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Some new variants of interval-valued Gronwall type inequalities on time scales

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Abstract

By using an efficient partial order and concept of gH-differentiability on interval-valued functions, we investigate some new variants of Gronwall type inequalities on time scales, which provide explicit bounds on unknown functions. Our results not only unify and extend some continuous inequalities, but also in discrete case, all are new.

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1 Introduction

The work of Thomas Gronwall [12] and its first generalization proved by Richard Bellman [3] motivate many researchers to obtain various generalizations and extensions [22, 23]. The Gronwall-Bellman type inequalities, are important tools to obtain various estimates in the theory of differential equations, see e.g. [2, 18, 22] and [23].

There are several mathematical models to study the behavior of the real-world systems such as static or dynamic, linear or nonlinear, continuous or discrete, deterministic or probabilistic. In many cases, the knowledge about the parameters of a real-world system is imprecise or uncertain because, generally, we cannot observe or measure these parameters accurately. In these situations, the parameters cannot be represented by real numbers. This shortcoming is overcome using fuzzy or interval models. Interval analysis is based on the representation of an uncertain variable as an interval of real numbers [20].

Interval analysis is a particular case and it has relevant applications in the treatment of the uncertainty that appears in the modeling of some real-world problems [11, 26]. In this direction, recently several classical integral inequalities have been extended to the interval-valued context [7, 24], for instance by using the concept of gH-differentiability the authors in [7] shows an Ostrowski's inequality for interval-valued functions. A note on Gronwall type inequality for interval-valued functions is presented by Román-Flores et al. [24].

It is well known that the dynamic inequalities play important role in the development of the qualitative theory of dynamic equations on time scales. The study of dynamic equations on time scales which goes back to its founder Stefan Hilger [15] becomes an area of mathematics and recently has received a lot of attention. Recently, Luplescu in [17] developed calculus for interval-valued functions on time scales, using the concept of generalized Hukuhara difference provided by Markov [19].

Among the more recent investigations, let us mention the work of Younus et al. [27], where authors obtained Gronwall type inequalities for interval-valued functions under the notion of Kulish-Mirankor partial order on a set of compact intervals. However, there are many other partial orders, which cannot be covered by Kulish-Mirankor partial order.

In the study of Gronwall inequalities, an important notion is an exponential function on time scales. The exponential function on time scales has been introduced by Hilger [14], and his definition seems to be commonly accepted [4, 5]. A different situation has a place in the case of hyperbolic and trigonometric functions, where Hilger's approach [13] differs

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from Bohner and Peterson's approach [4]. A newly improved exponential, hyperbolic and trigonometric functions base on Cayley transformation has been defined by Cieśliński [9], which not only preserve more properties of their continuous counterparts in comparison to the previous definitions, but also Cayley exponential function maps the imaginary axis into the unit circle and trigonometric functions satisfy the Pythagorean identities [9, 10].

In the main part of this paper, we first investigate some Gronwall type inequalities for real-valued functions on time scales by using the concept of Cayley exponential function, which generalizes some inequalities from [1, 6, 16]. By defining an efficient partial order on a set of compact intervals, we obtain new variants of Gronwall type inequality for interval-valued functions, which are more general than our previous results [27].

2 Preliminaries

2.1 Time scale calculus

In what follows, we recall some notions about the time scale analysis. An extensive study of the analysis on time scales can be found in [4]. For a time scale $\mathbb{T}$, the forward jump operator σ (shortly t^{σ}) is defined by $\sigma(t) := \inf\{s \in \mathbb{T}, s > t\}$, backward jump operator$\rho$ (shortly t^{ρ}) is defined by $\rho(t) := \sup\{s \in \mathbb{T}, s < t\}$ and the step size function $\mu : \mathbb{T} \to \mathbb{R}^{+}$ is given by $\mu(t) := t^{\sigma} - t$. We say a point $t \in \mathbb{T}$ is right dense if $\mu(t) = 0$, and right scattered if $\mu(t) > 0$. Furthermore, a point $t \in \mathbb{T}$ is said to be left dense if $t^{\rho} = t$ and left scattered if $t^{\rho} < t$. If $\mathbb{T}$ has a left-scattered maximum M, then $\mathbb{T}^{k}$ $= \mathbb{T} - \{M\}$; otherwise set $\mathbb{T}^{k} = \mathbb{T}$. Moreover, the delta derivative of a function $f : \mathbb{T} \to \mathbb{R}$ at a point $t \in \mathbb{T}^{k}$ is defined by

$$f^{\Delta}(t) = \lim_{\substack{s \to t \\ s \neq t^{\sigma}}} \frac{f(t^{\sigma}) - f(s)}{t^{\sigma} - s}.$$

A function f is called rd-continuous provided that it is continuous at right dense points in $\mathbb{T}$, and has a finite limit at left-dense points, and the set of rd-continuous functions are denoted by $C_{rd}(\mathbb{T}, \mathbb{R})$. The set of functions $C_{rd}^{1}(\mathbb{T}, \mathbb{R})$ includes the functions f whose derivative is in $C_{rd}(\mathbb{T}, \mathbb{R})$ too. For $s, t \in \mathbb{T}$ and a function $f \in C_{rd}(\mathbb{T}, \mathbb{R})$, the Δ-integral is defined to be

$$\int_{s}^{t} f(\tau)\Delta\tau = F(t) - F(s),$$

where $F \in C_{rd}^{1}(\mathbb{T}, \mathbb{R})$ is an anti-derivative of f, i.e., $F^{\Delta} = f$ on $\mathbb{T}^{k}$.

In order to define Cayley-exponential (shortly, C-exponential) function, Cieśliński [9], redefined a notion of regressivity as follows:

A function $f \in C_{rd}(\mathbb{T}, \mathbb{R})$ is called regressive if $\mu(t)f(t) \neq \pm 2$ for all $t \in \mathbb{T}^{k}$, and $f \in C_{rd}(\mathbb{T}, \mathbb{R})$ is called positively regressive if $|\mu(t)f(t)| < 2$ on $\mathbb{T}^{k}$. The set of regressive and positively regressive functions are denoted by $\mathcal{R}(\mathbb{T}, \mathbb{R})$ and $\mathcal{R}^{+}(\mathbb{T}, \mathbb{R})$(or shortly denoted by $\mathcal{R}$ and $\mathcal{R}^{+}$), respectively.

Under the binary operation $\oplus$, defined by $\alpha \oplus \beta = \frac{\alpha+\beta}{1+\frac{1}{4}\mu^{2}\alpha\beta}$, $\mathcal{R}^{+}$ is an Abelian group [9, Theorem 3.14]. However, the set $\mathcal{R}$ is not closed with respect to $\oplus$.

Let $f \in \mathcal{R}$ and $s \in \mathbb{T}$, then the Cayley-exponential function $E_{f}(\cdot, s)$ on a time scale $\mathbb{T}$ is defined to be the unique solution of the following initial value problem

$$\begin{cases} x^{\Delta}(t) = f(t)\left\langle x(t)\right\rangle \\ x(s) = 1, \end{cases}$$

where

$$\left\langle x(t)\right\rangle = \frac{x\left(t\right) + x\left(\sigma(t)\right)}{2}.$$

For $h \in \mathbb{R}^{+}$, the Cayley transformation ξ_{h} is defined as

$$\xi_{h}(z) \colon = \begin{cases} z, & h = 0 \\ \frac{1}{h}\operatorname{Log}\left(\frac{1+\frac{zh}{2}}{1-\frac{zh}{2}}\right), & h > 0, \end{cases}$$

and the Cayley-exponential function for $f \in \mathcal{R}$ is defined by

$$E_{f}(t, s) \colon = \exp\left\{\int_{s}^{t} \xi_{\mu(\tau)}(f(\tau))\Delta\tau\right\} \text{ for } s, t \in \mathbb{T}.$$

Lemma 2.1. *If $\alpha, \beta : \mathbb{T} \to \mathbb{C}$ are regressive, then the following properties hold:*

1. $E_\alpha(t^\sigma, t_0) = \frac{1+\frac{1}{2}\alpha(t)\mu(t)}{1-\frac{1}{2}\alpha(t)\mu(t)} E_\alpha(t, t_0)$,
2. $(E_\alpha(t, t_0))^{-1} = E_{-\alpha}(t, t_0) = \frac{1}{E_\alpha(t,t_0)}$,
3. $\overline{E_\alpha(t, t_0)} = E_{\bar{\alpha}}(t, t_0)$,
4. $E_\alpha(t, t_0)E_\alpha(t_0, t_1) = E_\alpha(t, t_1)$,
5. $E_\alpha(t, t_0)E_\beta(t, t_0) = E_{\alpha\oplus\beta}(t, t_0)$.

Lemma 2.2. *If $\alpha : \mathbb{T} \to \mathbb{R}$ is positively regressive and rd- continuous, then the C-exponential function E_α is positive.*

Lemma 2.3. *There is a bijection between C-exponential functions and delta exponential functions. Namely, $E_\alpha(t, t_0) = e_\beta(t, t_0)$, if $\alpha(t) = \frac{\beta(t)}{1+\frac{1}{2}\beta(t)\mu(t)}$, $\beta(t) = \frac{\alpha(t)}{1-\frac{1}{2}\alpha(t)\mu(t)}$, with $\alpha\mu \neq \pm 2$ and $\beta\mu \neq -1$.*

These lemmas are Theorem 3.10 and 3.13 in [9], and Theorem 3.2 in [10], respectively. For further details about these notions, one may consult [9, 10].

2.2 Interval-valued calculus on time scales

Let $\mathcal{K}_C$ be the set of all non-empty compact intervals of the real line $\mathbb{R}$. If $[a^-, a^+]$, $[b^-, b^+] \in \mathcal{K}_C$, then the usual interval operations, i.e. Minkowski addition and scalar multiplication, are defined by

$$[a^-, a^+] + [b^-, b^+] = [a^- + b^-, a^+ + b^+]$$

and

$$\lambda[a^-, a^+] = \begin{cases} [\lambda a^-, \lambda a^+] & \text{if } \lambda > 0 \\ \{0\} & \text{if } \lambda = 0 \\ [\lambda a^+, \lambda a^-] & \text{if } \lambda < 0, \end{cases}$$

respectively. If $\lambda = -1$, then $(-1)[a^-, a^+] = -[a^-, a^+] = [-a^+, -a^-]$. It is easy to see that $\lambda A = A\lambda$ for all $\lambda \in \mathbb{R}$.

For two intervals $[a^-, a^+]$, $[b^-, b^+] \in \mathcal{K}_C$, the generalized Hukuhara difference (gH-difference for short), is defined as follows [19, 25]:

$$[a^-, a^+] \ominus_g [b^-, b^+] = [\min\{a^- - b^-, a^+ - b^+\}, \max\{a^- - b^-, a^+ - b^+\}]. \tag{1}$$

For $A = [a^-, a^+] \in \mathcal{K}_C$, $len(A) = a^+ - a^-$ is called the *length* of interval A. Then, for $A = [a^-, a^+]$ and $B = [b^-, b^+]$, we have

$$A \ominus_g B = \begin{cases} [a^- - b^-, a^+ - b^+], \text{ if } len(A) \geq len(B), \\ [a^+ - b^+, a^- - b^-], \text{ if } len(A) < len(B). \end{cases} \tag{2}$$

If $A, B, C \in \mathcal{K}_C$ then

$$A \ominus_g B = C \Longleftrightarrow \begin{cases} A = B + C, \text{ if } len(A) \geq len(B), \\ B = A + (-C), \text{ if } len(A) < len(B). \end{cases} \tag{3}$$

Since, $\mathcal{K}_C$ is not totally order set. To compare the images of interval-valued functions in the context of optimization problems, several partial order relations exist in $\mathcal{K}_C$ (for e.g., see [7, 19, 21, 27])

Let us define a new partial order. For A, $B \in \mathcal{K}_C$, we say that

$$A \preceq_{LW} B \Leftrightarrow a^- \leq b^- \text{and } len\,(A) \leq len\,(B)\,. \tag{4}$$

and $A \prec_{LW} B \Leftrightarrow A \preceq_{LW} B$ and $A \neq B$.

It is easy to see that $A \preceq_{LW} B$ implies $A \preceq_{LU} B$, but the converse may not be true.

Under the Hausdorff-Pompeiu distance $D : \mathcal{K}_C \times \mathcal{K}_C \to [0, \infty)$ defined by $D(A, B) = \max\{|a^- - b^-|, |a^+ - b^+|\}$, $(\mathcal{K}_C, D)$ is a complete and separable metric space. Moreover, the limits and continuity can be characterized, in the metric space $(\mathcal{K}_C, D)$, by the gH-difference.

Next, we recall the basics of the calculus of interval-valued functions on time scales(see [17]).

For an interval-valued function $F : \mathbb{T} \to \mathcal{K}_C$, $A \in \mathcal{K}_C$ at $t_0 \in \mathbb{T}$ is called $\mathbb{T}$-limit, if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $D(F(t) \ominus_g A, \{0\}) \leq \varepsilon$ for all $t \in U_{\mathbb{T}}(t_0, \delta)$.

An interval-valued function $F:\mathbb{T}\to\mathcal{K}_C$ is called rd-continuous if it is continuous at all right-dense points in $\mathbb{T}$ and its left-sided $\mathbb{T}$-limits (in $\mathcal{K}_C$) exist at all left-dense points in $\mathbb{T}$.

Obviously, $F:\mathbb{T}\to\mathcal{K}_C$, with $F(t)=[f^-(t),f^+(t)]$ is rd-continuous if and only if f^- and f^+ are rd-continuous.

For $F:\mathbb{T}\to\mathcal{K}_C$ and $t\in\mathbb{T}^\kappa$, $F^\Delta(t)\in\mathcal{K}_C$ (provided it exists) is called delta generalized Hukuhara derivative (Δ_{gH}-derivative for short) if, for every $\varepsilon>0$, there is $\delta>0$ such that

$$D(F(t^\sigma)\ominus_g F(s),(t^\sigma-s)\,F^\Delta(t))\le\varepsilon\,|t^\sigma-s| \tag{5}$$

for all $s\in U_{\mathbb{T}}(t,\delta)$. Moreover, we say that F is delta generalized Hukuhara differentiable (Δ_{gH}-differentiable for short) on $\mathbb{T}^\kappa$ if $F^\Delta(t)\in\mathcal{K}_C$ exists at each point $t\in\mathbb{T}^\kappa$. The interval-valued function $F^\Delta:\mathbb{T}^\kappa\to\mathcal{K}_C$ is then called the Δ_{gH}-derivative of F on $\mathbb{T}^\kappa$.

Lemma 2.4. *Let $F:\mathbb{T}\to\mathcal{K}_C$ be such that $F(t)=[f^-(t),f^+(t)]$, $t\in\mathbb{T}$. If the real-valued functions f^- and f^+ are Δ-differentiable at $t_0\in\mathbb{T}^\kappa$, then F is Δ_{gH}-differentiable at $t_0\in\mathbb{T}^\kappa$ and*

$$F^\Delta(t)=\left[\min\{(f^-)^\Delta(t_0),(f^+)^\Delta(t_0)\},\max\{(f^-)^\Delta(t_0),(f^+)^\Delta(t_0)\}\right]. \tag{6}$$

The converse of Lemma 2.4 may not be true, that is, the Δ_{gH}-differentiability of F does not imply the Δ-differentiability of f^- and f^+ (see [8, 19]).

As a consequence of Lemma 2.4 we have, if $g:\mathbb{T}\to\mathbb{R}$ be a Δ-differentiable at $t_0\in\mathbb{T}^\kappa$ and $C=[a,b]$, $a<b$, *$F:\mathbb{T}\to\mathcal{K}_C$ be an interval valued function given by, $F(t)=C.g(t)$.* Then F is Δ_{gH}-differentiable at $t_0\in\mathbb{T}^\kappa$ and $F^\Delta(t)=C\cdot g^\Delta(t)$.

We say that an interval-valued function $F:\mathbb{T}\to\mathcal{K}_C$ is l-nondecreasing (l-nonincreasing) on $\mathbb{T}$ if the real function $t\mapsto len(F(t))$ nondecreasing (nonincreasing) on $\mathbb{T}$. If F is l-nondecreasing or l-nonincreasing on $\mathbb{T}$, then we say that F is l-monotone on $\mathbb{T}$.

Lemma 2.5. *Let $F:[a,b]_{\mathbb{T}}\to\mathcal{K}_C$ be such that $F(t)=[f^-(t),f^+(t)]$, $t\in[a,b]_{\mathbb{T}}$. If F is l-monotone on $[a,b]_{\mathbb{T}}$ and Δ_{gH}-differentiable on $[a,b)_{\mathbb{T}}$, then $(f^-)^\Delta(t)$ and $(f^+)^\Delta(t)$ exist for all $t\in[a,b)_{\mathbb{T}}$. Moreover, we have that:*

(i) $F^\Delta(t)=[(f^-)^\Delta(t),(f^+)^\Delta(t)]$ *for all $t\in[a,b)_{\mathbb{T}}$, if F is l-nondecreasing*;
(ii) $F^\Delta(t)=[(f^+)^\Delta(t),(f^-)^\Delta(t)]$ *for all $t\in[a,b)_{\mathbb{T}}$, if F is l-nonincreasing.*

For l-monotone Δ_{gH}-differentiable interval-valued function $F:[a,b]_{\mathbb{T}}\to\mathcal{K}_C$, F is called $\Delta_{1,gH}$-differentiable on $[a,b)_{\mathbb{T}}$ if

$$\text{(i)}\quad F^\Delta(t)=[(f^-)^\Delta(t),(f^+)^\Delta(t)]\text{ for all }t\in[a,b)_{\mathbb{T}}, \tag{7}$$

and $\Delta_{2,gH}$-differentiable on $[a,b)_{\mathbb{T}}$ if

$$\text{(ii)}\quad F^\Delta(t)=[(f^+)^\Delta(t),(f^-)^\Delta(t)]\text{ for all }t\in[a,b)_{\mathbb{T}}. \tag{8}$$

Let $F:\mathbb{T}\to\mathcal{K}_C$ be an interval-valued function and let $P:a=t_0<t_1<...<t_n=b$ be a partition of $[a,b)_{\mathbb{T}}$. In each interval $[t_{i-1},t_i)_{\mathbb{T}}$, where $1\le i\le n$, choose an arbitrary point ξ_i and form the sum

$$S=\sum_{i=1}^n(t_i-t_{i-1})F(\xi_i).$$

We call S a Riemann Δ-sum of F corresponding to the partition P.

A bounded interval-valued function $F:\mathbb{T}\to\mathcal{K}_C$ is Riemann Δ-integrable from a to b (or on $[a,b)_{\mathbb{T}}$) if there is $A\in\mathcal{K}_C$ such that for each $\varepsilon>0$ there is $\delta>0$ such that

$$D\left(S\ominus_g A,\{0\}\right)<\varepsilon$$

for every Riemann Δ-sum S of F corresponding to a partition $P\in\mathcal{P}([a,b)_{\mathbb{T}},\delta)$ independent of the way in which we choose $\xi_i\in[t_{i-1},t_i)_{\mathbb{T}}$, $i=1,2,\cdots,n$. It is easily seen that $A\in\mathcal{K}_C$ is unique. $A\in\mathcal{K}_C$ is called the Riemann Δ-integral of F from a to b, and we will denote it by $\int_a^b F(t)\Delta t$.

Let $F:T\to K_C$ be an interval-valued function such that $F(t)=[f^-(t),f^+(t)]$. Then F is Riemann Δ-integrable on $[a,b)_{\mathbb{T}}$ if and only if f^- and f^+ are Riemann Δ-integrable on $[a,b)_{\mathbb{T}}$. Moreover, we have

$$\int_a^b F(t)\Delta t=\left[\int_a^b f^-(t)\Delta t,\int_a^b f^+(t)\Delta t\right].$$

Lemma 2.6. *Let $F : \mathbb{T} \to \mathcal{K}_C$ be a bounded interval-valued function on $[a,b)_{\mathbb{T}}$. If F is Riemann Δ-integrable from a to b, then the interval-valued function $G : \mathbb{T} \to \mathcal{K}_C$, given by $G(t) = \int_a^t F(s)\,\Delta s$, $t \in [a,b)_{\mathbb{T}}$ is continuous on $[a,b]_{\mathbb{T}}$. Further, let $t_0 \in [a,b)_{\mathbb{T}}$ and let F be arbitrary at t_0 if t_0 is right-scattered, and let F be continuous at t_0 if t_0 is right-dense. Then G is Δ_{gH}-differentiable at t_0 and $G^{\Delta}(t_0) = F(t_0)$.*

Remark 2.7. *Let F is Riemann Δ-integrable from a to b, then the interval-valued function $G : \mathbb{T} \to \mathcal{K}_C$, given by $G(t) = \int_a^t F(s)\,\Delta s$, $t \in [a,b)_{\mathbb{T}}$ is l-nondecreasing on $[a,b]_{\mathbb{T}}$.*

3 Gronwall type inequalities

3.1 For real-valued functions

We start this section with a comparison results. For the convenience of notation, we let throughout $t_0 \in \mathbb{T}$, $\mathbb{T}_0 = [t_0,\infty) \cap \mathbb{T}$ and $\mathbb{T}_0^- = (-\infty, t_0] \cap \mathbb{T}$.

Lemma 3.1. *Let $f, y \in C_{rd}(\mathbb{T},\mathbb{R})$ and $p \in \mathcal{R}^+(\mathbb{T},\mathbb{R})$. Then*

$$y^{\Delta}(t) \le p(t)\langle y(t)\rangle + f(t) \quad \text{for all } t \in \mathbb{T}_0, \tag{9}$$

implies

$$y(t) \le y(t_0)E_p(t,t_0) + \textstyle\int_{t_0}^t f(s)\langle E_{-p}(s,t)\rangle\,\Delta s \tag{10}$$

for all $t \in \mathbb{T}_0$.

Proof. For $y \in C_{rd}(\mathbb{T},\mathbb{R})$ and $p \in \mathcal{R}^+(\mathbb{T},\mathbb{R})$, we have

$$\begin{aligned}(y(t)E_{-p}(t,t_0))^{\Delta} &= y^{\Delta}(t)E_{-p}(t^{\sigma},t_0) + y(t)(E_{-p}(t,t_0))^{\Delta}\\ &= y^{\Delta}(t)E_{-p}(t^{\sigma},t_0) - p(t)y(t)\langle E_{-p}(t,t_0)\rangle.\end{aligned}$$

Since $\langle E_{-p}(t,t_0)\rangle = \frac{E_{-p}(t,t_0)+E_{-p}(t^{\sigma},t_0)}{2}$ for all $t \in \mathbb{T}$, it follows that

$$\begin{aligned}(y(t)E_{-p}(t,t_0))^{\Delta} &= y^{\Delta}(t)E_{-p}(t^{\sigma},t_0) - p(t)y(t)\langle E_{-p}(t,t_0)\rangle\\ &= 2y^{\Delta}(t)\langle E_{-p}(t,t_0)\rangle - y^{\Delta}(t)E_{-p}(t,t_0)\\ &\quad -p(t)y(t)\langle E_{-p}(t,t_0)\rangle\\ &= \left[2y^{\Delta}(t) - p(t)y(t)\right]\langle E_{-p}(t,t_0)\rangle - y^{\Delta}(t)E_{-p}(t,t_0).\end{aligned}$$

By Lemmas 2.1, 2.2 and integration by parts [4, Theorem 1.77-(vi)], we can obtain

$$\begin{aligned}&y(t)E_{-p}(t,t_0) - y(t_0)\\ =\ &\int_{t_0}^t (y(s)E_{-p}(s,t_0))^{\Delta}\,\Delta s\\ =\ &\int_{t_0}^t \left(\left[2y^{\Delta}(s) - p(s)y(s)\right]\langle E_{-p}(s,t_0)\rangle - y^{\Delta}(s)E_{-p}(s,t_0)\right)\Delta s\\ =\ &\int_{t_0}^t \left[2y^{\Delta}(s) - p(s)y(s)\right]\langle E_{-p}(s,t_0)\rangle\,\Delta s - \int_{t_0}^t y^{\Delta}(s)E_{-p}(s,t_0)\,\Delta s\\ =\ &\int_{t_0}^t \left[2y^{\Delta}(s) - p(s)(y(s)+y(s^{\sigma}))\right]\langle E_{-p}(s,t_0)\rangle\,\Delta s\\ &-[y(t)E_{-p}(t,t_0) - y(t_0)].\end{aligned}$$

Inequality (9) yields that

$$y(t)E_{-p}(t,t_0) - y(t_0) = \int_{t_0}^t \left[y^{\Delta}(s) - p(s)\langle y(s)\rangle\right]\langle E_{-p}(s,t_0)\rangle\,\Delta s \le \int_{t_0}^t f(s)\langle E_{-p}(s,t_0)\rangle\,\Delta s$$

and hence the assertion follows by applying Lemma 2.1. □

Lemma 3.2. *Let $f, y \in C_{rd}(\mathbb{T},\mathbb{R})$ and $p \in \mathcal{R}^+(\mathbb{T},\mathbb{R})$. Then*

$$y^{\Delta}(t) \le -p(t)\langle y(t)\rangle + f(t) \text{ for all } t \in \mathbb{T}_0, \tag{11}$$

implies

$$y(t) \le y(t_0) E_{-p}(t,t_0) + \int_{t_0}^{t} f(s)\langle E_p(s,t)\rangle \Delta s \text{ for all } t \in \mathbb{T}_0 \tag{12}$$

and

$$y^{\Delta}(t) \le -p(t)\langle y(t)\rangle + f(t) \text{ for all } t \in \mathbb{T}_0^-, \tag{13}$$

implies

$$y(t) \ge y(t_0) E_{-p}(t,t_0) + \int_{t_0}^{t} f(s)\langle E_p(s,t)\rangle \Delta s \text{ for all } t \in \mathbb{T}_0^-. \tag{14}$$

Proof. For $y \in C_{rd}(\mathbb{T},\mathbb{R})$ and $p \in \mathcal{R}^+(\mathbb{T},\mathbb{R})$, we have

$$\begin{aligned}(y(t)E_p(t,t_0))^{\Delta} &= y^{\Delta}(t)E_p(t^{\sigma},t_0) + y(t)(E_p(t,t_0))^{\Delta}\\ &= y^{\Delta}(t)E_p(t^{\sigma},t_0) + p(t)y(t)\langle E_p(t,t_0)\rangle.\end{aligned}$$

Since $\langle E_p(t,t_0)\rangle = \frac{E_p(t,t_0)+E_p(t^{\sigma},t_0)}{2}$ for all $t \in \mathbb{T}$, it follows that

$$\begin{aligned}(y(t)E_p(t,t_0))^{\Delta} &= y^{\Delta}(t)E_p(t^{\sigma},t_0) + p(t)y(t)\langle E_p(t,t_0)\rangle\\ &= 2y^{\Delta}(t)\langle E_p(t,t_0)\rangle - y^{\Delta}(t)E_p(t,t_0)\\ &\quad + p(t)y(t)\langle E_p(t,t_0)\rangle\\ &= \left[2y^{\Delta}(t) + p(t)y(t)\right]\langle E_p(t,t_0)\rangle - y^{\Delta}(t)E_p(t,t_0).\end{aligned}$$

By Lemmas 2.1, 2.2 and integration by parts [4, Theorem 1.77-(vi)], we can obtain

$$\begin{aligned}& y(t)E_p(t,t_0) - y(t_0)\\ =\ & \int_{t_0}^{t} (y(s)E_p(s,t_0))^{\Delta}\Delta s\\ =\ & \int_{t_0}^{t} \left(\left[2y^{\Delta}(s) + p(s)y(s)\right]\langle E_p(s,t_0)\rangle - y^{\Delta}(s)E_p(s,t_0)\right)\Delta s\\ =\ & \int_{t_0}^{t} \left[2y^{\Delta}(s) + p(s)y(s)\right]\langle E_p(s,t_0)\rangle \Delta s - \int_{t_0}^{t} y^{\Delta}(s)E_p(s,t_0)\Delta s\\ =\ & \int_{t_0}^{t} \left[2y^{\Delta}(s) + p(s)(y(s)+y(s^{\sigma}))\right]\langle E_p(s,t_0)\rangle \Delta s\\ & - \left[y(t)E_p(t,t_0) - y(t_0)\right].\end{aligned}$$

Inequality (9) yields that

$$y(t)E_p(t,t_0) - y(t_0) = \int_{t_0}^{t}\left[y^{\Delta}(s) - p(s)\langle y(s)\rangle\right]\langle E_p(s,t_0)\rangle \Delta s \le \int_{t_0}^{t} f(s)\langle E_p(s,t_0)\rangle \Delta s$$

and hence the assertion (12) follows by applying Lemma 2.1. For the (14), note that the latter inequality is reversed if $t \in \mathbb{T}_0^-$. □

Theorem 3.3. *Suppose that $f, y \in C_{rd}(\mathbb{T},\mathbb{R})$, $p \in \mathcal{R}^+(\mathbb{T},\mathbb{R})$ and $p \ge 0$. Then*

$$y(t) \le f(t) + \int_{t_0}^{t} p(s)\langle y(s)\rangle \Delta s \text{ for all } t \in \mathbb{T}_0, \tag{15}$$

implies

$$y(t) \le f(t) + \int_{t_0}^{t} p(s)\langle f(s)\rangle\langle E_{-p}(s,t)\rangle \Delta s \text{ for all } t \in \mathbb{T}_0. \tag{16}$$

Proof. Define

$$z(t) = \int_{t_0}^{t} p(s)\langle y(s)\rangle \Delta s.$$

Taking delta derivative and using inequality (15), we get that

$$z^{\Delta}(t) = p(t)\langle y(t)\rangle \leq p(t)\langle z(t)\rangle + p(t)\langle f(t)\rangle .$$

Lemma 3.1, implies that

$$z(t) \leq \int_{t_0}^{t} p(s)\langle f(s)\rangle \langle E_{-p}(s,t)\rangle \Delta s,$$

and hence the claim follows because $y(t) \leq f(t) + z(t)$. □

Corollary 3.4. *Suppose that* $y \in C_{rd}(\mathbb{T},\mathbb{R})$,$p \in \mathcal{R}^{+}(\mathbb{T},\mathbb{R})$ *and* $p \geq 0$. *Then*

$$y(t) \leq \int_{t_0}^{t} p(s)\langle y(s)\rangle \Delta s \text{ for all } t \in \mathbb{T}_0, \tag{17}$$

implies

$$y(t) \leq 0 \text{ for all } t \in \mathbb{T}_0. \tag{18}$$

Corollary 3.5. *Suppose that* $y \in C_{rd}(\mathbb{T},\mathbb{R})$,$f_0 \in \mathbb{R}$,$p \in \mathcal{R}^{+}(\mathbb{T},\mathbb{R})$ *and* $p \geq 0$. *Then*

$$y(t) \leq f_0 + \int_{t_0}^{t} p(s)\langle y(s)\rangle \Delta s \text{ for all } t \in \mathbb{T}_0, \tag{19}$$

implies

$$y(t) \leq f_0 E_p(t,t_0) \text{ for all } t \in \mathbb{T}_0. \tag{20}$$

Corollary 3.6. *If* $p, q \in \mathcal{R}^{+}(\mathbb{T},\mathbb{R})$ *with* $p(t) \leq q(t)$ *for all* $t \in \mathbb{T}$, *then*

$$E_p(t,t_0) \leq E_q(t,t_0) \text{ for all } t \in \mathbb{T}_0. \tag{21}$$

Moreover,

$$\langle E_p(t,t_0)\rangle \leq \langle E_q(t,t_0)\rangle \text{ for all } t \in \mathbb{T}_0. \tag{22}$$

Similar to the proof of Theorem 3.3, we can obtain the following results and hence is omitted.

Theorem 3.7. *Suppose that* $f, g, y \in C_{rd}(\mathbb{T},\mathbb{R})$, $\alpha_0 \in \mathbb{R}$,$q \in \mathcal{R}^{+}(\mathbb{T},\mathbb{R})$ *and* $q \geq 0$. *Then*

$$y(t) \leq f(t) + \alpha_0 \int_{t_0}^{t} [q(s)\langle y(s)\rangle + g(s)] \Delta s \text{ for all } t \in \mathbb{T}_0, \tag{23}$$

implies

$$y(t) \leq f(t) + \alpha_0 \int_{t_0}^{t} [q(s)\langle f(s)\rangle + g(s)] \langle E_{-\alpha_0 q}(s,t)\rangle \Delta s \text{ for all } t \in \mathbb{T}_0. \tag{24}$$

An important consequence of Lemma 3.2, is as follows:

Theorem 3.8. *Suppose that* $f, g, y \in C_{rd}(\mathbb{T},\mathbb{R})$, $\alpha_0 \in \mathbb{R}$,$q \in \mathcal{R}^{+}(\mathbb{T},\mathbb{R})$ *and* $q \geq 0$. *Then*

$$y(t) \leq f(t) + \alpha_0 \int_{t}^{t_0} [q(s)\langle y(s)\rangle + g(s)] \Delta s \text{ for all } t \in \mathbb{T}_0^{-}, \tag{25}$$

implies

$$y(t) \leq f(t) + \alpha_0 \int_{t}^{t_0} [q(s)\langle f(s)\rangle + g(s)] \langle E_{\alpha_0 q}(s,t)\rangle \Delta s \text{ for all } t \in \mathbb{T}_0^{-}. \tag{26}$$

Remark 3.9. *For the continuous, discrete and time scales versions of above results, we refer to [1, 4, 6, 22, 23].*

3.2 For interval-valued functions

For an interval-valued function $F : \mathbb{T} \to \mathcal{K}_C$, we can define

$$\langle F(t)\rangle = \frac{F(t) + F(t^{\sigma})}{2}. \tag{27}$$

If $F : \mathbb{T} \to \mathcal{K}_C$ such that $F(t) = [f^{-}(t), f^{+}(t)]$, then (27) implies that

$$\langle F(t)\rangle = \left[\langle f^{-}(t)\rangle, \langle f^{+}(t)\rangle\right]. \tag{28}$$

By definition, we can get

$$len\langle F(t)\rangle = \langle lenF(t)\rangle . \tag{29}$$

Let us start this section with comparison results for interval-valued functions under LW-partial order:

Lemma 3.10. *Let $F, Y \in C_{rd}(\mathbb{T}, \mathcal{K}_C)$ and $p \in \mathcal{R}^+(\mathbb{T}, \mathbb{R})$.*

(a) If Y is $\Delta_{1,gH}$-differentiable on $\mathbb{T}_0$ and satisfies the interval dynamic inequality

$$Y^{\Delta}(t) \preceq_{LW} p(t)\langle Y(t)\rangle + F(t) \text{ for all } t \in \mathbb{T}_0, \tag{30}$$

then for $p \geq 0$,

$$Y(t) \preceq_{LW} E_p(t,t_0)\, Y(t_0) + \textstyle\int_{t_0}^{t} \langle E_{-p}(\tau,t)\rangle F(\tau)\,\Delta\tau \tag{31}$$

and for $p < 0, y_0^- \geq 0$ and $f^- \geq 0$, we have

$$Y(t) \preceq_{LW} E_{-p}(t,t_0)\, Y(t_0) + \int_{t_0}^{t} \langle E_p(\tau,t)\rangle F(\tau)\,\Delta\tau \tag{32}$$

for all $t \in \mathbb{T}_0$.

(b) If Y is $\Delta_{2,gH}$-differentiable on $\mathbb{T}_0$ and satisfies the interval dynamic inequality

$$-Y^{\Delta}(t) \preceq_{LW} p(t)\langle Y(t)\rangle + F(t) \text{ for all } t \in \mathbb{T}_0, \tag{33}$$

then for $p \geq 0$ we have

$$Y(t) \succeq_{LW} E_{-p}(t,t_0)\, Y(t_0) - \textstyle\int_{t_0}^{t} \langle E_p(\tau,t)\rangle F(\tau)\,\Delta\tau \tag{34}$$

and for $p < 0, y_0^- \geq 0$ and $f^- \leq 0$, we have

$$Y(t) \preceq_{LW} E_p(t,t_0)\, Y(t_0) - \int_{t_0}^{t} \langle E_{-p}(\tau,t)\rangle F(\tau)\,\Delta\tau \tag{35}$$

for all $t \in \mathbb{T}_0$.

Proof. Let $Y, F : \mathbb{T} \to \mathcal{K}_C$ be interval-valued functions such that $Y(t) = [y^-(t), y^+(t)]$ and $F(t) = [f^-(t), f^+(t)]$ and both are rd-continuous on $\mathbb{T}$.

(a) If $Y(t)$ is $\Delta_{1,gH}$-differentiable on $\mathbb{T}_0$, then by using Lemma 2.5 $Y^{\Delta}(t) = \left[(y^-)^{\Delta}(t), (y^+)^{\Delta}(t)\right]$.

First we consider the case if $p(t) \geq 0$ on $\mathbb{T}_0$ then we have

$$p(t)\langle Y(t)\rangle = [p(t)\langle y^-(t)\rangle, p(t)\langle y^+(t)\rangle]. \tag{36}$$

By using inequality (30) we obtain

$$\left[(y^-)^{\Delta}(t), (y^+)^{\Delta}(t)\right] \preceq_{LW} [p(t)\langle y^-(t)\rangle + f^-(t), p(t)\langle y^+(t)\rangle + f^+(t)].$$

Apply LW-order (4), we have

$$(y^-)^{\Delta}(t) \leq p(t)\langle y^-(t)\rangle + f^-(t) \tag{37}$$

and

$$(lenY(t))^{\Delta} \leq p(t)\langle lenY(t)\rangle + lenF(t). \tag{38}$$

By using Lemma 3.1, on (37) and (38) respectively, we obtain

$$y^-(t) \leq E_p(t,t_0)\, y^-(t_0) + \int_{t_0}^{t} \langle E_{-p}(\tau,t)\rangle f^-(\tau)\,\Delta\tau, \tag{39}$$

and

$$lenY(t) \leq E_p(t,t_0)\, lenY(t_0) + \textstyle\int_{t_0}^{t} \langle E_{-p}(\tau,t)\rangle\, lenF(\tau)\,\Delta\tau. \tag{40}$$

Inequalities (39) and (40) yields (31).

If $p(t) < 0$ on $\mathbb{T}_0$ so we have $p(t)Y(t) = [p(t)\langle y^+(t)\rangle, p(t)\langle y^-(t)\rangle]$. By using inequality (30) we obtain

$$\left[(y^-)^{\Delta}(t), (y^+)^{\Delta}(t)\right] \preceq_{LW} [p(t)\langle y^+(t)\rangle + f^-(t), p(t)\langle y^-(t)\rangle + f^+(t)].$$

Apply LW-order (4), we have

$$\left(y^-\right)^{\Delta}(t) \leq p(t)\left\langle y^+(t)\right\rangle + f^-(t) \leq p(t)\left\langle y^-(t)\right\rangle + f^-(t) \tag{41}$$

and

$$(lenY(t))^{\Delta} \leq (-p(t))\langle lenY(t)\rangle + lenF(t). \tag{42}$$

By using Lemma 3.1, on (41) and (42) respectively, we obtain

$$y^{-}(t) \leq E_p(t,t_0)\, y^{-}(t_0) + \int_{t_0}^{t} \langle E_{-p}(\tau,t)\rangle f^{-}(\tau)\,\Delta\tau, \tag{43}$$

and

$$lenY(t) \leq E_{-p}(t,t_0)\, lenY(t_0) + \int_{t_0}^{t} \langle E_p(\tau,t)\rangle\, lenF(\tau)\,\Delta\tau. \tag{44}$$

Since $p(t) < 0$ and $p \in \mathcal{R}^{+}$, it follows that $(-p) \in \mathcal{R}^{+}$ and $p \leq -p$, therefore, Lemma 2.1 and Corollary 3.6 implies that

$$y^{-}(t) \leq E_p(t,t_0)\, y^{-}(t_0) + \textstyle\int_{t_0}^{t} \langle E_{-p}(\tau,t)\rangle f^{-}(\tau)\,\Delta\tau \leq E_{-p}(t,t_0)\, y^{-}(t_0) + \int_{t_0}^{t} \langle E_p(\tau,t)\rangle f^{-}(\tau)\,\Delta\tau \tag{45}$$

Combining (44) and (45), we get

$$Y(t) \preceq_{LW} E_{-p}(t,t_0)\, Y(t_0) + \int_{t_0}^{t} \langle E_p(\tau,t)\rangle F(\tau)\,\Delta\tau.$$

(*b*) If $Y(t)$ is $\Delta_{2,gH}$-differentiable, then $Y^{\Delta}(t) = \left[(y^{+})^{\Delta}(t), (y^{-})^{\Delta}(t)\right]$ and for $p(t) \geq 0$, so we have $p(t)Y(t) = [p(t)\langle y^{-}(t)\rangle, p(t)\langle y^{+}(t)\rangle]$. Inequality (33) implies that

$$(-y^{-})^{\Delta}(t) \leq p(t)\langle y^{-}(t)\rangle + f^{-}(t) = (-p(t))\langle -y^{-}(t)\rangle + f^{-}(t)$$

and

$$(-lenY(t))^{\Delta} \leq (-p(t))\langle -lenY(t)\rangle + lenF(t).$$

It implies that

$$y^{-}(t) \geq E_{-p}(t,t_0)\, y^{-}(t_0) - \int_{t_0}^{t} \langle E_p(\tau,t)\rangle f^{-}(\tau)\,\Delta\tau \tag{46}$$

and

$$lenY(t) \geq E_{-p}(t,t_0)\, lenY(t_0) - \int_{t_0}^{t} \langle E_{-p}(\tau,t)\rangle\, lenF(\tau)\,\Delta\tau. \tag{47}$$

By using (46) and (47) in LW order, we can get (34). For $p(t) < 0$, similar to the second inequality of part (a), we can obtain (35). □

Similar to Lemma 3.10, by applying Lemma 3.2, we can get the following result.

Lemma 3.11. *Let $F, Y \in C_{rd}(\mathbb{T}, \mathcal{K}_C)$ and $p \in \mathcal{R}^{+}(\mathbb{T}, \mathbb{R})$.*
(*a*) *If Y is $\Delta_{1,gH}$-differentiable on $\mathbb{T}_0$ and satisfies the interval dynamic inequality*

$$Y^{\Delta}(t) \preceq_{LW} -p(t)\langle Y(t)\rangle + F(t) \;\; \text{for all } t \in \mathbb{T}_0, \tag{48}$$

then for $p \geq 0$, $y_0^{-} \geq 0$ and $f^{-} \geq 0$

$$Y(t) \preceq_{LW} E_p(t,t_0)\, Y(t_0) + \textstyle\int_{t_0}^{t} \langle E_{-p}(\tau,t)\rangle F(\tau)\,\Delta\tau \tag{49}$$

and for $p < 0$, we have

$$Y(t) \preceq_{LW} E_{-p}(t,t_0)\, Y(t_0) + \int_{t_0}^{t} \langle E_p(\tau,t)\rangle F(\tau)\,\Delta\tau \tag{50}$$

for all $t \in \mathbb{T}_0$.
(*b*) *If Y is $\Delta_{2,gH}$-differentiable on $\mathbb{T}_0$ and satisfies the interval dynamic inequality*

$$-Y^{\Delta}(t) \preceq_{LW} -p(t)\langle Y(t)\rangle + F(t) \;\; \text{for all } t \in \mathbb{T}_0, \tag{51}$$

then for $p \geq 0$, $y_0^{-} \geq 0$ and $f^{-} \leq 0$

$$Y(t) \succeq_{LW} E_{-p}(t,t_0)\, Y(t_0) - \textstyle\int_{t_0}^{t} \langle E_p(\tau,t)\rangle F(\tau)\,\Delta\tau \tag{52}$$

and for $p<0$, *we have*

$$Y(t) \succeq_{LW} E_p(t,t_0)\,Y(t_0) - \int_{t_0}^{t} \langle E_{-p}(\tau,t)\rangle\, F(\tau)\,\Delta\tau \tag{53}$$

for all $t \in \mathbb{T}_0$.

Now, we are assuming that all functions are bounded.

Theorem 3.12. *Let* $F, Y \in C_{rd}(\mathbb{T}_0, \mathcal{K}_C)$ *and* $p \in \mathcal{R}^+(\mathbb{T}_0, \mathbb{R})$, $p(t) \geq 0$ *for all* $t \in \mathbb{T}_0$, *such that the interval integral inequality*

$$Y(t) \preceq_{LW} F(t) + \int_{t_0}^{t} p(s)\,\langle Y(s)\rangle\,\Delta s \tag{54}$$

holds for all $t \in \mathbb{T}_0$. *Then*

$$Y(t) \preceq_{LW} F(t) + \int_{t_0}^{t} p(s)\,\langle E_p(t,s)\rangle\,\langle F(s)\rangle\,\Delta s \tag{55}$$

for all $t \in \mathbb{T}_0$.

Proof. Consider $Z(t) = \int_{t_0}^{t} p(\tau)\langle Y(\tau)\rangle\Delta\tau$. Since $p(\tau), \langle Y(\tau)\rangle$ are bounded and belong to C_{rd} class, therefore Lemma 2.6, implies that $Z(t)$ is Δ_{gH}-differentiable, moreover from Remark 2.7, it follows that $Z(t)$ is $\Delta_{1,gH}$-differentiable and $Z^{\Delta}(t) = p(t)\langle Y(t)\rangle$, $t \in \mathbb{T}_0$.

From inequality (54), we can see that $\langle Y(t)\rangle \leq_{LW} \langle F(t)\rangle + \langle Z(t)\rangle$. Clearly,

$$Z^{\Delta}(t) \preceq_{LW} p(t)\langle Z(t)\rangle + p(t)\langle F(t)\rangle.$$

Part (a) in Lemma 3.10 and $Z(t_0) = \{0\}$, implies that

$$Z(t) \preceq_{LW} \int_{t_0}^{t} p(s)\,\langle E_p(t,s)\rangle\,\langle F(s)\rangle\,\Delta s,$$

and hence the assertion (55) follows by inequality (54). □

Corollary 3.13. *Let* $Y \in C_{rd}(\mathbb{T}_0, \mathcal{K}_C)$, $p \in \mathcal{R}^+(\mathbb{T}_0, \mathbb{R})$, $p \geq 0$ *and* $X_0 \in \mathcal{K}_C$. *If*

$$Y(t) \preceq_{LW} X_0 + \int_{t_0}^{t} p(s)\,\langle Y(s)\rangle\,\Delta s \text{ for all } t \in \mathbb{T}_0, \tag{56}$$

then

$$Y(t) \preceq_{LW} X_0 E_p(t,t_0) \text{ for all } t \in \mathbb{T}_0. \tag{57}$$

Proof. By taking $F(t) = X_0$, in Theorem 3.12, we obtain (57). □

Corollary 3.14. *Let* $Y \in C_{rd}(\mathbb{T}_0, \mathcal{K}_C)$, $p \in \mathcal{R}^+(\mathbb{T}_0, \mathbb{R})$, $p \geq 0$ *and satisfies the interval integral inequality* $Y(t) \preceq_{LW} \int_{t_0}^{t} Y(t)\,p(t)\,\Delta t$ *for all* $t \in \mathbb{T}_0$, *then*$Y(t) \preceq_{LW} \{0\}$, *for all* $t \in \mathbb{T}_0$.

Similar to Theorem 3.12, we can obtain the following result.

Theorem 3.15. *Let* $F, Q, Y \in C_{rd}(\mathbb{T}_0, \mathcal{K}_C)$, $p \in \mathcal{R}^+(\mathbb{T}_0, \mathbb{R})$, $p \geq 0$ $b_0 \in \mathbb{R}^+$ *and satisfies the interval integral inequality*

$$Y(t) \preceq_{LW} F(t) + b_0 \int_{t_0}^{t} [p(\tau)\,Y(\tau) + Q(\tau)]\,\Delta\tau \text{ for all } t \in \mathbb{T}_0,$$

then

$$Y(t) \preceq_{LW} F(t) + b_0 \int_{t_0}^{t} (p(\tau)\,\langle F(\tau)\rangle + Q(\tau))\,\langle E_{pb_0}(t,\tau)\rangle\,\Delta\tau$$

for all $t \in \mathbb{T}_0$.

The next corollary is obtained by taking $F(t) = Q(t) = 0$ in Theorem 3.15.

Corollary 3.16. *Suppose* $Y(t) \in C_{rd}(\mathbb{T}_0, \mathcal{K}_C)$ *and* $p \in \mathcal{R}^+(\mathbb{T}_0, \mathbb{R})$, $p \geq 0$ $b_0 \in \mathbb{R}^+$ *and satisfies the interval integral inequality* $Y(t) \preceq_{LW} b_0 \int_{t_0}^{t} Y(\tau)\,p(\tau)\,\Delta\tau$ *for all* $t \in \mathbb{T}_0$, *then*$Y(t) \preceq_{LW} \{0\}$, *for all* $t \in \mathbb{T}_0$.

Remark 3.17. *If* $b_0 = 1$ *in Corollary 3.16 then we obtain Corollary 3.14.*

4 Conclusions

In this paper, some results of Gronwall type inequalities for interval-valued functions, which provide explicit bounds on unknown functions, are presented. The results can be useful in the study of the uniqueness of solution for interval-valued differential equations or interval-valued integro differential equations. The results also unify and extend some continuous inequalities and some new results of the discrete case are proved.

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