = \bigwedge_{A \in L^X} \bigwedge_{x_\lambda \in J(L^X)} \left\{ FI^{\mathcal{F}_1}(A)(x_\lambda) \mapsto FI^{\mathcal{F}_2}(\varphi^{\Rightarrow}(A))(\varphi(x)_\lambda) \right\}$ .
- (4)  $\mathbf{Fo}(\varphi) = \bigwedge_{A \in L^X} \bigwedge_{x_\lambda \in J(L^X)} \left\{ FCl^{\mathcal{F}_1}(A')(x_\lambda)' \mapsto FCl^{\mathcal{F}_2}(\varphi^{\Rightarrow}(A'))(\varphi(x)_\lambda)' \right\}$ .
- (5)  $\mathbf{Fo}(\varphi) = \bigwedge_{B \in L^Y} \bigwedge_{x_\lambda \in J(L^X)} \left\{ FQ_{x_\lambda}^{\mathcal{F}_1}(\varphi^{\Leftarrow}(B)) \mapsto FQ_{\varphi(x)_\lambda}^{\mathcal{F}_2}(B) \right\}$ .
- (6)  $\mathbf{Fo}(\varphi) = \bigwedge_{B \in L^Y} \bigwedge_{x_\lambda \in J(L^X)} \left\{ FN_{x_\lambda}^{\mathcal{F}_1}(\varphi^{\Leftarrow}(B)) \mapsto FN_{\varphi(x)_\lambda}^{\mathcal{F}_2}(B) \right\}$ .
- (7)  $\mathbf{Fo}(\varphi) = \bigwedge_{B \in L^Y} \bigwedge_{x_\lambda \in J(L^X)} \left\{ FI^{\mathcal{F}_1}(\varphi^{\Leftarrow}(B))(x_\lambda) \mapsto FI^{\mathcal{F}_2}(B)(\varphi(x)_\lambda) \right\}$ .
- (8)  $\mathbf{Fo}(\varphi) = \bigwedge_{B \in L^Y} \bigwedge_{x_\lambda \in J(L^X)} \left\{ FCl^{\mathcal{F}_1}(\varphi^{\Leftarrow}(B'))(x_\lambda)' \mapsto FCl^{\mathcal{F}_2}(B')(\varphi(x)_\lambda)' \right\}$ .

*Proof.* We prove only (5). Firstly,

$$\begin{aligned}
& \bigwedge_{B \in L^Y} \bigwedge_{x_\lambda \in J(L^X)} \left\{ FQ_{x_\lambda}^{\mathcal{F}_1}(\varphi^{\Leftarrow}(B)) \mapsto FQ_{\varphi(x)_\lambda}^{\mathcal{F}_2}(B) \right\} \\
&= \bigwedge_{B \in L^Y} \bigwedge_{x_\lambda \in J(L^X)} \left\{ \bigvee_{x_\lambda \not\leq A_1' \geq \varphi^{\Leftarrow}(B)'} \mathcal{F}_1(A_1) \mapsto \bigvee_{\varphi(x)_\lambda \not\leq B_1' \geq B'} \mathcal{F}_2(B_1) \right\} \\
&\geq \bigwedge_{B \in L^Y} \bigwedge_{x_\lambda \in J(L^X)} \left\{ \bigvee_{x_\lambda \not\leq A_1' \geq \varphi^{\Leftarrow}(B)'} \mathcal{F}_1(A_1) \mapsto \bigvee_{\varphi(x)_\lambda \not\leq \varphi^{\Rightarrow}(A_2)' \geq B'} \mathcal{F}_2(\varphi^{\Rightarrow}(A_2)) \right\} \\
&\geq \bigwedge_{B \in L^Y} \bigwedge_{x_\lambda \in J(L^X)} \left\{ \bigvee_{x_\lambda \not\leq A_1' \geq \varphi^{\Leftarrow}(B)'} \mathcal{F}_1(A_1) \mapsto \bigvee_{x_\lambda \not\leq A_2' \geq \varphi^{\Leftarrow}(B)'} \mathcal{F}_2(\varphi^{\Rightarrow}(A_2)) \right\} \\
&= \bigwedge_{B \in L^Y} \bigwedge_{x_\lambda \in J(L^X)} \bigwedge_{x_\lambda \not\leq A_1' \geq \varphi^{\Leftarrow}(B)'} \left\{ \mathcal{F}_1(A_1) \mapsto \bigvee_{x_\lambda \not\leq A_2' \geq \varphi^{\Leftarrow}(B)'} \mathcal{F}_2(\varphi^{\Rightarrow}(A_2)) \right\} \\
&\geq \bigwedge_{B \in L^Y} \bigwedge_{x_\lambda \in J(L^X)} \bigwedge_{x_\lambda \not\leq A_1' \geq \varphi^{\Leftarrow}(B)'} \left\{ \mathcal{F}_1(A_1) \mapsto \mathcal{F}_2(\varphi^{\Rightarrow}(A_1)) \right\} \\
&\geq \bigwedge_{A \in L^X} \left\{ \mathcal{F}_1(A) \mapsto \mathcal{F}_2(\varphi^{\Rightarrow}(A)) \right\} = \mathbf{Fo}(\varphi).
\end{aligned}$$

Thus we have to prove  $\bigwedge_{y_\mu \not\leq \varphi^{\Rightarrow}(A)'} FQ_{y_\mu}^{\mathcal{F}_2}(\varphi^{\Rightarrow}(A)) = \bigwedge_{\varphi(x)_\lambda \not\leq \varphi^{\Rightarrow}(A)'} FQ_{\varphi(x)_\lambda}^{\mathcal{F}_2}(\varphi^{\Rightarrow}(A))$ , for any  $A \in L^X$ . It is evident that

$$\bigwedge_{y_\mu \not\leq \varphi^{\Rightarrow}(A)'} FQ_{y_\mu}^{\mathcal{F}_2}(\varphi^{\Rightarrow}(A)) \leq \bigwedge_{\varphi(x)_\lambda \not\leq \varphi^{\Rightarrow}(A)'} FQ_{\varphi(x)_\lambda}^{\mathcal{F}_2}(\varphi^{\Rightarrow}(A)).$$

Now we prove

$$\bigwedge_{y_\mu \not\leq \varphi^{\Rightarrow}(A)'} FQ_{y_\mu}^{\mathcal{F}_2}(\varphi^{\Rightarrow}(A)) \geq \bigwedge_{\varphi(x)_\lambda \not\leq \varphi^{\Rightarrow}(A)'} FQ_{\varphi(x)_\lambda}^{\mathcal{F}_2}(\varphi^{\Rightarrow}(A)).$$

For any  $y_\mu \in J(L^Y)$  with  $y_\mu \not\leq \varphi^\Rightarrow(A)'$ , we have  $\mu \not\leq (\varphi^\Rightarrow(A)(y))' = \bigwedge_{\varphi(x)=y} A(x)'$ . Then there exists  $x \in X$  such that  $\varphi(x) = y$  and  $\mu \leq A(x)'$ . Then  $\mu \not\leq \bigwedge_{\varphi(x)=\varphi(z)} A(z)' = \varphi^\Rightarrow(A)(\varphi(x))'$ . Hence  $\varphi(x)_\mu \leq \varphi^\Rightarrow(A)'$ . Since

$$\bigwedge_{\varphi(x)_\lambda \not\leq \varphi^\Rightarrow(A)'} FQ_{\varphi(x)_\lambda}^{\mathcal{F}_2}(\varphi^\Rightarrow(A)) \leq FQ_{\varphi(x)_\lambda}^{\mathcal{F}_2}(\varphi^\Rightarrow(A)) = FQ_{y_\mu}^{\mathcal{F}_2}(\varphi^\Rightarrow(A)).$$

we get

$$\bigwedge_{\varphi(x)_\lambda \not\leq \varphi^\Rightarrow(A)'} FQ_{\varphi(x)_\lambda}^{\mathcal{F}_2}(\varphi^\Rightarrow(A)) \leq \bigwedge_{y_\mu \not\leq \varphi^\Rightarrow(A)'} FQ_{y_\mu}^{\mathcal{F}_2}(\varphi^\Rightarrow(A)).$$

Hence

$$\bigwedge_{y_\mu \not\leq \varphi^\Rightarrow(A)'} FQ_{y_\mu}^{\mathcal{F}_2}(\varphi^\Rightarrow(A)) = \bigwedge_{\varphi(x)_\lambda \not\leq \varphi^\Rightarrow(A)'} FQ_{\varphi(x)_\lambda}^{\mathcal{F}_2}(\varphi^\Rightarrow(A)).$$

To prove the following fact:

$$\bigwedge_{A \in L^X} \left\{ \bigwedge_{x_\lambda \not\leq A'} FQ_{x_\lambda}^{\mathcal{F}_1}(\varphi^\Leftarrow(\varphi^\Rightarrow(A))) \mapsto \bigwedge_{\varphi(x)_\lambda \not\leq \varphi^\Rightarrow(A)'} FQ_{\varphi(x)_\lambda}^{\mathcal{F}_2}(\varphi^\Rightarrow(A)) \right\} \geq \bigwedge_{B \in L^Y} \bigwedge_{x_\lambda \in J(L^X)} \left\{ FQ_{x_\lambda}^{\mathcal{F}_1}(\varphi^\Leftarrow(B)) \mapsto FQ_{\varphi(x)_\lambda}^{\mathcal{F}_1}(B) \right\}.$$

Assume that  $r \in M$  such that

$$r \ll \bigwedge_{B \in L^Y} \bigwedge_{x_\lambda \in J(L^X)} \left\{ FQ_{x_\lambda}^{\mathcal{F}_1}(\varphi^\Leftarrow(B)) \mapsto FQ_{\varphi(x)_\lambda}^{\mathcal{F}_2}(B) \right\}.$$

Then  $r \leq FQ_{x_\lambda}^{\mathcal{F}_1}(\varphi^\Leftarrow(B)) \mapsto FQ_{\varphi(x)_\lambda}^{\mathcal{F}_2}(B)$  for each  $B \in L^Y$  and  $x_\lambda \in J(L^X)$ . Based on Lemma 2.1 (1), we have  $r \wedge FQ_{x_\lambda}^{\mathcal{F}_1}(\varphi^\Leftarrow(B)) \leq FQ_{\varphi(x)_\lambda}^{\mathcal{F}_2}(B)$ . For any  $A \in L^X$  and  $\varphi(x)_\lambda \leq \varphi^\Rightarrow(A)'$ , we have  $\lambda \leq \varphi^\Rightarrow(A)(\varphi(x))' = \bigwedge_{\varphi(x)=\varphi(z)} A(z)'$ . Then there exists  $z \in X$  such that  $\varphi(z) = \varphi(x)$  and  $\lambda \not\leq A(z)'$ . It follows that  $z_\lambda \not\leq A'$ . From

$$r \wedge \bigwedge_{x_\lambda \not\leq A'} FQ_{x_\lambda}^{\mathcal{F}_1}(\varphi^\Leftarrow(\varphi^\Rightarrow(A))) \leq r \wedge FQ_{z_\lambda}^{\mathcal{F}_1}(\varphi^\Leftarrow(\varphi^\Rightarrow(A))) \leq FQ_{\varphi(z)_\lambda}^{\mathcal{F}_2}(\varphi^\Rightarrow(A)) = FQ_{\varphi(x)_\lambda}^{\mathcal{F}_2}(\varphi^\Rightarrow(A)),$$

we have  $r \wedge \bigwedge_{x_\lambda \not\leq A'} FQ_{x_\lambda}^{\mathcal{F}_1}(\varphi^\Leftarrow(\varphi^\Rightarrow(A))) \leq \bigwedge_{\varphi(x)_\lambda \not\leq \varphi^\Rightarrow(A)'} FQ_{\varphi(x)_\lambda}^{\mathcal{F}_2}(\varphi^\Rightarrow(A))$ . Based on Lemma 2.1 (1), we have  $r \leq \bigwedge_{x_\lambda \not\leq A'} FQ_{x_\lambda}^{\mathcal{F}_1}(\varphi^\Leftarrow(\varphi^\Rightarrow(A))) \mapsto \bigwedge_{\varphi(x)_\lambda \not\leq \varphi^\Rightarrow(A)'} FQ_{\varphi(x)_\lambda}^{\mathcal{F}_2}(\varphi^\Rightarrow(A))$ . Hence,

$$r \leq \bigwedge_{A \in L^X} \left\{ \bigwedge_{x_\lambda \not\leq A'} FQ_{x_\lambda}^{\mathcal{F}_1}(\varphi^\Leftarrow(\varphi^\Rightarrow(A))) \leq \bigwedge_{\varphi(x)_\lambda \not\leq \varphi^\Rightarrow(A)'} FQ_{\varphi(x)_\lambda}^{\mathcal{F}_2}(\varphi^\Rightarrow(A)) \right\}.$$

Since  $r$  is arbitrary, we obtain

$$\begin{aligned} & \bigwedge_{A \in L^X} \left\{ \bigwedge_{x_\lambda \not\leq A'} FQ_{x_\lambda}^{\mathcal{F}_1}(\varphi^\Leftarrow(\varphi^\Rightarrow(A))) \leq \bigwedge_{\varphi(x)_\lambda \not\leq \varphi^\Rightarrow(A)'} FQ_{\varphi(x)_\lambda}^{\mathcal{F}_2}(\varphi^\Rightarrow(A)) \right\} \\ & \geq \bigwedge_{B \in L^Y} \bigwedge_{x_\lambda \in J(L^X)} \left\{ \bigwedge_{x_\lambda \not\leq A'} FQ_{x_\lambda}^{\mathcal{F}_1}(\varphi^\Leftarrow(A)) \leq \bigwedge_{\varphi(x)_\lambda \not\leq \varphi^\Rightarrow(A)'} FQ_{\varphi(x)_\lambda}^{\mathcal{F}_2}(\varphi^\Rightarrow(A)) \right\}. \end{aligned}$$

But  $A \leq \varphi^\Leftarrow(\varphi^\Rightarrow(A))$  for any  $A \in L^X$ , then

$$\begin{aligned} \mathbf{Fo}(\varphi) &= \bigwedge_{A \in L^X} \left\{ \mathcal{F}_1(A) \mapsto \mathcal{F}_2(\varphi^\Rightarrow(A)) \right\} = \bigwedge_{A \in L^X} \left\{ \bigwedge_{x_\lambda \not\leq A'} \left\{ FQ_{x_\lambda}^{\mathcal{F}_1}(A) \mapsto \bigwedge_{y_\mu \not\leq \varphi^\Rightarrow(A)'} FQ_{y_\mu}^{\mathcal{F}_2}(\varphi^\Rightarrow(A)) \right\} \right\} \\ &= \bigwedge_{A \in L^X} \left\{ \bigwedge_{x_\lambda \not\leq A'} FQ_{x_\lambda}^{\mathcal{F}_1}(A) \mapsto \bigwedge_{\varphi(x)_\lambda \not\leq \varphi^\Rightarrow(A)'} FQ_{\varphi(x)_\lambda}^{\mathcal{F}_2}(\varphi^\Rightarrow(A)) \right\} \\ &\geq \bigwedge_{A \in L^X} \left\{ \bigwedge_{x_\lambda \not\leq A'} FQ_{x_\lambda}^{\mathcal{F}_1}(\varphi^\Leftarrow(\varphi^\Rightarrow(A))) \mapsto \bigwedge_{\varphi(x)_\lambda \not\leq \varphi^\Rightarrow(A)'} FQ_{\varphi(x)_\lambda}^{\mathcal{F}_2}(\varphi^\Rightarrow(A)) \right\} \\ &\geq \bigwedge_{B \in L^Y} \left\{ \bigwedge_{x_\lambda \in J(L^X)} FQ_{x_\lambda}^{\mathcal{F}_1}(\varphi^\Leftarrow(B) \mapsto FQ_{\varphi(x)_\lambda}^{\mathcal{F}_2}(B)) \right\}, \end{aligned}$$

which completes the proof.  $\square$

## 4 The application of $\mathbf{F}$ -continuity and $\mathbf{F}$ -irresoluteness degree in $\mathbf{F}$ -compactness, $\mathbf{F}$ -connectedness, $\mathbf{FT}_1$ , and $\mathbf{FT}_2$ degree

In this section we verify the relationship between  $\mathbf{F}$ -continuity and  $\mathbf{F}$ -irresoluteness degree with  $\mathbf{F}$ -compactness,  $\mathbf{F}$ -connectedness,  $\mathbf{FT}_1$ , and  $\mathbf{FT}_2$  degree.

**Theorem 4.1.** *Let  $\varphi : (X, \tau) \rightarrow (Y, \sigma)$  be a function between two  $(L, M)$ -fuzzy topological spaces and  $L = M$ . Then, for any  $A \in L^X$ ,  $\mathbf{FCM}_{\mathcal{F}_1}(A) \wedge \mathbf{Fi}(\varphi) \leq \mathbf{FCM}_{\mathcal{F}_2}(\varphi^{\Rightarrow}(A))$*

*Proof.* Assume that  $r \in M$  such that  $r \ll \mathbf{FCM}_{\mathcal{F}_1}(A) \wedge \mathbf{Fi}(\varphi)$ . Then  $r \ll \mathbf{Fi}(\varphi) = \bigwedge_{B \in L^Y} \left\{ \mathcal{F}_2(B) \mapsto \mathcal{F}_1(\varphi^{\Leftarrow}(B)) \right\}$ , and

$$r \ll \mathbf{FCM}_{\mathcal{F}_1}(A) = \bigwedge_{A \in L^X} \left\{ \left( \bigwedge_{A_1 \in \mathcal{A}} \mathcal{F}_1(A_1) \wedge \bigwedge_{x \in X} \left( A' \vee \bigvee_{A_1 \in \mathcal{A}} A_1 \right)(x) \right) \mapsto \bigvee_{\mathcal{V} \in 2(\mathcal{A})} \bigwedge_{x \in X} \left( A' \vee \bigvee_{A_1 \in \mathcal{V}} A_1 \right)(x) \right\}$$

Then for any  $B \in L^Y$  and  $\mathcal{A} \subseteq L^X$ , we obtain  $r \leq \mathcal{F}_2(B) \mapsto \mathcal{F}_1(\varphi^{\Leftarrow}(B))$  and

$$r \leq \left( \bigwedge_{A_1 \in \mathcal{A}} \mathcal{F}_1(A_1) \wedge \bigwedge_{x \in X} \left( A' \vee \bigvee_{A_1 \in \mathcal{A}} A_1 \right)(x) \right) \mapsto \bigvee_{\mathcal{V} \in 2(\mathcal{A})} \bigwedge_{x \in X} \left( A' \vee \bigvee_{A_1 \in \mathcal{V}} A_1 \right)(x).$$

By Lemma 2.1 (1), we have  $r \wedge \mathcal{F}_2(B) \leq \mathcal{F}_1(\varphi^{\Leftarrow}(B))$  for any  $B \in L^Y$ , and

$$r \wedge \bigwedge_{W \in \mathcal{A}} \mathcal{F}_1(W) \wedge \bigwedge_{x \in X} \left( A' \vee \bigvee_{W \in \mathcal{A}} W \right)(x) \leq \bigvee_{\mathcal{V} \in 2(\mathcal{A})} \bigwedge_{x \in X} \left( A' \vee \bigvee_{W \in \mathcal{V}} W \right)(x).$$

To prove

$$r \leq \mathbf{FCM}_{\mathcal{F}_2}(\varphi^{\Rightarrow}(A)) = \bigwedge_{W \in L^Y} \left\{ \left( \bigwedge_{B_1 \in W} \mathcal{F}_2(B_1) \wedge \bigwedge_{y \in Y} \left( \varphi^{\Rightarrow}(A)' \vee \bigvee_{B_1 \in W} B_1 \right)(y) \right) \mapsto \bigvee_{\mathcal{D} \in 2(W)} \bigwedge_{y \in Y} \left( \varphi^{\Rightarrow}(A)' \vee \bigvee_{B_1 \in \mathcal{D}} B_1 \right)(y) \right\},$$

for all  $W \subseteq L^Y$ , let  $\varphi^{\Leftarrow}(W) = \{\varphi^{\Leftarrow}(B_1) | B_1 \in W\} \subseteq L^X$ . Then by Lemma 2.23, we have

$$\begin{aligned} & r \wedge \bigwedge_{B_1 \in W} \mathcal{F}_2(B_1) \wedge \bigwedge_{y \in Y} \left( \varphi^{\Rightarrow}(A)' \vee \bigvee_{B_1 \in W} B_1 \right)(y) \wedge \bigwedge_{B_1 \in W} \mathcal{F}_1(\varphi^{\Leftarrow}(B_1)) \wedge \bigwedge_{y \in Y} \left( \varphi^{\Rightarrow}(A)' \vee \bigvee_{B_1 \in W} B_1 \right)(y) \\ &= r \wedge \bigwedge_{B_1 \in W} \mathcal{F}_1(\varphi^{\Leftarrow}(B_1)) \wedge \bigwedge_{x \in X} \left( A' \vee \bigvee_{B_1 \in W} \varphi^{\Leftarrow}(B_1) \right)(x) = r \wedge \bigwedge_{A_1 \in \varphi^{\Leftarrow}(W)} \mathcal{F}_1(A_1) \wedge \bigwedge_{x \in X} \left( A' \vee \bigvee_{A_1 \in \varphi^{\Leftarrow}(W)} A_1 \right)(x) \\ &\leq \bigvee_{\mathcal{V} \in 2(\varphi^{\Leftarrow}(W))} \bigwedge_{x \in X} \left( A' \vee \bigvee_{A_1 \in \mathcal{V}} A_1 \right)(x) = \bigvee_{\mathcal{D} \in 2(W)} \bigwedge_{x \in X} \left( A' \vee \bigvee_{B_1 \in \mathcal{D}} \varphi^{\Leftarrow}(B_1) \right)(x) = \bigvee_{\mathcal{D} \in 2(W)} \bigwedge_{y \in Y} \left( \varphi^{\Rightarrow}(A)' \vee \bigvee_{B_1 \in \mathcal{D}} B_1 \right)(y). \end{aligned}$$

By using Lemma 2.1 (1), we know

$$r \leq \left( \bigwedge_{B_1 \in W} \mathcal{F}_2(B_1) \wedge \bigwedge_{y \in Y} \left( \varphi^{\Rightarrow}(A)' \vee \bigvee_{B_1 \in W} B_1 \right)(y) \right) \mapsto \bigvee_{\mathcal{D} \in 2(W)} \bigwedge_{y \in Y} \left( \varphi^{\Rightarrow}(A)' \vee \bigvee_{B_1 \in \mathcal{D}} B_1 \right)(y).$$

Thus

$$r \leq \bigwedge_{W \subseteq L^Y} \left\{ \left( \bigwedge_{B_1 \in W} \mathcal{F}_2(B_1) \wedge \bigwedge_{y \in Y} \left( \varphi^{\Rightarrow}(A)' \vee \bigvee_{B_1 \in W} B_1 \right)(y) \right) \mapsto \bigvee_{\mathcal{D} \in 2(W)} \bigwedge_{y \in Y} \left( \varphi^{\Rightarrow}(A)' \vee \bigvee_{B_1 \in \mathcal{D}} B_1 \right)(y) \right\} = \mathbf{FCM}_{\mathcal{F}_2}(\varphi^{\Rightarrow}(A)).$$

Since  $r$  is arbitrary, we have  $\mathbf{FCM}_{\mathcal{F}_1}(A) \wedge \mathbf{Fi}(\varphi) \leq \mathbf{FCM}_{\mathcal{F}_2}(\varphi^{\Rightarrow}(A))$ . The proof is completed.  $\square$

From Theorem 4.1, the following corollary is true.

**Corollary 4.2.** *Let  $\varphi : (X, \tau) \rightarrow (Y, \sigma)$  be a surjective function between the two  $(L, M)$ -fuzzy topological spaces  $(X, \tau)$  and  $(Y, \sigma)$  and  $L = M$ . Then  $\mathbf{FCM}_{\mathcal{F}_1}(\perp_{L^X}) \wedge \mathbf{Fi}(\varphi) \leq \mathbf{FCM}_{\mathcal{F}_2}(\perp_{L^Y})$ .*

In general topology, it is well known that  $\varphi(A)$  is connected subset if  $A$  is connected and  $\varphi$  is continuous function. Now we extend this result to  $(L, M)$ -fuzzy topological space as follows.

**Theorem 4.3.** Let  $\varphi : X \rightarrow Y$  be any function between two  $(L, M)$ -fuzzy topological spaces  $(X, \tau)$  and  $(Y, \sigma)$ . Then, for any  $A \in L^X$ ,  $[\mathbf{Fcon}_{\mathcal{F}_2}(\varphi^{\Rightarrow}(A))]' \wedge \mathbf{Fi}(\varphi) \leq \mathbf{Fcon}_{\mathcal{F}_1}(A)'$ .

*Proof.* Assume that  $r \in M$  such that  $r \ll \mathbf{Fcon}_{\mathcal{F}_2}(\varphi^{\Rightarrow}(A))' \wedge \mathbf{Fi}(\varphi)$ . From Theorems 2.21 and 3.4 (2), we have

$$r \ll \mathbf{Fcon}_{\mathcal{F}_2}(\varphi^{\Rightarrow}(A))' = \bigvee_{\substack{\varphi^{\Rightarrow}(A) \wedge B_1 \neq \perp_{LY}, \\ \varphi^{\Rightarrow}(A) \wedge B_2 \neq \perp_{LY}, \\ \varphi^{\Rightarrow}(A) \wedge B_1 \wedge B_2 = \perp_{LY}, \\ \varphi^{\Rightarrow}(A) \leq B_1 \vee B_2}} \left\{ \mathcal{F}_2(B'_1) \vee \mathcal{F}_2(B'_2) \right\},$$

and

$$r \ll \mathbf{Fi}(\varphi) = \bigwedge_{B_3 \in L^Y} \left\{ \mathcal{F}_2(B'_3) \mapsto \mathcal{F}_1(\varphi^{\Leftarrow}(B_3)') \right\}.$$

Then there exist  $B_1, B_2 \in L^Y$  such that  $\varphi^{\Rightarrow}(A) \wedge B_1 \neq \perp_{LY}$ ,  $\varphi^{\Rightarrow}(A) \wedge B_2 \neq \perp_{LY}$ ,  $\varphi^{\Rightarrow}(A) \wedge B_1 \wedge B_2 = \perp_{LY}$ ,  $\varphi^{\Rightarrow}(A) \leq B_1 \vee B_2$  with  $r \leq \mathcal{F}_2(B'_1) \wedge \mathcal{F}_2(B'_2)$ , and  $r \leq \mathcal{F}_2(B'_3) \mapsto \mathcal{F}_1(\varphi^{\Leftarrow}(B_3)')$  for any  $B_3 \in L^Y$ . It follows that there exist  $B_1, B_2 \in L^Y$  such that  $A \wedge \varphi^{\Leftarrow}(B_1) \neq \perp_{LX}$ ,  $A \wedge \varphi^{\Leftarrow}(B_2) \neq \perp_{LX}$ ,  $A \wedge \varphi^{\Leftarrow}(B_1) \wedge \varphi^{\Leftarrow}(B_2) = \perp_{LX}$ ,  $A \leq \varphi^{\Leftarrow}(B_1) \vee \varphi^{\Leftarrow}(B_2)$  with

$$r \leq \mathcal{F}_2(B_1) \wedge \mathcal{F}_2(B_2), \quad (1)$$

and

$$r \wedge \mathcal{F}_2(B'_3) \leq \mathcal{F}_1(\varphi^{\Leftarrow}(B_3)'), \quad (2)$$

for any  $B_3 \in L^Y$ . From Eqs. (1) and (2), we have

$$r = r \wedge \mathcal{F}_2(B'_1) \wedge \mathcal{F}_2(B'_2) \leq \mathcal{F}_1(\varphi^{\Leftarrow}(B_1)') \wedge \mathcal{F}_1(\varphi^{\Leftarrow}(B_1)') \leq \bigvee_{\substack{A \wedge A_1 \neq \perp_{LX}, A \wedge A_2 \neq \perp_{LX}, \\ A \wedge A_1 \wedge A_2 \neq \perp_{LX}, A \leq A_1 \vee A_2}} \left\{ \mathcal{F}_1(A'_1) \vee \mathcal{F}_1(A'_2) \right\} = \mathbf{Fcon}_{\mathcal{F}_1}(A)'.$$

Since  $r$  is arbitrary, we have  $\mathbf{Fcon}_{\mathcal{F}_2}(\varphi^{\Rightarrow}(A))' \wedge \mathbf{Fi}(\varphi) \leq \mathbf{Fcon}_{\mathcal{F}_1}(A)'$ . □

It is easy to prove that  $\mathbf{FT}_1$  and  $\mathbf{FT}_2$  in our sense are preserved by  $\mathbf{F}$ -homeomorphisms. In the remainder of this section, we generalize this result into the  $(L, M)$ -fuzzy topology setting.

**Lemma 4.4.** Let  $\varphi : (X, \tau) \rightarrow (Y, \sigma)$  be a bijective function between two  $(L, M)$ -fuzzy topological spaces  $(X, \tau)$  and  $(Y, \sigma)$ . Then

$$(1) \quad \mathbf{FT}_1(X, \tau) \wedge \mathbf{Fo}(\varphi) \leq \mathbf{FT}_1(Y, \sigma).$$

$$(2) \quad \mathbf{FT}_2(X, \tau) \wedge \mathbf{Fo}(\varphi) \leq \mathbf{FT}_2(Y, \sigma).$$

*Proof.* We only prove (1) and the proof of (2) is similar. Assume that  $r \in M$  with

$$r \ll \mathbf{FT}_1(X, \tau) \wedge \mathbf{Fo}(\varphi) = \bigwedge_{a \not\leq b} \bigvee_{a \not\leq A \geq b} \mathcal{F}_1(A') \wedge \bigwedge_{A_1 \in L^X} \left( \mathcal{F}_1(A_1) \mapsto \mathcal{F}_2(\varphi^{\Rightarrow}(A_1)) \right).$$

Thus for each  $a_1, a_2 \in J(L^X)$  such that  $a_1 \not\leq a_2$ , there exists  $A \in L^X$  with  $a_1 \not\leq A \geq a_2$  and  $r \leq \mathcal{F}_1(A')$ . For  $A_1 \in L^X$ ,  $r \leq \mathcal{F}_1(A_1) \mapsto \mathcal{F}_2(\varphi^{\Rightarrow}(A_1))$ . From Lemma 2.1 (1), we obtain  $r \wedge \mathcal{F}_1(A_1) \leq \mathcal{F}_2(\varphi^{\Rightarrow}(A_1))$ . To verify

$$r \leq \mathbf{FT}_1(Y, \sigma) = \bigwedge_{b_1 \not\leq b_2} \bigvee_{b_1 \not\leq B \geq b_2} \mathcal{F}_2(B'),$$

let  $b_1, b_2 \in J(L^Y)$  such that  $b_1 \not\leq b_2$ . From the bijectivity of the function  $\varphi$ , there exist  $a_1, a_2 \in J(L^X)$  such that  $a_1 \not\leq a_2$  with  $b_1 = \varphi^{\Rightarrow}(a_1)$  and  $b_2 = \varphi^{\Leftarrow}(a_2)$ . Since  $a_1 \not\leq a_2$ , there exists  $A \in L^X$  such that  $a_1 \not\leq A \geq a_2$  with  $r \leq \mathcal{F}_1(A')$ . Thus  $b_1 = \varphi^{\Leftarrow}(a_1) \not\leq \varphi^{\Leftarrow}(A) \geq \varphi^{\Rightarrow}(a_2) = b_2$ . Since  $\varphi$  is a bijective function, we know

$$r = r \wedge \mathcal{F}_1(A') \leq \mathcal{F}_2(\varphi^{\Rightarrow}(A')) = \mathcal{F}_2(\varphi^{\Rightarrow}(A)').$$

Therefore,  $r \leq \bigwedge_{b_1 \not\leq b_2} \bigvee_{b_1 \not\leq B \geq b_2} \mathcal{F}_2(B') = \mathbf{FT}_1(Y, \sigma)$ . Since  $r$  is arbitrary, we proved that  $\mathbf{FT}_1(X, \tau) \wedge \mathbf{Fo}(\varphi) \leq \mathbf{FT}_1(Y, \sigma)$ . This complete the proof. □

From Lemma 4.4 and Definition 3.2 (1), we have the following theorem.

**Theorem 4.5.** *Let  $\varphi : (X, \tau) \rightarrow (Y, \sigma)$  be a bijective function between two  $(L, M)$ -fuzzy topological spaces  $(X, \tau)$  and  $(Y, \sigma)$ . Then*

- (1)  $\mathbf{FT}_1(X, \tau) \wedge \mathbf{F-Hom}(\varphi) \leq \mathbf{FT}_1(Y, \sigma)$ ,  $\mathbf{FT}_1(Y, \sigma) \wedge \mathbf{F-Hom}(\varphi) \leq \mathbf{FT}_1(X, \tau)$ .
- (2)  $\mathbf{FT}_2(X, \tau) \wedge \mathbf{F-Hom}(\varphi) \leq \mathbf{FT}_2(Y, \sigma)$ ,  $\mathbf{FT}_2(Y, \sigma) \wedge \mathbf{F-Hom}(\varphi) \leq \mathbf{FT}_2(X, \tau)$ .

From Theorem 4.5, the following corollary is true.

**Corollary 4.6.** *Let  $\varphi : (X, \tau) \rightarrow (Y, \sigma)$  be a bijective function between two  $(L, M)$ -fuzzy topological spaces  $(X, \tau)$  and  $(Y, \sigma)$ . Then*

- (1)  $\mathbf{FT}_1(X, \tau) \wedge \mathbf{F-Hom}(\varphi) = \mathbf{FT}_1(Y, \sigma) \wedge \mathbf{F-Hom}(\varphi)$ .
- (2)  $\mathbf{FT}_2(X, \tau) \wedge \mathbf{F-Hom}(\varphi) = \mathbf{FT}_2(Y, \sigma) \wedge \mathbf{F-Hom}(\varphi)$ .

## 5 Conclusions

This paper introduced a new description of  $(L, M)$ -fuzzy topological spaces based on sub-varieties of new functions degree. We defined and discussed new degrees of weak forms of functions in  $(L, M)$ -fuzzy topology with the help of  $(L, M)$ -fuzzy  $\mathbf{F}$ -open operator and implication operation. The properties of  $\mathbf{F}$ -openness,  $\mathbf{F}$ -continuity, and  $\mathbf{F}$ -irresoluteness degree of functions in  $(L, M)$ -fuzzy topology are investigated. Further, we proved that any function can be regard as  $\mathbf{F}$ -open,  $\mathbf{F}$ -continuous, and  $\mathbf{F}$ -irresolute to some degree. Moreover, its relationship with  $\mathbf{F}$ -compactness,  $\mathbf{F}$ -connectedness,  $\mathbf{FT}_1$ , and  $\mathbf{FT}_2$  degree have been constructed and analyzed systematically.

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