

## COUNTABLE COMPACTNESS AND THE LINDELÖF PROPERTY OF $L$ -FUZZY SETS

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ABSTRACT. In this paper, countable compactness and the Lindelöf property are defined for  $L$ -fuzzy sets, where  $L$  is a complete de Morgan algebra. They don't rely on the structure of the basis lattice  $L$  and no distributivity is required in  $L$ . A fuzzy compact  $L$ -set is countably compact and has the Lindelöf property. An  $L$ -set having the Lindelöf property is countably compact if and only if it is fuzzy compact. Many characterizations of countable compactness and the Lindelöf property are presented by means of open  $L$ -sets and closed  $L$ -sets when  $L$  is a completely distributive de Morgan algebra.

### 1. Introduction

In 1976, the concept of fuzzy compactness was introduced in  $[0, 1]$ -topological spaces by R. Lowen [2]. Subsequently its characterization was given by G.J. Wang in terms of  $\alpha$ -net in [10]. In 1988, it was extended to  $L$ -topological space [11], where  $L$  is a completely distributive de Morgan algebra. Recently a new definition of fuzzy compactness was presented in  $L$ -topological spaces [3], which doesn't depend on the structure of basis lattice  $L$  and no distributivity is required in  $L$ . When  $L$  is a completely distributive de Morgan algebra, it is equivalent to the notion of fuzzy compactness in [4, 5, 11, 13].

In this paper, our aim is to continue the research of countable compactness and the Lindelöf property of  $L$ -sets.

Throughout this paper  $(L, \vee, \wedge, ')$  is a complete de Morgan algebra,  $X$  a nonempty set.  $L^X$  is the set of all  $L$ -fuzzy sets on  $X$ . The smallest element and the largest element in  $L^X$  are denoted by  $\underline{0}$  and  $\underline{1}$ .

A complete lattice  $L$  is called a complete Heyting algebra if it satisfies the following infinite distributive law:

(ID) For all  $a \in L$  and all  $B \subseteq L$ ,  $a \wedge \bigvee B = \bigvee \{a \wedge b \mid b \in B\}$ .

An element  $a$  in  $L$  is called prime element if  $a \geq b \wedge c$  implies  $a \geq b$  or  $a \geq c$ .  $a$  in  $L$  is called co-prime element if  $a'$  is a prime element [1]. The set of all nonunit prime elements in  $L$  is denoted by  $P(L)$ . The set of all nonzero co-prime elements in  $L$  is denoted by  $M(L)$ .

According to [12], we know that when  $L$  is completely distributive, each element  $a$  in  $L$  has the greatest minimal family (the greatest maximal family), denoted

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by  $\beta(a)$  ( $\alpha(a)$ ). Obviously  $\beta^*(a) = \beta(a) \cap M(L)$  is a minimal family of  $a$  and  $\alpha^*(a) = \beta(a) \cap P(L)$  is a maximal family of  $a$ .

An  $L$ -topological space (or  $L$ -space for short) is a pair  $(X, \mathcal{T})$ , where  $\mathcal{T}$  is a subfamily of  $L^X$  which contains  $\underline{0}$ ,  $\underline{1}$  and is closed for any suprema and finite infima. Each Member of  $\mathcal{T}$  is called an open  $L$ -set and its quasi-complementation is called a closed  $L$ -set.

For a subfamily  $\Phi \subseteq L^X$ ,  $2^{(\Phi)}$  denotes the set of all finite subfamilies of  $\Phi$ .  $2^{[\Phi]}$  denotes the set of all countable subfamilies of  $\Phi$ .

**Definition 1.1** ([3]). Let  $(X, \mathcal{T})$  be an  $L$ -space.  $G \in L^X$  is called fuzzy compact if for every family  $\mathcal{U} \subseteq \mathcal{T}$ , we have:

$$\bigwedge_{x \in X} \left( G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x) \right) \leq \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} \bigwedge_{x \in X} \left( G'(x) \vee \bigvee_{A \in \mathcal{V}} A(x) \right).$$

**Definition 1.2** ([3]). Let  $(X, \mathcal{T})$  be an  $L$ -space,  $a \in L \setminus \{1\}$  and  $G \in L^X$ . A family  $\mathcal{U} \subseteq \mathcal{T}$  is said to be an  $a$ -shading of  $G$  if for any  $x \in X$ ,  $G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x) \not\leq a$ .  $\mathcal{U}$

is said to be a strong  $a$ -shading of  $G$  if  $\bigwedge_{x \in X} \left( G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x) \right) \not\leq a$ .

**Definition 1.3** ([3]). Let  $(X, \mathcal{T})$  be an  $L$ -space,  $a \in L \setminus \{0\}$  and  $G \in L^X$ . A family  $\mathcal{P} \subseteq \mathcal{T}$  is called an  $a$ -R-neighborhood family of  $G$  if for any  $x \in X$  with  $G(x) \geq a$ , there exists  $B \in \mathcal{P}$  such that  $B(x) \not\leq a$ .  $\mathcal{P}$  is called a strong  $a$ -R-neighborhood family of  $G$  if  $\bigvee_{x \in X} \left( G(x) \wedge \bigwedge_{B \in \mathcal{P}} B(x) \right) \not\leq a$ .

**Definition 1.4** ([3]). Let  $(X, \mathcal{T})$  be an  $L$ -space,  $L$  a completely distributive de Morgan algebra,  $a \in L \setminus \{0\}$  and  $G \in L^X$ . A family  $\mathcal{U} \subseteq \mathcal{T}$  is called a  $\beta_a$ -open cover of  $G$  if for any  $x \in X$  with  $a \notin \beta(G'(x))$ , there exists  $A \in \mathcal{U}$  such that  $a \in \beta(A(x))$ .

$\mathcal{U}$  is called a strong  $\beta_a$ -open cover of  $G$  if  $a \in \beta \left( \bigwedge_{x \in X} \left( G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x) \right) \right)$ .

**Definition 1.5** ([3, 9]). Let  $(X, \mathcal{T})$  be an  $L$ -space,  $L$  a completely distributive de Morgan algebra,  $a \in L \setminus \{0\}$  and  $G \in L^X$ . A family  $\mathcal{U} \subseteq \mathcal{T}$  is called a  $Q_a$ -open cover of  $G$  if for any  $x \in X$  with  $G(x) \not\leq a'$ , it follows that  $\bigvee_{A \in \mathcal{U}} A(x) \geq a$ .

## 2. Fuzzy Countable Compactness

**Definition 2.1.** Let  $(X, \mathcal{T})$  be an  $L$ -space,  $G \in L^X$  is called (fuzzy) countably compact if for every countable family  $\mathcal{U} \subseteq \mathcal{T}$ , it follows that

$$\bigwedge_{x \in X} \left( G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x) \right) \leq \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} \bigwedge_{x \in X} \left( G'(x) \vee \bigvee_{A \in \mathcal{V}} A(x) \right).$$

**Lemma 2.2.** *In Definition 2.1, if we let  $L = \{0, 1\}$ , then by the following fact*

$$\begin{aligned} \bigwedge_{x \in X} \left( G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x) \right) = 1 &\Leftrightarrow \forall x \in X, G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x) = 1 \\ &\Leftrightarrow \forall x \in X, G'(x) \neq 1 \text{ implies } \bigvee_{A \in \mathcal{U}} A(x) = 1 \\ &\Leftrightarrow \forall x \in X, G(x) = 1 \text{ implies } \bigvee_{A \in \mathcal{U}} A(x) = 1, \end{aligned}$$

*we know that fuzzy compactness in Definition 2.1 is a generalization of countably compactness in general topology.*

Obviously we have the following two theorems.

**Theorem 2.3.** *Let  $(X, \mathcal{T})$  be an L-space. Then  $G \in L^X$  is countably compact if and only if for every countable subfamily  $\mathcal{P} \subseteq \mathcal{T}'$ , it follows that*

$$\bigvee_{x \in X} \left( G(x) \wedge \bigwedge_{B \in \mathcal{P}} B(x) \right) \geq \bigwedge_{\mathcal{F} \in 2^{(\mathcal{P})}} \bigvee_{x \in X} \left( G(x) \wedge \bigwedge_{B \in \mathcal{F}} B(x) \right).$$

**Theorem 2.4.** *If  $G$  is fuzzy compact, then it is also countably compact.*

By Definition 1.2, Definition 1.3, Definition 2.1 and Theorem 2.3, we can easily prove the following result.

**Theorem 2.5.** *Let  $(X, \mathcal{T})$  be an L-space and  $G \in L^X$ . Then the following statements are equivalent:*

- (1)  *$G$  is countably compact.*
- (2) *For any  $a \in L \setminus \{1\}$ , each countable strong  $a$ -shading  $\mathcal{U}$  of  $G$  has a finite subfamily  $\mathcal{V}$  which is still a strong  $a$ -shading of  $G$ .*
- (3) *For any  $a \in L \setminus \{0\}$ , each countable strong  $a$ - $R$ -neighborhood family  $\mathcal{P}$  of  $G$  has a finite subfamily  $\mathcal{F}$  which is still a strong  $a$ - $R$ -neighborhood family of  $G$ .*

**Theorem 2.6.** *If  $G$  is countably compact and  $H$  is closed, then  $G \wedge H$  is countably compact.*

*Proof.* Since  $G$  is countably compact, for any countable family  $\mathcal{P}$  of closed  $L$ -sets, by Theorem 2.3 we have that

$$\begin{aligned}
& \bigvee_{x \in X} \left( (G \wedge H)(x) \wedge \bigwedge_{B \in \mathcal{P}} B(x) \right) \\
&= \bigvee_{x \in X} \left( G(x) \wedge \bigwedge_{B \in \mathcal{P} \cup \{H\}} B(x) \right) \geq \bigwedge_{\mathcal{F} \in 2^{\mathcal{P} \cup \{H\}}} \bigvee_{x \in X} \left( G(x) \wedge \bigwedge_{B \in \mathcal{F}} B(x) \right) \\
&= \left\{ \bigwedge_{\mathcal{F} \in 2^{\mathcal{P}}} \bigvee_{x \in X} \left( G(x) \wedge \bigwedge_{B \in \mathcal{F}} B(x) \right) \right\} \wedge \left\{ \bigwedge_{\mathcal{F} \in 2^{\mathcal{P}}} \bigvee_{x \in X} \left( G(x) \wedge \left( H(x) \wedge \bigwedge_{B \in \mathcal{F}} B(x) \right) \right) \right\} \\
&= \bigwedge_{\mathcal{F} \in 2^{\mathcal{P}}} \bigvee_{x \in X} \left( (G \wedge H)(x) \wedge \bigwedge_{B \in \mathcal{F}} B(x) \right).
\end{aligned}$$

This shows that  $G \wedge H$  is countably compact.  $\square$

**Lemma 2.7.** *Let  $L$  be a complete Heyting algebra,  $f : X \rightarrow Y$  be a map and  $f_L^- : L^X \rightarrow L^Y$  is the extension of  $f$ , then for any family  $\mathcal{P} \subseteq L^Y$ , we have:*

$$\bigvee_{y \in Y} \left( f_L^-(G)(y) \wedge \bigwedge_{B \in \mathcal{P}} B(y) \right) = \bigvee_{x \in X} \left( G(x) \wedge \bigwedge_{B \in \mathcal{P}} f_L^-(B)(x) \right).$$

*Proof.* This can be proved from the following equation.

$$\begin{aligned}
& \bigvee_{y \in Y} \left( f_L^-(G)(y) \wedge \bigwedge_{B \in \mathcal{P}} B(y) \right) = \bigvee_{y \in Y} \left( \left( \bigvee_{x \in f^{-1}(y)} G(x) \right) \wedge \bigwedge_{B \in \mathcal{P}} B(y) \right) \\
&= \bigvee_{y \in Y} \left( \bigvee_{x \in f^{-1}(y)} \left( G(x) \wedge \bigwedge_{B \in \mathcal{P}} B(f(x)) \right) \right) = \bigvee_{y \in Y} \bigvee_{x \in f^{-1}(y)} \left( G(x) \wedge \bigwedge_{B \in \mathcal{P}} f_L^-(B)(x) \right) \\
&= \bigvee_{x \in X} \left( G(x) \wedge \bigwedge_{B \in \mathcal{P}} f_L^-(B)(x) \right).
\end{aligned}$$

$\square$

**Theorem 2.8.** *Let  $L$  be a complete Heyting algebra,  $f : X \rightarrow Y$  be a map,  $\mathcal{T}_1$  be an  $L$ -topology on  $X$ ,  $\mathcal{T}_2$  be an  $L$ -topology on  $Y$  and  $f_L^- : L^X \rightarrow L^Y$  be continuous. If  $G$  is fuzzy countably compact in  $(X, \mathcal{T}_1)$ , then so is  $f_L^-(G)$  in  $(Y, \mathcal{T}_2)$ .*

*Proof.* For any countable family  $\mathcal{P} \subseteq \mathcal{T}'_2$ , by Lemma 2.7 and countable compactness of  $G$  we have that

$$\begin{aligned} & \bigvee_{y \in Y} \left( f_L^{\rightarrow}(G)(y) \wedge \bigwedge_{B \in \mathcal{P}} B(y) \right) = \bigvee_{x \in X} \left( G(x) \wedge \bigwedge_{B \in \mathcal{P}} f_L^{\leftarrow}(B)(x) \right) \\ & \geq \bigwedge_{\mathcal{F} \in 2^{(\mathcal{P})}} \bigvee_{x \in X} \left( G(x) \wedge \bigwedge_{B \in \mathcal{F}} f_L^{\leftarrow}(B)(x) \right) = \bigwedge_{\mathcal{F} \in 2^{(\mathcal{P})}} \bigvee_{y \in Y} \left( f_L^{\rightarrow}(G)(y) \wedge \bigwedge_{B \in \mathcal{F}} B(y) \right). \end{aligned}$$

Therefore  $f_L^{\rightarrow}(G)$  is countably compact.  $\square$

**Theorem 2.9.** *Let  $(X, T)$  be an  $L$ -space and  $G \in L^X$ . If  $L$  is a completely distributive de Morgan algebra, then the following conditions are equivalent.*

- (1)  $G$  is countably compact.
- (2) For any  $a \in L \setminus \{0\}$ , each countable strong  $a$ - $R$ -neighborhood family  $\mathcal{P}$  of  $G$  has a finite subfamily  $\mathcal{F}$  which is a strong  $a$ - $R$ -neighborhood family of  $G$ .
- (3) For any  $a \in L \setminus \{0\}$ , each countable strong  $a$ - $R$ -neighborhood family  $\mathcal{P}$  of  $G$  has a finite subfamily  $\mathcal{F}$  which is an  $a$ - $R$ -neighborhood family of  $G$ .
- (4) For any  $a \in L \setminus \{0\}$  and any countable strong  $a$ - $R$ -neighborhood family  $\mathcal{P}$  of  $G$ , there exists a finite subfamily  $\mathcal{F}$  of  $\mathcal{P}$  and  $b \in \beta(a)$  such that  $\mathcal{P}$  is a strong  $b$ - $R$ -neighborhood family of  $G$ .
- (5) For any  $a \in L \setminus \{0\}$  and any countable strong  $a$ - $R$ -neighborhood family  $\mathcal{P}$  of  $G$ , there exists a finite subfamily  $\mathcal{F}$  of  $\mathcal{P}$  and  $b \in \beta(a)$  such that  $\mathcal{P}$  is a  $b$ - $R$ -neighborhood family of  $G$ .
- (6) For any  $a \in M(L)$ , each countable strong  $a$ - $R$ -neighborhood family  $\mathcal{P}$  of  $G$  has a finite subfamily  $\mathcal{F}$  which is a strong  $a$ - $R$ -neighborhood family of  $G$ .
- (7) For any  $a \in M(L)$ , each countable strong  $a$ - $R$ -neighborhood family  $\mathcal{P}$  of  $G$  has a finite subfamily  $\mathcal{F}$  which is an  $a$ - $R$ -neighborhood family of  $G$ .
- (8) For any  $a \in M(L)$  and any countable strong  $a$ - $R$ -neighborhood family  $\mathcal{P}$  of  $G$ , there exists a finite subfamily  $\mathcal{F}$  of  $\mathcal{P}$  and  $b \in \beta^*(a)$  such that  $\mathcal{P}$  is a strong  $b$ - $R$ -neighborhood family of  $G$ .
- (9) For any  $a \in M(L)$  and any countable strong  $a$ - $R$ -neighborhood family  $\mathcal{P}$  of  $G$ , there exists a finite subfamily  $\mathcal{F}$  of  $\mathcal{P}$  and  $b \in \beta^*(a)$  such that  $\mathcal{P}$  is a  $b$ - $R$ -neighborhood family of  $G$ .
- (10) For any  $a \in L \setminus \{1\}$ , each countable strong  $a$ -shading  $\mathcal{U}$  of  $G$  has a finite subfamily  $\mathcal{V}$  which is a strong  $a$ -shading of  $G$ .
- (11) For any  $a \in L \setminus \{1\}$ , each countable strong  $a$ -shading  $\mathcal{U}$  of  $G$  has a finite subfamily  $\mathcal{V}$  which is an  $a$ -shading of  $G$ .
- (12) For any  $a \in L \setminus \{1\}$  and any countable strong  $a$ -shading  $\mathcal{U}$  of  $G$ , there exists a finite subfamily  $\mathcal{V}$  of  $\mathcal{U}$  and  $b \in \alpha(a)$  such that  $\mathcal{P}$  is a strong  $b$ -shading of  $G$ .
- (13) For any  $a \in L \setminus \{1\}$  and any countable strong  $a$ -shading  $\mathcal{U}$  of  $G$ , there exists a finite subfamily  $\mathcal{V}$  of  $\mathcal{U}$  and  $b \in \alpha(a)$  such that  $\mathcal{P}$  is a  $b$ -shading of  $G$ .
- (14) For any  $a \in P(L)$ , each countable strong  $a$ -shading  $\mathcal{U}$  of  $G$  has a finite subfamily  $\mathcal{V}$  which is a strong  $a$ -shading of  $G$ .
- (15) For any  $a \in P(L)$ , each countable strong  $a$ -shading  $\mathcal{U}$  of  $G$  has a finite subfamily  $\mathcal{V}$  which is an  $a$ -shading of  $G$ .

(16) For any  $a \in P(L)$  and any countable strong  $a$ -shading  $\mathcal{U}$  of  $G$ , there exists a finite subfamily  $\mathcal{V}$  of  $\mathcal{U}$  and  $b \in \alpha^*(a)$  such that  $\mathcal{P}$  is a strong  $b$ -shading of  $G$ .

(17) For any  $a \in P(L)$  and any countable strong  $a$ -shading  $\mathcal{U}$  of  $G$ , there exists a finite subfamily  $\mathcal{V}$  of  $\mathcal{U}$  and  $b \in \alpha^*(a)$  such that  $\mathcal{P}$  is a  $b$ -shading of  $G$ .

(18) For any  $a \in L \setminus \{0\}$ , each countable strong  $\beta_a$ -open cover  $\mathcal{U}$  of  $G$  has a finite subfamily  $\mathcal{V}$  which is a strong  $\beta_a$ -open cover of  $G$ .

(19) For any  $a \in L \setminus \{0\}$ , each countable strong  $\beta_a$ -open cover  $\mathcal{U}$  of  $G$  has a finite subfamily  $\mathcal{V}$  which is a  $\beta_a$ -open cover of  $G$ .

(20) For any  $a \in L \setminus \{0\}$  and any countable strong  $\beta_a$ -open cover  $\mathcal{U}$  of  $G$ , there exists a finite subfamily  $\mathcal{V}$  of  $\mathcal{U}$  and  $b \in L$  with  $a \in \beta(b)$  such that  $\mathcal{V}$  is a strong  $\beta_b$ -open cover of  $G$ .

(21) For any  $a \in L \setminus \{0\}$  and any countable strong  $\beta_a$ -open cover  $\mathcal{U}$  of  $G$ , there exists a finite subfamily  $\mathcal{V}$  of  $\mathcal{U}$  and  $b \in L$  with  $a \in \beta(b)$  such that  $\mathcal{V}$  is a  $\beta_b$ -open cover of  $G$ .

(22) For any  $a \in M(L)$ , each countable strong  $\beta_a$ -open cover  $\mathcal{U}$  of  $G$  has a finite subfamily  $\mathcal{V}$  which is a strong  $\beta_a$ -open cover of  $G$ .

(23) For any  $a \in M(L)$ , each countable strong  $\beta_a$ -open cover  $\mathcal{U}$  of  $G$  has a finite subfamily  $\mathcal{V}$  which is a  $\beta_a$ -open cover of  $G$ .

(24) For any  $a \in M(L)$  and any countable strong  $\beta_a$ -open cover  $\mathcal{U}$  of  $G$ , there exists a finite subfamily  $\mathcal{V}$  of  $\mathcal{U}$  and  $b \in M(L)$  with  $a \in \beta^*(b)$  such that  $\mathcal{V}$  is a strong  $\beta_b$ -open cover of  $G$ .

(25) For any  $a \in M(L)$  and any countable strong  $\beta_a$ -open cover  $\mathcal{U}$  of  $G$ , there exists a finite subfamily  $\mathcal{V}$  of  $\mathcal{U}$  and  $b \in M(L)$  with  $a \in \beta^*(b)$  such that  $\mathcal{V}$  is a  $\beta_b$ -open cover of  $G$ .

(26) For any  $a \in L \setminus \{0\}$  and any  $b \in \beta(a) \setminus \{0\}$ , each countable  $Q_a$ -open cover of  $G$  has a finite subfamily which is a  $Q_b$ -open cover of  $G$ .

(27) For any  $a \in L \setminus \{0\}$  and any  $b \in \beta(a) \setminus \{0\}$ , each countable  $Q_a$ -open cover of  $G$  has a finite subfamily which is a  $\beta_b$ -open cover of  $G$ .

(28) For any  $a \in L \setminus \{0\}$  and any  $b \in \beta(a) \setminus \{0\}$ , each countable  $Q_a$ -open cover of  $G$  has a finite subfamily which is a strong  $\beta_b$ -open cover of  $G$ .

(29) For any  $a \in M(L)$  and any  $b \in \beta^*(a)$ , each countable  $Q_a$ -open cover of  $G$  has a finite subfamily which is a  $Q_b$ -open cover of  $G$ .

(30) For any  $a \in M(L)$  and any  $b \in \beta^*(a)$ , each countable  $Q_a$ -open cover of  $G$  has a finite subfamily which is a  $\beta_b$ -open cover of  $G$ .

(31) For any  $a \in M(L)$  and any  $b \in \beta^*(a)$ , each countable  $Q_a$ -open cover of  $G$  has a finite subfamily which is a strong  $\beta_b$ -open cover of  $G$ .

*Proof.* By Theorem 2.5 we can obtain (1)  $\Leftrightarrow$  (2). (2)  $\Rightarrow$  (3) is obvious. To prove (3)  $\Rightarrow$  (4), suppose that  $a \in L \setminus \{0\}$  and  $\mathcal{P}$  is a countable strong  $a$ -R-neighborhood family of  $G$ , then  $\bigvee_{x \in X} \left( G(x) \wedge \bigwedge_{B \in \mathcal{P}} B(x) \right) \not\geq a$ , take  $c \in \beta(a)$

such that  $\bigvee_{x \in X} \left( G(x) \wedge \bigwedge_{B \in \mathcal{P}} B(x) \right) \not\geq c$ , obviously  $\mathcal{P}$  is a strong  $c$ -R-neighborhood

family of  $G$ , by (3) we know that  $\mathcal{P}$  has a finite subfamily  $\mathcal{F}$  which is a  $c$ - $R$ -neighborhood family of  $G$ . Take  $b \in \beta(a)$  such that  $c \in \beta(b)$ ; then  $\mathcal{F}$  is a strong  $b$ - $R$ -neighborhood family of  $G$ . (4) is shown. (4)  $\Rightarrow$  (5)  $\Rightarrow$  (2) is obvious. Similarly we can prove that (2)  $\Rightarrow$  (6)  $\Rightarrow$  (7)  $\Rightarrow$  (8)  $\Rightarrow$  (9)  $\Rightarrow$  (1). By Theorem 2.5 we can obtain (1)  $\Leftrightarrow$  (10). (10)  $\Rightarrow$  (11) is obvious. To prove (11)  $\Rightarrow$  (12), suppose that  $a \in L \setminus \{1\}$  and  $\mathcal{U}$  is a countable strong  $a$ -shading of  $G$ , then  $\bigwedge_{x \in X} \left( G'(x) \vee \bigvee_{B \in \mathcal{U}} B(x) \right) \not\leq a$ .

Take  $c \in \alpha(a)$  such that  $\bigwedge_{x \in X} \left( G'(x) \vee \bigvee_{B \in \mathcal{U}} B(x) \right) \not\leq c$ ; obviously  $\mathcal{U}$  is a strong  $c$ -shading of  $G$  and by (11) we know that  $\mathcal{U}$  has a finite subfamily  $\mathcal{V}$  which is a  $c$ -shading of  $G$ . Take  $b \in \alpha(a)$  such that  $c \in \alpha(b)$ ; then  $\mathcal{V}$  is a strong  $b$ -shading of  $G$ , (12) is shown. (12)  $\Rightarrow$  (13)  $\Rightarrow$  (10) is obvious. Similarly we can prove that (14)  $\Rightarrow$  (15)  $\Rightarrow$  (16)  $\Rightarrow$  (17)  $\Rightarrow$  (18)  $\Rightarrow$  (10). Similarly we can also prove the other results.  $\square$

**Lemma 2.10.** *The condition (7) in Theorem 2.9 is an equivalent condition of fuzzy countable compactness in [7, 8]. Therefore Definition 2.1 is a generalization of fuzzy countable compactness in [7, 8].*

### 3. The Fuzzy Lindelöf Property

**Definition 3.1.** Let  $(X, \mathcal{T})$  be an  $L$ -space,  $G \in L^X$  is said to have the (fuzzy) Lindelöf property if for every family  $\mathcal{U} \subseteq \mathcal{T}$ , it follows that

$$\bigwedge_{x \in X} \left( G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x) \right) \leq \bigvee_{\mathcal{V} \in 2^{[\mathcal{U}]}} \bigwedge_{x \in X} \left( G'(x) \vee \bigvee_{A \in \mathcal{V}} A(x) \right).$$

**Lemma 3.2.** *In Definition 3.1, if we let  $L = \{0, 1\}$ , then by Remark 2.2 we know that the fuzzy Lindelöf property in Definition 3.1 is a generalization of the Lindelöf property in general topology.*

Obviously we have the following theorem.

**Theorem 3.3.** *Let  $(X, \mathcal{T})$  be an  $L$ -space and  $G \in L^X$  has the Lindelöf property. Then  $G$  is fuzzy compact if and only if it is fuzzy countably compact.*

Analogous to countable compactness, we have the following results.

**Theorem 3.4.** *Let  $(X, \mathcal{T})$  be an  $L$ -space. Then  $G \in L^X$  has the Lindelöf property if and only if for every subfamily  $\mathcal{P} \subseteq \mathcal{T}'$ , it follows that*

$$\bigvee_{x \in X} \left( G(x) \wedge \bigwedge_{B \in \mathcal{P}} B(x) \right) \geq \bigwedge_{\mathcal{F} \in 2^{[\mathcal{P}]}} \bigvee_{x \in X} \left( G(x) \wedge \bigwedge_{B \in \mathcal{F}} B(x) \right).$$

**Theorem 3.5.** *Let  $(X, \mathcal{T})$  be an  $L$ -space and  $G \in L^X$ . Then the following statements are equivalent:*

- (1)  $G$  has the Lindelöf property.
- (2) For any  $a \in L \setminus \{1\}$ , each strong  $a$ -shading  $\mathcal{U}$  of  $G$  has a countable subfamily  $\mathcal{V}$  which is still a strong  $a$ -shading of  $G$ .

(3) For any  $a \in L \setminus \{0\}$ , each strong  $a$ - $R$ -neighborhood family  $\mathcal{P}$  of  $G$  has a countable subfamily  $\mathcal{F}$  which is still a strong  $a$ - $R$ -neighborhood family of  $G$ .

**Theorem 3.6.** *If  $G$  has the Lindelöf property and  $H$  is closed, then  $G \wedge H$  has the Lindelöf property.*

**Theorem 3.7.** *Let  $L$  be a complete Heyting algebra,  $f : X \rightarrow Y$  be a map,  $\mathcal{T}_1$  be an  $L$ -topology on  $X$ ,  $\mathcal{T}_2$  be an  $L$ -topology on  $Y$  and  $f_L^- : L^X \rightarrow L^Y$  be continuous. If  $G$  has the Lindelöf property in  $(X, \mathcal{T}_1)$ , then  $f(G)$  has the Lindelöf property in  $(Y, \mathcal{T}_2)$ .*

**Theorem 3.8.** *Let  $(X, \mathcal{T})$  be an  $L$ -space and  $G \in L^X$ . If  $L$  is a completely distributive de Morgan algebra, then the following conditions are equivalent.*

- (1)  $G$  has the Lindelöf property.
- (2) For any  $a \in L \setminus \{0\}$ , each strong  $a$ - $R$ -neighborhood family  $\mathcal{P}$  of  $G$  has a countable subfamily  $\mathcal{F}$  which is a strong  $a$ - $R$ -neighborhood family of  $G$ .
- (3) For any  $a \in L \setminus \{0\}$ , each strong  $a$ - $R$ -neighborhood family  $\mathcal{P}$  of  $G$  has a countable subfamily  $\mathcal{F}$  which is an  $a$ - $R$ -neighborhood family of  $G$ .
- (4) For any  $a \in L \setminus \{0\}$  and any strong  $a$ - $R$ -neighborhood family  $\mathcal{P}$  of  $G$ , there exists a countable subfamily  $\mathcal{F}$  of  $\mathcal{P}$  and  $b \in \beta(a)$  such that  $\mathcal{P}$  is a strong  $b$ - $R$ -neighborhood family of  $G$ .
- (5) For any  $a \in L \setminus \{0\}$  and any strong  $a$ - $R$ -neighborhood family  $\mathcal{P}$  of  $G$ , there exists a countable subfamily  $\mathcal{F}$  of  $\mathcal{P}$  and  $b \in \beta(a)$  such that  $\mathcal{P}$  is a  $b$ - $R$ -neighborhood family of  $G$ .
- (6) For any  $a \in M(L)$ , each strong  $a$ - $R$ -neighborhood family  $\mathcal{P}$  of  $G$  has a countable subfamily  $\mathcal{F}$  which is a strong  $a$ - $R$ -neighborhood family of  $G$ .
- (7) For any  $a \in M(L)$ , each strong  $a$ - $R$ -neighborhood family  $\mathcal{P}$  of  $G$  has a countable subfamily  $\mathcal{F}$  which is an  $a$ - $R$ -neighborhood family of  $G$ .
- (8) For any  $a \in M(L)$  and any strong  $a$ - $R$ -neighborhood family  $\mathcal{P}$  of  $G$ , there exists a countable subfamily  $\mathcal{F}$  of  $\mathcal{P}$  and  $b \in \beta^*(a)$  such that  $\mathcal{P}$  is a strong  $b$ - $R$ -neighborhood family of  $G$ .
- (9) For any  $a \in M(L)$  and any strong  $a$ - $R$ -neighborhood family  $\mathcal{P}$  of  $G$ , there exists a countable subfamily  $\mathcal{F}$  of  $\mathcal{P}$  and  $b \in \beta^*(a)$  such that  $\mathcal{P}$  is a  $b$ - $R$ -neighborhood family of  $G$ .
- (10) For any  $a \in L \setminus \{1\}$ , each strong  $a$ -shading  $\mathcal{U}$  of  $G$  has a countable subfamily  $\mathcal{V}$  which is still a strong  $a$ -shading of  $G$ .
- (11) For any  $a \in L \setminus \{1\}$ , each strong  $a$ -shading  $\mathcal{U}$  of  $G$  has a countable subfamily  $\mathcal{V}$  which is an  $a$ -shading of  $G$ .
- (12) For any  $a \in L \setminus \{1\}$  and any strong  $a$ -shading  $\mathcal{U}$  of  $G$ , there exists a countable subfamily  $\mathcal{V}$  of  $\mathcal{U}$  and  $b \in \alpha(a)$  such that  $\mathcal{P}$  is a strong  $b$ -shading of  $G$ .
- (13) For any  $a \in L \setminus \{1\}$  and any strong  $a$ -shading  $\mathcal{U}$  of  $G$ , there exists a countable subfamily  $\mathcal{V}$  of  $\mathcal{U}$  and  $b \in \alpha(a)$  such that  $\mathcal{P}$  is a  $b$ -shading of  $G$ .
- (14) For any  $a \in P(L)$ , each strong  $a$ -shading  $\mathcal{U}$  of  $G$  has a countable subfamily  $\mathcal{V}$  which is still a strong  $a$ -shading of  $G$ .
- (15) For any  $a \in P(L)$ , each strong  $a$ -shading  $\mathcal{U}$  of  $G$  has a countable subfamily  $\mathcal{V}$  which is an  $a$ -shading of  $G$ .



(16) For any  $a \in P(L)$  and any strong  $a$ -shading  $\mathcal{U}$  of  $G$ , there exists a countable subfamily  $\mathcal{V}$  of  $\mathcal{U}$  and  $b \in \alpha^*(a)$  such that  $\mathcal{P}$  is a strong  $b$ -shading of  $G$ .

(17) For any  $a \in P(L)$  and any strong  $a$ -shading  $\mathcal{U}$  of  $G$ , there exists a countable subfamily  $\mathcal{V}$  of  $\mathcal{U}$  and  $b \in \alpha^*(a)$  such that  $\mathcal{P}$  is a  $b$ -shading of  $G$ .

(18) For any  $a \in L \setminus \{0\}$ , each strong  $\beta_a$ -open cover  $\mathcal{U}$  of  $G$  has a countable subfamily  $\mathcal{V}$  which is a strong  $\beta_a$ -open cover of  $G$ .

(19) For any  $a \in L \setminus \{0\}$ , each strong  $\beta_a$ -open cover  $\mathcal{U}$  of  $G$  has a countable subfamily  $\mathcal{V}$  which is a  $\beta_a$ -open cover of  $G$ .

(20) For any  $a \in L \setminus \{0\}$  and any strong  $\beta_a$ -open cover  $\mathcal{U}$  of  $G$ , there exists a countable subfamily  $\mathcal{V}$  of  $\mathcal{U}$  and  $b \in L$  with  $a \in \beta(b)$  such that  $\mathcal{V}$  is a strong  $\beta_b$ -open cover of  $G$ .

(21) For any  $a \in L \setminus \{0\}$  and any strong  $\beta_a$ -open cover  $\mathcal{U}$  of  $G$ , there exists a countable subfamily  $\mathcal{V}$  of  $\mathcal{U}$  and  $b \in L$  with  $a \in \beta(b)$  such that  $\mathcal{V}$  is a  $\beta_b$ -open cover of  $G$ .

(22) For any  $a \in M(L)$ , each strong  $\beta_a$ -open cover  $\mathcal{U}$  of  $G$  has a countable subfamily  $\mathcal{V}$  which is a strong  $\beta_a$ -open cover of  $G$ .

(23) For any  $a \in M(L)$ , each strong  $\beta_a$ -open cover  $\mathcal{U}$  of  $G$  has a countable subfamily  $\mathcal{V}$  which is a  $\beta_a$ -open cover of  $G$ .

(24) For any  $a \in M(L)$  and any strong  $\beta_a$ -open cover  $\mathcal{U}$  of  $G$ , there exists a countable subfamily  $\mathcal{V}$  of  $\mathcal{U}$  and  $b \in M(L)$  with  $a \in \beta^*(b)$  such that  $\mathcal{V}$  is a strong  $\beta_b$ -open cover of  $G$ .

(25) For any  $a \in M(L)$  and any strong  $\beta_a$ -open cover  $\mathcal{U}$  of  $G$ , there exists a countable subfamily  $\mathcal{V}$  of  $\mathcal{U}$  and  $b \in M(L)$  with  $a \in \beta^*(b)$  such that  $\mathcal{V}$  is a  $\beta_b$ -open cover of  $G$ .

(26) For any  $a \in L \setminus \{0\}$  and any  $b \in \beta(a) \setminus \{0\}$ , each  $Q_a$ -open cover of  $G$  has a countable subfamily which is a  $Q_b$ -open cover of  $G$ .

(27) For any  $a \in L \setminus \{0\}$  and any  $b \in \beta(a) \setminus \{0\}$ , each  $Q_a$ -open cover of  $G$  has a countable subfamily which is a  $\beta_b$ -open cover of  $G$ .

(28) For any  $a \in L \setminus \{0\}$  and any  $b \in \beta(a) \setminus \{0\}$ , each  $Q_a$ -open cover of  $G$  has a countable subfamily which is a strong  $\beta_b$ -open cover of  $G$ .

(29) For any  $a \in M(L)$  and any  $b \in \beta^*(a)$ , each countable  $Q_a$ -open cover of  $G$  has a finite subfamily which is a  $Q_b$ -open cover of  $G$ .

(30) For any  $a \in M(L)$  and any  $b \in \beta^*(a)$ , each  $Q_a$ -open cover of  $G$  has a countable subfamily which is a  $\beta_b$ -open cover of  $G$ .

(31) For any  $a \in M(L)$  and any  $b \in \beta^*(a)$ , each  $Q_a$ -open cover of  $G$  has a countable subfamily which is a strong  $\beta_b$ -open cover of  $G$ .

**Lemma 3.9.** *The conditions (6) and (7) in Theorem 3.8 are two equivalent conditions of the fuzzy Lindelöf property in [6, 8]. Therefore Definition 3.1 is a generalization of the fuzzy Lindelöf property in [6, 8].*

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