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Negations and aggregation operators based on a new hesitant fuzzy partial ordering

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Abstract

Based on a new hesitant fuzzy partial ordering proposed by Garmendia et al. [7], in this paper a fuzzy disjunction D on the set H of finite and nonempty subsets of the unit interval and a t-conorm S on the set $\bar{B}$ of equivalence class on the set of finite bags of unit interval based on this partial ordering are introduced respectively. Then, hesitant fuzzy negations N_n on H and μ_n on $\bar{B}$ are proposed. Particularly, their De Morgan's laws are investigated with respect to binary operations C and D on H, as well as T and S on $\bar{B}$ respectively, where C is a commutative fuzzy conjunction on $(H, \leq_H)$ and T is a t-norm on $(\bar{B}, \leq_B)$. Finally, the new hesitant fuzzy aggregation operators are presented on H and $\bar{B}$ and their more general forms are given. Moreover, the validity of the aggregation operations is illustrated by a numerical example on decision making.

Keywords: Hesitant fuzzy sets, finite subsets of the unit interval, partial ordering, t-conorm, negation, aggregation operation.

1 Introduction

As the generalization of fuzzy sets proposed by Zadeh [27], hesitant fuzzy sets allow us the possibility of assigning more than a value of the unit interval to an object of a universe of discourse. The theory of hesitant fuzzy sets has been found to be useful to deal with uncertainty of informations when there is doubt or hesitation. It is well known that fuzzy negations and aggregation operators are significant mathematical tools in approximate reasoning [9, 23] and decision making [11, 10, 13, 21, 24]. The flourishing achievements on hesitant fuzzy negations and hesitant aggregation operators (see [3, 4, 9, 11, 10, 12, 13, 14, 16, 15, 17, 18, 22, 23, 21, 24]) have been obtained.

Partial order relation is the necessity for defining negations and aggregation operators on the set of hesitant fuzzy elements. Different hesitant fuzzy negations and aggregation operators have been generated by different hesitant partial orderings (see [3, 4, 16, 15]). It is very vital that how to compare the hesitant fuzzy elements with different cardinalities on the set H of nonempty and finite subsets of $[0, 1]$. Xu and Xia [25] have proposed a partial ordering on H and investigated distance and similarity measures. Based on Xu-Xia-partial ordering, Santos et al. [16] have explored typical hesitant fuzzy negations on H. Thereafter, a constructive method of typical hesitant triangular norms and the notion of aggregation functions for typical hesitant fuzzy elements regarding Xu-Xia-partial order have been proposed in [15]. Bedregal et al. [3] have proposed a partial ordering based on α-normalization and then studied the typical hesitant triangular norms and aggregation functions. In addition, typical hesitant fuzzy negations based on this partial order have been presented in [4].

In [7], on one hand, Garmendia et al. have pointed out that these partial orderings given above seem acceptable in some occasions but can be too radical in others. In order to compare two hesitant fuzzy elements with different cardinalities, L.Garmendia et al. proposed a more balanced partial ordering on H and presented that H with this partial ordering is a bounded partially ordered set instead of lattice. However, $\bar{B}$(i.e., the set of equivalence classes

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on the set of finite bags of unit interval) and some subsets of H have the lattice structure. On the other hand, they investigated t-norms on H and on $\bar{B}$. Nevertheless, hesitant fuzzy negations and aggregation operations based on this more balanced partial ordering on the H and on $\bar{B}$ have not been studied. Hence, in this paper we will utilize this partial ordering to construct hesitant fuzzy negations and aggregation operations on H and on $\bar{B}$, which would be beneficial to research for hesitant fuzzy approximate reasoning and decision making.

The rest of the paper is arranged as follows. Section 2 briefly reviews some related definitions and properties. Based on the partial ordering proposed by Garmendia et al. [7], Section 3 investigates t-conorms on $(H, \leq_H)$ and $(\bar{B}, \leq_B)$. Section 4 proposes negations on $(H, \leq_H)$ and $(\bar{B}, \leq_B)$ and investigates their algebraic properties. Section 5 defines new aggregation operators and their more general forms on $(H, \leq_H)$ and $(\bar{B}, \leq_B)$. Section 6 utilizes a special aggregation operator on $(H, \leq_H)$ to deal with decision making. Section 7 summarizes the conclusion.

2 Preliminaries

The following definitions and preliminaries are required in the sequel of our work and hence presented in brief. In order to simplify notations, throughout this paper, suppose that any hesitant fuzzy element $A = \{a_1, a_2, ..., a_n\} \in H$ always satisfies increasing order, *i.e.*, $a_1 < a_2 < ... < a_n$.

Definition 2.1. [7] *Let $A = \{a_1, a_2, ..., a_n\}$ be a finite subset of the unit interval and $r \in N$. $A_r \in [0, 1]^{rn}$ is the vector of rn coordinates defined as* $A_r = (\overbrace{a_1, ..., a_1}^{rtimes}, \overbrace{a_2, ..., a_2}^{rtimes}, ..., \overbrace{a_n, ..., a_n}^{rtimes})$.

Definition 2.2. [7] *Let $A = \{a_1, a_2, ..., a_n\}$ and $B = \{b_1, b_2, ..., b_m\}$ be two finite subsets of the unit interval. $lcm(n, m)$ represents the least common multiple of* n *and* m. *Rewriting* $A_{\frac{lcm(n,m)}{n}} = (c_1, c_2, ..., c_{lcm(n,m)})$ *and* $B_{\frac{lcm(n,m)}{m}} = (d_1, d_2, ..., d_{lcm(n,m)})$, *$A \leq_H B$ if and only if $c_i \leq d_i$ for all $i = 1, 2, ..., lcm(n, m)$.*

Lemma 2.3. [7] *Let $A = \{a_1, a_2, ..., a_n\}$ and $B = \{b_1, b_2, ..., b_m\}$ be two finite subsets of the unit interval and $r \in N$. $A \leq_H B$ if and only if every coordinate of $A_{r\frac{lcm(n,m)}{n}}$ is smaller than or equal to the corresponding coordinate of $B_{r\frac{lcm(n,m)}{m}}$.*

In [7], it is proved that $(H, \leq_H)$ is a bounded partially ordered set instead of a lattice structure (the smallest element and biggest element are $\{0\}$ and $\{1\}$ respectively) and the subset $(H_R, \leq_H)$ of the partially ordered set $(H, \leq_H)$ is defined in the following way.

Definition 2.4. [7] Let $R = r_1, r_2, ..., r_n, ...$ be a sequence of natural numbers with r_{i+1} a multiple of r_i for every $i \geq 0$, H_{r_i} is the set of finite subsets of unit interval of cardinality r_i. $H_R = \bigcup_{i\geq 1}^{\infty} H_{r_i}$.

It is notable that every element of $A = \{a_1, a_2, ..., a_n\}$ in $(H, \leq_H)$ is different. Now, it is allowed that some elements of A in $(H, \leq_H)$ are repeated, the so called bags or multisets [26]. A finite bag of the unit interval can be represented as a vector $\vec{v} = (a_1, a_2, ..., a_n)$ of $[0, 1]^n$ and we will always assume that $a_1 \leq a_2 \leq ... \leq a_n$. The set of finite bags of [0,1] will be denoted by B.

Definition 2.5. [7] *Let $\vec{v} = (a_1, a_2, ..., a_n) \in B$ and $r \in N$, $\vec{v}_r$ is defined by* $\vec{v}_r = (\overbrace{a_1, ..., a_1}^{r \ \ times}, \overbrace{a_2, ..., a_2}^{r \ \ times}, ..., \overbrace{a_n, ..., a_n}^{r \ \ times})$.

Definition 2.6. [7] *Let $\vec{u} = (a_1, a_2, ..., a_n)$ and $\vec{v} = (b_1, b_2, ..., b_m)$ be two finite bags of the unit interval and $lcm(n, m)$ is the least common multiple of* n *and* m. *Rewriting* $\vec{u}_{\frac{lcm(n,m)}{n}} = (c_1, c_2, ..., c_{lcm(n,m)})$ *and* $\vec{v}_{\frac{lcm(n,m)}{m}} = (d_1, d_2, ..., d_{lcm(n,m)})$, *$\vec{u} \leq_B \vec{v}$ if and only if $c_i \leq d_i$ for all $i = 1, 2, ..., lcm(n, m)$.*

Proposition 2.7. [7] *The relation $\leq_B$ on B is a preoder (i.e., it is reflexive and transitive).*

Lemma 2.8. [7] *Given two bags $\vec{u}$ and $\vec{v}$ of B, $\vec{u} \leq_B \vec{v}$ and $\vec{v} \leq_B \vec{u}$ if and only if there exists $\vec{w} \in B$ and $r, s \in N$ such that $\vec{u} = \vec{w}_r$ and $\vec{v} = \vec{w}_s$.*

Definition 2.9. [7] *On B consider the equivalence relation $\sim$ for all $\vec{u}, \vec{v} \in B$ by $\vec{u} \sim \vec{v}$ if and only if there exists $\vec{w} \in B$ and $r, s \in N$ such that $\vec{u} = \vec{w}_r$ and $\vec{v} = \vec{w}_s$ and denote the quotient $B/\sim$ by $\bar{B}$.*

The vector of a class with the smallest number of coordinates will be called its canonical representative. In [7], $\leq_B$ is compatible with $\sim$ and $(\bar{B}, \leq_B)$ is a lattice.

In [1] t-norms are defined on bounded partially ordered sets. The concept of general conjunction is provided by [6].

Definition 2.10. [6] *An operation C: $[0,1]^2 \to [0,1]$ is a fuzzy conjunction if*
(i) It is increasing with respect to each variable.
(ii) $C(1,1) = 1, C(0,0) = C(0,1) = C(1,0) = 0$.

Definition 2.11. [7] *Let $P = (P, \leq_P, 0, 1)$ be a bounded partially ordered set. An operation C: $P^2 \to P$ is a fuzzy conjunction if*
(i) It is increasing with respect to each variable.
(ii) $C(1,1) = 1, C(0,0) = C(0,1) = C(1,0) = 0$.

Definition 2.12. [1] *A t-norm T on a bounded partially ordered set $P = (P, \leq_P, 0, 1)$ is a binary operation on P that for all $x, y, z \in P$ satisfies*
(i) $T(x,1) = x$ (neutral element)
(ii) If $x \leq_P y$,then $T(x,z) \leq_P T(y,z)$ (monotonicity)
(iii) $T(x,y) = T(y,x)$ (commutativity)
(iv) $T(x, T(y,z)) = T(T(x,y), z)$.

Definition 2.13. [7] *Let T be a t-norm on $[0,1]$. On H the binary operation C is defined in the following way. For two elements $A = \{a_1, a_2, ..., a_n\}$ and $B = \{b_1, b_2, ..., b_m\}$ of H, rewriting $A_{\frac{lcm(n,m)}{n}} = (c_1, c_2, ..., c_{lcm(n,m)})$ and $B_{\frac{lcm(n,m)}{m}} = (d_1, d_2, ..., d_{lcm(n,m)})$,*

$$C(A,B) = \{T(c_1, d_1), T(c_2, d_2), ..., T(c_{lcm(n,m)}, d_{lcm(n,m)})\}. \tag{1}$$

Definition 2.14. [7] *Let T be a t-norm on $[0,1]$. On $\bar{B}$ the binary operation T is defined by the following way. For two elements $[\vec{u}] = [(a_1, a_2, ..., a_n)]$ and $[\vec{v}] = [(b_1, b_2, ..., b_m)]$ of $\bar{B}$, rewriting $\vec{u}_{\frac{lcm(n,m)}{n}} = (c_1, c_2, ..., c_{lcm(n,m)})$ and $\vec{v}_{\frac{lcm(n,m)}{m}} = (d_1, d_2, ..., d_{lcm(n,m)})$,*

$$T([\vec{u}], [\vec{v}]) = [(T(c_1, d_1), T(c_2, d_2), ..., T(c_{lcm(n,m)}, d_{lcm(n,m)}))]. \tag{2}$$

In [7], it is shown that C on $(H, \leq_H)$ is a commutative fuzzy conjunction instead of a t-norm and T on $(\bar{B}, \leq_B)$ is a t-norm. Moreover, C is proved that it is a t-norm on the subset H_R of H.

3 T-conorms on $(H, \leq_H)$ and $(\bar{B}, \leq_B)$

In this section, the binary operations D on $(H, \leq_H)$ and S on $(\bar{B}, \leq_B)$ associated with a t-conorm S on $[0,1]$ are introduced respectively. The following properties are investigated: D on H is a commutative fuzzy disjunction instead of a t-conorm. However, D on H_R is a t-conorm. Moreover, S on $\bar{B}$ is a t-conorm.

Definition 3.1. *Let $P = (P, \leq_P, 0, 1)$ be a bounded partially ordered set. An operation **D**: $P^2 \to P$ is called a fuzzy disjunction if*
(i) it is increasing with respect to each variable.
(ii) $D(0,0) = 0, D(0,1) = D(1,0) = D(1,1) = 1$.

Definition 3.1 presents the notion of a general fuzzy disjunction on a bounded partially ordered set. The fuzzy disjunction on a bounded partially ordered set is precisely a classical fuzzy disjunction introduced by Batyrshin et al. [2] when the bounded partially ordered set is the interval [0,1].

Let us present the definition of a t-conorm on a bounded partially ordered set.

Definition 3.2. *A t-conorm S on a bounded partially ordered set $P = (P, \leq_P, 0, 1)$ is a binary operation on P that for all $x, y, z \in P$ satisfies*
(i) $S(x,0) = x$ (neutral element)
(ii) If $x \leq_P y$,then $S(x,z) \leq_P S(y,z)$ (monotonicity)
(iii) $S(x,y) = S(y,x)$ (commutativity)
(iv) $S(x, S(y,z)) = S(S(x,y), z)$.

From a t-conorm S on $[0,1]$, an operation D can be defined on H.

Definition 3.3. *Let S be a t-conorm on $[0,1]$. On $\boldsymbol{H}$ the binary operation D is defined in the following way. For two elements $A=\{a_1,a_2,...,a_n\}$ and $B=\{b_1,b_2,...,b_m\}$ of H, rewriting $A_{\frac{lcm(n,m)}{n}}=(c_1,c_2,...,c_{lcm(n,m)})$ and $B_{\frac{lcm(n,m)}{m}}=(d_1,d_2,...,d_{lcm(n,m)})$,*

$$D(A,B)=\{S(c_1,d_1),S(c_2,d_2),...,S(c_{lcm(n,m)},d_{lcm(n,m)})\}. \tag{3}$$

Example 3.4. *If $A=\{0.3,0.6\}$ and $B=\{0.2,0.5,0.6\}$ and S is the Product t-conorm ($a\oplus b=a+b-ab$), then $A_3=(0.3,0.3,0.3,0.6,0.6,0.6)$, $B_2=(0.2,0.2,0.5,0.5,0.6,0.6)$ and*

$$D(A,B)=\{0.44,0.44,0.65,0.8,0.84,0.84\}=\{0.44,0.65,0.8,0.84\}.$$

Notably the last equality is obtained by transforming the multiset into a set.

Lemma 3.5. *Let $A=\{a_1,a_2,...,a_n\}$ and $B=\{b_1,b_2,...,b_m\}$ be two finite subsets of the unit interval and $r\in N$. Rewriting $A_{r\frac{lcm(n,m)}{n}}=(c_1,c_2,...,c_{rlcm(n,m)})$ and $B_{r\frac{lcm(n,m)}{m}}=(d_1,d_2,...,d_{rlcm(n,m)})$,*

$$D(A,B)=\{S(c_1,d_1),S(c_2,d_2),...,S(c_{rlcm(n,m)},d_{rlcm(n,m)})\}. \tag{4}$$

Proposition 3.6. *Let S be a t-conorm on $[0,1]$. Then D is a commutative fuzzy disjunction on $(H,\leq_H)$.*

Proof. (i) Obviously, it follows that D satisfies commutativity by Definition of D.

(ii) Let $A=\{a_1,a_2,...,a_n\}$, $B=\{b_1,b_2,...,b_m\}$ and $C=\{c_1,c_2,...,c_p\}$ be finite subsets of the unit interval. Rewriting $A_{\frac{lcm(n,m,p)}{n}}=(d_1,d_2,...,d_{lcm(n,m,p)})$, $B_{\frac{lcm(n,m,p)}{m}}=(e_1,e_2,...,e_{lcm(n,m,p)})$ and $C_{\frac{lcm(n,m,p)}{p}}=(f_1,f_2,...,f_{lcm(n,m,p)})$. If $A\leq_H B$, then, from Lemma 2.3, $d_i\leq e_i$ for all $i=1,2,...,lcm(n,m,p)$, so $S(d_i,f_i)\leq S(e_i,f_i)$. From Lemma 3.5, it follows that $D(A,C)\leq D(B,C)$.

(iii) Conditions $D(\{0\},\{0\})=\{0\}$, $D(\{0\},\{1\})=D(\{1\},\{0\})=D(\{1\},\{1\})=\{1\}$ are easily proved. □

Since it does not satisfy associativity, the fuzzy disjunction D is not a t-conorm on $(H,\leq_H)$. In fact, we could show the following counter-example:

Let $A=\{0.5,0.6\}$, $B=\{0.4,0.5,0.7\}$, $C=\{0.2,0.3,0.5\}$ and S is a Product t-conorm ($a\oplus b=a+b-ab$). Then

$$D(D(A,B),C)=\{0.76,0.8,0.825,0.86,0.9,0.94\},\text{ and } D(A,D(B,C))=\{0.76,0.825,0.86,0.94\}.$$

Clearly, $D(D(A,B),C)\neq D(A,D(B,C))$. Therefore, D is not a t-conorm on $(H,\leq_H)$.

Proposition 3.7. *Let $R=r_1,r_2,...,r_n,...$ be a sequence of natural numbers with r_{i+1} a multiple of r_i for every $i\geq 0$, H_{r_i} is the set of finite subsets of unit interval of cardinality r_i and $H_R=\bigcup_{i\geq 1}^{\infty}H_{r_i}$. Then $D:H_R^2\rightarrow H_R$ is a t-conorm on H_R.*

On $(\bar{B},\leq_B)$ we can also derive a binary operation S from a t-conorm on $[0,1]$ in a similar way as Definition 3.3. In this case, S is a t-conorm.

Definition 3.8. *Let S be a t-conorm on $[0,1]$. On $\bar{B}$ the binary operation S is defined by the following way. For two elements $[\vec{u}]=[(a_1,a_2,...,a_n)]$ and $[\vec{v}]=[(b_1,b_2,...,b_m)]$ of $\bar{B}$, rewriting $\vec{u}_{\frac{lcm(n,m)}{n}}=(c_1,c_2,...,c_{lcm(n,m)})$ and $\vec{v}_{\frac{lcm(n,m)}{m}}=(d_1,d_2,...,d_{lcm(n,m)})$,*

$$S([\vec{u}],[\vec{v}])=[(S(c_1,d_1),S(c_2,d_2),...,S(c_{lcm(n,m)},d_{lcm(n,m)}))]. \tag{5}$$

Example 3.9. *If $\vec{u}=(0.3,0.6)$ and $\vec{v}=(0.2,0.5,0.6)$ and S is the Product t-conorm ($a\oplus b=a+b-ab$), then $\vec{u}_3=(0.3,0.3,0.3,0.6,0.6,0.6)$, $\vec{v}_2=(0.2,0.2,0.5,0.5,0.6,0.6)$ and $S([\vec{u}],[\vec{v}])=[(0.44,0.44,0.65,0.8,0.84,0.84)]$.*

Proposition 3.10. *Let S be a t-conorm on $[0,1]$. Then S is a t-conorm on$(\bar{B},\leq_B)$.*

Proof. Let $\vec{u}=\{a_1,a_2,...,a_n\}$, $\vec{v}=\{b_1,b_2,...,b_m\}$ and $\vec{w}=\{c_1,c_2,...,c_p\}$ be finite bags of the unit interval. Rewriting $\vec{u}_{\frac{lcm(n,m,p)}{n}}=(d_1,d_2,...,d_{lcm(n,m,p)})$, $\vec{v}_{\frac{lcm(n,m,p)}{m}}=(e_1,e_2,...,e_{lcm(n,m,p)})$ and $\vec{w}_{\frac{lcm(n,m,p)}{p}}=(f_1,f_2,...,f_{lcm(n,m,p)})$.

(i) It is trivial that $S([\vec{u}],[\vec{0}])=[\vec{u}]$.

(ii) Monotonicity:If $\vec{u}\leq_B\vec{v}$, then $d_i\leq e_i$ for all $i=1,2,...,lcm(n,m,p)$ and from this $S(d_i,f_i)\leq S(e_i,f_i)$. Hence, $S(\vec{u},\vec{w})\leq S(\vec{v},\vec{w})$.

(iii) Commutativity is easily proved from the commutativity of S.
(iv) Associativity:

$$\begin{aligned}S([\vec{u}],S([\vec{v}],[\vec{w}])) &= S([(d_1,d_2,...,d_{lcm(n,m,p)})],S([(e_1,e_2,...,e_{lcm(n,m,p)})],[(f_1,f_2,...,f_{lcm(n,m,p)})]))\\ &= S([(d_1,d_2,...,d_{lcm(n,m,p)})],[(S(e_1,f_1),S(e_2,f_2),...,S(e_{lcm(n,m,p)},f_{lcm(n,m,p)}))])\\ &= [(S(d_1,S(e_1,f_1)),S(d_2,S(e_2,f_2)),...,S(d_{lcm(n,m,p)},S(e_{lcm(n,m,p)},f_{lcm(n,m,p)})))]\\ &= [(S(S(d_1,e_1),f_1),S(S(d_2,e_2),f_2),...,S(S(d_{lcm(n,m,p)},e_{lcm(n,m,p)}),f_{lcm(n,m,p)}))]\\ &= S([(S(d_1,e_1),S(d_2,e_2),...,S(d_{lcm(n,m,p)},e_{lcm(n,m,p)}))],[(f_1,f_2,...,f_{lcm(n,m,p)})])\\ &= S(S([\vec{u}],[\vec{v}]),[\vec{w}]).\end{aligned}$$

□

4 Negations on $(H,\leq_H)$ and $(\bar{B},\leq_B)$

In this section, we will define negation operations N_n on $(H,\leq_H)$ and μ_n on $(\bar{B},\leq_B)$ associated with a negation n on $[0,1]$ and investigate whether binary operations C and D (T and S) with respect to negation N_n (μ_n) satisfy De Morgan's law or not. In the following, we will recall definition of negation on $[0,1]$.

Definition 4.1. [20] *A decreasing function $n:[0,1]\to[0,1]$ such that $n(0)=1$ and $n(1)=0$ is said to be negation. If, additionally, $n(n(x))=x$ holds for all $x\in[0,1]$, it is said to be a strong negation.*

This definition can be extended to any bounded partially ordered set.

Definition 4.2. [8] *Let A be a set and $\leq_A$ be a partial order in A such that $(A,\leq_A)$ has a minimum element $\mathrm{Min}_{\leq_A}$ and a maximum element $\mathrm{Max}_{\leq_A}$. A negation in $(A,\leq_A)$ is a function $N:A\to A$ such that N is decreasing, $N(\mathrm{Min}_{\leq_A})=\mathrm{Max}_{\leq_A}$ and $N(\mathrm{Max}_{\leq_A})=\mathrm{Min}_{\leq_A}$. If , additionally, $N(N(x))=x$ holds for all $x\in A$, it is said to be a strong negation.*

Definition 4.3. *Let $A=\{a_1,a_2,...,a_n\}\in H$, and n be a negation in $[0,1]$. The operation N_n associated with n is defined as*

$$N_n(A)=\{n(a_n),n(a_{n-1}),...,n(a_1)\}. \tag{6}$$

Lemma 4.4. *Let $A=\{a_1,a_2,...,a_n\}\in H$, and n is a negation in $[0,1]$. Rewriting $A_r=(\overbrace{a_1,...,a_1}^{rtimes},\overbrace{a_2,...,a_2}^{rtimes},...,\overbrace{a_n,...,a_n}^{rtimes})=(c_1,c_2,...,c_{rn})$, $r\in N$,*

$$N_n(A)=\{n(c_{rn}),...,n(c_2),n(c_1)\}. \tag{7}$$

Proposition 4.5. *Let $N_n:H\to H$ be the operation associated with a negation n in $[0,1]$. Then N_n is a negation in $(H,\leq_H)$.*

Proof. (i) $N_n(\{0\})=\{1\}$ and $N_n(\{1\})=\{0\}$ are easily proved by definition of N_n.
(ii) Let $=\{a_1,a_2,...,a_n\}$ and $B=\{b_1,b_2,...,b_m\}$ be any two elements of H. Rewriting $A_{\frac{lcm(n,m)}{n}}=(c_1,c_2,...,c_{lcm(n,m)})$ and $B_{\frac{lcm(n,m)}{m}}=(d_1,d_2,...,d_{lcm(n,m)})$, applying Lemma 4.4, so $A\leq_H B\Rightarrow c_i\leq d_i\Rightarrow n(c_i)\geq n(d_i)\Rightarrow N_n(B)\leq_H N_n(A)$. Hence, N_n is a negation on $(H,\leq_H)$. □

Proposition 4.6. *Let $N_n:H\to H$ be the operation associated with a negation n in $[0,1]$. Then N_n is involutive, that is, $N_n(N_n(A))=A$, $\forall A\in H$ if and only if n is strong.*

Proof. " $\Leftarrow$ " if n is strong, then $n(n(x))=x$, $\forall x\in[0,1]$. Hence, $\forall A=\{a_1,a_2,...,a_n\}$ $\in H$,

$$N_n(N_n(A))=N_n(\{n(a_n),n(a_{n-1}),...,n(a_1)\})=\{n(n(a_1)),...,n(n(a_{n-1})),n(n(a_n))\}=\{a_1,a_2,...,a_n\}=A.$$

" $\Rightarrow$ " if negation n is not strong, there exists $u\in[0,1]$ such that $n(n(u))\neq u$. Without loss of generality, assuming u is unique, let $B=\{u,b_2,...,b_m\}\in H$ and $n(n(b_i))=b_i$ for all $i=2,3,...,m$. Then $N_n(N_n(B))=\{n(n(u)),b_2,...,b_m\}\neq B$. It is contradictory with N_n which is involutive. □

Proposition 4.7. *Let $N_n : H \to H$ be the operation associated with a negation n in $[0,1]$. Then, N_n is a strong negation in H if and only if n is strong.*

Proof. It is directly proved from Propositions 4.5 and 4.6. □

Proposition 4.8. *Let $N_n : H \to H$ be the operation associated with an injective negation n in $[0,1]$. Then N_n is a negation in H_R.*

Proof. On the one hand N_n is closed in H_R because n is an injective negation, on the other hand H_R is the subset of the H, therefore from Proposition 4.5, the N_n is a negation in H_R. □

Remark 4.9. *In Proposition 4.8, n must be injective, otherwise N_n may not be closed in H_R. Indeed, we have the following counter-example: Assuming*

$$n(x) = \begin{cases} 1-2x, & 0 \le x \le 0.5; \\ 0, & 0.5 \le x \le 1 \end{cases}$$

Obviously, n is a negation in $[0,1]$ but is not injective. Let $A = \{0.3, 0.6, 0.7\} \in H_R$. Then $N_n(A) = \{n(0.7), n(0.6), n(0.3)\}$ $= \{0, 0, 0.4\} = \{0, 0.4\}$, it follows from the definition of H_R that $N_n(A) = \{0, 0.4\} \notin H_R$.

Proposition 4.10. *Let $N_n : H \to H$ be the operation associated with an injective negation n in $[0,1]$. Then N_n is strong in H_R if and only if n is strong in $[0,1]$.*

Proof. Since H_R is the subset of H, from Proposition 4.6, N_n is involutive in H_R if and only if n is strong. Additionally, from Proposition 4.8, N_n is strong in H_R if and only if n is strong in $[0,1]$. □

On $(\mathbf{\bar{B}}, \le_B)$ we can also derive a negation operation μ_n from a negation operation n on $[0,1]$ in a similar way as in Definition 4.3.

Definition 4.11. *Let n be a negation operation in $[0,1]$. The operation μ_n on $\bar{B}$ is defined by the following way: for any element $[\vec{u}] = [(a_1, a_2, ..., a_n)] \in \bar{B}$,*

$$\mu_n([\vec{u}]) = [(n(a_n), n(a_{n-1}), ..., n(a_1))]. \tag{8}$$

Lemma 4.12. *Let $[\vec{u}] = [(a_1, a_2, ..., a_n)] \in \bar{B}$, and n be a negation in $[0,1]$. Rewriting $\vec{u}_r = (\overbrace{a_1, ..., a_1}^{rtimes}, \overbrace{a_2, ..., a_2}^{rtimes}, ..., \overbrace{a_n, ..., a_n}^{rtimes}) = (c_1, c_2, ..., c_{rn})$, $r \in N$,*

$$\mu_n([\vec{u}]) = [(n(c_{rn}), ..., n(c_2), n(c_1))]. \tag{9}$$

Proposition 4.13. *Let $\mu_n : \bar{B} \to \bar{B}$ be the operation associated with a negation n in $[0,1]$. Then μ_n is a negation in $(\bar{B}, \le_B)$.*

Proof. (1) $\mu_n([\vec{0}]) = [\vec{1}]$ and $\mu_n([\vec{1}]) = [\vec{0}]$ are easily proved by definition of μ_n.
(2) Let $[\vec{u}] = [(a_1, a_2, ..., a_n)]$ and $[\vec{v}] = [(b_1, b_2, ..., b_m)]$ be any two elements of $\bar{B}$. Rewriting $\vec{u}_{\frac{lcm(n,m)}{n}} = (c_1, c_2, ..., c_{lcm(n,m)})$ and $\vec{v}_{\frac{lcm(n,m)}{m}} = (d_1, d_2, ..., d_{lcm(n,m)})$, applying Lemma 4.12, so $\vec{u} \le_B \vec{v} \Rightarrow c_i \le d_i \Rightarrow n(c_i) \ge n(d_i) \Rightarrow \mu_n([\vec{v}]) \le_B \mu_n([\vec{u}])$. Hence, μ_n is a negation on $(\bar{B}, \le_B)$. □

Proposition 4.14. *Let $\mu_n : \bar{B} \to \bar{B}$ be the operation associated with a negation n in $[0,1]$. Then μ_n is involutive, that is, $\mu_n(\mu_n([\vec{u}])) = [\vec{u}]$, $\forall [\vec{u}] \in \bar{B}$ if and only if n is strong.*

Proof. " $\Leftarrow$ " if n is strong, then $n(n(x)) = x$, $\forall x \in [0,1]$. For all $[\vec{u}] = [(a_1, a_2, ..., a_n)] \in \bar{B}$,

$$\mu_n(\mu_n([\vec{u}])) = \mu_n([(n(a_n), n(a_{n-1}), ..., n(a_1))]) = [(n(n(a_1)), ..., n(n(a_{n-1})), n(n(a_n)))] = [(a_1, a_2, ..., a_n)] = [\vec{u}].$$

" $\Rightarrow$ " if negation n is not strong, there exist $u \in [0,1]$ such that $n(n(u)) \ne u$. Without loss of generality, assuming u is unique, let $[\vec{v}] = [(u, b_2, ..., b_m)] \in \bar{B}$ and $n(n(b_i)) = b_i$ for all $i = 2, 3, ..., m$. Then $\mu_n(\mu_n([\vec{v}])) = [(n(n(u)), b_2, ..., b_m)] \ne [\vec{v}]$, it is contradictory with μ_n which is involutive. □

Proposition 4.15. *Let $\mu_n : \bar{B} \to \bar{B}$ be the operation associated with a negation n in $[0,1]$. Then μ_n is strong in $(\bar{B}, \le_B)$ if and only if n is strong.*

Proof. It is directly proved from Propositions 4.13 and 4.14. □

In [7], binary operations C on H and T on $\bar{B}$ associated with t-norm T in $[0,1]$ are defined respectively. In the following we will analyze whether C and D satisfies De Morgan's law or not with respect to N_n.

Proposition 4.16. *Let N_n be an operation associated with the strong negation n, T and S, t-norm and dual t-conorm, respectively, with respect to n. For any two elements $A = \{a_1, a_2, ..., a_n\}$ and $B = \{b_1, b_2, ..., b_m\}$ of H, rewriting $A_{\frac{lcm(n,m)}{n}} = (c_1, c_2, ..., c_{lcm(n,m)})$ and $B_{\frac{lcm(n,m)}{m}} = (d_1, d_2, ..., d_{lcm(n,m)})$, C and D are defined as*

$$C(A,B) = \{T(c_1,d_1), T(c_2,d_2), ..., T(c_{lcm(n,m)}, d_{lcm(n,m)})\},$$
$$D(A,B) = \{S(c_1,d_1), S(c_2,d_2), ..., S(c_{lcm(n,m)}, d_{lcm(n,m)})\},$$

it holds that

$$N_n(C(A,B)) = D(N_n(A), N_n(B)) \text{ and } N_n(D(A,B)) = C(N_n(A), N_n(B)). \tag{10}$$

Proof.

$$\begin{aligned} N_n(C(A,B)) &= \{n(T(c_{lcm(n,m)}, d_{lcm(n,m)})), ..., n(T(c_2,d_2)), n(T(c_1,d_1))\} \\ &= \{S(n(c_{lcm(n,m)}), n(d_{lcm(n,m)})), ..., S(n(c_2), n(d_2)), S(n(c_1), n(d_1))\} \\ &= D(N_n(A), N_n(B)). \end{aligned}$$

The proof of $N_n(D(A,B)) = C(N_n(A), N_n(B))$ is similar. □

Example 4.17. *Assuming $n(x) = 1-x$ and T and S are Product t-norm ($a \otimes b = ab$) and Product t-conorm ($a \oplus b = a+b-ab$) respectively. Obviously, T and S are dual with respect to the strong negation n. For two elements $A = \{0.1, 0.3\}$ and $B = \{0.2, 0.3, 0.4\}$ of H, rewriting $A_3 = \{0.1, 0.1, 0.1, 0.3, 0.3, 0.3\}$ and $B_2 = \{0.2, 0.2, 0.3, 0.3, 0.4, 0.4\}$. Then*

$$N_n(C(A,B)) = N_n(\{0.02, 0.02, 0.03, 0.09, 0.12, 0.12\}) = N_n(\{0.02, 0.03, 0.09, 0.12\}) = \{0.88, 0.91, 0.97, 0.98\}$$

and

$$D(N_n(A), N_n(B)) = D(\{0.7, 0.9\}, \{0.6, 0.7, 0.8\}) = \{0.88, 0.88, 0.91, 0.97, 0.98, 0.98\} = \{0.88, 0.91, 0.97, 0.98\}.$$

Clearly, $N_n(C(A,B)) = D(N_n(A), N_n(B))$. Similarly, $N_n(D(A,B)) = C(N_n(A), N_n(B))$.

Remark 4.18. *If n is not strong, the Proposition 4.16 may not be true. Indeed, we can show the following counter-example: Assuming*

$$n(x) = \begin{cases} 1-2x, & 0 \leq x \leq 0.4; \\ 0, & 0.4 < x \leq 1 \end{cases}$$

and although the other conditions of Example 4.17. Then

$$N_n(C(A,B)) = N_n(\{0.02, 0.02, 0.03, 0.09, 0.12, 0.12\}) = N_n(\{0.02, 0.03, 0.09, 0.12\}) = \{0.76, 0.82, 0.94, 0.96\}$$

and

$$D(N_n(A), N_n(B)) = D(\{0.4, 0.8\}, \{0.2, 0.4, 0.6\}) = \{0.52, 0.52, 0.64, 0.88, 0.92, 0.92\} = \{0.52, 0.64, 0.88, 0.92\}.$$

Clearly, $N_n(\boldsymbol{C}(A,B)) \neq D(N_n(A), N_n(B))$.

Proposition 4.19. *Let μ_n be an operation associated with the strong negation n, T and S, t-norm and dual t-conorm, respectively, with respect to n. For two any elements $[\vec{u}] = [(a_1, a_2, ..., a_n)]$ and $[\vec{v}] = [(b_1, b_2, ..., b_m)]$ of $\bar{B}$, rewriting $\vec{u}_{\frac{lcm(n,m)}{n}} = (c_1, c_2, ..., c_{lcm(n,m)})$ and $\vec{v}_{\frac{lcm(n,m)}{m}} = (d_1, d_2, ..., d_{lcm(n,m)})$,$T$ and S are defined as*

$$T([\vec{u}], [\vec{v}]) = [(T(c_1,d_1), T(c_2,d_2), ..., T(c_{lcm(n,m)}, d_{lcm(n,m)}))],$$
$$S([\vec{u}], [\vec{v}]) = [(S(c_1,d_1), S(c_2,d_2), ..., S(c_{lcm(n,m)}, d_{lcm(n,m)}))],$$

it holds that

$$\mu_n(T([\vec{u}], [\vec{v}])) = S(\mu_n([\vec{u}]), \mu_n([\vec{v}])) \text{ and } \mu_n(S([\vec{u}], [\vec{v}])) = T(\mu_n([\vec{u}]), \mu_n([\vec{v}])). \tag{11}$$

Proof.

$$\begin{aligned}\mu_n(T([\vec{u}],[\vec{v}])) &= [(n(T(c_{lcm(n,m)},d_{lcm(n,m)})),...,n(T(c_2,d_2)),n(T(c_1,d_1)))] \\ &= [(S(n(c_{lcm(n,m)}),n(d_{lcm(n,m)})),...,S(n(c_2),n(d_2)),S(n(c_1),n(d_1)))] \\ &= S(\mu_n([\vec{u}]),\mu_n([\vec{v}])).\end{aligned}$$

The proof of $\mu_n(S([\vec{u}],[\vec{v}])) = T(\mu_n([\vec{u}]),\mu_n([\vec{v}]))$ is similar. □

Example 4.20. Assuming $n(x) = 1 - x$ and T and S are Łukasiewicz t-norm ($a \otimes_{Lu} b = (a + b - 1) \vee 0$) and Łukasiewicz t-conorm ($a \oplus_{Lu} b = (a + b) \wedge 1$) respectively. Obviously, T and S are dual with respect to the strong negation n. For two elements $[\vec{u}] = [(0.5, 0.7)]$ and $[\vec{v}] = [(0.5, 0.6, 0.8)]$ of $\bar{B}$, rewriting $[\vec{u}_3] = [(0.5, 0.5, 0.5, 0.7, 0.7, 0.7)]$ and $[\vec{v}_2] = [(0.5, 0.5, 0.6, 0.6, 0.8, 0.8)]$. Then

$$\mu_n(T([\vec{u}],[\vec{v}])) = \mu_n([(0,0,0.1,0.3,0.5,0.5)]) = [(0.5,0.5,0.7,0.9,1,1)]$$

and

$$S(\mu_n([\vec{u}]),\mu_n(\vec{v})) = S([(0.3,0.5)],[(0.2,0.4,0.5)]) = [(0.5,0.5,0.7,0.9,1,1)].$$

Clearly, $\mu_n(T([\vec{u}],[\vec{v}])) = S(\mu_n([\vec{u}]),\mu_n(\vec{v}))$. Similarly, $\mu_n(S([\vec{u}],[\vec{v}])) = T(\mu_n([\vec{u}]),\mu_n([\vec{v}]))$.

Remark 4.21. *If n is not strong, Proposition 4.19 may not be true. Indeed, we have the following counter-example: Assuming*

$$n(x) = \begin{cases} 1-2x, & 0 \le x \le 0.4; \\ 0, & 0.4 < x \le 1 \end{cases},$$

and also the other conditions of Example 4.20. Then

$$\mu_n(T([\vec{u}],[\vec{v}])) = \mu_n([(0,0,0.1,0.3,0.5,0.5)]) = [(0,0,0.4,0.8,1,1)]$$

and

$$S(\mu_n([\vec{u}]),\mu_n([\vec{v}])) = S([(0,0)],[(0,0,0)]) = [(0,0,0,0,0,0)].$$

Clearly, $\mu_n(T([\vec{u}],[\vec{v}])) \neq S(\mu_n([\vec{u}]),\mu_n(\vec{v}))$.

5 Aggregation operations on $(H, \le_H)$ and $(\bar{B}, \le_B)$

In this section, new aggregation operations will be proposed on $(H, \le_H)$ and $(\bar{B}, \le_B)$ and they will be extended to more general form. Next, let us recall some conceptions about aggregation operation.

Definition 5.1. [5] *An n-ary aggregation function is an increasing function $M : [0,1]^n \to [0,1]$ such that $M(0,...,0) = 0$ and $M(1,...,1) = 1$.*

This definition can be extended to the bounded partially ordered sets.

Definition 5.2. [19] *Let A be a set and $\le_A$ be a partial order in A such that $(A, \le_A)$ has a minimum element $\text{Min}_{\le_A}$ and a maximum element $\text{Max}_{\le_A}$. An aggregation operation in $(A, \le_A)$ is a function $M : A^n \to A$ such that M is increasing, $M(\text{Min}_{\le_A},...,\text{Min}_{\le_A}) = \text{Min}_{\le_A}$ and $M(\text{Max}_{\le_A},...,\text{Max}_{\le_A}) = \text{Max}_{\le_A}$.*

Definition 5.3. Let M be an n-ary aggregation function in $[0,1]$. For $A_i = \{a_1^i, a_2^i, ..., a_{m_i}^i\} \in H$ and $e = lcm(m_1, m_2, ..., m_n)$, rewriting $(A_i)_{\frac{e}{m_i}} = (c_1^i, c_2^i, ..., c_e^i)$, $i = 1, 2, ..., n$, an n-ary function $M : H^n \to H$ is defined as

$$M(A_1, A_2, ..., A_n) = \{M(c_1^1,...,c_1^n), M(c_2^1,...,c_2^n), ..., M(c_e^1,...,c_e^n)\}. \tag{12}$$

Lemma 5.4. *Let M be an n-ary aggregation function in $[0,1]$. For $A_i = \{a_1^i, a_2^i, ..., a_{m_i}^i\} \in H$ and $e = lcm(m_1, m_2, ..., m_n)$, rewriting $(A_i)_{r\frac{e}{m_i}} = (c_1^i, c_2^i, ..., c_{re}^i)$, $i = 1, 2, ..., n$, $r \in N$,*

$$M(A_1, A_2, ..., A_n) = \{M(c_1^1,...,c_1^n), M(c_2^1,...,c_2^n), ..., M(c_{re}^1,...,c_{re}^n)\}. \tag{13}$$

Proposition 5.5. *Let $M : H^n \to H$ be the n-ary operation associated with an aggregation function M in $[0,1]$. Then, M is an aggregation operation in H.*

Proof. (i) $M(\{0\},\{0\},...,\{0\}) = \{0\}$ and $M(\{1\},\{1\},...,\{1\}) = \{1\}$.

(ii) Assume $A_i = \{a^i_1, a^i_2, ..., a^i_{m_i}\}$ and $B_i = \{b^i_1, b^i_2, ..., b^i_{p_i}\}$ are elements of H for all $i = 1,2,...,n$. Let $\delta = lcm(m_1, m_2, ..., m_n)$, $\varepsilon = lcm(p_1, p_2, ..., p_n)$ and $\eta = lcm(\delta, \varepsilon)$, rewriting $(A_i)_{\frac{\eta}{m_i}} = (c^i_1, c^i_2, ..., c^i_\eta)$ and $(B_i)_{\frac{\eta}{p_i}} = (d^i_1, d^i_2, ..., d^i_\eta)$, then from Lemma 5.4

$$M(A_1, A_2, ..., A_n) = \{M(c^1_1, ..., c^n_1), M(c^1_2, ..., c^n_2), ..., M(c^1_\eta, ..., c^n_\eta)\}$$

and

$$M(B_1, B_2, ..., B_n) = \{M(d^1_1, ..., d^n_1), M(d^1_2, ..., d^n_2), ..., M(d^1_\eta, ..., d^n_\eta)\}.$$

For all $i = 1,2,...,n$, $A_i \leq_H B_i \Rightarrow c^i_j \leq d^i_j \Rightarrow M(c^1_j, ..., c^n_j) \leq M(d^1_j, ..., d^n_j)$ for all $j = 1,2,...,\eta$. Hence, $M(A_1, A_2, ..., A_n) \leq_H M(B_1, B_2, ..., B_n)$. So M is an aggregation function in H. ☐

Example 5.6. *Let $A_1 = \{0.2, 0.3\}$, $A_2 = \{0.4, 0.5\}$ and $A_3 = \{0.1, 0.3, 0.4\}$ be elements of H, and $M(x_1, x_2, x_3) = \frac{x_1+x_2+x_3}{3}$. Rewriting*

$$(A_1)_3 = \{0.2, 0.2, 0.2, 0.3, 0.3, 0.3\}, \quad (A_2)_3 = \{0.4, 0.4, 0.4, 0.5, 0.5, 0.5\}, \quad and \quad (A_3)_2 = \{0.1, 0.1, 0.3, 0.3, 0.4, 0.4\}.$$

Then $M(A_1, A_2, A_3) = \{M(0.2, 0.4, 0.1), M(0.2, 0.4, 0.1), M(0.2, 0.4, 0.3), M(0.3, 0.5, 0.3), M(0.3, 0.5, 0.4), M(0.3, 0.5, 0.4)\}$ $= \{\frac{7}{30}, \frac{7}{30}, 0.3, \frac{11}{30}, 0.4, 0.4\} = \{\frac{7}{30}, 0.3, \frac{11}{30}, 0.4\}$.

Corollary 5.7. *Let $M_1 \leq M_2 \leq ... \leq M_n \leq ...$ be a sequence of n-ary aggregation functions in $[0,1]$. For $A_i = \{a^i_1, a^i_2, ..., a^i_{m_i}\} \in H$ and $e = lcm(m_1, m_2, ..., m_n)$, rewriting $(A_i)_{\frac{e}{m_i}} = (c^i_1, c^i_2, ..., c^i_e)$, $i = 1,2,...,n$, an n-ary function $M : H^n \to H$ is an aggregation operation in H if M is defined as*

$$M(A_1, A_2, ..., A_n) = \{M_1(c^1_1, ..., c^n_1), M_2(c^1_2, ..., c^n_2), ..., M_e(c^1_e, ..., c^n_e)\}. \tag{14}$$

Proof. The proof is similar to that of Proposition 5.5. ☐

Proposition 5.8. *Let $M : H^n \to H$ be the n-ary operation associated with an injective aggregation function M in $[0,1]$. Then M is an aggregation operation in H_R.*

Proof. Firstly, M is closed in H_R because M is an injective aggregation function M in $[0,1]$. Secondly H_R is the subset of H. Therefore, it follows from Proposition 5.5 that M is an aggregation operation in H_R. ☐

Remark 5.9. *In Proposition 5.8, the M must be injective, otherwise it may be not closed in H_R. In fact, we can show the following counter-example: Assuming*

$$f(u,v) = \begin{cases} u, & v = 1 \\ v, & u = 1 \\ 0, & otherwise \end{cases},$$

it is a t-norm in $[0,1]$, so it is an aggregation function, which is not injective. Let $A = \{0.1, 1\}$ and $B = \{0.1, 0.3, 0.4, 1\}$ be two elements of H_R, rewriting $A_2 = \{0.1, 0.1, 1, 1\}$. Then $M(A,B) = \{f(0.1, 0.1), f(0.1, 0.3), f(1, 0.4), f(1,1)\} = \{0, 0, 0.4, 1\} = \{0, 0.4, 1\}$, but $\{0, 0.4, 1\} \notin H_R$ here.

On $(\bar{B}, \leq_B)$ we can also derive an n-ary operation G from an aggregation function M on $[0,1]$ in a similar way as in Definition 5.3.

Definition 5.10. *Let M be an n-ary aggregation function in $[0,1]$. For $[\vec{u_i}] = [(a^i_1, a^i_2, ..., a^i_{m_i})] \in \bar{B}$ and $e = lcm(m_1, m_2, ..., m_n)$, rewriting $(\vec{u_i})_{\frac{e}{m_i}} = (c^i_1, c^i_2, ...,$
$c^i_e)$, $i = 1,2,...,n$. An n-ary function $G : \bar{B}^n \to \bar{B}$ is defined as*

$$G([\vec{u_1}], [\vec{u_2}], ..., [\vec{u_n}]) = [(M(c^1_1, ..., c^n_1), M(c^1_2, ..., c^n_2), ..., M(c^1_e, ..., c^n_e))]. \tag{15}$$

Proposition 5.11. *Let $G : \bar{B}^n \to \bar{B}$ be the n-ary operation associated with an aggregation function M on $[0,1]$. Then G is an aggregation operation in $\bar{B}$.*

Proof. (i) $G([\vec{0}],[\vec{0}],...,[\vec{0}])=[\vec{0}]$ and $G([\vec{1}],[\vec{1}],...,[\vec{1}])=[\vec{1}]$.
(ii) Assume that $[\vec{u_i}]=[(a_1^i,a_2^i,...,a_{m_i}^i)]$ and $[\vec{v_i}]=[(b_1^i,b_2^i,...,b_{p_i}^i)]$ are elements of $\bar{B}$, for all $i=1,2,...,n$. Let $\delta=lcm(m_1,m_2,...,m_n)$, $\varepsilon=lcm(p_1,p_2,...,p_n)$ and $\eta=lcm(\delta,\varepsilon)$, rewriting $(\vec{u_i})_{\frac{\eta}{m_i}}=(c_1^i,c_2^i,...,c_\eta^i)$ and $(\vec{v_i})_{\frac{\eta}{p_i}}=(d_1^i,d_2^i,...,d_\eta^i)$, then

$$G([\vec{u_1}],[\vec{u_2}],...,[\vec{u_n}])=[(M(c_1^1,...,c_1^n),M(c_2^1,...,c_2^n),...,M(c_\eta^1,...,c_\eta^n))]$$

and

$$G([\vec{v_1}],[\vec{v_2}],...,[\vec{v_n}])=[(M(d_1^1,...,d_1^n),M(d_2^1,...,d_2^n),...,M(d_\eta^1,...,d_\eta^n))].$$

For all $i=1,2,...,n$, $\vec{u_i}\le_B\vec{v_i}\Rightarrow c_j^i\le d_j^i\Rightarrow M(c_j^1,...,c_j^n)\le M(d_j^1,...,d_j^n)$ for all $j=1,2,...,\eta$. Hence, $G([\vec{u_1}],[\vec{u_2}],...,[\vec{u_n}])$ $\le_B G([\vec{v_1}],[\vec{v_2}],...,[\vec{v_n}])$. So G is an aggregation function in $\bar{B}$. □

Example 5.12. Let $[\vec{u_1}]=[(0.1,0.2)]$, $[\vec{u_2}]=[(0.1,0.3)]$ and $[\vec{u_3}]=[(0.1,0.3,0.3)]$ be elements of $\bar{B}$, and $M(x_1,x_2,x_3)=\frac{x_1+x_2+x_3}{3}$. Rewriting

$$(\vec{u_1})_3=(0.1,0.1,0.1,0.2,0.2,0.2),\quad (\vec{u_2})_3=(0.1,0.1,0.1,0.3,0.3,0.3),\text{ and }(\vec{u_3})_2=(0.1,0.1,0.3,0.3,0.3,0.3).$$

Then $G([\vec{u_1}],[\vec{u_2}],[\vec{u_3}])=[(M(0.1,0.1,0.1),M(0.1,0.1,0.1),M(0.1,0.1,0.3),M(0.2,0.3,0.3),M(0.2,0.3,0.3),M(0.3,0.3,0.3))]=[(0.1,0.1,\frac{1}{6},\frac{8}{30},\frac{8}{30},0.3)]$.

Corollary 5.13. *Let $M_1\le M_2\le...\le M_n\le...$ be a sequence of n-ary aggregation functions in $[0,1]$. For $[\vec{u_i}]=[(a_1^i,a_2^i,...,a_{m_i}^i)]\in\{\bar{B}$ and $e=lcm(m_1,m_2,...,m_n)$, rewriting $(\vec{u_i})_{\frac{e}{m_i}}=(c_1^i,c_2^i,...,c_e^i)$, $i=1,2,...,n$. An n-ary function $G:\{\bar{B}^n\to\{\bar{B}$ is an aggregation operation in $\{\bar{B}$ if $\{G$ is defined as*

$$\{G([\vec{u_1}],[\vec{u_2}],...,[\vec{u_n}])=[(M_1(c_1^1,...,c_1^n),M_2(c_2^1,...,c_2^n),...,M_n(c_e^1,...,c_e^n))].\tag{16}$$

Proof. The proof is similar to that of Proposition 5.11. □

6 Decision making based on hesitant fuzzy information

In this section, we will utilize a special aggregation operation M and the score function value of hesitant fuzzy element [24](i.e. score function value of hesitant fuzzy element h is $s(h)=\frac{1}{\sharp h}\sum_{\gamma\in h}\gamma$, where $\sharp h$ is the number of elements in h, and for two hesitant fuzzy elements $h_1,h_2\in H$, if $s(h_1)<s(h_2)$, then $h_1<h_2$; if $s(h_1)=s(h_2)$, then $h_1=h_2$.) to deal with decision making based on hesitant fuzzy information.

Example 6.1. [11, 24] *The enterprise's board of directors, which includes five members, is to plan the development of large projects (strategy initiatives) for the following five years. Suppose there are four possible projects $Y_i(i=1,2,3,4)$ to be evaluated. It is necessary to compare these projects to select the most important of them as well as order them from the point of view of their importance, taking into account four attributes suggested by the Balanced Scorecard methodology (it should be noted that all of them are of the maximization type):G_1: financial perspective, G_2: the customer satisfaction, G_3: internal business process perspective, and G_4: learning and growth perspective. And suppose that the weight vector of the attributes is* $\mathrm{w}=(0.2,0.3,0.15,0.35)^T$.

In the following, we use the proposed method to determine the optimal project.

*Step*1. The decision matrix $H=(h_{ij})_{4\times4}$ is given in *Table*1, where $h_{ij}(i,j=1,2,3,4)$ is in the form of hesitant fuzzy elements.

*Step*2. Let $M(x_1,x_2,x_3,x_4)=0.2\times x_1+0.3\times x_2+0.15\times x_3+0.35\times x_4$. Then by utilizing aggregation operation M of Definition 23, the hesitant fuzzy elements $h_i(i=1,2,3,4)$ for the projects $Y_i(i=1,2,3,4)$ could be obtained. That is,

$$\begin{aligned}h_1=M(h_{11},h_{12},h_{13},h_{14})&=M((0.2,0.4,0.7),(0.2,0.6,0.8),(0.2,0.3,0.6,0.7,0.9),(0.3,0.4,0.5,0.7,0.8))\\&=(0.235,0.235,0.235,0.285,0.285,0.445,0.525,0.525,0.525,0.61,0.73,0.73,0.795,0.795,0.795)\\&=(0.235,0.285,0.445,0.525,0.61,0.73,0.795),\end{aligned}$$

$$\begin{aligned}h_2=M(h_{21},h_{22},h_{23},h_{24})&=M((0.2,0.4,0.7,0.9),(0.1,0.2,0.4,0.5),(0.3,0.4,0.6,0.9),(0.5,0.6,0.8,0.9))\\&=(0.29,0.41,0.63,0.78),\end{aligned}$$

$$h_3 = M(h_{31}, h_{32}, h_{33}, h_{34}) = M((0.3, 0.5, 0.6, 0.7), (0.2, 0.4, 0.5, 0.6), (0.3, 0.5, 0.7, 0.8), (0.2, 0.5, 0.6, 0.7)) \\ = (0.235, 0.47, 0.585, 0.685),$$

$$h_4 = M(h_{41}, h_{42}, h_{43}, h_{44}) = M((0.3, 0.5, 0.6), (0.2, 0.4), (0.5, 0.6, 0.7), (0.8, 0.9)) \\ = (0.475, 0.475, 0.53, 0.625, 0.66, 0.66) = (0.475, 0.53, 0.625, 0.66).$$

Step 3. Calculate the score values $s(h_i)(i = 1, 2, 3, 4)$ of h_i:

$$s(h_1) = 0.5179, s(h_2) = 0.5275, s(h_3) = 0.4938, s(h_4) = 0.5725.$$

Step 4. By ranking $s(h_i)(i = 1, 2, 3, 4)$ of h_i, $h_4 > h_2 > h_1 > h_3$, which means that Y_4 is the optimal project. This result is consistent with the conclusion in [24], so the validity of this method is verified.

Table 1 Hesitant fuzzy decision matrix

	G_1	G_2	G_3	G_4
Y_1	$(0.2, 0.4, 0.7)$	$(0.2, 0.6, 0.8)$	$(0.2, 0.3, 0.6, 0.7, 0.9)$	$(0.3, 0.4, 0.5, 0.7, 0.8)$
Y_2	$(0.2, 0.4, 0.7, 0.9)$	$(0.1, 0.2, 0.4, 0.5)$	$(0.3, 0.4, 0.6, 0.9)$	$(0.5, 0.6, 0.8, 0.9)$
Y_3	$(0.3, 0.5, 0.6, 0.7)$	$(0.2, 0.4, 0.5, 0.6)$	$(0.3, 0.5, 0.7, 0.8)$	$(0.2, 0.5, 0.6, 0.7)$
Y_4	$(0.3, 0.5, 0.6)$	$(0.2, 0.4)$	$(0.5, 0.6, 0.7)$	$(0.8, 0.9)$

7 Conclusions

The paper has introduced two binary operations D on $(H, \leq_H)$ and S on $(\bar{B}, \leq_B)$ respectively. It is shown that D is not a t-conorm on $(H, \leq_H)$. However, D is a t-conorm on $(H_R, \leq_H)$ and S is a t-conorm on $(\bar{B}, \leq_B)$. In addition, we have proposed two negations N_n on $(H, \leq_H)$ and μ_n on $(\bar{B}, \leq_B)$ respectively and investigated their De Morgan's laws with respect to C and D on $(H, \leq_H)$ as well as T and S on $(\bar{B}, \leq_B)$. Two aggregation operations and their general forms have been provided respectively on $(H, \leq_H)$ and on $(\bar{B}, \leq_B)$. Moreover, the validity of the aggregation operation on $(H, \leq_H)$ has been illustrated by a numerical example on decision making.

The next research directions are to find reasonable ways to construct triangular norms on H from triangular norms on $[0, 1]$ and investigate the application of negations and aggregation operators based on the hesitant fuzzy partial ordering introduced by [7].

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