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The Wijsman structure of a quantale-valued metric space

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Abstract

We define and study a quantale-valued Wijsman structure on the hyperspace of all non-empty closed sets of a quantale-valued metric space. We show its admissibility and that the metrical coreflection coincides with the quantale-valued Hausdorff metric and that, for a metric space, the topological coreflection coincides with the classical Wijsman topology. We further define an index of compactness and show that the indices of compactness of the quantale-valued metric space and of the hyperspaces equipped with the quantale-valued Hausdorff metric and with the quantale-valued Wijsman structure coincide.

Keywords: L-gauge space, L-metric space, L-Hausdorff metric, L-hyperspace, Wijsman topology.

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1 Introduction

In [21] a convergence notion for closed, non-empty subsets of a metric space (X, d) was introduced, which proved to be fundamental in the study of hyperspaces [3]. A sequence of such sets A_n converges to a set A if the sequence of values of the distance functionals $d(x, A_n)$ converges to the value of the distance functional $d(x, A)$ for each $x \in X$. This convergence concept is induced by a topology, the *Wijsman topology* on $CL(X)$, the set of all non-empty closed subsets of the metric space (X, d). Important results for the Wijsman topology are e.g. the following [3].

- The Wijsman topology on $CL(X)$ is the weak topology determined by the family of distance functionals $\{d(x, \cdot) : x \in X\}$.
- The Wijsman topology is admissible, Hausdorff and completely regular.
- The Wijsman topology is metrizable if and only if (X, d) is separable.
- If (X, d) is totally bounded, then the Wijsman topology is metrized by the Hausdorff metric.

In [7] a Wijsman topology on the hyperspace of all non-empty closed subsets of a fuzzy metric spaces was studied and some fundamental results were obtained. However, the authors of [7] admit that "the situation presents some differences with respect to the classical case of metric spaces as the Wijsman topology is not admissible". In this sense, the generalization of the classical Wijsman topology of a metric space to the case of fuzzy metric spaces is not quite satisfactory and the authors of [7] are looking for a better definition of the Wijsman topology in the realm of fuzzy metric spaces. In this paper, we give a possible solution. Firstly, noting that a fuzzy metric space in the sense of Kramosil and Michálek [13] can be identified with a quantale-valued metric space if we use the quantale of distance distribution functions, we present our theory in the – as we believe – more appropriate framework of quantale-valued metric spaces. Secondly, we follow an idea of Lowen [14, 15] and instead of using topologies we rather use the concept of quantale-valued gauges as introduced in [11]. A natural candidate for the Wijsman topology is then the topological coreflection of our quantale-valued Wisjman structure. We therefore, as a first step, have to develop the theory for

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the topological coreflection of a quantale-valued gauge space. Finally, in particular in the probabilistic case, i.e. in the fuzzy metric case, we show the appropriateness of our construction by showing

(1) that the quantale-valued Wijsman structure can be obtained in the natural way via initial constructions;

(2) the admissibility of the quantale-valued Wijsman structure;

(3) that the quantale-valued metrical coreflection is the quantale-valued Hausdorff metric as studied in the probabilistic resp. fuzzy case in [7, 20] and in the general quantale-valued case in [2];

(4) that in the case of Lawvere's quantale the topological coreflection is the classical Wijsman topology [3];

(5) that the connection with compactness yields generalizations of classical results to the quantale-valued setting.

The paper is organised as follows. In a preliminary section we fix the notation and collect the necessary theory about lattices and quantales. Then we study quantale-valued metric spaces and their open and closed sets in the next section. In Sections 4 and 5 we introduce quantale-valued gauge spaces and study their topological coreflection. In Section 6 we collect the necessary results about the quantale-valued Hausdorff metric on the set of all non-empty closed sets of a quantale-valued metric space. After these preparations, Section 7 presents the core of the paper and studies the quantale-valued Wijsman structure. The final Section 8 then generalizes the definitions of index of compactness from the theory of approach spaces [14] and obtains generalizations of classical results, such as the equivalence of the total boundedness of a metric space with the total boundedness of the hyperspace of non-empty closed subsets equipped with the Hausdorff metric.

2 Preliminaries

Let L be a complete lattice with $\top \neq \bot$ for the top element $\top$ and the bottom element $\bot$. In any complete lattice L, we can define the *well-below relation* $\alpha \lhd \beta$, α is *well-below* β, if for all subsets $D \subseteq L$ such that $\beta \leq \bigvee D$ there is $\delta \in D$ such that $\alpha \leq \delta$. Then $\alpha \leq \beta$ whenever $\alpha \lhd \beta$ and $\alpha \lhd \bigvee_{j\in J} \beta_j$ iff $\alpha \lhd \beta_i$ for some $i \in J$. A complete lattice is *completely distributive* if we have $\alpha = \bigvee\{\beta \,:\, \beta \lhd \alpha\}$ for any $\alpha \in L$. Similarly, we can define the *well-above relation*, β is *well-above* α, $\alpha \prec \beta$ if for all subsets $D \subseteq L$ such that $\bigwedge D \leq \alpha$ there is $\delta \in D$ with $\delta \leq \beta$. Then $\alpha \prec \beta$ implies $\alpha \leq \beta$ and $\bigwedge_{j\in J} \beta_j \prec \alpha$ iff $\beta_j \prec \alpha$ for some $j \in J$ and L is completely distributive iff $\alpha = \bigwedge\{\beta \in L \,:\, \alpha \prec \beta\}$ for any $\alpha \in L$. Clearly, in a complete lattice L we have $\alpha \lhd \beta$ iff $\beta \prec^{op} \alpha$ in the opposite order. For more results on lattices we refer to [5].

The triple $\mathsf{L} = (L, \leq, *)$, where $(L, \leq)$ is a complete lattice, is called a *commutative and integral quantale* if $(L, *)$ is a commutative semigroup, and $*$ is distributive over arbitrary joins, i.e. if $(\bigvee_{j\in J} \alpha_j) * \beta = \bigvee_{j\in J}(\alpha_j * \beta)$ and if $\alpha * \top = \top * \alpha = \alpha$ for all $\alpha, \beta_j \in L$, $j \in J$. In any such quantale we can define an implication $\alpha \to \beta = \bigvee\{\gamma \in L \,:\, \alpha * \gamma \leq \beta\}$. Then $\alpha * \beta \leq \gamma$ iff $\alpha \leq \beta \to \gamma$.

Example 2.1. *(1)* **t-norms.** *A* t-norm *is a binary operation $*$ on the unit interval $[0, 1]$ which is associative, commutative, non-decreasing in each argument and which has 1 as the unit. The triple $([0, 1], \leq, *)$ can be considered as a quantale if the t-norm is left-continuous. The three most commonly used (left-continuous) t-norms are the minimum t-norm, $\alpha * \beta = \alpha \wedge \beta$, the product t-norm, $\alpha * \beta = \alpha \cdot \beta$, and the Lukasiewicz t-norm, $\alpha * \beta = (\alpha + \beta - 1) \vee 0$.*

(2) **Lawvere's quantale.** *The interval $[0, \infty]$ with the opposite order and addition as the quantale operation $\alpha * \beta = \alpha + \beta$ (extended by $\alpha + \infty = \infty + a = \infty$ for all $\alpha, \beta \in [0, \infty]$) is a quantale, see e.g. [4].*

(3) **Distance distribution functions.** *A function $\varphi : [0, \infty] \longrightarrow [0, 1]$, which is non-decreasing, left-continuous on $(0, \infty)$ in the sense that $\varphi(x) = \bigvee\{\varphi(y) \,:\, y < x\}$ for all $x \in (0, \infty)$, and satisfies $\varphi(0) = 0$ is called a* distance distribution function [19]. *The set of all distance distribution functions is denoted by Δ^+. For example, for each $0 \leq t \leq \infty$ and $0 \leq \epsilon \leq 1$, the functions*

$$f_{t,\epsilon}(s) = \begin{cases} 0 & \textit{if } 0 \leq s \leq t \\ \epsilon & \textit{if } t < s \leq \infty \end{cases}$$

are in Δ^+. The set Δ^+ is ordered pointwise, i.e. for $\varphi, \psi \in \Delta^+$ we define $\varphi \leq \psi$ if for all $x \geq 0$ we have $\varphi(x) \leq \psi(x)$. The bottom element of Δ^+ is $\varepsilon_\infty = f_{\infty,0}$ and the top element is $\varepsilon_0 = f_{0,1}$ and the set Δ^+ with this order then becomes a completely distributive lattice [4].

A binary operation, $*: \Delta^+ \times \Delta^+ \longrightarrow \Delta^+$, which is commutative, associative, non-decreasing in each place and satisfies the boundary condition $\varphi * \varepsilon_0 = \varphi$ for all $\varphi \in \Delta^+$, is called a triangle function [19]. *A triangle function is called sup-continuous* [19], *if $(\bigvee_{i\in I} \varphi_i) * \psi = \bigvee_{i\in I}(\varphi_i * \psi)$ for all $\varphi_i, \psi \in \Delta^+$, $(i \in I)$, i.e. if $(\Delta^+, \leq, *)$ is a quantale.*

We will later use the triangle function $\circledast$ induced by a t-norm $*$, defined by $\varphi \circledast \psi(x) = \bigvee_{u+v=x} \varphi(u) * \psi(v)$ for all $x \in [0,\infty]$, see [19].

(4) **Frames.** *A frame is a quantale with $* = \wedge$.*

A *value quantale* [4] is a commutative and integral quantale $\mathsf{L} = (L, \leq, *)$ with an underlying completely distributive lattice $(L, \leq)$ such that $\alpha \vee \beta \lhd \top$ whenever $\alpha, \beta \lhd \top$. Examples for value quantales are $([0,\infty], \geq, +)$ or $(\Delta^+, \leq, *)$ with a sup-continuous triangle function, see [4]. It should be noted that Flagg [4] uses the opposite order.

We will consider in this paper only commutative, integral quantales L with completely distributive underlying lattices.

For a set X we denote the filters on X by $\mathbb{F}, \mathbb{G}, ...$ and the set of these filters by $\mathbb{F}(X)$. The set of all ultrafilters on X is denoted by $\mathbb{U}(X)$.

We assume some familiarity with category theory and refer to the textbooks [1] and [17] for more details and notation. In particular, we denote, for a category $\mathcal{C}$, by $|\mathcal{C}|$ the class of its objects. A *construct* is a category $\mathcal{C}$ with a faithful functor $U : \mathcal{C} \longrightarrow SET$, from $\mathcal{C}$ to the category of sets. We always consider a construct as a category whose objects are structured sets (S, ξ) and morphisms are suitable mappings between the underlying sets. A construct is called *topological* if it allows *initial constructions*, i.e. if for every source $(f_i : S \longrightarrow (S_i, \xi_i))_{i\in I}$ there is a unique structure ξ on S, such that a mapping $g : (T, \eta) \longrightarrow (S, \xi)$ is a morphism if and only if for each $i \in I$ the composition $f_i \circ g : (T, \eta) \longrightarrow (S_i, \xi_i)$ is a morphism.

3 L-metric spaces and their open and closed sets

In the sequel, let $\mathsf{L} = (L, \leq, *)$ be a commutative and integral quantale, where $(L, \leq)$ is completely distributive. For a set X we denote its power set by $P(X)$.

Definition 3.1. *An* L-metric space *is a pair (X, d) of a set X and an* L-metric $d : X \times X \longrightarrow L$ *which satisfies the following properties.*
(LM1) $d(x, x) = \top$ for all $x \in X$ (reflexivity)*, and*
(LM2) $d(x, y) * d(y, z) \leq d(x, z)$ for all $x, y, z \in X$(transitivity).
An L-metric space is called symmetric *if $d(x, y) = d(y, x)$ for all $x, y \in X$. It is called* separated *if $x = y$ whenever $d(x, y) = \top$.*
A mapping between two L-metric spaces, $f : (X, d_X) \longrightarrow (Y, d_Y)$ is called an L-metric morphism *if $d_X(x_1, x_2) \leq d_Y(f(x_1), f(x_2))$ for all $x_1, x_2 \in X$.*

We denote the category of L-metric spaces with L-metric morphisms by L-MET. We further denote the set of all L-metrics on a set X by L-$\mathsf{MET}(X)$. We note that for $d_j \in \mathsf{L}\text{-}\mathsf{MET}(X)$ $(j \in J)$, we have that the pointwise infimum $\bigwedge_{j\in J} d_j \in \mathsf{L}\text{-}\mathsf{MET}(X)$. As there is also a largest L-metric on X, namely $d(x, y) = \top$ for all $x, y \in X$, the set $\mathsf{L}\text{-}\mathsf{MET}(X)$ is a complete lattice.

In case $L = \{0, 1\}$, an L-metric space is a preordered set. In case of Lawvere's quantale, an L-metric space is a quasimetric space. If $\mathsf{L} = (\Delta^+, \leq, *)$, an L-metric space is a probabilistic quasimetric space, see [4].

Often, L-metric spaces are called *continuity spaces*, in particular if L is a value quantale [4]. Other names are L-*categories*, e.g. [9], L-*preordered sets*, e.g. [24], or, in the symmetric case, *fuzzy partially ordered sets* [22, 23].

Example 3.2. *A commutative and integral quantale $\mathsf{L} = (L, \leq, *)$ becomes a symmetric, separated L-metric space if we define $d_{\mathsf{L}}(\alpha, \beta) = (\alpha \to \beta) \wedge (\beta \to \alpha)$, $(x, y \in L)$. In the special case of Lawvere's quantale $\mathsf{L} = ([0,\infty], \geq, +)$ we have $\alpha \to \beta = (\beta - \alpha) \vee 0$ and $d_{\mathsf{L}}(\alpha, \beta) = |\alpha - \beta|$ is the standard metric on $[0,\infty]$. As the probabilistic case, $\mathsf{L} = (\Delta^+, \leq, \circledast)$, is important, we mention the implication in Δ^+ with respect to the quantale operation $\circledast$ and its relation to the implication on $[0, 1]$ with respect to the t-norm $*$. We have for $\varphi, \psi \in \Delta^+$, for each $t \in [0,\infty]$, see* [10, 8]*,*
$\varphi \overset{\circledast}{\to} \psi(t) = \sup_{s<t} \inf_u (\varphi(u) \overset{*}{\to} \psi(s + u))$. □

Example 3.3. [13] *Consider $\mathsf{L} = (\Delta^+, \leq, \circledast)$ with a left-continuous t-norm $*$. Then a symmetric and separated L-metric space (X, d) is a* probabilistic metric space *under the triangle function $\circledast$. We define $M_d : X \times X \times [0,\infty] \longrightarrow [0, 1]$ by $M_d(x, y, t) = d(x, y)(t)$. Then we have*

- $M_d(x,y,0) = 0$ *for all* $x, y \in X$*;*
- $M_d(x,y,t) = M_d(y,x,t)$*;*
- $M_d(x,y,t) * M_d(y,z,s) \le M_d(x,z,t+s)$*;*
- $M_d(x,y,t) = 1$ *for all* $t > 0$ *implies* $x = y$*;*
- $M_d(x,y,\cdot)$ *is left-continuous.*

These are the axioms of a fuzzy metric space in the definition of Kramosil and Michálek, a KM-fuzzy metric space*,* [13]*. Conversely, for a KM-fuzzy metric space* (X, M)*, if we define* $d_M(x,y) = M(x,y,\cdot)$*, then* (X, d_M) *is a symmetric and separated* L*-metric space. In this sense, we can identify KM-fuzzy metric spaces with* L*-metric spaces.*

We can define open and closed sets for an L-metric space (X, d).

For $\epsilon \lhd \top$ and $x \in X$ we define the *ϵ-ball at x* by $B^d(x,\epsilon) = \{y \in X \ : \ d(x,y) \rhd \epsilon\}$. If L is a value quantale, the set $\{B^d(x,\epsilon) \ : \ \epsilon \lhd \top\}$ is a filter basis and we call the generated filter, $\mathbb{U}^d_x$, the *neighbourhood filter of* x. We call $A \subseteq X$ *open* (in (X,d)) if for all $x \in A$ there is $\epsilon \lhd \top$ such that $B^d(x,\epsilon) \subseteq A$.

Proposition 3.4. [4] *Let* L *be a value quantale and* (X,d) *be an* L*-metric space. Let* $x \in X$ *and* $\epsilon \lhd \top$*. Then* $B^d(x,\epsilon)$ *is open.*

We denote τ_d the topology on X with basis $\{B^d(x,\epsilon) \ : \ x \in X, \epsilon \lhd \top\}$. Note that for $\epsilon = \bot$ we have $B^d(x,\bot) = X$. As $x \in B^d(x,\epsilon)$ for all $\epsilon \lhd \top$ we have $X = \bigcup_{x\in X} B^d(x,\epsilon)$, we have also $\{B^d(x,\epsilon) \ : \ x \in X, \bot \neq \epsilon \lhd \top\}$ as a basis for τ_d.

Remark 3.5. *We consider the case* $\mathsf{L} = (\Delta^+, \le, \circledast)$ *and a KM-fuzzy metric space* (X,M)*. The underlying topology,* τ_M*, has as basis the sets* $B^M(x,\epsilon,t) = \{y \in X \ : \ M(x,y,t) > 1-\epsilon\}$*,* $0 < \epsilon < 1$*,* $t > 0$*, see* [6]*. We show that* $\tau_M = \tau_{d_M}$*.*

Let $\varepsilon_\infty \neq \varphi \lhd \varepsilon_0$*. Then there are* $t \in [0,\infty]$ *and* $0 < \epsilon < 1$ *such that* $B^M(x,\epsilon,t) \subseteq B^{d_M}(x,\varphi)$*. To see this, we note that for* $\varphi \lhd \varepsilon_0$ *we have* $\varphi \le f_{t,1-\epsilon}$ *for some* $t > 0$ *and* $0 < \epsilon < 1$*, see e.g.* [11]*. For* $y \in B^M(x,\epsilon,t)$ *then* $d_M(x,y)(t) = M(x,y,t) > 1-\epsilon$ *which implies* $d_M(x,y) \rhd f_{t,1-\epsilon} \ge \varphi$*, i.e.* $y \in B^{d_M}(x,\varphi)$*.*

Conversely, if $t > 0$ *and* $0 < \epsilon < 1$*, there is* $\varphi \lhd \varepsilon_0$ *such that* $B^{d_M}(x,\varphi) \subseteq B_M(x,\epsilon,t)$*. To see this, we choose* $\varphi = f_{t,1-\epsilon} \lhd \varepsilon_0$*, see Lemma 2.11(5) in* [11]*. If* $y \in B^{d_M}(x,\varphi)$*, then* $d_M(x,y) \rhd f_{t,1-\epsilon}$*. Hence* $M(x,y,t) = d_M(x,y)(t) > 1-\epsilon$*, i.e.* $y \in B^M(x,\epsilon,t)$*.*

From this it is not difficult to show that $A \subseteq X$ *is open in* (X,τ_M) *if and only if* A *is open in* (X,τ_{d_M})*.*

For an L-metric space (X,d) we further define $\overline{c^d} = (c^d_\alpha : \mathbb{F}(X) \longrightarrow P(X))_{\alpha\in L}$ by

$$x \in c^d_\alpha(\mathbb{F}) \iff \bigvee_{F\in\mathbb{F}} \bigwedge_{y\in F} d(x,y) \ge \alpha.$$

Then $(X, \overline{c^d})$ is an L-convergence tower space [12]. In particular, we have

$$x \in c^d_\top(\mathbb{F}) \iff \forall \epsilon \lhd \top \exists F_\epsilon \in \mathbb{F} \text{ s.t. } \forall y \in F_\epsilon : d(x,y) \rhd \epsilon \iff \mathbb{F} \ge \mathbb{U}^d_x.$$

For $A \subseteq X$ we define the d-closure, $\overline{A}^d$, of A by

$$x \in \overline{A}^d \iff \exists \mathbb{F} \in \mathbb{F}(X) \text{ such that } A \in \mathbb{F}, x \in c^d_\top(\mathbb{F}).$$

Proposition 3.6. *Let* L *be a value quantale, let* (X,d) *be an* L*-metric space and let* $A \subseteq X$*. Then* $x \in \overline{A}^d$ *iff* $\bigvee_{a\in A} d(x,a) = \top$*.*

Proof. Let $x \in \overline{A}^d$. Then there is $\mathbb{F} \in \mathbb{F}(X)$ with $A \in \mathbb{F}$ and $\bigvee_{F\in\mathbb{F}} \bigwedge_{y\in F} d(x,y) = \top$. For $\epsilon \lhd \top$ there is $F_\epsilon \in \mathbb{F}$ such that $d(x,y) \ge \epsilon$ for all $y \in F_\epsilon$. The set $F_\epsilon \cap A \in \mathbb{F}$ and we conclude $\bigvee_{a\in A} d(x,a) \ge \bigvee_{a \in F_\epsilon \cap A} d(x,a) \ge \epsilon$. This is true for all $\epsilon \lhd \top$ and hence, by complete distributivity, $\bigvee_{a\in A} d(x,a) = \top$.

Let now $\bigvee_{a\in A} d(x,a) = \top$. For $U \in \mathbb{U}^d_x$ there is $\epsilon \lhd \top$ such that $B^d(x,\epsilon) \subseteq U$. From $\bigvee_{a\in A} d(x,a) = \top \rhd \epsilon$, there is $a_\epsilon \in A$ auch that $d(x,a_\epsilon) \rhd \epsilon$, i.e. $a_\epsilon \in B^d(x,\epsilon)$. Hence $U \cap A \neq \emptyset$ and thus there is a filter, $\mathbb{F}$, finer than $\mathbb{U}^d_x$ which contains A. Clearly $x \in c^d_\top(\mathbb{F})$ and hence $x \in \overline{A}^d$. □

Proposition 3.7. *Let* L *be a value quantale, let* (X,d) *be an* L*-metric space and let* $A, B \subseteq X$*. Then the following properties are true.*
(1) $\overline{\emptyset}^d = \emptyset$;
(2) $A \subseteq \overline{A}^d$;
(3) $\overline{A \cup B}^d = \overline{A}^d \cup \overline{B}^d$;
(4) $\overline{\overline{A}^d}^d = \overline{A}^d$.

Proof. We only prove (4). Let $x \in \overline{\overline{A}^d}^d$. Then $\bigvee_{a \in \overline{A}^d} d(x,a) = \top$. For $\epsilon \lhd \top$ there is $a \in \overline{A}^d$ such that $d(x,a) \geq \epsilon$. As $a \in \overline{A}^d$, we know $\bigvee_{y \in A} d(a,y) = \top$ and we conclude $\epsilon = \epsilon * \top \leq d(x,a) * \bigvee_{y \in A} d(a,y) = \bigvee_{y \in A} d(x,a) * d(a,y) \leq \bigvee_{y \in A} d(x,y)$. This is true for all $\epsilon \lhd \top$ and by complete distributivity $\bigvee_{y \in A} d(x,y) = \top$, i.e. $x \in \overline{A}^d$. □

For an L-metric space (X,d) we call $A \subseteq X$ *closed* (in (X,d)) if $\overline{A}^d \subseteq A$. Then $A \subseteq X$ is closed iff for all $\mathbb{U} \in \mathbb{U}(X)$ we have $x \in A$ whenever $A \in \mathbb{U}$ and $x \in c^d_\top(\mathbb{U})$.

Proposition 3.8. *Let* L *be a value quantale and let* (X,d) *be an* L*-metric space and let* $A \subseteq X$*. Then* A *is open iff its complement* A^c *is closed.*

Proof. Let A be open and let $A^c \in \mathbb{F}$ and $x \in c^d_\top(\mathbb{F})$. Assume $x \notin A^c$. Then $x \in A$ and hence there is $\epsilon \lhd \top$ such that $B^d(x,\epsilon) \subseteq A$. There is $F_\epsilon \in \mathbb{F}$ such that $F_\epsilon \subseteq B^d(x,\epsilon) \subseteq A$ and we have $A \in \mathbb{F}$, a contradiction to $A^c \in \mathbb{F}$.

Let now A^c be closed and let $x \in A$ and assume that for all $\epsilon \lhd \top$ we have $B^d(x,\epsilon) \not\subseteq A$. Then for all $\epsilon \lhd \top$ we have $A^c \cap B^d(x,\epsilon) \neq \emptyset$, i.e. the filter generated by this filter base, $\mathbb{F} = \mathbb{U}^d_x \vee [A^c]$, exists and we have $A^c \in \mathbb{F}$ and $x \in c^d_\top(\mathbb{F})$. As A^c is closed, then $x \in A^c$, a contradiction. □

Proposition 3.9. *Let* L *be a value quantale and let* (X,d) *be an* L*-metric space. Then*

(1) $X, \emptyset$ *are closed;*

(2) $A \cup B$ *is closed whenever* A, B *are closed;*

(3) $\bigcap_{j \in J} A_j$ *is closed whenever* A_j *is closed for all* $j \in J$*.*

4 L-gauge spaces

Definition 4.1. [11] *Let* $\mathcal{H} \subseteq \mathsf{L}\text{-}\mathsf{MET}$ *and* $d \in \mathsf{L}\text{-}\mathsf{MET}(X)$*.*

(1) d *is called* locally supported by $\mathcal{H}$ *if for all* $x \in X$, $\alpha \lhd \top$, $\bot \prec \omega$ *there is* $e^{\alpha,\omega}_x \in \mathcal{H}$ *such that* $e^{\alpha,\omega}_x(x,\cdot) * \alpha \leq d(x,\cdot) \vee \omega$;

(2) $\mathcal{H}$ *is called* locally directed *if for all finite subsets* $\mathcal{H}_0 \subseteq \mathcal{H}$, $\bigwedge_{d \in \mathcal{H}_0} d$ *is locally supported by* $\mathcal{H}$;

(3) $\mathcal{H}$ *is called* locally saturated *if for* $d \in \mathsf{L}\text{-}\mathsf{MET}(X)$ *we have* $d \in \mathcal{H}$ *whenever* d *is locally supported by* $\mathcal{H}$.

(4) The set $\widehat{\mathcal{H}} = \{d \in \mathsf{L}\text{-}\mathsf{MET}(X) \ : \ d \text{ is locally supported by } \mathcal{H}\}$ *is called the* local saturation of $\mathcal{H}$.

For Lawvere's quantale, a locally supporting family is called *locally dominating* in [14].

Definition 4.2. [11] *Let* X *be a set.* $\mathcal{G} \subseteq \mathsf{L}\text{-}\mathsf{MET}(X)$ *is called an* L-gauge *if* $\mathcal{G}$ *is a filter in* $\mathsf{L}\text{-}\mathsf{MET}(X)$ *and* $\mathcal{G}$ *is locally saturated. The pair* $(X,\mathcal{G})$ *is then called an* L-gauge space. *A mapping between two* L*-gauge spaces,* $f : (X,\mathcal{G}) \longrightarrow (X',\mathcal{G}')$ *is called an* L-gauge morphism *if* $d' \circ (f \times f) \in \mathcal{G}$ *whenever* $d' \in \mathcal{G}'$*. The category with* L*-gauge spaces as objects and* L*-gauge morphisms as morphisms is denoted by* $\mathsf{L}\text{-}\mathsf{GS}$*.*

In case of Lawvere's quantale, L-gauge spaces are approach spaces defined by means of gauges, [14].

For $(X,\mathcal{G}) \in |\mathsf{L}\text{-}\mathsf{GS}|$ we call $\mathcal{H} \subseteq \mathsf{L}\text{-}\mathsf{MET}(X)$ a *basis for the* L*-gauge* $\mathcal{G}$ if its local saturation $\widehat{\mathcal{H}} = \mathcal{G}$. It is shown in [11] that if L is a value quantale and if $\emptyset \neq \mathcal{H} \subseteq \mathsf{L}\text{-}\mathsf{MET}(X)$ is locally directed, then its local saturation $\mathcal{G} = \widehat{\mathcal{H}}$ is an L-gauge with $\mathcal{H}$ as basis.

The category $\mathsf{L}\text{-}\mathsf{GS}$ is topological with initial constructions as follows [11]. For a family of mappings, $f_j : X \longrightarrow X_j$ $(j \in J)$, and $(X_j, \mathcal{G}_j) \in |\mathsf{L}\text{-}\mathsf{GS}|$ we define the initial L-gauge on X as the L-gauge with basis

$$\mathcal{H} = \{ \bigwedge_{j \in K} d_j \circ (f_j \times f_j) \ : \ K \subseteq J \text{ finite }, d_j \in \mathcal{G}_j \forall j \in J \}.$$

We showed in [11] that the category $\mathsf{L\text{-}MET}$ is isomorphic to a coreflective subcategory of $\mathsf{L\text{-}GS}$. More precisely, let $(X,d) \in |\mathsf{L\text{-}MET}|$ and define $\mathcal{G}^d = [d] = \{e \in \mathsf{L\text{-}MET}(X) \ : \ d \leq e\}$. Then $(X, \mathcal{G}^d) \in |\mathsf{L\text{-}GS}|$. Furthermore, for $f : (X,d) \longrightarrow (X',d')$ an L-metric morphism we have that $f : (X, \mathcal{G}^d) \longrightarrow (X', \mathcal{G}^{d'})$ is an L-gauge morphism, i.e. we have a functor $E : \left\{ \begin{array}{ccc} \mathsf{L\text{-}MET} & \longrightarrow & \mathsf{L\text{-}GS} \\ (X,d) & \longmapsto & (X,\mathcal{G}^d) \\ f & \longmapsto & f \end{array} \right.$. This functor is injective on objects.

Let now $(X,\mathcal{G}) \in |\mathsf{L\text{-}GS}|$ and define $d^{\mathcal{G}} : X \times X \longrightarrow L$ by $d^{\mathcal{G}}(x,y) = \bigwedge_{d \in \mathcal{G}} d(x,y)$. Then $(X, d^{\mathcal{G}}) \in |\mathsf{L\text{-}MET}|$. For $(X,\mathcal{G}), (X',\mathcal{G}') \in |\mathsf{L\text{-}GS}|$ and an L-gauge morphism $f : (X,\mathcal{G}) \longrightarrow (X',\mathcal{G}')$ then $f : (X, d^{\mathcal{G}}) \longrightarrow (X', d^{\mathcal{G}'})$ is an L-metric morphism. Hence we can define a functor $F : \left\{ \begin{array}{ccc} \mathsf{L\text{-}GS} & \longrightarrow & \mathsf{L\text{-}MET} \\ (X,\mathcal{G}) & \longmapsto & (X,d^{\mathcal{G}}) \\ f & \longmapsto & f \end{array} \right.$. For $(X,d) \in |\mathsf{L\text{-}MET}|$ we have $d^{(\mathcal{G}^d)} = d$, i.e. $E(F((X,d))) = (X,d)$. For $(X,\mathcal{G}) \in |\mathsf{L\text{-}GS}|$ we have $\mathcal{G} \subseteq \mathcal{G}^{(d^{\mathcal{G}})}$, i.e. $F(E((X,\mathcal{G}))) \leq (X,\mathcal{G})$.

We call $(X, d^{\mathcal{G}})$ the L-*metric coreflection* of $(X,\mathcal{G})$.

5 Topological spaces as L-gauge spaces

In the sequel, we will need the following concept.

Definition 5.1. [5] *Let L be a complete lattice. The well-below relation $\lhd$ is called* multiplicative *if for all $\alpha, \beta, \epsilon \in L$ we have that $\epsilon \lhd \alpha \wedge \beta$ whenever $\epsilon \lhd \alpha$ and $\epsilon \lhd \beta$.*

Example 5.2. *If the complete lattice L is linearly ordered, then $\lhd$ is multiplicative. Let $\alpha \wedge \beta \leq \bigvee D$. Then $\alpha \leq \bigvee D$ or $\beta \leq \bigvee D$ and hence there is $\delta \in D$ such that $\epsilon \leq \delta$.*

Example 5.3. *For $(L, \leq) = (\Delta^+, \leq)$, the well-below relation is multiplicative. Let $\varphi \lhd \psi = \bigvee \{f_{\delta,\epsilon} \ : \ f_{\delta,\epsilon} \lhd \psi\}$ and $\varphi \lhd \eta = \bigvee \{f_{\delta',\epsilon'} \ : \ f_{\delta',\epsilon'} \lhd \eta\}$. Then there are δ, ϵ such that $\varphi \lhd f_{\delta,\epsilon} \lhd \psi$ and δ', ϵ' such that $\varphi \lhd f_{\delta',\epsilon'} \lhd \eta$. Hence $\epsilon < \psi(\delta)$ and $\epsilon' < \eta(\delta')$ and therefore also $\epsilon \wedge \epsilon' < \psi \wedge \eta(\delta \vee \delta')$. We conclude $\varphi \leq f_{\delta,\epsilon} \wedge f_{\delta',\epsilon'} = f_{\delta \vee \delta', \epsilon \wedge \epsilon'} \lhd \psi \wedge \eta$.*

Proposition 5.4. *Let $\lhd$ be multiplicative and let $d_1, d_2 \in \mathsf{L\text{-}MET}(X)$ and $\epsilon \lhd \top$. Then $B^{d_1 \wedge d_2}(x,\epsilon) = B^{d_1}(x,\epsilon) \cap B^{d_2}(x,\epsilon)$.*

Proof. We have $(d_1 \wedge d_2)(x,y) \rhd \epsilon$ if and only if $d_1(x,y) \rhd \epsilon$ and $d_2(x,y) \rhd \epsilon$. □

For two topologies τ_1, τ_2 on X we denote by $\tau_1 \vee \tau_2$ the topology with $\tau_1 \cup \tau_2$ as subbase.

Corollary 5.5. *Let $\lhd$ be multiplicative, let $d_1, d_2 \in \mathsf{L\text{-}MET}(X)$ and let τ be a topology on X with $\tau_{d_1}, \tau_{d_2} \subseteq \tau$. Then $\tau_{d_1 \wedge d_2} \subseteq \tau_{d_1} \vee \tau_{d_2} \subseteq \tau$.*

Proof. Let $x \in A \in \tau_{d_1 \wedge d_2}$. Then there is $\epsilon \lhd \top$ such that $B^{d_1 \wedge d_2}(x,\epsilon) = B^{d_1}(x,\epsilon) \cap B^{d_2}(x,\epsilon) \subseteq A$. As $B^{d_1}(x,\epsilon) \cap B^{d_2}(x,\epsilon) \in \tau_{d_1} \vee \tau_{d_2}$, we see that A is the union of $\tau_{d_1} \vee \tau_{d_2}$-open sets and hence $A \in \tau_{d_1} \vee \tau_{d_2}$. □

Proposition 5.6. *Let L be a value quantale and let the well-below relation be multiplicative. Let $(X,\tau) \in |\mathsf{TOP}|$. Define $\mathcal{G}_\tau = \widehat{\mathcal{H}}_\tau$ with $\mathcal{H}_\tau = \{d \in \mathsf{L\text{-}MET}(X) \ : \ \tau_d \subseteq \tau\}$. Then $(X, \mathcal{G}_\tau) \in |\mathsf{L\text{-}GS}|$.*

Proof. If $d_1, d_2 \in \mathcal{H}_\tau$, then $d_1 \wedge d_2 \in \mathcal{H}_\tau$. Hence $\mathcal{H}_\tau$ is locally directed. □

Proposition 5.7. *Let L be a value quantale and let the well-below relation be multiplicative. Let $(X,\tau), (Y,\sigma) \in |\mathsf{TOP}|$ and let $f : (X,\tau) \longrightarrow (Y,\sigma)$ be continuous. Then $f : (X,\mathcal{G}_\tau) \longrightarrow (Y, \mathcal{G}_\sigma)$ is an $\mathsf{L\text{-}GS}$-morphism.*

Proof. Let $d' \in \mathcal{G}_\sigma$. Then $\tau_{d'} \subseteq \sigma$ and we need to show that $\tau_{d' \circ (f \times f)} \subseteq \tau$. Let $A \in \tau_{d' \circ (f \times f)}$ and let $x \in A$. Then there is $\epsilon \lhd \top$ such that $B^{d' \circ (f \times f)}(x,\epsilon) \subseteq A$. For $z \in f^{-1}(B^{d'}(f(x),\epsilon))$ we have $f(z) \in B^{d'}(f(x),\epsilon)$, i.e. $d' \circ (f \times f)(x,z) \rhd \epsilon$. We conclude $z \in B^{d' \circ (f \times f)}(x,\epsilon)$. Hence $x \in f^{-1}(B^{d'}(f(x),\epsilon)) \subseteq B^{d' \circ (f \times f)}(x,\epsilon) \subseteq A$. From the continuity of f we know that $f^{-1}(B^{d'}(f(x),\epsilon)) \in \tau$. Therefore, A is the union of τ-open sets, i.e. $A \in \tau$. □

We can thus define a functor

$$G : \left\{ \begin{array}{ccc} \mathsf{TOP} & \longrightarrow & \mathsf{L\text{-}GS} \\ (X,\tau) & \longmapsto & (X,\mathcal{G}_\tau) \\ f & \longmapsto & f \end{array} \right.$$

We show that this functor is injective on objects. Let $(X,\tau) \in |\mathsf{TOP}|$ and let $A \subseteq X$. We denote the topological closure of A by $\overline{A}$ or by $\overline{A}^{\tau}$, if we need to emphasize the topology τ. We define

$$d_A^{\tau}(x,y) = \begin{cases} \bot & \text{if } y \in \overline{A}, x \notin \overline{A} \\ \top & \text{otherwise.} \end{cases}$$

Then $d_A^{\tau} \in \mathsf{L}\text{-}\mathsf{MET}(X)$. Clearly $d_A^{\tau}(x,x) = \top$. We show $d_A^{\tau}(x,y) * d_A^{\tau}(y,z) \leq d_A^{\tau}(x,z)$. Let $d_A^{\tau}(x,z) = \bot$. Then $z \in \overline{A}$ and $x \notin \overline{A}$. If $d_A^{\tau}(x,y) = \top$, then, as $x \notin \overline{A}$, we have $y \notin \overline{A}$ and hence $d_A^{\tau}(y,z) = \bot$.

Furthermore, $\tau_{d_A^{\tau}} \subseteq \tau$, i.e. $d_A^{\tau} \in \mathcal{G}_{\tau}$. We show $\overline{B} \subseteq \overline{B}^{\tau_{d_A^{\tau}}}$ for all $B \subseteq X$. To this end, we define $\delta^{\tau}(x,B) = \begin{cases} \top & \text{if } x \in \overline{B} \\ \bot & \text{otherwise} \end{cases}$. Let $x \in \overline{B}$. Then $\top = \delta^{\tau}(x,B) \leq (\bigwedge_{b\in B} \delta^{\tau}(b,A)) \to \delta^{\tau}(x,A)$. We distinguish two cases. *Case 1:* $\bigwedge_{b\in B} \delta^{\tau}(b,A) = \bot$ and $\delta^{\tau}(x,A) = \bot$. Then there is $b \in B$ such that $b \notin \overline{A}$ and we have $x \notin \overline{A}$. Hence $d_A^{\tau}(x,b) = \top$ and therefore $\bigvee_{b\in B} d_A^{\tau}(x,y) = \top$ which means $x \in \overline{B}^{\tau_{d_A^{\tau}}}$. *Case 2:* $\bigwedge_{b\in B} \delta^{\tau}(b,A) = \top$. Then also $\delta^{\tau}(x,A) = \top$, and hence we have $x \in \overline{A}$ and for all $b \in B$, $b \in \overline{A}$. Again this implies $\top = \bigvee_{b\in B} d_A^{\tau}(x,B)$, i.e. $x \in \overline{B}^{\tau_{d_A^{\tau}}}$.

Let now $\tau \neq \sigma$ be two topologies on X. Without loss of generality, there is $A \subseteq X$ and $x \in X$ such that $x \in \overline{A}^{\sigma}$ but $x \notin \overline{A}^{\tau}$. Then $d_A^{\tau} \in \mathcal{G}_{\tau}$ but $d_A^{\tau} \notin \mathcal{G}_{\sigma}$. We show $\overline{A}^{\sigma} \not\subseteq \overline{A}^{d_A^{\tau}}$. From $A \subseteq \overline{A}^{\tau}$ we get $\bigvee_{a\in A} d_A^{\tau}(x,a) = \bot$ and hence $x \notin \overline{A}^{d_A^{\tau}}$.

We are now going to show that this embedding is very nice by showing its coreflectivity.

Let $(X,\mathcal{G}) \in |\mathsf{L}\text{-}\mathsf{GS}|$ and let $A \subseteq X$. We define $\overline{A}^{\mathcal{G}} = \{x \in X \,:\, x \in \overline{A}^{\tau_e} \text{ for all } e \in \mathcal{G}\}$. Clearly, $\overline{A}^{\mathcal{G}} = \bigcap_{e\in\mathcal{G}} \overline{A}^{\tau_e}$.

Proposition 5.8. *Let $(X,\mathcal{G}) \in |\mathsf{L}\text{-}\mathsf{GS}|$ and let $\mathcal{H}$ be an L-gauge basis for $\mathcal{G}$, i.e. $\widehat{\mathcal{H}} = \mathcal{G}$. Then $\overline{A}^{\mathcal{G}} = \bigcap_{e\in\mathcal{H}} \overline{A}^{\tau_e}$.*

Proof. As $\mathcal{H} \subseteq \mathcal{G}$, clearly $\overline{A}^{\mathcal{G}} \subseteq \bigcap_{e\in\mathcal{H}} \overline{A}^{\tau_e}$. Let now $d \in \mathcal{G}$. Then d is locally supported by $\mathcal{H}$, i.e. for all $x \in X, \alpha \lhd \top, \omega \succ \bot$ there is $e_{\omega}^{\alpha,x} \in \mathcal{H}$ such that $e_{\omega}^{\alpha,x}(x,\cdot) * \alpha \leq d(x,\cdot) \vee \omega$. For $x \in \overline{A}^{\tau_{e_{\omega}^{\alpha,x}}}$ we have $\alpha = \alpha * \top = \alpha * \bigvee_{a\in A} e_{\omega}^{\alpha,x}(x,a) = \bigvee_{a\in A} \alpha * e_{\omega}^{\alpha,x}(x,a) \leq \bigvee_{a\in A}(d(x,a) \vee \omega) = (\bigvee_{a\in A} d(x,a)) \vee \omega$. Taking the join over all $\alpha \lhd \top$ and the meet over all $\omega \succ \bot$, yields $\top = \bigvee_{a\in A} d(x,a)$, i.e. $x \in \overline{A}^{\tau_d}$.

We conclude $\bigcap_{e\in\mathcal{H}} \overline{A}^{\tau_e} \subseteq \overline{A}^{\tau_{e_{\omega}^{\alpha,x}}} \subseteq \overline{A}^{\tau_d}$. This is true for all $d \in \mathcal{G}$ and hence $\bigcap_{e\in\mathcal{H}} \overline{A}^{\tau_e} \subseteq \bigcap_{d\in\mathcal{G}} \overline{A}^{\tau_d} = \overline{A}^{\mathcal{G}}$. $\square$

We note that for $d, e \in \mathsf{L}\text{-}\mathsf{MET}(X)$, $d \leq e$ implies $\overline{A}^{\tau_d} \subseteq \overline{A}^{\tau_e}$. This follows directly with Proposition 3.6.

Proposition 5.9. *Let L be a value quantale, let $(X,\mathcal{G})$ be an L-gauge space and let $A, B \subseteq X$. Then the following properties are true.*

(1) $\overline{\emptyset}^{\mathcal{G}} = \emptyset$;

(2) $A \subseteq \overline{A}^{\mathcal{G}}$;

(3) $\overline{A \cup B}^{\mathcal{G}} = \overline{A}^{\mathcal{G}} \cup \overline{B}^{\mathcal{G}}$;

(4) $\overline{\overline{A}^{\mathcal{G}}}^{\mathcal{G}} \subseteq \overline{A}^{\mathcal{G}}$.

Proof. We only show (4). Let $x \in \overline{\overline{A}^{\mathcal{G}}}^{\mathcal{G}}$ and assume $x \notin \overline{A}^{\mathcal{G}}$. Then there is $e \in \mathcal{G}$ such that $x \notin \overline{A}^{\tau_e}$. Then also $x \notin \overline{\overline{A}^{\tau_e}}^{\tau_e}$ and hence also $x \notin \overline{\overline{A}^{\mathcal{G}}}^{\tau_e}$, a contradiction. $\square$

We denote $\tau_{\mathcal{G}}$ the topology on X which belongs to the closure operator $\overline{\cdot}^{\mathcal{G}}$.

Proposition 5.10. *Let L be a value quantale, let $(X,\mathcal{G}), (X',\mathcal{G}')$ be L-gauge spaces and let $f : (X,\mathcal{G}) \longrightarrow (X',\mathcal{G}')$ be an L-gauge morphism. Then for all $A \subseteq X$ we have $f(\overline{A}^{\mathcal{G}}) \subseteq \overline{f(A)}^{\mathcal{G}'}$, i.e. $f : (X,\tau_{\mathcal{G}}) \longrightarrow (X',\tau_{\mathcal{G}'})$ is continuous.*

Proof. Let $x' \in f(\overline{A}^{\mathcal{G}})$. Then there is $x \in \overline{A}^{\mathcal{G}}$ such that $f(x) = x'$. Hence for all $e \in \mathcal{G}$, we have $\bigvee_{a\in A} e(x,a) = \top$ and $f(x) = x'$. Let $e' \in \mathcal{G}'$. Then $\top = \bigvee_{a\in A} e' \circ (f\times f)(x,a) = \bigvee_{a\in A} e'(f(x),f(a)) = \bigvee_{b\in f(A)} e'(x',b)$. Hence $x' \in \overline{f(A)}^{\tau_{e'}}$. This is true for all $e' \in \mathcal{G}'$ and we conclude $x' \in \overline{f(A)}^{\mathcal{G}'}$. $\square$

Hence we can define a functor

$$H : \begin{cases} \mathsf{L}\text{-}\mathsf{GS} & \longrightarrow & \mathsf{TOP} \\ (X,\mathcal{G}) & \longmapsto & (X,\tau_{\mathcal{G}}) \\ f & \longmapsto & f \end{cases}$$

Proposition 5.11. *Let L be a value quantale, let $\lhd$ be multiplicative and let $(X,\mathcal{G})$ be an L-gauge space. Then $\mathcal{G} \subseteq \mathcal{G}_{\tau_{\mathcal{G}}}$, i.e. $G \circ H \leq id_{\mathsf{L}\text{-}\mathsf{GS}}$.*

Proof. Let $d \in \mathcal{G}$. Then, for all $A \subseteq X$, we have $\overline{A}^{\mathcal{G}} \subseteq \overline{A}^{d}$, i.e. $\tau_d \subseteq \tau_{\mathcal{G}}$. Hence $d \in \mathcal{G}_{\tau_{\mathcal{G}}}$. □

Proposition 5.12. *Let* L *be a value quantale, let* $\lhd$ *be multiplicative and let* (X,τ) *be a topological space. Then* $\tau = \tau_{(\mathcal{G}_\tau)}$, *i.e.* $H \circ G = id_{\mathsf{TOP}}$.

Proof. If $d \in \mathcal{G}_\tau$, then $\tau_d \subseteq \tau$ and hence, for all $A \subseteq X$, we have $\overline{A}^{\tau} \subseteq \bigcap_{d \in \mathcal{G}_\tau} \overline{A}^{\tau_d} = \overline{A}^{\mathcal{G}_\tau}$. Therefore, $\tau_{(\mathcal{G}_\tau)} \subseteq \tau$.

Let now $x \in \overline{A}^{\mathcal{G}_\tau}$. As $d^\tau_A \in \mathcal{G}_\tau$, then $\bigvee_{a \in A} d^\tau_A(x,a) = \top$. Hence there is $a \in A \subseteq \overline{A}^{\tau}$ such that $d^\tau_A(x,a) = \top$. Therefore $x \in \overline{A}^{\tau}$. This shows $\overline{A}^{\mathcal{G}_\tau} \subseteq \overline{A}^{\tau}$ for all $A \subseteq X$ and this means $\tau \subseteq \tau_{(\mathcal{G}_\tau)}$. □

We call $(X, \tau_{\mathcal{G}})$ the *topological coreflection* of the L-gauge space $(X,\mathcal{G})$. The following result describes the neighbourhoods in the topological coreflection.

Proposition 5.13. *Let* L *be a value quantale, let* $(X,\mathcal{G})$ *be an* L*-gauge space and let* $\widehat{\mathcal{H}} = \mathcal{G}$. *Then* $U \in \mathbb{U}^{\tau_{\mathcal{G}}}_x$ *iff there is* $d \in \mathcal{H}$ *and* $\epsilon \lhd \top$ *such that* $B^d(x,\epsilon) \subseteq U$.

Proof. We have $U \in \mathbb{U}^{\tau_{\mathcal{G}}}_x$ if and only if $x \notin \overline{U^c}^{\mathcal{G}}$. By Proposition 5.8, this is equivalent to $x \notin \overline{U^c}^{d}$ for some $d \in \mathcal{H}$, which in turn is equivalent to $U \in \mathbb{U}^{\tau_d}_x$ for some $d \in \mathcal{H}$. The latter is equivalent to $B^d(x,\epsilon) \subseteq U$ for some $\epsilon \lhd \top$ and $d \in \mathcal{H}$. □

Remark 5.14. *If the* L*-gauge basis* $\mathcal{H}$ *consists of* symmetric L*-metrics, then the collection* $\{U^d(\epsilon) \,:\, d \in \mathcal{H}, \epsilon \lhd \top\}$ *with* $U^d(\epsilon) = \{(x,y) \in X \times X \,:\, d(x,y) \rhd \epsilon\}$ *is the basis for a uniformity* $\mathcal{U}$ *with* $\mathcal{U}(x) = \mathbb{U}^{\tau_{\mathcal{G}}}_x$ *for all* $x \in X$. *Hence, in this case, the topological coreflection* $(X,\tau_{\mathcal{G}})$ *is uniformizable and therefore also completely regular.*

Remark 5.15. *For* L*-gauge spaces there is a natural concept of convergence. Let* $(X,\mathcal{G})$ *be an* L*-gauge space. For a filter* $\mathbb{F} \in \mathbb{F}(X)$, $\alpha \in L$ *and* $x \in X$ *we define*

$$x \in c^{\mathcal{G}}_\alpha(\mathbb{F}) \iff \bigwedge_{d \in \mathcal{G}} \bigvee_{F \in \mathbb{F}} \bigwedge_{y \in \mathbb{F}} d(x,y) \geq \alpha.$$

Then $(X, \overline{c^{\mathcal{G}}})$ *is an* L*-convergence tower space and we have* $c^{\mathcal{G}}_\alpha(\mathbb{F}) = \bigwedge_{d\in\mathcal{G}} c^d_\alpha(\mathbb{F})$. *We shall study this convergence concept for* L*-gauge spaces in more detail elsewhere. Here we remark the following. We have, for* $\alpha = \top$, *that* $x \in c^{\mathcal{G}}_\top(\mathbb{F})$ *if and only if for all* $\epsilon \lhd \top$, *for all* $d \in \mathcal{G}$, *there is* $F \in \mathbb{F}$ *such that for all* $y \in F$ *we have* $d(x,y) \rhd \epsilon$. *This is equivalent to: for all* $\epsilon \lhd \top$, *for all* $d \in \mathcal{G}$ *we have* $B^d(x,\epsilon) \in \mathbb{F}$. *Hence,* $x \in c^{\mathcal{G}}_\top(\mathbb{F})$ *is equivalent to the convergence of* $\mathbb{F}$ *to* x *in the topological coreflection* $(X,\tau_{\mathcal{G}})$. □

6 The quantale-valued Hausdorff metric

The results of this section can be found in [2], where the quantale-valued Hausdorff metric is studied from a categorical point of view. We state the proofs in order to be self-contained.

For an L-metric space (X,d), we denote the set of all non-empty closed subsets by $CL(X) = \{A \subseteq X \,:\, A \neq \emptyset,\ A \text{ closed in } (X,d)\}$. We define an L-metric on $CL(X)$ by $H_d(A,B) = \bigwedge_{x\in X} d_{\mathsf{L}}(d(x,A), d(x,B))$, where $d(x,A) = \bigvee_{a \in A} d(x,a)$ and $d_{\mathsf{L}}(\alpha,\beta) = (\alpha \to \beta) \wedge (\beta \to \alpha)$. Clearly, for $A = B \in CL(X)$, we have $H_d(A,A) = \top$ and by definition of d_{L} also the symmetry, $H_d(A,B) = H_d(B,A)$, as well as the triangle inequality, $H_d(A,B) * H_d(B,C) \leq H_d(A,C)$ is clear. As we defined the function H_d only for closed sets, H_d is also separated. For, let $H_d(A,B) = \top$. Then for all $x \in X$ we have $d(x,A) \to d(x,B) = \top$ and $d(x,B) \to d(x,A) = \top$, i.e. for all $x \in X$ we have $d(x,A) = d(x,B)$. Hence, for all $x \in A$ we have, as A is closed, $\top = d(x,A) = d(x,B)$ and hence, $x \in \overline{B}^{d} = B$, again as also B is closed. Hence $A \subseteq B$. Similarly we get $B \subseteq A$ and hence $A = B$. We call H_d the L-*Hausdorff metric* of the L-metric space (X,d).

Note that we have just shown that the sets $A \in CL(X)$ can be identified with their distance functionals $d(\cdot, A)$, i.e. that we have for $A,B \in CL(X)$, $A = B$ if and only if $d(x,A) = d(x,B)$ for all $x \in X$.

We now derive a different characterization of the L-Hausdorff metric.

Lemma 6.1. [2] *Let* (X,d) *be an* L*-metric space and let* $A, B \subseteq X$. *Then*

$$\bigwedge_{x \in X} (d(x,A) \to d(x,B)) = \bigwedge_{a \in A} d(a,B).$$

Proof. For $a \in A$ we have $d(a,A) \geq d(a,a) = \top$ and hence $\bigwedge_{x\in X}(d(x,A) \to d(x,B)) \leq \bigwedge_{a\in A}(\top \to d(a,B)) = \bigwedge_{a\in A} d(a,B)$. For the converse inequality, we denote $\delta = \bigwedge_{a\in A} d(a,B)$. Then for all $x \in X$ we have, for $a \in A$, that $d(x,a) * \delta \leq \bigvee_{b\in B}(d(x,a) * d(a,b)) \leq d(x,B)$. Taking the join over all $a \in A$ we obtain $d(x,A) * \delta \leq d(x,B)$, i.e. $\delta \leq d(x,A) \to d(x,B)$. Taking the meet over all $x \in X$ completes the proof. □

Proposition 6.2. *Let (X,d) be an L-metric space and let $A,B \in CL(X)$. Then $H_d(A,B) = \bigwedge_{a\in A} d(a,B) \wedge \bigwedge_{b\in B} d(A,b)$.*

Remark 6.3. *For $L = \Delta^+$ we obtain for the KM-fuzzy metric space $(X,M,\circledast)$ the Hausdorff fuzzy metric of* [18]*, see also* [20]*. We note in this respect that the meet of distance distribution functions $\varphi_j \in \Delta^+, j \in J$ in Δ^+ is computed by $\bigwedge_{j\in J}\varphi_j(t) = \sup_{s<t}\inf_{j\in J}\varphi_j(s)$ with $\sup\emptyset = 0$, see* [10].

The next result shows that, as long as the one point subsets $\{x\} \subseteq X$ of an L-metric space are d-closed, we can embed (X,d) into $(CL(X),H_d)$. We note that $\{x\}$ to be d-closed is equivalent to $x = y$ whenever $\top = d(x,\{y\}) = d(x,y)$, i.e. to (X,d) being separated.

Proposition 6.4. *For a separated L-metric space (X,d) we have for all $x,y \in X$ that $H_d(\{x\},\{y\}) = d(x,y)$.*

Proof. We have $H_d(\{x\},\{y\}) = \bigwedge_{a\in\{x\}} d(a,\{y\}) \wedge \bigwedge_{b\in\{y\}} d(\{x\},b) = d(x,y)$. □

7 The Wijsman L-gauge of an L-metric space

In this section, we always assume that L is a value quantale. For the special case of Lawvere's quantale, the results of this section have already been obtained in [14].

For an L-metric space (X,d) we consider again on L the *canonical* L*-metric* $d_{\mathsf{L}}(\alpha,\beta) = (\alpha \to \beta) \wedge (\beta \to \alpha)$. For a finite subset $F \subseteq X$ we define, for $A,B \in CL(X)$, $d_F(A,B) = \bigwedge_{x\in F} d_{\mathsf{L}}(d(x,A),d(x,B))$. It is then not difficult to show that d_F is a symmetric L-metric on $CL(X)$. Furthermore, we define $\mathcal{H} = \{d_F \ : \ F \subseteq X \text{ finite}\}$. Then $\mathcal{H}$ is locally directed and hence a symmetric basis for an L-gauge and we denote $\mathcal{G}_W = \widehat{\mathcal{H}}$. We call $\mathcal{G}_W$ the L-*Wijsman structure* of the L-metric space (X,d). We are now going to justify this name.

Proposition 7.1. *Let (X,d) be an L-metric space. Then $\mathcal{G}_W$ is the initial L-gauge on $CL(X)$ for the source*

$$\left(d(x,\cdot) : \left\{ \begin{array}{ccc} CL(X) & \longrightarrow & (L,[d_{\mathsf{L}}]) \\ A & \longmapsto & d(x,A) \end{array} \right. \right)_{x\in X}.$$

Proof. An L-gauge basis for the initial L-gauge is given by

$$\{\bigwedge_{x\in F} d_{\mathsf{L}} \circ (d(x,\cdot) \times d(x,\cdot)) \ : \ F \subseteq X \text{ finite}\}.$$

The equality $\bigwedge_{x\in F} d_{\mathsf{L}} \circ (d(x,\cdot) \times d(x,\cdot))(A,B) = \bigwedge_{x\in F} d_{\mathsf{L}}(d(x,A),d(x,B))$ then completes the proof. □

For the next result we note that $d(\cdot,A)$ is a mapping from X to L, i.e. an element of the product L^X. We equip L^X with the product L-gauge $\pi - d_{\mathsf{L}}$ of $(L,[d_{\mathsf{L}}])$.

Proposition 7.2. *Let (X,d) be an L-metric space. Then $\mathcal{G}_W$ is the initial L-gauge on $CL(X)$ for the source*

$$\varphi : \left\{ \begin{array}{ccc} CL(X) & \longrightarrow & (L^X,\pi - d_{\mathsf{L}}) \\ A & \longmapsto & d(\cdot,A) \end{array} \right. .$$

Proof. A basis for the product L-gauge on L^X is given by $\{\bigwedge_{x\in F} d_{\mathsf{L}} \circ (pr_x \times pr_x) \ : \ F \subseteq X \text{ finite}\}$, where $p_x : L^X \longrightarrow L, a \longmapsto a(x)$ are the projections. Here, for $a,b \in L^X$ and $F \subseteq X$ finite, we have $\bigwedge_{x\in F} d_{\mathsf{L}} \circ (pr_x \times pr_x)(a,b) = \bigwedge_{x\in F} d_{\mathsf{L}}(a(x),b(x))$. An element of the L-gauge basis of the initial gauge is therefore given by $d(A,B) = (\bigwedge_{x\in F} d_{\mathsf{L}} \circ (pr_x \times pr_x)) \circ (\varphi \times \varphi)(A,B) = (\bigwedge_{x\in F} d_{\mathsf{L}} \circ (pr_x \times pr_x))(d(\cdot,A),d(\cdot,B)) = \bigwedge_{x\in F} d_{\mathsf{L}}(d(x,A),d(x,B)) = d_F(A,B)$. □

Proposition 7.3. *Let (X,d) be an L-metric space. The L-metric coreflection of the L-Wijsman structure is the L-Hausdorff metric on $CL(X)$.*

Proof. We have $d^{\mathcal{G}_W}(A,B) = \bigwedge_{F\subseteq X \text{ finite}} d_F(A,B) = \bigwedge_{x\in X} d_{\mathsf{L}}(d(x,A),d(x,B)) = H_d(A,B)$. □

The next result is a direct consequence of Proposition 5.13.

Proposition 7.4. *Let (X,d) be an L-metric space. A basis for the neighbourhoods of $A \in CL(X)$ in the topological coreflection $(CL(X), \tau_{\mathcal{G}_W})$ of $(CL(X), \mathcal{G}_W)$ is given by the collection $\{\{B \in CL(X) : d_F(A,B) \rhd \epsilon\} : F \subseteq X \text{ finite}, \epsilon \lhd \top\}$.*

We note that the L-gauge basis for $\mathcal{G}_W$ is symmetric. Hence the topological coreflection is uniformizable with basis of the uniformity the collection of sets of the form $\{(A,B) \in CL(X) \times CL(X) : d_F(A,B) \rhd \epsilon\}$, $F \subseteq X$ finite, $\epsilon \lhd \top$.

Remark 7.5. *For Lawvere's quantale, $\mathsf{L} = ([0,\infty], \geq, +)$, we see that in the case of a metric space, the topological coreflection of the gauge space $(CL(X), \mathcal{G}_W)$ is the classical Wijsman topology, see* [3]*, page 42, Example 2.1(5). This was already shown in* [14].

Remark 7.6. *Identifying a KM-fuzzy metric space $(X, M, *)$ with the $(\Delta^+, \leq, \circledast)$-metric space (X, d_M), the topological coreflection $(X, \tau_{\mathcal{G}_W})$ seems to be a natural candidate for the Wijsman topology of a KM-fuzzy metric space. However, directly using the corresponding $(\Delta^+, \leq, \circledast)$-Wijsman structure has advantages, as is shown e.g. with the next result.*

Following [16] we call, for an L-metric space (X,d), an L-gauge on $CL(X)$ *admissible* if the mapping $\psi : (X, \mathcal{G}_d) \longrightarrow (CL(X), \mathcal{G}), x \longmapsto \{x\}$ is an embedding in L-GS. Here, as always, we denote $\mathcal{G}_d = [d] = \{e \in \mathsf{L}\text{-}\mathsf{MET}(X) : d \leq e\}$. In order for the mapping ψ to be well-defined, we need to ensure that the one-point subsets $\{x\}$ are closed, i.e. we have to assume that the L-metric space is separated.

Proposition 7.7. *Let (X,d) be a separated and symmetric L-metric space. Then the L-Wijsman structure $\mathcal{G}_W$ on $CL(X)$ is admissible.*

Proof. We first show that $\psi : (X, [d]) \longrightarrow (CL(X), \mathcal{G}_W)$ is an L-gauge morphism. We note first that from $d(z,x) * d(x,y) \leq d(z,y)$ we conclude $d(x,y) \leq d(z,x) \to d(z,y)$. Let now $d_F \in \mathcal{G}_W$. Then, for $x, y \in X$ we have

$$\begin{aligned} d_F \circ (\psi \times \psi)(x,y) &= d_F(\psi(x), \psi(y)) \\ &= d_F(\{x\}, \{y\}) \\ &= \bigwedge_{z \in F} d_{\mathsf{L}}(d(z,x), d(z,y)) \\ &= \bigwedge_{z \in F} (d(z,x) \to d(z,y)) \wedge (d(z,y) \to d(z,x)) \\ &\geq d(x,y) \wedge d(y,x) \ = \ d(x,y). \end{aligned}$$

Hence $d_F \circ (\psi \times \psi) \in [d]$.

Next we show that $\psi^{-1} : (\psi(X), \mathcal{G}_W|_{\psi(X)}) \longrightarrow (X, [d]), \{x\} \longmapsto x$ is an L-gauge morphism. We have $d \circ (\psi^{-1} \times \psi^{-1})(\{x\}, \{y\}) = d(x,y)$ and hence, with the finite set $F = \{x\}$, we obtain

$$d_{\{x\}}(\{x\}, \{y\}) = d_{\mathsf{L}}(d(x,x), d(x,y)) = d(x,y).$$

Therefore $d \circ (\psi^{-1} \times \psi^{-1})$ is locally supported by $\mathcal{G}_W|_{\psi(X)}$ and the proof is complete. □

8 Compactness of $(CL(X), \mathcal{G}_W)$

For an L-gauge space $(X, \mathcal{G})$, we define the following *index of compactness*.

$$\chi_c((X,\mathcal{G})) = \bigwedge_{\varphi \in \mathcal{G}^X} \bigvee_{Y \subseteq X \text{ finite}} \bigwedge_{z \in X} \bigvee_{y \in Y} \varphi(y)(y,z).$$

For Lawvere's quantale, we obtain the index of compactness in [14]. Again, the results of this section specialize to results that have been obtained in [14] in the case of Lawvere's quantale.

We first show that it is sufficient to consider a basis for the L-gauge.

Proposition 8.1. *Let $(X, \mathcal{G})$ be an L-gauge space and let $\widehat{\mathcal{H}} = \mathcal{G}$. Then*

$$\chi_c((X,\mathcal{G})) = \bigwedge_{\varphi \in \mathcal{H}^X} \bigvee_{Y \subseteq X \text{ finite}} \bigwedge_{z \in X} \bigvee_{y \in Y} \varphi(y)(y,z).$$

Proof. As $\mathcal{H} \subseteq \widehat{\mathcal{H}} = \mathcal{G}$, we immediately obtain $\leq$. For the converse inequality, we fix $\varphi \in \mathcal{G}^X$. Then for each $x \in X$, $\varphi(x) \in \mathcal{G}$ and hence, for $\alpha \lhd \top$ and $\bot \prec \omega$, there is $e_x^{\alpha,\omega} \in \mathcal{H}$ such that $e_x^{\alpha,\omega}(x,\cdot) * \alpha \leq \varphi(x)(x,\cdot) \vee \omega$. If follows with the distributivity of the quantale operation over finite meets and the complete distributivity of L,

$$\left(\bigvee_{F \text{ finite}} \bigwedge_{z\in X} \bigvee_{y \in F} \varphi(y)(y,z)\right) \vee \omega \geq \left(\bigvee_{F \text{ finite}} \bigwedge_{z\in X} \bigvee_{y \in F} e_y^{\alpha,\omega}(y,z)\right) * \alpha \geq \left(\bigwedge_{\psi \in \mathcal{H}^X} \bigvee_{F \text{ finite}} \bigwedge_{z\in X} \bigvee_{y \in F} \psi(y)(y,z)\right) * \alpha$$

Taking joins over all $\alpha \lhd \top$ and meets over all $\bot \prec \omega$ yields the desired inequality. □

In order to motivate the term "index of compactness", we show the following result.

Proposition 8.2. *A topological space (X,τ) is compact if and only if $\chi_c((X,\mathcal{G}_\tau)) = \top$.*

Proof. Let first (X,τ) be compact and let $\epsilon \lhd \top$ and $\varphi : X \longrightarrow \mathcal{H}_\tau = \{d \in \mathsf{L}\text{-}\mathsf{MET}(X) \ : \ \tau_d \subseteq \tau\}$. Then $\bigcup_{y\in X} B^{\varphi(y)}(y,\epsilon) = X$ and as $\tau_{\varphi(y)} \subseteq \tau$, the sets $B^{\varphi(y)}(y,\epsilon)$ form an open cover of X. Hence there is a finite subset $Y \subseteq X$ with $X = \bigcup_{y\in Y} B^{\varphi(y)}(y,\epsilon)$, i.e. for every $z \in X$ there is $y_z \in Y$ such that $\varphi(y)(y,z) \rhd \epsilon$. We conclude $\bigvee_{Y\subseteq X \text{ finite}} \bigwedge_{z\in X} \bigvee_{y\in Y} \varphi(y)(y,z) \geq \epsilon$. This is true for any $\varphi \in \mathcal{H}_\tau^X$ and hence $\chi_c((X,\mathcal{G}_\tau)) \geq \epsilon$. As $\epsilon \rhd \top$ was arbitrary, the complete distributivity finally yields $\chi_c((X,\mathcal{G}_\tau)) = \top$.

Let now $\chi_c((X,\mathcal{G}_\tau)) = \top$ and let $\{A_y \ : \ y \in X\}$ be an open cover and assume without loss of generality $y \in A_y$ for all $y \in X$. Define $B_y = A_y^c$. Then B_y is closed and as shown after Proposition 5.7

$$d_{B_y}(u,v) = \begin{cases} \bot & v \in B_y, u \in A_y \\ \top & \text{otherwise} \end{cases}$$

defines an L-metric with $d_{B_y} \in \mathcal{G}_\tau$. Furthermore, for $\bot \neq \epsilon \lhd \top$ we have $A_y = B^{d_{B_y}}(y,\epsilon)$. With $\varphi(y) = d_{B_y}$ we then have

$$\bigvee_{Y\subseteq X \text{ finite}} \bigwedge_{z\in X} \bigvee_{y\in Y} d_{B_y}(y,z) \rhd \epsilon$$

and hence there is a finite set $Y \subseteq X$ such that $X = \bigcup_{y\in Y} B^{d_{B_y}}(y,\epsilon) = \bigcup_{y\in Y} A_y$ and the proof is complete. □

For an L-metric space (X,d), we define $\chi_c((X,d)) = \chi_c((X,[d]))$.

Proposition 8.3. *Let (X,d) be an L-metric space. Then*

$$\chi_c((X,d)) = \bigvee\{\alpha \in L \ : \ \textit{there is } F \subseteq X \textit{ finite s.t. } X = \bigcup_{x\in F} B^d(x,\alpha)\}.$$

Proof. If $\epsilon \lhd \chi_c((X,d))$, then there is a finite subset $F \subseteq X$ such that for all $z \in X$ there is $x \in F$ with $d(x,z) \rhd \epsilon$. The latter is equivalent to $z \in B^d(x,\epsilon)$ and hence there is a finite subset $F \subseteq X$ such that $X = \bigcup_{x\in F} B^d(x,\epsilon)$. Hence $\epsilon \lhd \bigvee\{\alpha \in L \ : \ \text{there is } F \subseteq X \text{ finite s.t. } X = \bigcup_{x\in F} B^d(x,\alpha)\}$ and we get from the complete distributivity $\chi_c((X,d)) \leq \bigvee\{\alpha \in L \ : \ \text{there is } F \subseteq X \text{ finite s.t. } X = \bigcup_{x\in F} B^d(x,\alpha)\}$.

Conversely, let $\epsilon \lhd \bigvee\{\alpha \in L \ : \ \text{there is } F \subseteq X \text{ finite s.t. } X = \bigcup_{x\in F} B^d(x,\alpha)\}$. Then there is $\alpha \in L$ such that $X = \bigcup_{x\in F} B^d(x,\alpha)$ for some finite set $F \subseteq X$ and $\epsilon \leq \alpha$. We conclude that there is a finite set $F \subseteq X$ such that for all $z \in X$ there is $x \in F$ with $d(x,z) \rhd \alpha \geq \epsilon$ and hence $\chi_c((X,d)) \geq \epsilon$ and again from the complete distributivity we conclude $\bigvee\{\alpha \in L \ : \ \text{there is } F \subseteq X \text{ finite s.t. } X = \bigcup_{x\in F} B^d(x,\alpha)\} \leq \chi_c((X,d))$. □

Remark 8.4. *For Lawvere's quantale, we see that for a metric space (X,d) we have $\chi_c((X,d)) = 0$ if and only if (X,d) is totally bounded. So, for a general L, we call an L-metric space (X,d)* totally bounded *if $\chi_c((X,d)) = \top$.*

We are now in the position to prove the main result of this section.

Theorem 8.5. *Let $\lhd$ be multiplicative and let (X,d) be a separated and symmetric L-metric space. Then*

$$\chi_c((X,d)) = \chi_c((CL(X),H_d)) = \chi_c((CL(X),\mathcal{G}_W)).$$

Proof. We first show $\chi_c((CL(X),\mathcal{G}_W)) \le \chi_c((X,d))$. For each $A \in CL(X)$ we choose a point $x_A \in A$. For a finite subset $\mathcal{B} \subseteq CL(X)$ we define $Y_{\mathcal{B}} = \{x_A \,:\, A \in \mathcal{B}\}$. Then for $A \in \mathcal{B}$ we find $d_{\{x_A\}}(A,\{z\}) = d_{\mathsf{L}}(d(x_a,A), d(x_A,\{z\}) = d(x_A,\{z\})$ and hence

$$\bigwedge_{z\in X}\bigvee_{x\in Y_{\mathcal{B}}} d(x,z) = \bigwedge_{z\in X}\bigvee_{A\in\mathcal{B}} d_{\{x_A\}}(A,\{z\}) \ge \bigwedge_{C\in CL(X)}\bigvee_{A\in\mathcal{B}} d_{\{x_A\}}(A,C).$$

With the basis $\mathcal{H} = \{d_F \,:\, F \subseteq X \text{ finite}\}$ of the L-gauge $\mathcal{G}_W$ and $\psi \in \mathcal{H}^X$ defined by $\psi(A) = d_{\{x_A\}}$, we conclude

$$\begin{aligned}\chi_c((CL(X),\mathcal{G}_W)) &\le \bigvee_{\mathcal{B}\subseteq CL(X)\text{ finite}}\bigwedge_{C\in CL(X)}\bigvee_{A\in\mathcal{B}} d_{\{x_A\}}(A,C)\\ &\le \bigvee_{\mathcal{B}\subseteq CL(X)\text{ finite}}\bigwedge_{z\in X}\bigvee_{x\in Y_{\mathcal{B}}} d(x,z)\\ &\le \bigvee_{F\subseteq X\text{ finite}}\bigwedge_{z\in X}\bigvee_{x\in F} d(x,z) \;=\; \chi_c((X,d)).\end{aligned}$$

Next we show $\chi_c((CL(X),H_d)) \le \chi_c((CL(X),\mathcal{G}_W))$. For any mapping $\varphi \in \mathcal{H}^X$ we have $\bigwedge_{F\subset X \text{ finite}} d_F(A,C) \le \varphi(A)(A,C)$. Hence we conclude with Proposition 7.3

$$\begin{aligned}\chi_c((CL(X),H_d)) &= \bigvee_{\mathcal{F}\subseteq CL(X)\text{ finite}}\bigwedge_{C\in CL(X)}\bigvee_{A\in\mathcal{F}} H_d(A,C)\\ &= \bigvee_{\mathcal{F}\subseteq CL(X)\text{ finite}}\bigwedge_{C\in CL(X)}\bigvee_{A\in\mathcal{F}}\bigwedge_{F\subseteq X\text{ finite}} d_F(A,C)\\ &\le \bigvee_{\mathcal{F}\subseteq CL(X)\text{ finite}}\bigwedge_{C\in CL(X)}\bigvee_{A\in\mathcal{F}} \varphi(A)(A,C).\end{aligned}$$

This is true for any $\varphi \in \mathcal{H}^X$ from which the claim follows.

Finally, we show that $\chi_c((X,d)) \le \chi_c((CL(X),H_d))$. We use Proposition 8.3. Let $\alpha \lhd \chi_c((X,d))$. Then there is a finite set $F \subseteq X$ with $X = \bigcup_{x\in F} B^d(x,\alpha)$. For $A \in CL(X)$ we define $F_A = \{x \in F \,:\, A \cap B^d(x,\alpha) \neq \emptyset\}$. Then $F_A \neq \emptyset$ and is a finite subset of X. Furthermore, for $a \in A$ there is $x_A \in F_A$ such that $d(a,x_A) \rhd \alpha$ and hence $d(a,F_A) \rhd \alpha$. Also, for $x \in F_A$ there is $a \in A$ such that $d(x,a) \rhd \alpha$ and hence, by symmetry, also $d(A,x) \rhd \alpha$. From $\lhd$ being multiplicatice we conclude

$$H_d(A,F_A) = \bigwedge_{a\in A} d(a,F_A) \wedge \bigwedge_{x\in F_A} d(A,x) \rhd \alpha,$$

i.e. $A \in B^{H_d}(F_A,\alpha)$. As $A \in CL(X)$ was arbitrary, we conclude $CL(X) = \bigcup_{B\subseteq F \text{ finite}} B^{H_d}(B,\alpha)$ and hence $\chi_c((CL(X),H_d)) \ge \alpha$. From the complete distributivity, the claim follows. □

As a further application, we can compare the distance functionals in (X,H_d) and $(X,\mathcal{G}_W)$. For an L-gauge space $(X,\mathcal{G})$ we define its L-distance functional by

$$\delta^{\mathcal{G}}(x,A) = \bigwedge_{d\in\mathcal{G}}\bigvee_{a\in A} d(x,a) = \bigwedge_{d\in\mathcal{G}} d(x,A), \qquad x \in X, A \subseteq X.$$

This distance functional is also called the L-*approach distance* of the L-gauge space, see [14, 11]. It is not difficult to show that if $\widehat{\mathcal{H}} = \mathcal{G}$, then $\delta^{\mathcal{G}}(x,A) = \bigwedge_{d\in\mathcal{H}} d(x,A)$ for all $x \in X, A \subseteq X$.

Proposition 8.6. *Let (X,d) be a symmetric L-metric space, let $A \in CL(X)$ and $\mathcal{B} \subseteq CL(X)$. Then*

$$\delta^{\mathcal{G}_W}(A,\mathcal{B}) * \chi_c((X,d)) * \chi_c((X,d)) \le H_d(A,\mathcal{B}) \le \delta^{\mathcal{G}_W}(A,\mathcal{B}).$$

Proof. Let $\chi_c((X,d)) \rhd \alpha$, $A \in CL(X)$ and $\mathcal{B} \subseteq CL(X)$. Then there is a finite set $F \subseteq X$ such that $X = \bigcup_{x\in F} B^d(x,\alpha)$, i.e. for all $x \in X$ there is $y_x \in F$ such that $d(x,y_x) \rhd \alpha$. We then conclude for $B \in \mathcal{B}$, $\alpha * d(y_x,B) \le d(x,y_x) * d(y_x,B) \le d(x,B)$ and $\alpha * d(x,A) \le d(y_x,x) * d(x,A) \le d(y_x,A)$, i.e. we have

$$\alpha * \alpha \le (d(x,A) \to d(y_x,A)) * (d(y_x,B) \to d(x,B)).$$

As in general $(\alpha \to \beta) * (\gamma \to \delta) \le (\beta \to \gamma) \to (\alpha \to \delta)$ we obtain

$$\alpha * \alpha \le (d(y_x, A) \to d(y_x, B)) \to (d(x, A) \to d(x, B)),$$

i.e. we have

$$\alpha * \alpha * (d(y_x, A) \to d(y_x, B)) \le d(x, A) \to d(x, B).$$

Similarly we obtain

$$\alpha * \alpha * (d(y_x, B) \to d(y_x, A)) \le d(x, B) \to d(x, A),$$

and we conclude that for any $x \in X$ there is $y_x \in F$ such that

$$\alpha * \alpha * d_{\mathsf{L}}(d(y_x, A), d(y_x, B)) \le d_{\mathsf{L}}(d(x, A), d(x, B)).$$

Hence,

$$\begin{aligned} \alpha * \alpha * \delta^{\mathcal{G}_W}(A, \mathcal{B}) &= \alpha * \alpha * \bigwedge_{G \subseteq X \text{ finite}} \bigvee_{B \in \mathcal{B}} \bigwedge_{y \in G} d_{\mathsf{L}}(d(y, A), d(y, B)) \\ &\le \bigvee_{B \in \mathcal{B}} \bigwedge_{y \in F} (d_{\mathsf{L}}(d(y, A), d(y, B)) * \alpha * \alpha) \\ &\le \bigvee_{B \in \mathcal{B}} \bigwedge_{x \in X} d_{\mathsf{L}}(d(x, A), d(x, B)) \;=\; H_d(A, \mathcal{B}). \end{aligned}$$

This is true for any $\alpha \lhd \chi_c((X, d))$ and from the complete distributivity we obtain the first inequality.

The second inequality follows from Proposition 7.3. In fact, we have

$$\bigvee_{B \in \mathcal{B}} H_d(A, B) = \bigvee_{B \in \mathcal{B}} \bigwedge_{F \subseteq X \text{finite}} d_F(A, B) \le \bigwedge_{F \subseteq X \text{ finite}} \bigvee_{B \in \mathcal{B}} d_F(A, B) \;=\; \delta^{\mathcal{G}_W}(A, \mathcal{B}).$$

□

Corollary 8.7. *Let the symmetric* L*-metric space* (X, d) *be totally bounded. Then the distance functionals of* $(CL(X), H_d)$ *and of* $(CL(X), \mathcal{G}_W)$ *coincide.*

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