Partially continuous pretopological and topological operators for intuitionistic fuzzy sets

E. Marinov[1] and K. Atanassov[2]

[1,2] *Department of Bioinformatics and Mathematical Modelling, Institute of Biophysics and Biomedical Engineering, Bulgarian Academy of Sciences, 105 Acad. G. Bonchev Str, Sofia 1113, Bulgaria*

evgeniy.marinov@biomed.bas.bg, krat@bas.bg

Abstract

In this paper, pretopological and topological operators are introduced based on partially continuous linear transformations of the membership and non-membership functions for intuitionistic fuzzy sets. They turn out to be a generalization of the topological operators for intuitionistic fuzzy sets. On the other hand it is a generalization of the fuzzy set pretopological operators introduced by Wenzhong and Kimfung.

Keywords: Intuitionistic fuzzy set, pretopology, topology.

1 Introduction

Three main notions will be discussed in the presented paper - intuitionistic fuzzy sets, pretopological and topological spaces. We employ linear maps to introduce specific pretopological closure and interior operators for intuitionistic fuzzy sets (IFS) (see [2, 3, 4]) and α-cuts for their topological counterparts. As shown in [38], such operators can be applied to Geographic Information Systems. The results from this work will be further extended in a next research in the framework of interval valued intuitionistic fuzzy sets (IVIFS) (see [6, 7]) and a software implementation will be also proposed.

In 1968, Chang [13] introduced the notion of fuzzy topological space and examined its properties. Gayyar, Kerre, Ramadan [18] and Demirci [16, 17] introduced the concepts of fuzzy closure and fuzzy interior in the fuzzy topological space, and obtained some properties of them. We are going to state the axioms of fuzzy (pre)topological spaces, extended by Çoker [12] to intuitionistic fuzzy topological spaces. He gave concrete examples of such spaces, which can be applied to modeling real world problems.

There has been extensive research about intuitionistic fuzzy topology since its introduction by Çoker. Several M.Sc. and Ph.D. theses, and many papers by researchers have been published in this domain. For example, there are at least three books discussing topics related to topological metric spaces for IFSs [5, 10, 35, 36].

Singh and Srivastava [34] examined the separation axioms. In 2003, Lupiañez [25] defined new notions of Hausdorffness in the intuitionistic fuzzy sense, and obtained some new properties, in particular on convergence, whereas in Lupiañez [26], he introduced normality and regularity in the intuitionistic fuzzy sense and obtained relations between these concepts and also with the fuzzy notion. Again Lupiañez [24] introduced a new concept of compactness and a definition of paracompactness for intuitionistic fuzzy topological spaces, and obtained several preservation properties. Park [32, 30] introduced intuitionistic fuzzy metric spaces and Saadati [33] made extensive research on the topologically complete intuitionistic fuzzy metrizable and normed spaces; Kutlu et al. [22], Kutlu [23] made extensive research of temporal intuitionistic fuzzy Šostak topology. In 2002, Mondal and Samanta introduced the concept of intuitionistic gradations of openness [31] which is a generalization of the concept of gradation of openness defined by Chattopadyay [14]. Jin and Seok [21] investigated the categorical aspects of intuitionistic fuzzy topology and more precisely they obtained two types of adjoint functors between the category of intuitionistic fuzzy topological spaces in Mondal and Samanta's

sense, and the category of intuitionistic fuzzy topological spaces in Šostak's sense; Cong-hua and Xiao-ke [15] introduced the concept of intuitionistic I-fuzzy quasi-coincident neighbourhood systems of intuitionistic fuzzy points and studied the relation between the category of intuitionistic I-fuzzy topological spaces and the category of intuitionistic I-fuzzy quasi-coincident neighborhood spaces; Thakur and Rekha [37] introduced and studied the concept of intuitionistic fuzzy g-continuous mappings in intuitionistic fuzzy topological spaces.

Many authors extend the topological notions and theorems into the framework of IFSs but our goal in this paper is to introduce pretopological and topological operators in a more constructive way, allowing the application of computer programs, and simultaneously showing many properties and their mathematical correctness.

The notion *fuzzy set* (FS) was introduced by Lotfi Zadeh in 1965 (cf. Zadeh [39]). A fuzzy set is an object whose element's membership degree is not precisely defined. Fuzzy sets provide a better representation of reality than the classical mathematical binary representation of whether an element does or does not belong to a set. The membership in fuzzy sets is gradual, taking values in the range between "no" (0) and "yes" (1).

Since the introduction of fuzzy sets there have been some generalizations. Most of them consist of replacing the range $[0, 1]$ by more general algebraic structures satisfying the axioms for a lattice (cf. Birkhoff [11]) - they are called *L-fuzzy sets* (cf. Goguen [19]). An extension of fuzzy sets is the *intuitionistic fuzzy* set (IFS), introduced in 1983 [3, 4], where the corresponding lattice takes a natural form of triangular representation (described in the next section). In addition to the membership function of a FS, there is another function, expressing the notion of non-membership degree with the same domain X and range $[0, 1]$, so that the sum of the membership and non-membership degrees may not exceed 1. That is, in the framework of IFSs we have an additional degree of information, expressing the lack of knowledge/information, that makes the theory invaluable to extend the uncertainty of the limited level of crisp and even fuzzy precision in real world situations and preferences.

The operations of inclusion, union, intersection, complement are extended from the ordinary to the intuitionistic fuzzy sets. These operations are actually needed the notion of *pretopological* and *topological spaces* to be introduced (Arkhangelskii and Fedorchuk [1]). Many mathematicians and scientists actively employ concepts of topology to model and understand real-world structures and problems. A rich variety of results also has emerged in other areas of applied mathematics stemming from pure topological investigations. As topology originally grew up from geometry, it is often described as a rubber-sheet geometry - that means, literally, the study of position or location of points (elements) belonging to a given set called *topological space*. Distances are not always relevant in the framework of topology but the notion of *proximity* is a very important concept, which is established by specifying a collection of subsets of the considered topological space called *open sets*. Open sets and their counterparts - *closed sets* in a topological space are often defined by interior and closure operators. What a topologist can do is to identify and use the properties of objects that different shapes have in common. Often, the properties that are significant are those that are preserved when we treat objects as deformable, as opposed when we treat them as rigid bodies. Such specific situations emerge in many areas of applied mathematics, physics, biology, geographic information systems and system theory.

In systems described mainly by logic and algebra, topological axioms are too restrictive. In 1967 Hammer [20] showed that by some extensions of the principles of general topology it is possible to model almost any formal system. He introduces the *isotonic spaces* which are later called *pretopogical spaces* [1, Ch. 2.5]. In 1981, Robert Badard [9] introduced the *fuzzy pretopological spaces* and showed many of their properties.

2   Introduction to intuitionistic fuzzy sets

A fuzzy set (FS) in X (cf. Zadeh [39]) is given by

$$A = \{\langle x, \mu_A(x)\rangle | x \in X\} \tag{1}$$

where $\mu_A(x) \in [0, 1]$ is the *membership function* of the FS A. The intuitionistic fuzzy sets (IFSs, cf. [2, 3, 4]), are extensions of FSs with the form

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in X\} \tag{2}$$

where: $\mu_A : X \rightarrow [0, 1]$ and $\nu_A : X \rightarrow [0, 1]$ such that $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ and $\mu_A(x), \nu_A(x) \in [0, 1]$ denote a *degree of membership* and a *degree of non-membership* of $x \in A$, respectively. An additional concept for each IFS in X, $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$, a *degree of uncertainty* of $x \in A$. It expresses a lack of knowledge of whether x belongs to A or not (cf. [3]). It is obvious that $0 \leq \pi_A(x) \leq 1$, for each $x \in X$. Uncertainty degree turns out to be relevant for both - applications and the development of theory of IFSs.

Talking about *partial ordering* in IFSs, we will by default mean $(IFS(X), \leq)$ where $\leq$ stands for the standard partial ordering in IFS(X). The partially ordered set is called *poset*. For any two A and $B \in IFS(X)$: $A \leq B$ is satisfied if and only if $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$ for any $x \in X$. On Fig. 1 one may see the triangular representation of

the two chosen A and B in a particular point $x \in X$, where $f_A(x)$ stands for the point on the plane with coordinates $(\mu_A(x), \nu_A(x))$. The poset $(IFS(X), \leq)$ is actually a lattice (cf. [29]), which means that for every subset $S \subseteq IFS(X)$ there are $\inf(S), \sup(S) \in IFS(X)$ such that $A_0(S)$ is the *greatest lower bound* (infimum) and $A_1(S)$ is the *least upper bound* (supremum) of $S \in IFS(X)$.

Definition 2.1. *In the framework of the above notions, if we take $S = IFS(X)$, then for this paper let us define*

$$O_X^*(\text{ or } O^*(X)) := \inf(IFS(X)) = \{\langle x, 0, 1 \rangle \mid x \in X\},$$

corresponding to the point $\langle 0, 1 \rangle$ on Fig. 1 and

$$E_X^*(\text{ or } E^*(X)) = \sup(IFS(X)) = \{\langle x, 1, 0 \rangle \mid x \in X\},$$

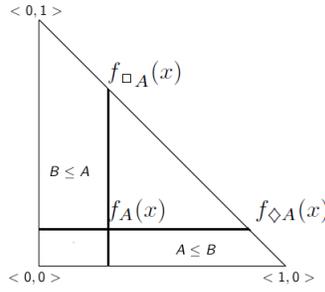*corresponding to the point $\langle 1, 0 \rangle$ on Fig. 1.*

Figure 1: Triangular representation of the the intuitionistic fuzzy sets A and $B \in IFS(X)$ in a particular point $x \in X$, where $f_A(x)$ stands for the point on the plane with coordinates $(\mu_A(x), \nu_A(x))$.

Let us recall the definitions and some properties of the modal operators on intuitionistic fuzzy sets as introduced originally in [2]. For more detailed descriptions and properties the reader may refer to [4], Ch. 4.1., although we introduce now some new statements and consider from various points of view. "Necessity" and "possibility" operators (denoted $\square$ and $\Diamond$, respectively) applied to an intuitionistic fuzzy set $A \in IFS(X)$ have been defined as:

$$\square A = \{\langle x, \mu_A(x), 1 - \mu_A(x) \rangle | x \in X\},$$
$$\Diamond A = \{\langle x, 1 - \nu(x), \nu_A(x) \rangle | x \in X\}$$

From the above definition it is evident that

$$\star \colon IFS(X) \longrightarrow FS(X) \tag{3}$$

where $\star$ is the prefix operator $\star \in \{\square, \Diamond\}$, operating on the class of intuitionistic fuzzy sets. Let us take any $A, B \in IFS(X)$ and define $A \leq_\square B$ iff $\mu_A \leq \mu_B$ on X, respectively $A \leq_\Diamond B$ iff $\nu_A \geq \nu_B$ on X. Obviously both $\leq_\square$ and $\leq_\Diamond$ are reflexive and transitive. That is, they are both quasi-orderings in IFS(X) which will be called *quasi $\square$-ordering* and *quasi $\Diamond$-ordering*, respectively. For more information and examples of quasi-orderings, the reader may consult the book of Birkhoff [11], Ch. II.1. and such orderings concerning IFSs, [29].

Let us remind the first topological operators introduced in [4, Ch. 4.2.], For every $A \in IFS(X)$,

$$\mathcal{C}(A) = \{\langle x, K_A, L_A \rangle | x \in X\} \tag{4}$$

where

$$K_A := \sup_{y \in X} \mu_A(y), L_A := \inf_{y \in X} \nu_A(y) \tag{5}$$

and

$$\mathcal{I}(A) = \{\langle x, k_A, l_A \rangle | x \in X\} \tag{6}$$

where

$$k_A = \inf_{y \in X} \mu_A(y), l_A = \sup_{y \in X} \nu_A(y) \tag{7}$$

The following operators are defined in [8], as extensions of the two topological operators $\mathcal{C}$ and $\mathcal{I}$:

$$\mathcal{C}_\mu(A) = \{\langle x, K_A, \min(1 - K_A, \nu_A(x))\rangle | x \in X\} \tag{8}$$

$$\mathcal{C}_\nu(A) = \{\langle x, \mu_A(x), L_A\rangle | x \in X\} \tag{9}$$

$$\mathcal{I}_\mu(A) = \{\langle x, k_A, \nu_A(x)\rangle | x \in X\} \tag{10}$$

$$\mathcal{I}_\nu(A) = \{\langle x, \min(1 - l_A, \mu_A(x)), l_A\rangle | x \in X\} \tag{11}$$

The geometrical interpretations of these operators applied on the IFS A are shown in [4, Figs. 4.8-4.11].

3 Pretopological, topological operators and their fuzzy representatives

Let us define first the preclosure and preinterior operators. We consider the universe X. X is a pretopological space in respect to the *preclosure* operator $\mathfrak{c} \colon \mathcal{X} \to \mathcal{X}$, where $\mathcal{X}$ can be $\mathcal{P}(X), FS(X)$ or $IFS(X)$ iff for any $A, B \in \mathcal{X}$ the following axioms are satisfied (cf [1, Ch. 2.5], and [9]):

1. $\mathfrak{c}(\emptyset) = \emptyset$

2. $A \subseteq \mathfrak{c}(A)$

3. $\mathfrak{c}(A \cup B) = \mathfrak{c}(A) \cup \mathfrak{c}(B)$

If in addition to the above stated axioms the operator $\mathfrak{c}$ is *idempotent*, that is $\mathfrak{c}(A) = \mathfrak{c}(\mathfrak{c}(A))$, then $\mathfrak{c}$ is called *closure* operator in $\mathcal{X}$.

Definition 3.1 (Fixed point)**.** *Let $f : Z \longrightarrow Z$ be a function, where Z is an arbitrary set. Then $z_0 \in Z$ is called fixed point if $f(z_0) = z_0$.*

Remark 3.2. *The preclosure operator $\mathfrak{c}$ is idempotent iff for every $A \in \mathcal{X} : \mathfrak{c}(A)$ is a fixed point for $\mathfrak{c}$, i.e. $\mathfrak{c}(\mathfrak{c}(A)) = \mathfrak{c}(A)$.*

An example of (idempotent) closure operator is given in [4] and [3, Ch. 1.6] in the case of IFSs, defined by

$$C(A) = \{\langle x, \sup_{y \in X} \mu_A(y), \inf_{y \in X} \nu_A(y)\rangle \mid x \in X\}.$$

Proposition 3.3. *The preclosure operator $\mathfrak{c}$ is non-decreasing in respect to the partial ordering $\subseteq$ in $\mathcal{X}$. That is, for all $A, B \in \mathcal{X}$, $A \subseteq B \Rightarrow \mathfrak{c}(A) \subseteq \mathfrak{c}(B)$.*

Proof. Since $B = A \cup B$ and from the second axiom for preclosure $\mathfrak{c}(A \cup B) = \mathfrak{c}(A) \cup \mathfrak{c}(B)$, then $\mathfrak{c}(B) = \mathfrak{c}(A \cup B) = \mathfrak{c}(A) \cup \mathfrak{c}(B) \supseteq \mathfrak{c}(A)$. $\qquad\square$

Definition 3.4 (see [1, Ch. 2.5])**.** *For the preclosure $\mathfrak{c}$ defined on $\mathcal{X}$ we say that a set $A \in \mathcal{X}$ is closed iff $\mathfrak{c}(A) = A$. That is, the closed sets are exactly the fixed points of $\mathfrak{c}$ and $\tau^{\mathfrak{c}} = \{A \mid A \in \mathcal{X} \ \& \ \mathfrak{c}(A) = A\}$, is the topology generated by the preclosure operator $\mathfrak{c}$. If $\mathcal{X}$ is $\mathcal{P}(X), FS(X)$ or $IFS(X)$, then τ is called crisp topology, fuzzy topology or intuitionistic fuzzy topology, respectively.*

There is a very important property for the notion of closed sets, namely,

Theorem 3.5 (see [1, Ch. 2.5])**.** *For the preclosure operator $\mathfrak{c}$ defined on $\mathcal{X}$ and every family of closed sets $B_j \in \tau^{\mathfrak{c}}, j \in J$ their intersection is also a closed set. That is, $(\bigcap_{j \in J} B_j) \in \tau^{\mathfrak{c}}$.*

For every preclosure operator on $\mathcal{X}$, and any $B \subseteq \mathcal{X}$ we define the *closure* of B,

$$Cl_{\mathfrak{c}}(B) = \bigcap \{B_1 \mid B_1 \supseteq B \ \& \ \mathfrak{c}(B_1) = B_1\}. \tag{12}$$

But Theorem 3.5 implies that $Cl_{\mathfrak{c}}(B)$ is closed set for every $B \subseteq \mathcal{X}$. And obviously $Cl_{\mathfrak{c}}(B)$ is the smallest closed set containing B.

Remark 3.6. *For every $B \subseteq \mathcal{X}$ and every preclosure operator $\mathfrak{c}$ in $\mathcal{X}$,*

$$B = \mathfrak{c}^0(B) \subseteq \mathfrak{c}^1(B) \subseteq \mathfrak{c}^2(B) \subseteq \cdots \subseteq Cl_{\mathfrak{c}}(B)$$

and $Cl_{\mathfrak{c}}(B)$ is the smallest closed set containing $\mathfrak{c}^m(B)$, for all $m \in \mathbb{N}$.

Analogically to $\mathfrak{c}$, we define $\mathfrak{i}$, the *preinterior* and *interior* operators in $\mathcal{X}$. $\mathfrak{i}$ is preinterior operator if the following axioms are satisfied (see [1, Ch. 2.5]),

1. $\mathfrak{i}(\mathcal{X}) = \mathcal{X}$

2. $\mathfrak{i}(A) \subseteq A$

3. $\mathfrak{i}(A \cap B) = \mathfrak{i}(A) \cap \mathfrak{i}(B)$

If in addition to the above stated axioms the operator $\mathfrak{i}$ is *idempotent*, that is $\mathfrak{i}(A) = \mathfrak{i}(\mathfrak{i}(A))$, then $\mathfrak{i}$ is called *interior* operator in $\mathcal{X}$ and $\mathcal{X}$ equipped with this operator has a topological structure.

An example of (idempotent) interior operator is given in [4] and [3, Ch. 1.6] in the case of IFSs, defined by

$$I(A) = \{\langle x, \inf_{y \in X} \mu_A(y), \sup_{y \in X} \nu_A(y) \rangle \mid x \in X\}.$$

The proof of the next proposition is analogical to Proposition 3.3.

Proposition 3.7 (see [1, Ch. 2.5])**.** *The preinterior operator $\mathfrak{i}$ is non-decreasing in respect to the partial ordering $\subseteq$ in $\mathcal{X}$. That is, for all $A, B \in \mathcal{X}$, $A \subseteq B \Rightarrow \mathfrak{i}(A) \subseteq \mathfrak{i}(B)$.*

Definition 3.8 ([1, see Ch. 2.5])**.** *For the preinterior $\mathfrak{i}$ defined on $\mathcal{X}$ we say that a set $A \in \mathcal{X}$ is open iff $\mathfrak{i}(A) = A$. That is, the open sets are exactly the fixed points of $\mathfrak{i}$ and $\tau_{\mathfrak{i}} = \{A \mid A \in \mathcal{X} \ \& \ \mathfrak{i}(A) = A\}$, is the topology generated by the preinterior operator $\mathfrak{i}$. If $\mathcal{X}$ is $\mathcal{P}(X)$ or $FS(X)$, then τ is called crisp topology or fuzzy topology, respectively.*

Now, similarly to Theorem 1, we can prove

Theorem 3.9 (see [1, Ch. 2.5])**.** *For the preinterior $\mathfrak{i}$ defined on $\mathcal{X}$ and every family of open sets $A_j \in \tau_{\mathfrak{i}}, j \in J$ their union is also an open set. That is, $\bigcup_{j \in J} A_j \in \tau_{\mathfrak{i}}$.*

For every preinterior operator on $\mathcal{X}$, and any $A \subseteq \mathcal{X}$ we define the *interior* of A,

$$Int_{\mathfrak{i}}(A) = \bigcup \{A_0 \mid A_0 \subseteq A \ \& \ \mathfrak{i}(A_0) = A_0\}. \tag{13}$$

But Theorem 3.9 implies that $Int_{\mathfrak{i}}(A)$ is an open set for every $A \subseteq \mathcal{X}$. And obviously $Int_{\mathfrak{i}}(A)$ is the largest open set contained in A.

Remark 3.10. *For every $A \subseteq \mathcal{X}$ and preinterior operator $\mathfrak{i}$ in $\mathcal{X}$,*

$$A \supseteq \mathfrak{i}^0(A) \supseteq \mathfrak{i}^1(A) \supseteq \mathfrak{i}^2(A) \supseteq \cdots \supseteq Int_{\mathfrak{i}}(A).$$

and $Int_{\mathfrak{i}}(A)$ is the largest open set contained in $\mathfrak{i}^m(A)$ for all $m \in \mathbb{N}$.

Every (pre)closure operator has its correspondent (pre)interior operator and vice versa. Let $\mathfrak{c}$ be a preclosure operator in $\mathcal{X}$ and let us define $\delta(A) := \neg\mathfrak{c}(\neg A)$. Then δ is a preinterior operator in $\mathcal{X}$, i.e. the axioms for preinterior are satisfied:

1. $\delta(X) = \neg\mathfrak{c}(\neg X) = \neg\mathfrak{c}(\emptyset) = \neg\emptyset = X$;

2. $\mathfrak{c}(\neg A) \supseteq \neg A$ and therefore $A \supseteq \neg\mathfrak{c}(\neg A) = \delta(A)$;

3. $\delta(A \cap B) = \delta(A) \cap \delta(B)$.

Indeed, for the last axiom $\delta(A \cap B) = \neg\mathfrak{c}(\neg(A \cap B)) = \neg\mathfrak{c}(\neg A \cup \neg B)$. But $\mathfrak{c}(\neg A \cup \neg B) = \mathfrak{c}(\neg A) \cup \mathfrak{c}(\neg B)$ and therefore,

$$\delta(A \cap B) = \neg\mathfrak{c}(\neg A) \cap \neg\mathfrak{c}(\neg B) = \delta(A) \cap \delta(B).$$

Moreover, if $\mathfrak{c}$ is a topological closure, i.e. $\mathfrak{c}(A) = \mathfrak{c}^2(A)$ for every $A \in \mathcal{X}$, then δ is also idempotent.

Analogically, if $\mathfrak{i}$ is a (pre)interior operator, then $\mathfrak{c}(A) := \neg\mathfrak{i}(\neg A)$ is its corresponding (pre)closure. It is clear now that the family of open sets is composed exactly of the complements of the above defined closed sets if we consider the pair (pre)closure - (pre)interior operators as conjugate pair operators. Obviously, we have shown the validity of the following proposition.

Proposition 3.11 (see [1, Ch. 2.5])**.** *If* $\mathfrak{i}$ *and* $\mathfrak{c}$ *is a conjugate pair of preinterior and preclosure operators in* $\mathcal{X}$, *then*

$$\tau^{\mathfrak{c}} = \{\neg A \mid A \in \tau_{\mathfrak{i}}\} \ and \ \tau_{\mathfrak{i}} = \{\neg B \mid B \in \tau^{\mathfrak{c}}\}.$$

In [28] are introduced some examples of fuzzy pretopological and topological spaces. Let us remind the definitions from [28] and then, in the next section we state generalizations in the framework of IFSs.
For every $\alpha \in [0, 1]$ we have the closure operator:

$$\mathcal{C}_\alpha : FS(X) \longrightarrow FS(X),$$

such that

$$\mu_{\mathcal{C}_\alpha(A)}(x) = \begin{cases} \mu_A(x) & \text{if } \mu_A(x) \le \alpha \\ 1 & \text{if } \alpha < \mu_A(x) \end{cases} \tag{14}$$

Moreover, the equality sign should really be in the first case in (14). Otherwise, if it were in the second case, if $\alpha = 0$ and $\mu_A \equiv 0$ ($A = \emptyset$), we would get that $\mathcal{C}_0(\emptyset) = X \ne \emptyset$. For every $\alpha \in [0, 1]$ (cf. [28])we define the interior operator:

$$\mathcal{I}_\alpha : FS(X) \longrightarrow FS(X),$$

such that

$$\mu_{\mathcal{I}_\alpha(A)}(x) = \begin{cases} 0 & \text{if } \mu_A(x) < \alpha \\ \mu_A(x) & \text{if } \alpha \le \mu_A(x) \end{cases} \tag{15}$$

It is clear now that $\mathcal{C}_\alpha(A) = \neg \mathcal{I}_{1-\alpha}(\neg A)$. Therefore, the pair $(\mathcal{C}_\alpha, \mathcal{I}_{1-\alpha})$ is a conjugate pair of closure-interior operators defining the same topological structure in $FS(X)$.
Let us remind also the generalizations of the above operators. For every $\alpha, \beta, \gamma \in [0, 1]$ we define the preclosure and preinterior operators:

$$\mathcal{I}_\alpha^\gamma, \mathcal{C}_\beta^\gamma(A) : FS(X) \longrightarrow FS(X),$$

such that (see Figure 2)

$$\mu_{\mathcal{C}_\beta^\gamma(A)}(x) = \begin{cases} \mu_A(x) & \text{if } \mu_A(x) \le \beta \\ \frac{1}{1-\gamma}(\mu_A(x) - \beta) + \beta & \text{if } \beta < \mu_A(x) \le 1 - \gamma(1-\beta) \\ 1 & \text{if } 1 - \gamma(1-\beta) < \mu_A(x) \le 1 \end{cases} \tag{16}$$

$$\mu_{\mathcal{I}_\alpha^\gamma(A)}(x) = \begin{cases} 0 & \text{if } 0 \le \mu_A(x) < \gamma.\alpha, \\ \frac{1}{1-\gamma}(\mu_A(x) - \alpha) + \alpha & \text{if } \gamma.\alpha \le \mu_A(x) < \alpha, \\ \mu_A(x) & \text{if } \alpha \le \mu_A(x) \le 1. \end{cases} \tag{17}$$

The above definition is well defined even for $\gamma = 1$ (although $\frac{1}{1-\gamma}(\mu_A(x) - \alpha) + \alpha$ is not defined) since in this case the condition in the second expression will become $\alpha \le \mu_A(x) < \alpha$. And therefore it will not be satisfied for any value of $\mu_A(x), x \in X$.

Theorem 3.12 (see [1, Ch. 2.5])**.** $(\mathcal{C}_{1-\alpha}^\gamma, \mathcal{I}_\alpha^\gamma)$ *is a pair of conjugate preclosure-preinterior operators, which define the same topology*

$$\tau_{\mathcal{I}_\alpha^\gamma} = \{\neg B \mid B \in \tau^{\mathcal{C}_{1-\alpha}^\gamma}\} \tag{18}$$

in $FS(X)$.
     Moreover, from the definition of $\tau_{\mathcal{I}_\alpha^\gamma}$ *if follows that*

$$\tau_{\mathcal{I}_\alpha^\gamma} = \{A \mid A \in FS(A) \ \& \ \mu_A(x) \in \{0\} \cup [\alpha, 1]\} \tag{19}$$

$$\tau^{\mathcal{C}_\beta^\gamma} = \{B \mid B \in FS(B) \ \& \ \mu_A(B) \in \{1\} \cup [0, \beta]\}. \tag{20}$$
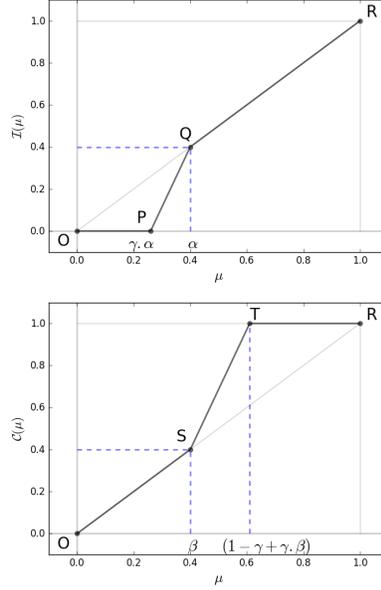
Figure 2: The first graphic represents the partially continuous map corresponding to the definition of the interior operator $\mathcal{I}_\alpha^\gamma$ (17). The second graphic represents the map corresponding to the definition of the closure operator $\mathcal{C}_\alpha^\gamma$ (16).

4 Partially linear IF (pre)topological operators

In this section we generalize the notion of (pre)interior and (pre)closure operators from [28] and show that in the intuitionistic fuzzy sense they are much richer source for research and versatile results. That is, $\mathfrak{i} : IFS(X) \longrightarrow IFS(X)$, is a preinterior operator if for all $A, B \in IFS(X)$,

1. $\mathfrak{i}(E_X^*) = E_X^*$

2. $\mathfrak{i}(A) \subseteq A$

3. $\mathfrak{i}(A \cap B) = \mathfrak{i}(A) \cap \mathfrak{i}(B)$

We introduce now a new form of topological operators for IFSs, which are analogical to $\mathcal{I}_\alpha^\gamma$ (17) and $\mathcal{C}_\alpha^\gamma$ (16).

$$\mathcal{I}_{\mu;\alpha,\beta}^{\gamma_\alpha,\gamma_\beta} : IFS(X) \longrightarrow IFS(X),$$

shown in Fig. 3 such that for every

$$0 \le \alpha, \beta \le 1, \alpha + \beta \le 1 \text{ and } 0 \le \gamma_\alpha, \gamma_\beta \le 1, \tag{21}$$

$$\mu_{\mathcal{I}_{\mu;\alpha,\beta}^{\gamma_\alpha,\gamma_\beta}(A)}(x) = \begin{cases} 0 & \text{if } 0 \le \mu_A(x) < \gamma_\alpha.\alpha \\ \frac{1}{1-\gamma_\alpha}(\mu_A(x) - \alpha) + \alpha & \text{if } \gamma_\alpha.\alpha \le \mu_A(x) < \alpha \\ \mu_A(x) & \text{if } \alpha \le \mu_A(x) \le 1 \end{cases} \tag{22}$$

$$\nu_{\mathcal{I}_{\mu;\alpha,\beta}^{\gamma_\alpha,\gamma_\beta}(A)}(x) = \begin{cases} \min((1-\gamma_\beta)\nu_A(x) + \beta\gamma_\beta, 1 - \mu_{\mathcal{I}_{\mu;\alpha,\beta}^{\gamma_\alpha,\gamma_\beta}(A)}(x)) & \text{if } 0 \le \nu_A(x) \le \beta \\ \nu_A(x) & \text{if } \beta < \nu_A(x) \le 1 \end{cases} \tag{23}$$

Let us note that (22) is well defined even if $\gamma_\alpha = 1$. In case $\gamma_\alpha = 1$ the condition $\gamma_\alpha.\alpha \le \mu_A(x) < \alpha$ is never satisfied. If we define the following auxiliary functions

$$f_\alpha^{\gamma_\alpha}(t) = \begin{cases} 0 & \text{if } 0 \le t < \gamma_\alpha.\alpha, \\ \frac{1}{1-\gamma_\alpha}(t-\alpha) + \alpha & \text{if } \gamma_\alpha.\alpha \le t < \alpha, \\ t & \text{if } \alpha \le t < 1. \end{cases} \tag{24}$$

$$g_\beta^{\gamma_\beta}(t) = \begin{cases} (1-\gamma_\beta)t + \beta\gamma_\beta & \text{if } 0 \le t \le \beta, \\ t & \text{if } \beta < t \le 1. \end{cases} \tag{25}$$

then the preinterior operator $\mathcal{I}_{\mu;\alpha,\beta}^{\gamma_\alpha,\gamma_\beta}$ can be defined in a more compact form as

$$\mu_{\mathcal{I}_{\mu;\alpha,\beta}^{\gamma_\alpha,\gamma_\beta}(A)}(x) = f_\alpha^{\gamma_\alpha}(\mu_A(x)) \tag{26}$$

$$\nu_{\mathcal{I}_{\mu;\alpha,\beta}^{\gamma_\alpha,\gamma_\beta}(A)}(x) = \min(g_\beta^{\gamma_\beta}(\nu_A(x)), 1 - \mu_{\mathcal{I}_{\mu;\alpha,\beta}^{\gamma_\alpha,\gamma_\beta}(A)}(x)) \tag{27}$$
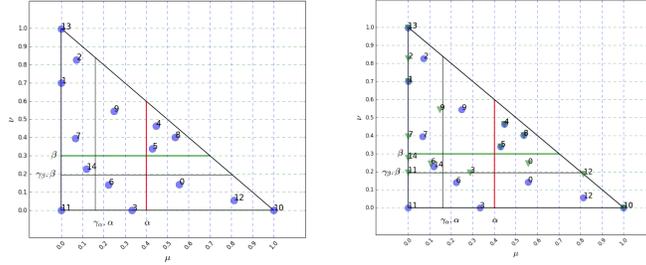

Figure 3: Example of a triangular representation of the the intuitionistic fuzzy sets A with a universum of 13 elements and corresponding topological parameters $\alpha, \beta, \gamma_\alpha, \gamma_\beta$ and the result of the application of the preinterior operator $\mathcal{I}_{\mu;\alpha,\beta}^{\gamma_\alpha,\gamma_\beta}$ on $A \in IFS(X)$

Remark 4.1. *The equations (23) and (27) are equivalent representations of $\nu_{\mathcal{I}_{\mu;\alpha,\beta}^{\gamma_\alpha,\gamma_\beta}(A)}$.*

Proof. For $x \in X$ such that $0 \le \nu_{\mathcal{I}_{\mu;\alpha,\beta}^{\gamma_\alpha,\gamma_\beta}(A)}(x) \le \beta$, the above remark is obvious. Let us suppose that $\beta < \nu_{\mathcal{I}_{\mu;\alpha,\beta}^{\gamma_\alpha,\gamma_\beta}(A)}(x) \le 1$. Taking into account that $\mu_{\mathcal{I}_{\mu;\alpha,\beta}^{\gamma_\alpha,\gamma_\beta}(A)}(x) \le \mu_A(x)$ and $1 - \mu_A(x) \ge \nu_A(x)$, then $1 - \mu_{\mathcal{I}_{\mu;\alpha,\beta}^{\gamma_\alpha,\gamma_\beta}(A)}(x) \ge 1 - \mu_A(x) \ge \nu_A(x)$. And therefore, $\min(g_\beta^{\gamma_\beta}(\nu_A(x)), 1 - \mu_{\mathcal{I}_{\mu;\alpha,\beta}^{\gamma_\alpha,\gamma_\beta}(A)}(x)) = \min(\nu_A(x), 1 - \mu_{\mathcal{I}_{\mu;\alpha,\beta}^{\gamma_\alpha,\gamma_\beta}(A)}(x)) = \nu_A(x)$ for $\beta < \nu_{\mathcal{I}_{\mu;\alpha,\beta}^{\gamma_\alpha,\gamma_\beta}(A)}(x) \le 1$. The remark is proved. $\square$

On Figure 4 are plotted the graphs corresponding to the definitions of $\mu_{\mathcal{I}_{\mu;\alpha,\beta}^{\gamma_\alpha,\gamma_\beta}}$ and $\nu_{\mathcal{I}_{\mu;\alpha,\beta}^{\gamma_\alpha,\gamma_\beta}}$ and we have the following

Proposition 4.2. *For every $A \in IFS(X)$, the above defined $\mathcal{I}_{\mu;\alpha,\beta}^{\gamma_\alpha,\gamma_\beta}(A)$ through the membership degree $\mu_{\mathcal{I}_{\mu;\alpha,\beta}^{\gamma_\alpha,\gamma_\beta}(A)}$ and non-membership degree $\nu_{\mathcal{I}_{\mu;\alpha,\beta}^{\gamma_\alpha,\gamma_\beta}(A)}$, is indeed a valid element of $IFS(X)$.*

Proof. An easy check shows that $\mu_{\mathcal{I}_{\mu;\alpha,\beta}^{\gamma_\alpha,\gamma_\beta}(A)}(x), \nu_{\mathcal{I}_{\mu;\alpha,\beta}^{\gamma_\alpha,\gamma_\beta}(A)}(x) \in [0,1]$ for all $x \in X$. It is enough to check that

$$\mu_{\mathcal{I}_{\mu;\alpha,\beta}^{\gamma_\alpha,\gamma_\beta}(A)}(x) + \nu_{\mathcal{I}_{\mu;\alpha,\beta}^{\gamma_\alpha,\gamma_\beta}(A)}(x) \le 1$$

for all $x \in X$. Suppose that $\nu_A(x) \le \beta$, then $\mu_{\mathcal{I}_{\mu;\alpha,\beta}^{\gamma_\alpha,\gamma_\beta}(A)}(x) + \nu_{\mathcal{I}_{\mu;\alpha,\beta}^{\gamma_\alpha,\gamma_\beta}(A)}(x) = \mu_{\mathcal{I}_{\mu;\alpha,\beta}^{\gamma_\alpha,\gamma_\beta}(A)}(x) + \min((1-\gamma_\beta)\nu_A(x) + \beta\gamma_\beta, 1 - \mu_{\mathcal{I}_{\mu;\alpha,\beta}^{\gamma_\alpha,\gamma_\beta}(A)}(x)) \le \mu_{\mathcal{I}_{\mu;\alpha,\beta}^{\gamma_\alpha,\gamma_\beta}(A)}(x) + 1 - \mu_{\mathcal{I}_{\mu;\alpha,\beta}^{\gamma_\alpha,\gamma_\beta}(A)}(x) = 1$ by the definition of $\nu_{\mathcal{I}_{\mu;\alpha,\beta}^{\gamma_\alpha,\gamma_\beta}(A)}$. And if $\nu_A(x) > \beta$, then $\nu_{\mathcal{I}_{\mu;\alpha,\beta}^{\gamma_\alpha,\gamma_\beta}(A)}(x) = \nu_A(x)$ and since $\mu_{\mathcal{I}_{\mu;\alpha,\beta}^{\gamma_\alpha,\gamma_\beta}(A)}(x) \le \mu_A(x)$, we have that

$$\nu_{\mathcal{I}_{\mu;\alpha,\beta}^{\gamma_\alpha,\gamma_\beta}(A)}(x) + \mu_{\mathcal{I}_{\mu;\alpha,\beta}^{\gamma_\alpha,\gamma_\beta}(A)}(x) \le \nu_A(x) + \mu_A(x) \le 1.$$

The proposition is proved. $\square$

Proposition 4.3. *The operator $\mathcal{I}_{\mu;\alpha,\beta}^{\gamma_\alpha,\gamma_\beta}$ is a valid preinterior operator in $IFS(X)$, i.e. it satisfies the three corresponding axioms.*

Proof. Let us suppose that $A = X^* \in IFS(X)$. Then from the definition of (22), $\mu_{\mathcal{I}_{\mu;\alpha,\beta}^{\gamma_\alpha,\gamma_\beta}(A)} \equiv 1$ and therefore from (23),

$$\nu_{\mathcal{I}_{\nu;\alpha,\beta}^{\gamma_\alpha,\gamma_\beta}(A)} = \min((1-\gamma_\beta)0 + \beta\gamma_\beta, 1-1) = 0.$$

Hence, $\mathcal{I}_{\mu;\alpha,\beta}^{\gamma_\alpha,\gamma_\beta}(X^*) = X^*$ and the first axiom is satisfied. The second axiom follows from the fact that $\mu_{\mathcal{I}_{\mu;\alpha,\beta}^{\gamma_\alpha,\gamma_\beta}(A)} \leq \mu_A$ and $\nu_{\mathcal{I}_{\mu;\alpha,\beta}^{\gamma_\alpha,\gamma_\beta}(A)} \geq \nu_A$. The third axiom is a trivial check. The proposition is proved. $\square$

From the definition of $\mathcal{I}_{\mu;\alpha,\beta}^{\gamma_\alpha,\gamma_\beta}$ and more intuitively from Fig. 4 and Fig. 3, we obtain almost directly the following remark.
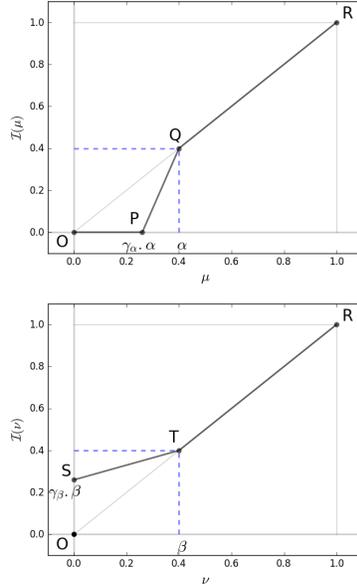

Figure 4: Auxiliary functions for the membership $f_\alpha^{\gamma_\alpha}(\mu) \sim \mathcal{I}(\mu)$ and the non-membership $g_\beta^{\gamma_\beta}(\nu) \sim \mathcal{I}(\nu)$ functions of the family of preinterior operators depending on the parameters $\alpha, \beta, \gamma_\alpha, \gamma_\beta$.

Remark 4.4. *The family of open IFSs of the pretopological space $(IFS(X), \tau_{\mathcal{I}_{\mu;\alpha,\beta}^{\gamma_\alpha,\gamma_\beta}})$, i.e. the fixed points of the operator $\mathcal{I}_{\mu;\alpha,\beta}^{\gamma_\alpha,\gamma_\beta}$, consists of $A \in IFS(X)$ such that for all $x \in X$, $\mathcal{I}_{\mu;\alpha,\beta}^{\gamma_\alpha,\gamma_\beta}(A) = A$ iff $(\alpha \leq \mu_A(x) \leq 1 - \beta$ and $\beta \leq \nu_A(x) \leq 1 - \alpha)$ or $(\mu_A(x) = 0$ and $\nu_A(x) \geq \beta)$.*

Proposition 4.5. *For any $\alpha, \beta \in [0,1]$ and $\gamma_\alpha, \gamma_\beta \in (0,1]$ we have that $\lim_{n\to\infty}(\mathcal{I}_{\mu;\alpha,\beta}^{\gamma_\alpha,\gamma_\beta})^n = \mathcal{I}_{\mu;\alpha,\beta}^{1,1}$ and hence $\tau_{\mathcal{I}_{\mu;\alpha,\beta}^{\gamma_\alpha,\gamma_\beta}} = \tau_{\mathcal{I}_{\mu;\alpha,\beta}^{1,1}}$.*

Proof. We have to show that for every $A \in IFS(X)$ and $x \in X$:

$$\lim_{n\to\infty}\mu_{(\mathcal{I}_{\mu;\alpha,\beta}^{\gamma_\alpha,\gamma_\beta})^n(A)}(x) = \mu_{\mathcal{I}_{\mu;\alpha,\beta}^{1,1}(A)}(x)$$

$$\lim_{n\to\infty}\nu_{(\mathcal{I}_{\mu;\alpha,\beta}^{\gamma_\alpha,\gamma_\beta})^n(A)}(x) = \nu_{\mathcal{I}_{\mu;\alpha,\beta}^{1,1}(A)}(x).$$

The above expressions follow geometrically from Fig 4 but let us write them out analytically too.
Let us first show the first equality. It is enough to check it for $\mu_A(x) < \alpha$ since for $\mu_A(x) \geq \alpha$ it is trivial by the

definition of $\mathcal{I}_{\mu;\alpha,\beta}^{\gamma_\alpha,\gamma_\beta}$. If $\mu_A(x) \leq \gamma_\alpha.\alpha$, then $\mu_{\mathcal{I}_{\mu;\alpha,\beta}^{\gamma_\alpha,\gamma_\beta}(A)}(x) = 0 = \mu_{\mathcal{I}_{\mu;\alpha,\beta}^{1,1}(A)}(x)$. Thereby, suppose that $\gamma_\alpha.\alpha < \mu_A(x) < \alpha$. Then since

$$\mu_{(\mathcal{I}_{\mu;\alpha,\beta}^{\gamma_\alpha,\gamma_\beta})^n(A)} = \frac{1}{(1-\gamma_\alpha)^n}(\mu_A(x) - \alpha) + \alpha,$$

we are looking for $n \in \mathbb{N}$ such that the above expression is less or equal than $\gamma_\alpha.\alpha$. That is,

$$\frac{1}{(1-\gamma_\alpha)^n}(\mu_A(x) - \alpha) + \alpha \leq \gamma_\alpha.\alpha$$

or

$$\alpha(1-\gamma_\alpha) \leq \frac{1}{(1-\gamma_\alpha)^n}(\alpha - \mu_A(x)) \equiv (1-\gamma_\alpha)^{n+1} \leq \frac{\alpha - \mu_A(x)}{\alpha}.$$

But since $0 < \frac{\alpha - \mu_A(x)}{\alpha} < 1$ and $0 < 1-\gamma_\alpha < 1$, taking the natural logarithm on both sides of the above inequality we get: $(n+1)\ln(1-\gamma_\alpha) \leq \ln\frac{\alpha - \mu_A(x)}{\alpha}$. And since both sides are negative numbers, it is equivalent to, $n+1 \geq \frac{\ln\frac{\alpha - \mu_A(x)}{\alpha}}{\ln(1-\gamma_\alpha)}$. Then for $n \geq \frac{\ln\frac{\alpha - \mu_A(x)}{\alpha}}{\ln(1-\gamma_\alpha)} - 1$ we have that $\mu_{(\mathcal{I}_{\mu;\alpha,\beta}^{\gamma_\alpha,\gamma_\beta})^n(A)}(x) = \mu_{\mathcal{I}_{\mu;\alpha,\beta}^{1,1}(A)}(x)$. The equality $\lim_{n\to\infty} \nu_{(\mathcal{I}_{\mu;\alpha,\beta}^{\gamma_\alpha,\gamma_\beta})^n(A)}(x) = \nu_{\mathcal{I}_{\mu;\alpha,\beta}^{1,1}(A)}(x)$ can be analogically checked taking into account that $\nu_{(\mathcal{I}_{\mu;\alpha,\beta}^{\gamma_\alpha,\gamma_\beta})^n(A)}(x) \geq \nu_A(x)$ for every $A \in IFS(X)$ and every $x \in X$ and applying analogical equivalent inequality chains. The proposition is proved. $\qquad\square$

Since $\mathcal{I}_{\mu;\alpha,\beta}^{1,1}$ is very important let us write it down separately:

$$\mu_{\mathcal{I}_{\mu;\alpha,\beta}^{1,1}(A)}(x) = \begin{cases} 0 & \text{if } 0 \leq \mu_A(x) < \alpha \\ \mu_A(x) & \text{if } \alpha \leq \mu_A(x) \leq 1 \end{cases} \tag{28}$$

$$\nu_{\mathcal{I}_{\mu;\alpha,\beta}^{1,1}(A)}(x) = \begin{cases} \min(\beta, 1 - \mu_{\mathcal{I}_{\mu;\alpha,\beta}^{1,1}(A)}(x)) & \text{if } 0 \leq \nu_A(x) \leq \beta \\ \nu_A(x) & \text{if } \beta < \nu_A(x) \end{cases} \tag{29}$$

Proposition 4.6. $\mathcal{I}_{\mu;\alpha,\beta}^{1,1}$ *is idempotent, i.e. for all A in $IFS(X), \mathcal{I}_{\mu;\alpha,\beta}^{1,1}(A)$ is a fixed point of $\mathcal{I}_{\mu;\alpha,\beta}^{1,1}$, which means that it is a preinterior operator satisfying the condition for interior operator.*

Proof. It is enough to show that for every $A \in IFS(X)$ and $x \in X$ the functions $\mu_{(\mathcal{I}_{\mu;\alpha,\beta}^{1,1})^n(A)}$ and $\nu_{(\mathcal{I}_{\mu;\alpha,\beta}^{1,1})^n(A)}$ are constant on X for all natural numbers $n > 1$. More specifically,

$$\mu_{(\mathcal{I}_{\mu;\alpha,\beta}^{1,1})^n(A)} = \mu_{\mathcal{I}_{\mu;\alpha,\beta}^{1,1}(A)} \text{ and } \nu_{(\mathcal{I}_{\mu;\alpha,\beta}^{1,1})^n(A)} = \nu_{\mathcal{I}_{\mu;\alpha,\beta}^{1,1}(A)}.$$

Let us take any $x \in X$. If $\mu_A(x) \leq \alpha$ then

$$\mu_{\mathcal{I}_{\mu;\alpha,\beta}^{1,1}(A)}(x) = 0 \text{ and } \{\mu_{\mathcal{I}_{\mu;\alpha,\beta}^{1,1}(A)}\}^n(x) = 0$$

for every $n \geq 1$. If $\mu_A(x) > \alpha$ then

$$\mu_{\mathcal{I}_{\mu;\alpha,\beta}^{1,1}(A)}(x) = \mu_A(a) \text{ and } \{\mu_{\mathcal{I}_{\mu;\alpha,\beta}^{1,1}(A)}\}^n(x) = \mu_A(x)$$

for every $n \geq 1$. Therefore, $\mu_{\mathcal{I}_{\mu;\alpha,\beta}^{1,1}(A)}$ is idempotent.

Analogically, $\nu_{\mathcal{I}_{\mu;\alpha,\beta}^{1,1}(A)}(x)$ is idempotent too. And therefore, $\mathcal{I}_{\mu;\alpha,\beta}^{1,1}$ is an interior operator. $\qquad\square$

Let us define the axioms for a (pre)clusure for IFS(X). As in Section 3 $\mathfrak{c} : IFS(X) \longrightarrow IFS(X)$, is an *intuitionistic fuzzy preclosure* operator if for all $A \in IFS(X)$,

1. $\mathfrak{c}(O_X^*) = O_X^*$

2. $A \subseteq \mathfrak{c}(A)$

3. $\mathfrak{c}(A \cup B) = \mathfrak{c}(A) \cup \mathfrak{c}(B)$

And in addition, if $\mathfrak{c}$ is idempotent then it is an *intuitionistic fuzzy closure* operator.

Let us define

$$\mathcal{C}_{\nu;\alpha,\beta}^{\gamma_\alpha,\gamma_\beta} : IFS(X) \longrightarrow IFS(X),$$

such that (21) is satisfied and

$$\nu_{\mathcal{C}_{\nu;\alpha,\beta}^{\gamma_\alpha,\gamma_\beta}(A)}(x) = \begin{cases} 0 & \text{if } 0 \le \nu_A(x) < \gamma_\alpha.\alpha \\ \frac{1}{1-\gamma_\alpha}(\nu_A(x) - \alpha) + \alpha & \text{if } \gamma_\alpha.\alpha \le \nu_A(x) < \alpha \\ \nu_A(x) & \text{if } \alpha \le \nu_A(x) \le 1 \end{cases} \tag{30}$$

$$\mu_{\mathcal{C}_{\nu;\alpha,\beta}^{\gamma_\alpha,\gamma_\beta}(A)}(x) =$$
$$\begin{cases} \min((1-\gamma_\beta)\mu_A(x) + \beta\gamma_\beta, 1 - \nu_{\mathcal{C}_{\nu;\alpha,\beta}^{\gamma_\alpha,\gamma_\beta}(A)}(x)) & \text{if } 0 \le \mu_A(x) \le \beta \\ \mu_A(x) & \text{if } \beta < \mu_A(x) \le 1 \end{cases} \tag{31}$$

We also remark the following corresponding representations of the above defined functions by the auxiliary functions $f_\alpha^{\gamma_\alpha}$ and $g_\beta^{\gamma_\beta}$, introduced in (24)

$$\nu_{\mathcal{C}_{\nu;\alpha,\beta}^{\gamma_\alpha,\gamma_\beta}(A)}(x) = f_\alpha^{\gamma_\alpha}(\nu_A(x)) \tag{32}$$

$$\mu_{\mathcal{C}_{\nu;\alpha,\beta}^{\gamma_\alpha,\gamma_\beta}(A)}(x) = \min(g_\beta^{\gamma_\beta}(\nu_A(x)), 1 - \nu_{\mathcal{C}_{\nu;\alpha,\beta}^{\gamma_\alpha,\gamma_\beta}(A)}(x)) \tag{33}$$

The above defined operation is a preclosure operator and analogically to Proposition 4.3, Proposition 4.5 and Proposition 4.6, let us state the following three propositions without proofs.

The action of the preclosure operator $\mathcal{C}_{\nu;\alpha,\beta}^{\gamma_\alpha,\gamma_\beta}$ is plotted on Fig. 5.
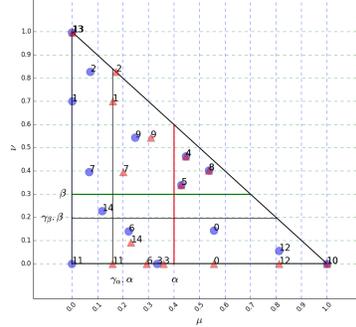

Figure 5: Example of the action of the preclosure operator $\mathcal{C}_{\nu;\alpha,\beta}^{\gamma_\alpha,\gamma_\beta}$ on $A \in IFS(X)$ from Fig. 3

.

Proposition 4.7. *The operator $\mathcal{C}_{\nu;\alpha,\beta}^{\gamma_\alpha,\gamma_\beta}$ is a valid preclosure operator in $IFS(X)$, i.e. it satisfies the three corresponding axioms.*

Proposition 4.8. *For any $\alpha, \beta \in [0,1]$ and $\gamma_\alpha, \gamma_\beta \in (0,1]$ we have that $\lim_{n\to\infty} (\mathcal{C}_{\nu;\alpha,\beta}^{\gamma_\alpha,\gamma_\beta})^n = \mathcal{C}_{\nu;\alpha,\beta}^{1,1}$ and hence $\tau^{\mathcal{C}_{\nu;\alpha,\beta}^{\gamma_\alpha,\gamma_\beta}} = \tau^{\mathcal{C}_{\nu;\alpha,\beta}^{1,1}}$.*

Proposition 4.9. *$\mathcal{C}_{\nu;\alpha,\beta}^{1,1}$ is idempotent, i.e. for all A in $IFS(X), \mathcal{C}_{\nu;\alpha,\beta}^{1,1}(A)$ is a fixed point of $\mathcal{C}_{\nu;\alpha,\beta}^{1,1}$, which means that it is a preclosure operator satisfying the condition for closure operator.*

Analogically to Theorem 3.12, let us state the following,

Theorem 4.10. *For any $A \in IFS(X)$ we have that $\mathcal{C}_{\nu;\beta,\alpha}^{\gamma_\beta,\gamma_\alpha}(A) = \neg\mathcal{I}_{\mu;\alpha,\beta}^{\gamma_\alpha,\gamma_\beta}(\neg A)$, i.e. $(\mathcal{C}_{\nu;\beta,\alpha}^{\gamma_\beta,\gamma_\alpha}, \mathcal{I}_{\mu;\alpha,\beta}^{\gamma_\alpha,\gamma_\beta})$ is a pair of conjugate preclosure-preinterior operators,*

which define the same topology $\tau_{\mathcal{I}^{\gamma_\alpha,\gamma_\beta}_{\mu;\alpha,\beta}} = \{\neg B \mid B \in \tau^{\mathcal{C}^{\gamma_\beta,\gamma_\alpha}_{\nu;\beta,\alpha}}\}$ in $IFS(X)$. Moreover, from the definition of $\tau_{\mathcal{I}^{\gamma_\alpha,\gamma_\beta}_{\mu;\alpha,\beta}}$ if follows that

$$\tau_{\mathcal{I}^{\gamma_\alpha,\gamma_\beta}_{\mu;\alpha,\beta}} = \{A \mid A \in IFS(X) \,\&\, (\mu_A,\nu_A)(x) \in (\alpha,1-\beta] \times [\beta,1-\alpha] \cup \{0\} \times [\beta,1]\} \tag{34}$$

$$\tau^{\mathcal{C}^{\gamma_\alpha,\gamma_\beta}_{\nu;\alpha,\beta}} = \{B \mid B \in IFS(X) \,\&\, (\mu_B,\nu_B)(x) \in [\alpha,1-\beta] \times (\beta,1-\alpha] \cup [\alpha,1] \times \{0\}\}. \tag{35}$$

From the above theorem and Remark 4.4 we have the following remark.

Remark 4.11. *The family of closed IFSs of the pretopological space* $(IFS(X), \tau^{\mathcal{C}^{\gamma_\alpha,\gamma_\beta}_{\mu;\alpha,\beta}})$, *i.e. the fixed points of the operator* $\mathcal{C}^{\gamma_\alpha,\gamma_\beta}_{\mu;\alpha,\beta}$, *consists of* $A \in IFS(X)$ *such that for all* $x \in X$, $\mathcal{C}^{\gamma_\alpha,\gamma_\beta}_{\mu;\alpha,\beta}(A) = A$ *iff* $(\alpha \le \nu_B(x) \le 1-\beta$ *and* $\beta \le \mu_B(x) \le 1-\alpha)$ *or* $(\mu_A(x) \ge \alpha$ *and* $\nu_A(x) = 0)$.

5   Generalization of the IF (pre)topological operators

In this section we provide some generalization of the already mentioned topological operators in Section 2. Let us denote $\bar{\alpha} = (\alpha_0,\alpha_1)$ and $\bar{\beta} = (\beta_0,\beta_1)$, where $\alpha_i,\beta_i \in [0,1]$ for $i \in \{0,1\}$ and $\alpha_0 \le \alpha_1, \beta_0 \le \beta_1$. We also take $\gamma_{\bar\alpha}, \gamma_{\bar\beta} \in [0,1]$ and based on an arbitrary $A \in IFS(X)$, let us define the operator, $\mathcal{I}^{\gamma_{\bar\alpha},\gamma_{\bar\beta}}_{\mu;\bar\alpha,\bar\beta} : IFS(X) \longrightarrow IFS(X)$, by the introduction of the following auxiliary functions, similarly to (24) and (25).

$$f^{\gamma_{\bar\alpha}}_{\bar\alpha}(t) = \begin{cases} t & \text{if } 0 \le t < \alpha_0 \\ \alpha_0 & \text{if } \alpha_0 \le t < \alpha_0 + \gamma_{\bar\alpha}(\alpha_1 - \alpha_0) \\ \frac{1}{1-\gamma_{\bar\alpha}}(t-\alpha_1) + \alpha_1 & \text{if } \alpha_0 + \gamma_{\bar\alpha}(\alpha_1-\alpha_0) \le t < \alpha_1 \\ t & \text{if } \alpha_1 \le t \le 1 \end{cases} \tag{36}$$

$$g^{\gamma_{\bar\beta}}_{\bar\beta}(t) = \begin{cases} t & \text{if } 0 \le t < \beta_0, \\ (1-\gamma_{\bar\beta})t + \beta_1\gamma_{\bar\beta} & \text{if } \beta_0 \le t < \beta_1, \\ t & \text{if } \beta_1 \le t \le 1. \end{cases} \tag{37}$$

then the preinterior operator $\mathcal{I}^{\gamma_{\bar\alpha},\gamma_{\bar\beta}}_{\mu;\bar\alpha,\bar\beta}$ can be defined in a more compact form as

$$\mu_{\mathcal{I}^{\gamma_{\bar\alpha},\gamma_{\bar\beta}}_{\mu;\bar\alpha,\bar\beta}}(A)(x) = f^{\gamma_{\bar\alpha}}_{\bar\alpha}(\mu_A(x)) \tag{38}$$

$$\nu_{\mathcal{I}^{\gamma_{\bar\alpha},\gamma_{\bar\beta}}_{\mu;\bar\alpha,\bar\beta}(A)}(x) = \min(g^{\gamma_{\bar\beta}}_{\bar\beta}(\nu_A(x)), 1 - \mu_{\mathcal{I}^{\gamma_{\bar\alpha},\gamma_{\bar\beta}}_{\mu;\bar\alpha,\bar\beta}(A)}(x)) \tag{39}$$

Analogically to (32) and (33), we define the membership and non-membership functions of the preclosure operator

$$\mathcal{C}^{\gamma_{\bar\beta},\gamma_{\bar\alpha}}_{\nu;\bar\beta,\bar\alpha} : IFS(X) \longrightarrow IFS(X),$$

and for any $A \in IFS(X)$ we have that $\mathcal{C}^{\gamma_{\bar\beta},\gamma_{\bar\alpha}}_{\nu;\bar\beta,\bar\alpha}(A) = \neg\mathcal{I}^{\gamma_{\bar\alpha},\gamma_{\bar\beta}}_{\mu;\bar\alpha,\bar\alpha}(\neg A)$, i.e. $(\mathcal{C}^{\gamma_{\bar\beta},\gamma_{\bar\alpha}}_{\nu;\bar\beta,\bar\alpha}, \mathcal{I}^{\gamma_{\bar\alpha},\gamma_{\bar\beta}}_{\mu;\bar\alpha,\bar\alpha})$ is a pair of conjugate preclosure-preinterior operators, which define the same topology $\tau_{\mathcal{I}^{\gamma_{\bar\alpha},\gamma_{\bar\beta}}_{\mu;\bar\alpha,\bar\beta}} = \{\neg B \mid B \in \tau^{\mathcal{C}^{\gamma_{\bar\beta},\gamma_{\bar\alpha}}_{\nu;\bar\beta,\bar\alpha}}\}$ in $IFS(X)$.

Proposition 5.1. *The operators* $\mathcal{I}^{\gamma_{\bar\alpha},\gamma_{\bar\beta}}_{\mu;\bar\alpha,\bar\beta}$ *and* $\mathcal{C}^{\gamma_{\bar\beta},\gamma_{\bar\alpha}}_{\nu;\bar\beta,\bar\alpha}$ *are generalizations of the corresponding operators* $\mathcal{I}^{\gamma_\alpha,\gamma_\beta}_{\mu;\alpha,\beta}$ *and* $\mathcal{C}^{\gamma_\beta,\gamma_\alpha}_{\nu;\beta,\alpha}$.

Proof. It is enough to take $\bar\alpha = (0,\alpha)$, $\bar\beta = (0,\beta)$, $\gamma_{\bar\alpha} = \gamma_\alpha$, $\gamma_{\bar\beta} = \gamma_\beta$ and the validity of the proposition follows directly from the definition of the corresponding operators. □

Let us now show that the operators defined in this section also generalize the operators from [4], Ch. 4.2, stated in Section 2.

Proposition 5.2. *The operators* $\mathcal{I}^{\gamma_{\bar\alpha},\gamma_{\bar\beta}}_{\mu;\bar\alpha,\bar\beta}$ *and* $\mathcal{C}^{\gamma_{\bar\beta},\gamma_{\bar\alpha}}_{\nu;\bar\beta,\bar\alpha}$ *are generalizations of the corresponding operators* $\mathcal{I}$ *from* (6) *and* $\mathcal{C}$ *from* (4).

Proof. Let us take an arbitrary set $A \in IFS(X)$. To show the validity of the interior operators generalization it is enough to take $\bar{\alpha} = (k_A, 1)$, $\gamma_{\bar{\alpha}} = 1$ and $\bar{\beta} = (0, l_A)$, $\gamma_{\bar{\beta}} = 1$, i.e. $\mathcal{I} = \mathcal{I}_{\mu;(k_A,1),(0,l_A)}^{1,1}$. To show the validity of the closure operators generalization it is enough to take $\bar{\alpha} = (0, K_A)$, $\gamma_{\bar{\alpha}} = 1$ and $\bar{\beta} = (L_A, 1)$, $\gamma_{\bar{\beta}} = 1$, i.e. $\mathcal{C} = \mathcal{C}_{\nu;(0,K_A),(L_A,1)}^{1,1}$. $\qquad\square$

Proposition 5.3. *The operator $\mathcal{I}_{\mu;\bar{\alpha},\bar{\beta}}^{\gamma_{\bar{\alpha}},\gamma_{\bar{\beta}}}$ is a generalization of the operators $\mathcal{I}_\mu$ from (10) and $\mathcal{I}_\nu$ from (11).*

Proof. Let us take an arbitrary set $A \in IFS(X)$. From the definition of the corresponding operators it follows that $\mathcal{I}_\mu = \mathcal{I}_{\mu;(k_A,1),(0,0)}^{1,1}$ and $\mathcal{I}_\nu = \mathcal{I}_{\mu;(1-l_A,1),(0,l_A)}^{1,1}$. $\qquad\square$

Proposition 5.4. *The operator $\mathcal{C}_{\nu;\bar{\alpha},\bar{\beta}}^{\gamma_{\bar{\alpha}},\gamma_{\bar{\beta}}}$ is a generalization of the operators $\mathcal{C}_\mu$ from (8) and $\mathcal{C}_\nu$ from (9).*

Proof. Let us take an arbitrary set $A \in IFS(X)$. From the definition of the corresponding operators it follows that $\mathcal{C}_\mu = \mathcal{C}_{\mu;(0,K_A),(1-K_A,1)}^{1,1}$ and $\mathcal{C}_\nu = \mathcal{C}_{\mu;(0,0),(L_A,1)}^{1,1}$. $\qquad\square$

The operators discussed in the paper can be applied for the estimation of different geometrical and real world objects. For example, area of forest, lake, sea and many others. In particular, these operators will give the results discussed in the papers [27] and [38]. One specific case is the area affected by a forest fire that is usually uncertain (fuzzy or intuitionistic fuzzy). Let us take X to be the area of the fire region and its surroundings to be the universe for $A \in IFS(X)$, where A stands for the region of the fire. During the fire because of the smoke screen there may be regions that can not be clearly specified as belonging or not the the fire region and that would be the regions with high degree of uncertainty (π). If we take any of the (pre)closure operators (let us denote it by $\mathcal{C}$) described in this paper, $A \in IFS(X)$ (where X stands for the whole area of the considered map) and compute its boundary $\partial A = \mathcal{C}(A) \cap \mathcal{C}(\bar{A})$, that can be considered as an IF estimation of the uncertain boundary (it is shown in [38] how it can be applied in the fuzzy case). In a next research we are going to investigate the results of the uncertain IF boundary in a real study case for forest fire.

6 Conclusion

In this paper we introduced pretopological and topological operators based on partially continuous linear transformations of the membership and non-membership functions for intuitionistic fuzzy sets. They turn out to be a generalization of the topological operators for intuitionistic fuzzy sets, introduced in the book [4] and generalization of the fuzzy set pretopological and topological operators from [28]. The mathematical correctness of the operators has also been proved and we showed that these operators are conjugate. Although there has been extensive research about intuitionistic fuzzy topologies and their properties, the goal of our paper is the introduction of (pre)topological operators in a more constructive way, allowing application of computer programs. As shown in [38] such kind of operators can be applied to Geographical Information Systems. We will just mention that possible applications of the operators introduced in the paper are, e.g., processing of GIS data and in particular, evaluations of forest fire regions (see, e.g., [27]), of artificial satellite trajectories and others; processing of intercriteria analysis results (see, e.g. [4]), etc. that will be object of particular authors research in near future. In a further research, we are going to extend the results from this paper for (pre)topologies in the framework of interval valued intuitionistic fuzzy sets (IVIFS) and give a software implementation for interactive manipulation of the proposed operators.

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