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A new family of (A, N)-implications: Construction and properties

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Abstract

This paper deals with a new family of (A, N)-implications that derived from fuzzy negation and continuous monotone function, and some properties, like the left neutrally, the exchange principle, the ordering property and the laws of contraposition, of the proposed family of (A, N)-implications are studied. Moreover, the T-conditionality and the Modus tollens for the new family of (A, N)-implications are also studied and some results are obtained.

Keywords: Fuzzy implication, fuzzy negation, T-conditionality, modus tollens.

1 Introduction

Fuzzy implications are the generalization of the classical (Boolean) implications to the unit interval $[0, 1]$, which play important roles in the field of fuzzy logic, fuzzy control, decision theory, expert systems, etc. The main way of generating fuzzy implications is from aggregation functions and fuzzy negation [2, 3, 4, 5, 6, 7, 8, 12, 13, 15]. The fuzzy implications generated from this method are called (A, N)-implications [19, 20]. Other way of generating fuzzy implications is from additive generating functions or from some initial implication(s) [9, 10, 14, 16, 17, 18, 21, 23]. Fuzzy implications are used to perform any fuzzy "if-then" rule in fuzzy systems and inference processes, through Modus Ponens and Modus Tollens [24], so, depending on the context, and on the proper rule and its behavior, different implications with different properties can be needed. Thus, it is necessary to construct various fuzzy implication operators.

This paper, inspired by the construction method of aggregation functions in [1] and an approach of generating fuzzy implications in [23], proposed a new family of (A,N)-implications. The proposed fuzzy implications are generated from fuzzy negation and continuous strictly monotone function. When the continuous strictly monotone function is an identity function, the proposed method can generate a fuzzy implication only from fuzzy negation, and some properties such as (NP), (OP), (EP), (IP), (CP) of the proposed fuzzy implications are studied. Furthermore, the T-conditionality and the Modus tollens with the proposed fuzzy implications are also studied, and some results are obtained.

The paper is organized as follows. In Section 2, we recall the basic concepts and definitions used in the paper. In Section 3, a new family of (A, N)-implications are proposed, and some basic properties of the proposed fuzzy implications are studied. In Section 4, the T-conditional with the proposed fuzzy implications are studied. In Section 5, the Modus tollens with the proposed fuzzy implications are studied, and some results are obtained. Finally, some conclusions are given in Section 6.

2 Preliminaries

For the convenience of reading, in this section, we recall some definitions and results that will be used in the rest of the paper.

Definition 2.1. [4] *A function $I : [0, 1]^2 \to [0, 1]$ is called a fuzzy implication, if it satisfies, for all $x, x_1, x_2, y, y_1, y_2 \in [0, 1]$, the following conditions:*

if $x_1 < x_2$, then $I(x_1, y) \geq I(x_2, y)$, i.e., $I(\cdot, y)$ is decreasing, (I1)

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if $y_1 < y_2$*, then* $I(x, y_1) \leq I(x, y_2)$*, i.e.,* $I(x, \cdot)$ *is increasing,* (I2)
$I(0,0) = 1,\ I(1,1) = 1,\ I(1,0) = 0.$ (I3)

From Definition 2.1, I satisfies the following properties, called left and right boundary condition, respectively:

$$I(0, y) = 1, y \in [0,1], \tag{LB}$$

$$I(x, 1) = 1, x \in [0,1]. \tag{RB}$$

The set of all fuzzy implications will be denoted by FI.

Definition 2.2. [4] *An implication* I *is said to satisfy*
(i) the left neutrality property if $I(1, y) = y$ *for all* $y \in [0,1]$*.* (NP)
(ii) the exchange principle, if $I(x, I(y,z)) = I(y, I(x,z))$ *for all* $x, y, z \in [0,1]$*.* (EP)
(iii) the identity principle, if $I(x,x) = 1$ *for all* $x \in [0,1]$*.* (IP)
(iv) the ordering property, if $I(x, y) = 1 \Leftrightarrow x \leq y$ *for all* $x, y \in [0,1]$*.* (OP)
(v) the contrapositive symmetry with respect to a fuzzy negation N*, if*
$I(x, y) = I(N(y), N(x))$*, for all* $x, y \in [0,1]$*.* (CP(N))
(vi) the left contrapositive symmetry with respect to a fuzzy negation N, if
$I(x, N(y)) = I(y, N(x))$*, for all* $x, y \in [0,1]$*.* (R-CP(N))
(vii) the law of importation if there exists a t-norm T *such that*
$I(T(x,y), z) = I(x, I(y, z))$ *for all* $x, y, z \in [0,1]$ (LI)

Definition 2.3. [4] *A function* $N : [0,1] \to [0,1]$ *is called a fuzzy negation if* $N(0) = 1, N(1) = 0$*, and* N *is decreasing.*

Definition 2.4. [4] *(i) A fuzzy negation* N *is called strong if it is an involution, i.e.,* $N(N(x)) = x$ *for all* $x \in [0,1]$*. A fuzzy negation* N *is called strict if it is continuous and strictly decreasing. A fuzzy negation* N *is said to be non-vanishing if* $N(x) = 0 \Leftrightarrow x = 1$*.*

(ii) The classical (standard) fuzzy negation is $N(x) = 1 - x, x \in [0,1]$*, denoted by* $N_C(x)$*. The least fuzzy negation is* $N(x) = \begin{cases} 1, & \text{if } x = 0, \\ 0, & \text{if } x > 0, \end{cases}$ *denoted by* N_{D_1}*. The greatest fuzzy negation is* $N(x) = \begin{cases} 1, & \text{if } x < 1, \\ 0, & \text{if } x = 1, \end{cases}$ *denoted by* N_{D_2}*.*

(iii) Let $I \in FI$*. The function* $N_I : [0,1] \to [0,1]$ *defined by* $N_I(x) = I(x, 0)$ *is called the natural negation of* I*.*

Theorem 2.5. [4] *A fuzzy negation* N *is strong if and only if there exists a strictly increasing, continuous function* $g : [0,1] \to [0, \infty)$ *such that* $g(0) = 0$ *and* $N(x) = g^{-1}(g(1) - g(x)), x \in [0,1]$*, where the function* g *is called an additive generator of* N*.*

Lemma 2.6. [4] *Let* $I : [0,1]^2 \to [0,1]$ *be a fuzzy implication and* N *a fuzzy negation. If* I *satisfies (NP) and CP(N), then* $N = N_I$ *is a strong negation, where* $N_I(x) = I(x, 0)$*.*

Definition 2.7. [4] *A function* $\varphi : [0,1] \to [0,1]$ *is an automorphism if it is continuous and strictly increasing and satisfies the boundary conditions* $\varphi(0) = 0$ *and* $\varphi(1) = 1$*, i.e., if it is an increasing bijection in [0,1]. By* Φ *we denote the family of all automorphism from* $[0,1]$ *to* $[0,1]$*.*

Definition 2.8. [4] *Let* $\varphi : [0,1] \to [0,1]$ *be an automorphism, we say that two functions* $f,\ g : [0,1]^n \to [0,1]$ *are* $\varphi-$*conjugate if* $g = f_\varphi$ *, where*

$$f_\varphi(x_1, x_2, \cdots, x_n) = \varphi^{-1}(f(\varphi(x_1), \varphi(x_2), \cdots, \varphi(x_n))).$$

Definition 2.9. [4] *(i) A t-norm is an associative, commutative and increasing function* $T : [0,1]^2 \to [0,1]$ *verifying the condition* $T(x, 1) = x, x \in [0,1]$*.*

(ii) A t-norm is positive, if $T(x,y) = 0$ *then either* $x = 0$ *or* $y = 0$*.*

Example 2.10. [4] *The following are examples of four basic t-norms:*

$T_M(x,y) = min(x,y),$ $\quad T_P(x,y) = xy,$

$T_{LK}(x,y) = max(x + y - 1, 0),$ $\quad T_D(x,y) = \begin{cases} min(x,y), & \text{if } x = 1 \text{ or } y = 1, \\ 0, & \text{if } otherwise. \end{cases}$

Definition 2.11. [4] *A t-conorm is an associative, commutative and increasing function* $S : [0,1]^2 \to [0,1]$ *verifying the condition* $S(x, 0) = x, x \in [0,1]$*.*

Example 2.12. [4] *The following are examples of two basic t-conorms:*
$S_M(x, y) = max(x, y)$, $S_{LK}(x, y) = min(x + y, 1)$.

Note that given an automorphism φ, the φ-conjugate of a t-norm T, that is T_φ, and the φ-conjugate of an implication I, that is I_φ, are again a t-norm and an implication, respectively [4].

Definition 2.13. [4] *Let T be a t-norm, a function $N_T : [0, 1] \to [0, 1]$ defined as $N_T(x) = sup\{y \in [0, 1] \mid T(x, y) = 0\}, x \in [0, 1]$ is called the natural negation of T.*

Example 2.14. [4] *The following are examples of natural negation of basic t-norm.*
$N_T = N_{D_1}$, *where T is positive t-norm.* $N_{T_{LK}} = N_C$, $N_{T_D} = N_{D_2}$, $N_{T_{nM}} = N_C$.

3 A new family of (A,N)-implications. Basic properties

3.1 An approach for constructing a new family of (A,N)-implications

A fact: Given a fuzzy implication I, we can obtain a fuzzy negation N_I, where $N_I = I(x, 0)$. A question that how to generate a fuzzy implication from a given fuzzy negation arises immediately. Unfortunately, the study on this question is rare. In paper [1], a method of constructing semicopulas from fuzzy negation was proposed. That is, given a fuzzy negation N, a semicopula F_N be obtained by $F_N(x, y) = max(0, x \wedge y - N(x \vee y)), x, y \in [0, 1]$, where the symbols $\wedge$, $\vee$ are the operator min, max, respectively. Inspired by this idea, we construct a binary function $I_N : [0, 1]^2 \to [0, 1]$, which is represented as

$$I_N(x, y) = min(1, N(x) \wedge y - N(N(x) \vee y) + 1), x, y \in [0, 1].$$

It is easy to prove that I_N is a fuzzy implication. If we consider a strictly increasing and continuous function $g : [0, 1] \to [0, \infty)$ such that $g(0) = 0$, then we can obtain a binary function $I_{g,N} : [0, 1]^2 \to [0, 1]$ defined as following.

Definition 3.1. *Let $g : [0, 1] \to [0, \infty)$ be a strictly increasing, continuous function with $g(0) = 0$, N a fuzzy negation. The binary function $I_{g,N} : [0, 1]^2 \to [0, 1]$ defined as*

$$I_{g,N}(x, y) = g^{-1}(min(g(1), g(N(x) \wedge y) - g(N(N(x) \vee y)) + g(1))),\ x,\ y \in [0, 1], \tag{1}$$

will be called an (g, N)-operator derived from g and N. The function g itself is called an $g-$generator of $I_{g,N}$.

Theorem 3.2. *Let $g : [0, 1] \to [0, \infty)$ be a strictly increasing, continuous function with $g(0) = 0$, and N a fuzzy negation, then the binary function $I_{g,N} : [0, 1]^2 \to [0, 1]$ defined by*

$$I_{g,N}(x, y) = g^{-1}(min(g(1), g(N(x) \wedge y) - g(N(N(x) \vee y)) + g(1))), \quad x, y \in [0, 1],$$

is a fuzzy implication.

Proof. It is easy to prove that $I_{g,N}$ is deceasing with the first variable and increasing with the second variable.

$$I_{g,N}(1, 1) = g^{-1}(min(g(1), g(0)) - g(0) + g(1))) = 1,$$

$$I_{g,N}(0, 0) = g^{-1}(min(g(1), g(0)) - g(0) + g(1))) = 1,$$

$$I_{g,N}(1, 0) = g^{-1}(min(g(1), g(0)) - g(1) + g(1))) = 0.$$

□

Remark 3.3. *(i) If the generator $g(x) = x$, then the fuzzy implication $I_{g,N}$ can be constructed only from fuzzy negation N, i.e., $I(x, y) = min(1, N(x) \wedge y - N(N(x) \vee y) + 1)$, $x,\ y \in [0, 1]$, that is, given a fuzzy negation N, a fuzzy implication I can be obtained.*

(ii) The fuzzy implication $I_{g,N}$ defined in Definition 3.1 is a (A, N)-implication (for detail, see [20]). Actually, let $A(x, y) = g^{-1}(min(g(1), g(x \wedge y) - g(N(x \vee y)) + g(1)))$, $x,\ y \in [0, 1]$, which is an aggregation function, then $I_{g,N}(x, y) = A(N(x), y)$, for all $x, y \in [0, 1]$.

(iii) If $N(x) = N_{D_2}(x) = \begin{cases} 1, & if\ x < 1 \\ 0, & if\ x = 1 \end{cases}$, then $I_{g,N}(x, y) = \begin{cases} 0, & if\ x = 1, y < 1 \\ 1, & otherwise \end{cases}$. For any generator g, $I_{g,N}$ satisfies (EP), (IP), but it does not satisfy (NP), (OP).

(iv) If $N(x)=N_{D_1}(x)=\begin{cases}1, & if\ x=0\\ 0, & if\ x>0\end{cases}$*, then* $I_{g,N}(x,y)=\begin{cases}0, & if\ x>0, y=0\\ 1, & otherwise\end{cases}$*. For any generator,* $I_{g,N}$ *satisfies (EP), (IP), but it does not satisfy (NP), (OP).*

(v) From Definition 3.1, the fuzzy implication $I_{g,N}$ *can be shown as*

$$I_{g,N}(x,y)=\begin{cases}1, & if\ x\le y,\ \ y\ge N(x),\\ 1, & if\ y\le N(x),\ \ y\ge N(N(x)),\\ g^{-1}(g(N(x))-g(N(y))+g(1)), & if\ x\ge y,\ \ y\ge N(x),\\ g^{-1}(g(y)-g(N(N(x)))+g(1)), & if\ y\le N(x),\ \ y\le N(N(x)).\end{cases}$$

If N *is strong, then* $I_{g,N}$ *can be shown as*

$$I_{g,N}(x,y)=\begin{cases}1, & if\ x\le y,\\ g^{-1}(g(N(x))-g(N(y))+g(1)), & if\ x\ge y,\ \ y\ge N(x),\\ g^{-1}(g(y)-g(x)+g(1)), & if\ x\ge y,\ \ y\le N(x).\end{cases}$$

Example 3.4. *(i) Let us consider an* $g-$*generator* $g(x)=kx, k>0$*, and* $N(x)=1-x$*, then the corresponding implication is given by*

$$I_{g,N}(x,y)=min(1,1-x+y),\quad x,y\in[0,1],$$

which is the Lukasiewicz implication I_{LK}*, it satisfies (NP), (EP), (IP), (OP), and it is continuous.*

(ii) Let us consider an $g-$*generator* $g(x)=x$*, and* $N(x)=1-x^2$*, then the corresponding implication is given by*

$$I_{g,N}(x,y)=\begin{cases}1, & if\ x\le y, y\ge 1-x^2\ or\ y<1-x^2, y\ge 2x^2-x^4,\\ y^2-x^2+1, & if\ x>y,\ \ y\ge 1-x^2,\\ y+(1-x^2)^2, & if\ y<2x^2-x^4, y<1-x^2,\end{cases}$$

which satisfies (IP), but it does not satisfy (OP).

(iii) Let us consider an $g-$*generator* $g(x)=kx, k>0$*, and* $N(x)=\sqrt{1-x}$*, then the corresponding implication is given by*

$$I_{g,N}(x,y)=\begin{cases}1, & if\ x\le y, y\ge N(x)\ or\ y<N(x), y\ge N(N(x)),\\ \sqrt{1-x}-\sqrt{1-y}+1, & if\ x>y,\ \ y\ge N(x),\\ y+\sqrt{1-\sqrt{1-x}}+1, & if\ y<N(N(x)), y<N(x),\end{cases}$$

which does not satisfy (IP), (OP).

(iv) Let us consider an $g-$*generator* $g(x)=kx, k>0$*, and* $N(x)=(1-x^{\alpha})^{\frac{1}{\alpha}}, \alpha=2$*, then the corresponding implication is given by*

$$I_{g,N}(x,y)=\begin{cases}1, & if\ x\le y,\\ (1-x^2)^{\frac{1}{2}}-(1-y^2)^{\frac{1}{2}}+1, & if\ x>y,\ \ y\ge N(x),\\ y-x+1, & if\ x>y, y<N(x),\end{cases}$$

which satisfies (IP), (OP), (EP).

(v) Let us consider an $g-$*generator* $g(x)=kx, k>0$*, and* $N(x)=\frac{1-x}{1+\lambda x}, \lambda=3$*, then the corresponding implication is given by*

$$I_{g,N}(x,y)=\begin{cases}1, & if\ x\le y,\\ \frac{1-x}{1+3x}-\frac{1-y}{1+3y}+1, & if\ x>y,\ \ y\ge N(x),\\ y-x+1, & if\ x>y, y<N(x),\end{cases}$$

which satisfies (IP), (OP), (EP).

3.2 Basic properties of the proposed fuzzy implications

In this section, the basic properties such as (NP), (IP), (OP), etc. of the proposed fuzzy implications are studied. The conditions under which these properties are obtained are different from the one in [20].

From Remark 3.3, we know that the proposed fuzzy implication $I_{g,N}$ may or not satisfy (NP), (IP), (OP), so, we study the conditions under which $I_{g,N}$ satisfy (NP), (IP), (OP), etc.

3.2.1 The left neutrality property (NP)

Proposition 3.5. *The fuzzy implication $I_{g,N}$ satisfies (NP) if and only if g is an additive generator of N, i.e., $N(x) = g^{-1}(g(1) - g(x))$, $x \in [0, 1]$.*

Proof.
$$\begin{aligned} I_{g,N} \text{ satisfies (NP)} &\Leftrightarrow I_{g,N}(1, x) = x,\ x \in [0, 1] \\ &\Leftrightarrow g^{-1}(g(1) - g(N(x))) = x,\ x \in [0, 1] \\ &\Leftrightarrow N(x) = g^{-1}(g(1) - g(x)),\ x \in [0, 1]. \end{aligned}$$
□

Remark 3.6. *From Proposition 3.5, if g is an additive generator of N, then the natural negation of $I_{g,N}$ is N, and $I_{g,N}$ can be represented by*

$$I_{g,N}(x, y) = min\{1, g^{-1}(g(1) + g(y) - g(x))\},\ \ x, y \in [0, 1].$$

In fact, from $N(x) = g^{-1}(g(1) - g(x))$, $x \in [0, 1]$, we have

$$\begin{aligned} I_{g,N}(x, y) &= g^{-1}(min(g(1), g(N(x) \wedge y) - g(N(N(x) \vee y)) + g(1))) \\ &= g^{-1}(min(g(1), g(N(x)) - g(N(y)) + 1))\ \ or\ \ g^{-1}(min(g(1), g(y) - g(x) + 1)) \\ &= g^{-1}(min(g(1), g(1) + g(y) - g(x))) \\ &= min(1, g^{-1}(g(1) + g(y) - g(x))), \end{aligned}$$

then $N_{I_{g,N}}(x) = I_{g,N}(x, 0) = g^{-1}(g(1) - g(x)) = N(x)$, i.e., $N_{I_{g,N}} = N$.

3.2.2 The identity principle (IP)

Proposition 3.7. *The fuzzy implication $I_{g,N}$ satisfies (IP) if and only if $\{x \in [0, 1]|x < N(x)\} \cap \{x \in [0, 1]|x < N(N(x))\} = \emptyset$, where the symbol $\emptyset$ is denoted an empty set.*

Proof. Let $I_{g,N}$ satisfy (IP), i.e., $I_{g,N}(x, x) = 1$ for all $x \in [0, 1]$.

Suppose $\{x \in [0, 1]|x < N(x)\} \cap \{x \in [0, 1]|x < N(N(x))\} \neq \emptyset$, then there exists a $x_0 \in (0, 1)$, such that $x_0 < N(x_0)$ and $x_0 < N(N(x_0))$, thus

$$\begin{aligned} I_{g,N}(x_0, x_0) &= g^{-1}(min(g(1), g(N(x_0) \wedge x_0) - g(N(N(x_0) \vee x_0)) + g(1))) \\ &= g^{-1}(min(g(1), g(x_0) - g(N(N(x_0))) + g(1))) \\ &= g^{-1}(g(x_0) - g(N(N(x_0))) + g(1)) \\ &< g^{-1}(g(1)) = 1, \end{aligned}$$

a contradiction to the fact that $I_{g,N}(x, x) = 1$ for all $x \in [0, 1]$.

Conversely. Let $\{x \in [0, 1]|x < N(x)\} \cap \{x \in [0, 1]|x < N(N(x))\} = \emptyset$, then $x < N(x) \Rightarrow x \geq N(N(x))$.

If $x < N(x)$, then

$$\begin{aligned} I_{g,N}(x, x) &= g^{-1}(min(g(1), g(N(x) \wedge x) - g(N(N(x) \vee x)) + g(1))) \\ &= g^{-1}(min(g(1), g(x) - g(N(N(x))) + g(1))) \\ &= g^{-1}(g(1)) \\ &= 1. \end{aligned}$$

If $x \geq N(x)$, then

$$\begin{aligned} I_{g,N}(x, x) &= g^{-1}(min(g(1), g(N(x) \wedge x) - g(N(N(x) \vee x)) + g(1))) \\ &= g^{-1}(min(g(1), g(N(x)) - g(N(x)) + g(1))) \\ &= g^{-1}(g(1)) \\ &= 1. \end{aligned}$$

Thus $I_{g,N}$ satisfies (IP). □

3.2.3 The ordering property (OP)

Theorem 3.8. *The fuzzy implication $I_{g,N}$ satisfies (OP) if and only if N is strong.*

Proof. Let $I_{g,N}$ satisfy (OP), then $I_{g,N}$ satisfies (IP). From Proposition 3.7, we have $x < N(x) \Rightarrow x \geq N(N(x))$, $x \in [0,1]$.

Suppose $N(x)$ is not strong, then there exists $x_0 \in (0,1)$ such that $x_0 \neq N(N(x_0))$.

(i) If $x_0 < N(x_0)$, then $x_0 > N(N(x_0))$, so there exists $y_0 \in [0,1]$ such that $N(N(x_0)) < y_0 < x_0 < N(x_0)$, thus

$$\begin{aligned} I_{g,N}(x_0,y_0) &= g^{-1}(min(g(1), g(N(x_0) \wedge y_0) - g(N(N(x_0) \vee y_0)) + g(1))) \\ &= g^{-1}(min(g(1), g(y_0) - g(N(N(x_0))) + g(1))) \\ &= g^{-1}(g(1)) \\ &= 1, \end{aligned}$$

it contradicts the fact that $I_{g,N}$ satisfies (OP).

(ii) If $x_0 = N(x_0)$, then $N(x_0) = N(N(x_0)) = x_0$, a contradiction.

(iii) If $x_0 > N(x_0)$, then $N(N(x_0)) \geq N(x_0)$, consider the following two cases:

Case 1. $N(N(x_0)) = N(x_0)$, then

$$\begin{aligned} I_{g,N}(x_0,N(x_0)) &= g^{-1}(min(g(1), g(N(x_0) \wedge N(x_0)) - g(N(N(x_0) \vee N(x_0))) + g(1))) \\ &= g^{-1}(min(g(1), g(N(x_0)) - g(N(x_0)) + g(1))) \\ &= g^{-1}(g(1)) \\ &= 1, \end{aligned}$$

a contradiction.

Case 2. $N(N(x_0)) > N(x_0)$, from Proposition 3.7, then $N(x_0) \geq N(N(N(x_0)))$. From the proof of (i), $N(x_0) > N(N(N(x_0)))$ is not hold, then $N(x_0) = N(N(N(x_0)))$.

If $x_0 > N(N(x_0))$, then there exists $y_0 \in [0,1]$, such that $N(N(x_0)) < y_0 < x_0$. Hence $y_0 > N(x_0)$, since $N(x_0) = N(N(N(x_0)))$, then $N(y_0) = N(x_0)$. However,

$$\begin{aligned} I_{g,N}(x_0,y_0) &= g^{-1}(min(g(1), g(N(x_0) \wedge y_0) - g(N(N(x_0) \vee y_0)) + g(1))) \\ &= g^{-1}(min(g(1), g(N(x_0)) - g(N(y_0)) + g(1))) \\ &= g^{-1}(min(g(1), g(1))) \\ &= 1, \end{aligned}$$

a contradiction.

If $x_0 < N(N(x_0))$, then there exists $y_0 \in [0,1]$, such that $x_0 < y_0 < N(N(x_0))$. So, $y_0 > N(x_0)$, since $N(x_0) = N(N(N(x_0)))$, then $N(y_0) = N(x_0)$, however,

$$\begin{aligned} I_{g,N}(y_0,x_0) &= g^{-1}(min(g(1), g(N(y_0) \wedge x_0) - g(N(N(y_0) \vee x_0)) + g(1))) \\ &= g^{-1}(min(g(1), g(N(x_0)) - g(N(x_0)) + g(1))) \\ &= g^{-1}(min(g(1), g(1))) \\ &= 1, \end{aligned}$$

a contradiction.

From above, we obtain $x = N(N(x))$ for all $x \in [0,1]$, i.e., N is strong.

Conversely. If N is strong, let $x \leq y$, then

$$\begin{aligned} I_{g,N}(x,y) &= g^{-1}(min(g(1), g(N(x) \wedge y) - g(N(N(x) \vee y)) + g(1))) \\ &= \begin{cases} g^{-1}(min(g(1), g(N(x)) - g(N(y)) + g(1))), & y \geq N(x) \\ g^{-1}(min(g(1), g(y) - g(N(N(x))) + g(1))), & y < N(x) \end{cases} \\ &= \begin{cases} 1, & y \geq N(x) \\ g^{-1}(min(g(1), g(y) - g(x) + g(1))), & y < N(x) \end{cases} \\ &= 1. \end{aligned}$$

Let $I_{g,N}(x, y) = 1$. If $y \geq N(x)$, then $g^{-1}(min(g(1), g(N(x)) - g(N(y)) + g(1))) = 1$, so, $min(g(1), g(N(x)) - g(N(y)) + g(1)) = g(1)$, thus $g(N(x)) \geq g(N(y))$, i.e., $x \leq y$. If $y < N(x)$, obviously, $x \leq y$. Thus, $I_{g,N}$ satisfies (OP) when N is strong. □

Corollary 3.9. *The fuzzy implication $(I_{g,N})_\varphi$ satisfies (OP) if and only if N is strong.*

3.2.4 The nature negation

Proposition 3.10. *Let $N_{I_{g,N}}$ be the nature negation of the fuzzy implication $I_{g,N}$, then $N_{I_{g,N}}$ is strong with an additive generator g if and only if N is strong.*

Proof. $N_{I_{g,N}}(x) = I_{g,N}(x, 0) = g^{-1}(g(1) - g(N(N(x))))$. If N is strong, then $N_{I_{g,N}}(x) = g^{-1}(g(1) - g(x))$. Since the function $g : [0, 1] \to [0, \infty)$ is continuous strictly increasing with $g(0) = 0$, from Theorem 2.5, then $N_{I_{g,N}}$ is strong, and g is an additive generator of $N_{I_{g,N}}$.

Conversely, since g is an additive generator of $N_{I_{g,N}}$, then $N_{I_{g,N}}(x) = g^{-1}(g(1) - g(x))$. By

$$N_{I_{g,N}}(x) = g^{-1}(g(1) - g(N(N(x)))),$$

we have $N(N(x)) = g^{-1}(g(1)-g(N_{I_{g,N}}(x)))$, then $N(N(x)) = g^{-1}(g(1)-g(g^{-1}(g(1)-g(x)))) = x$, thus N is strong. □

3.2.5 The law of contraposition (CP)

Proposition 3.11. *Let N' be a fuzzy negation, and N a continuous fuzzy negation, if $I_{g,N}$ satisfies CP(N'), then $N = N'$, and N is strong.*

Proof. Let $x, y \in [0, 1]$, since $I_{g,N}$ satisfies CP(N'), then

$$I_{g,N}(x, y) = I_{g,N}(N'(y), N'(x)), \tag{2}$$

taking $y = 0$ in equation (2), then $I_{g,N}(x, 0) = I_{g,N}(1, N'(x))$, i.e.,

$$g^{-1}(g(1) - g(N(N(x)))) = g^{-1}(g(1) - g(N(N'(x)))),$$

thus,

$$N(N(x)) = N(N'(x)) \text{ for all } x \in [0, 1]. \tag{3}$$

Taking $x = 1$ in equation (2), then $I_{g,N}(1, y) = I_{g,N}(N'(y), 0)$, i.e.,

$$g^{-1}(g(1) - g(N(y))) = g^{-1}(g(1) - g(N(N(N'(y))))),$$

thus

$$N(y) = N(N(N'(y))) \text{ for all } y \in [0, 1] \tag{4}$$

From (3), (4), we obtain $N(x) = N(N(N(x)))$, thus N is strong. From (3), we get $N = N'$. □

Proposition 3.12. *Let N be a strong fuzzy negation, then the fuzzy implication $I_{g,N}$ satisfies (CP) only with N.*

Proof. From Remark 3.3 (v), it is easy to prove $I_{g,N}(N(y), N(x)) = I_{g,N}(x, y)$. By Proposition 3.11, $I_{g,N}$ satisfies (CP) only with N. □

3.2.6 The exchange principle (EP)

Proposition 3.13. *Let N be a continuous fuzzy negation. Then the fuzzy implication $I_{g,N}$ satisfies (EP) if and only if N is strong with an additive generator g.*

Proof. Let $I_{g,N}$ satisfy (EP), then

$$I_{g,N}(x, I_{g,N}(y,z)) = I_{g,N}(y, I_{g,N}(x,z)), \text{ for all } x,y,z \in [0,1].$$

Let $z=0$, we obtain $I_{g,N}(x, N_{I_{g,N}}(y)) = I_{g,N}(y, N_{I_{g,N}}(x)), x, y \in [0,1]$. Let $y=1$, then $I_{g,N}(x,0) = I_{g,N}(1, N_{I_{g,N}}(x)), x \in [0,1]$, hence

$$N(N(x)) = N(N_{I_{g,N}}(x)), x \in [0,1],$$

thus

$$N(N(x)) = N(g^{-1}(g(1) - g(N(N(x))))), x \in [0,1].$$

Let $\widetilde{N}(x) = g^{-1}(g(1) - g(x)),\ x \in [0,1]$, obviously, $\widetilde{N}(x)$ is a strong fuzzy negation, then

$$N(N(x)) = N(\widetilde{N} \circ (N(N(x)))),\ x \in [0,1].$$

Let $u = N(N(x))$, since N is a continuous fuzzy negation, then $u = N(\widetilde{N}(u)) \in [0,1]$, note that $\widetilde{N}(\widetilde{N}(u)) = u$, then $N = \widetilde{N}$, i.e., N is strong with an additive generator g.

Conversely, let g be an additive generator of N, then $N(x) = g^{-1}(g(1) - g(x)), x \in [0,1]$, from Remark 3.3 (v), we obtain

$$I_{g,N}(x, I_{g,N}(y,z)) = \begin{cases} g^{-1}(g(z) - g(y) - g(x) + 2g(1)), & y \geq z \text{ and } g(x) \geq g(z) - g(y) + g(1), \\ 1, & \text{otherwise}, \end{cases}$$

$$I_{g,N}(y, I_{g,N}(x,z)) = \begin{cases} g^{-1}(g(z) - g(x) - g(y) + 2g(1)), & x \geq z \text{ and } g(y) \geq g(z) - g(x) + g(1), \\ 1, & \text{otherwise}. \end{cases}$$

Consider the following two cases:

Case 1. If $x \leq z, y \leq z$, then $I_{g,N}(y, I_{g,N}(x,z)) = 1 = I_{g,N}(x, I_{g,N}(y,z))$.

Case 2. If $x \leq z, y \geq z$, then $I_{g,N}(y, I_{g,N}(x,z)) = 1$. Since $g(y) - g(z) \geq g(1) - g(x)$ is not hold, then $I_{g,N}(x, I_{g,N}(y,z)) = 1$, thus $I_{g,N}(y, I_{g,N}(x,z)) = I_{g,N}(x, I_{g,N}(y,z))$.

Case 3. If $x \geq z, y \leq z$, similar to case 2, then $I_{g,N}(y, I_{g,N}(x,z)) = I_{g,N}(x, I_{g,N}(y,z))$.

Case 4. If $x \geq z, y \geq z$, obviously then $I_{g,N}(y, I_{g,N}(x,z)) = I_{g,N}(x, I_{g,N}(y,z))$.

From above discussion, $I_{g,N}$ satisfies (EP). □

Corollary 3.14. *Let N be a strong fuzzy negation with an additive generator g, then $I_{g,N}$ satisfies R-CP($N_{I_{g,N}}$).*

Corollary 3.15. *If N is a continuous fuzzy negation, then the following items are equivalent:*

(i) $I_{g,N}$ satisfies (EP).

(ii) $I_{g,N}$ is an (S,N)-implication generated from a continuous Archimedean t-conorm

$$S(x,y) = g^{-1}(min(g(1), g(x) + g(y)))$$

and a fuzzy negation N.

(iii) $I_{g,N}$ satisfies (LI) with a t-norm T.

3.3 Other properties of the proposed fuzzy implications

In this section, we show that the g-generators of the proposed fuzzy implications are unique up to a positive multiplicative constant, the result is same as the one in [11] (see Proposition 2.2, [11]), but the proof is a little different from the proof of Proposition 2.2 in [11].

Proposition 3.16. *Let $g_1, g_2 : [0,1] \to [0,\infty)$ be two strictly increasing, continuous functions, with $g_1(0) = 0$, $g_2(0) = 0$, and N_1, N_2 two continuous fuzzy negation. Then the fuzzy implication I_{g_1,N_1} and the fuzzy implication I_{g_2,N_2} are equal if and only if $N_1 = N_2$ and there exists a constant $k > 0$ such that $g_2 = kg_1$.*

Proof. Let $N_1 = N_2$ and $g_2 = kg_1$ for some positive constant k, then it is straight forward to show that $I_{g_1,N_1} = I_{g_2,N_2}$.

Conversely, let $I_{g_1,N_1} = I_{g_2,N_2}$. Since

$$I_{g_1,N_1}(x,y) = \begin{cases} 1, & y \geq N_1(x), x \leq y \ \ or, y < N_1(x), y \geq N_1(N_1(x)), \\ g_1^{-1}(g_1(N_1(x)) - g_1(N_1(y)) + g_1(1)), & y \geq N_1(x), x \geq y, \\ g_1^{-1}(g_1(y) - g_1(N_1(N_1(x))) + g_1(1)), & y < N_1(x), y < N_1(N_1(x)), \end{cases}$$

$$I_{g_2,N_2}(x,y)=\begin{cases}1, & y\geq N_2(x), x\leq y \ \ or, y<N_2(x), y\geq N_2(N_2(x)),\\ g_2^{-1}(g_2(N_2(x))-g_2(N_2(y))+g_2(1)), & y\geq N_2(x), x\geq y,\\ g_2^{-1}(g_2(y)-g_2(N_2(N_2(x)))+g_2(1)), & y<N_2(x), y<N_2(N_2(x)).\end{cases}$$

From above, obviously, $N_1 = N_2$.

Let $N_1 = N_2 = N$, then for $x \geq y \geq N(x)$, we have

$$g_1^{-1}(g_1(N(x))-g_1(N(y))+g_1(1)) = g_2^{-1}(g_2(N(x))-g_2(N(y))+g_2(1)).$$

Let $h_1, h_2 : [0,1] \to [0,1]$ be two function defined by

$$h_1(x)=\frac{g_1(x)}{g_1(1)}, h_2(x)=\frac{g_2(x)}{g_2(1)},$$

respectively, then we obtain

$$h_1^{-1}(h_1(N(x))-h_1(N(y))+1) = h_2^{-1}(h_2(N(x))-h_2(N(y))+1),$$

thus

$$h_2\circ h_1^{-1}(h_1(N(x))-h_1(N(y))+1) = h_2\circ h_1^{-1}(h_1(N(x))) - h_2\circ h_1^{-1}(h_1(N(y)))+1.$$

Let $\varphi = h_2 \circ h_1^{-1}$, then $\varphi : [0,1] \to [0,1]$ is a continuous strictly increasing function such that

$$\varphi(h_1(N(x))-h_1(N(y))+1) = \varphi(h_1(N(x))) - \varphi(h_1(N(y)))+1. \tag{5}$$

Let $x = 1$, thus, for $0 \leq y \leq 1$, we obtain

$$\varphi(1-h_1(N(y))) = 1-\varphi(h_1(N(y))).$$

Taking $z = h_1(N(y))$, then

$$\varphi(1-z) = 1-\varphi(z) \ \text{ for all } \ z\in[0,1]. \tag{6}$$

Thus

$$\varphi(\frac{1}{2}) = \frac{1}{2}, \ i.e., \ \ h_1^{-1}(\frac{1}{2}) = h_2^{-1}(\frac{1}{2}).$$

Let $N(y) = h_1^{-1}(\frac{1}{2}) = h_2^{-1}(\frac{1}{2})$ in (5), then

$$\varphi(h_1(N(x))+\frac{1}{2}) = \varphi(h_1(N(x)))+\frac{1}{2}, \ \text{wehere } \ N(x)\leq h_1^{-1}(\frac{1}{2}).$$

Taking $z = h_1(N(x))$, then

$$\varphi(z+\frac{1}{2}) = \varphi(z)+\frac{1}{2}, \ \ z\in[0,\frac{1}{2}], \tag{7}$$

from (6) and (7), we obtain

$$\varphi(\frac{1}{4}) = \frac{1}{4}, \ \ i.e., \ \ h_1^{-1}(\frac{1}{4}) = h_2^{-1}(\frac{1}{4}).$$

Similarly, taking $N(y) = h_1^{-1}(\frac{1}{4}) = h_2^{-1}(\frac{1}{4})$ in (5) and $z = h_1(N(x))$, we obtain

$$\varphi(z+\frac{3}{4}) = \varphi(z)+\frac{3}{4}, \ \ z\in[0,\frac{1}{4}], \tag{8}$$

from (6) and (8), we obtain $\varphi(\frac{1}{8}) = \frac{1}{8}$, iterating this reasoning we obtain

$$\varphi(z+\frac{2^n-1}{2^n}) = \varphi(z)+\frac{2^n-1}{2^n}, \ \ z\in[0,\frac{1}{2^n}], \tag{9}$$

and

$$\varphi(\frac{1}{2^n}) = \frac{1}{2^n},\ \ \varphi(\frac{2^n-1}{2^n}) = \frac{2^n-1}{2^n},$$

where $n \in N^*$ (set of positive integer). So, the interval $[0, \frac{1}{2}]$ can be written as

$$[0, \frac{1}{2^n}] \cup \cdots \cup [\frac{1}{8}, \frac{1}{4}] \cup [\frac{1}{4}, \frac{1}{2}].$$

Let us consider the midpoint of the interval $[\frac{1}{4}, \frac{1}{2}]$, which is $\frac{3}{8}$, since $\frac{3}{8} \in (\frac{1}{4}, \frac{1}{2})$, then

$$\varphi(\frac{3}{8} + \frac{1}{2}) = \varphi(\frac{3}{8}) + \frac{1}{2}.$$

Note that $\varphi(\frac{3}{8} + \frac{1}{2}) = \varphi(\frac{7}{8}) = \frac{7}{8}$, then $\varphi(\frac{3}{8}) = \frac{3}{8}$. The same can be done for other intervals: $[0, \frac{1}{2^n}], \cdots, [\frac{1}{16}, \frac{1}{8}], [\frac{1}{8}, \frac{1}{4}]$, that is, $\varphi(\frac{1}{2^{n+1}}) = \frac{1}{2^{n+1}}, \cdots, \varphi(\frac{3}{32}) = \frac{3}{32}, \varphi(\frac{3}{16}) = \frac{3}{16}$. So,

$$[0, \frac{1}{2}] = [0, \frac{1}{2^{n+1}}] \cup [\frac{1}{2^{n+1}}, \frac{1}{2^n}] \cup \cdots \cup [\frac{1}{8}, \frac{3}{16}] \cup [\frac{3}{16}, \frac{1}{4}] \cup [\frac{1}{4}, \frac{3}{8}] \cup [\frac{3}{8}, \frac{1}{2}].$$

For the midpoint of the interval $[\frac{1}{4}, \frac{3}{8}]$, which is $\frac{5}{16} \in (\frac{1}{4}, \frac{1}{2})$, we have

$$\varphi(\frac{5}{16} + \frac{1}{2}) = \varphi(\frac{5}{16}) + \frac{1}{2}.$$

Since $\varphi(\frac{5}{16} + \frac{1}{2}) = \varphi(\frac{13}{16}) = 1 - \varphi(\frac{3}{16})$, note that $\frac{3}{16}$ is the midpoint of the interval $[\frac{1}{8}, \frac{1}{4}]$, then $\varphi(\frac{3}{16}) = \frac{3}{16}$, thus $\varphi(\frac{5}{16}) = \frac{5}{16}$. The same can be done for other interval. keep doing the same reasoning, and note that φ is symmetric on the point $(\frac{1}{2}, \frac{1}{2})$, then the set $\{x \in [0,1] | \varphi(x_0) = x_0\}$ is dense, also, considering the continuity of φ , then φ must an identity function on $[0, 1]$, i.e.,

$$\varphi(z) = z \ \text{ for } \ z \in [0, 1],$$

so, $h_1^{-1}(z) = h_2^{-1}(z)$, and hence $h_1^{-1}(h_1(N(y))) = h_2^{-1}(h_1(N(y)))$, then $h_2(N(y)) = h_1(N(y))$, since $N(y) \in [0, 1]$, we have $h_2(x) = h_1(x), x \in [0, 1]$, and so

$$\frac{g_1(x)}{g_1(1)} = \frac{g_2(x)}{g_2(1)}.$$

Let $k = \frac{g_1(1)}{g_2(1)}$, obviously, $k > 0$, and we have $g_1(x) = kg_2(x)$. □

4 T-Conditionality with the proposed fuzzy implications

The functional inequality $T(x, I(x, y)) \leq y,\ x, y \in [0, 1]$, which is called T-conditional, is a very important property of implications, the reason is that such pair of implication I and t-norm T which do have this property may be used in the compositional rule of inference [24]. So, in this section, we will consider the T-conditionality for the proposed fuzzy implications.

Definition 4.1. [4, 22] *An implication I and a t-norm T satisfy T-conditionality if and only if*

$$T(x, I(x, y)) \leq y,\ \ x, y \in [0, 1]. \quad (TC)$$

Proposition 4.2. [4] *If $I \in FI$ is such that there exist $x, y \in (0, 1)$ with $x > y$ and $I(x, y) = 1$, then I does not satisfy (TC) with any t-norm T.*

Proposition 4.3. [4,22] *Let $I \in FI$ and a t-norm T satisfy (TC), then*
(i) $N_I \leq N_T$, the natural negation of T.
(ii) If $N_I = N_{D_1}$, the least fuzzy negation, then T has zero-divisors.
(iii) If N_I is a non-vanishing negation and T is continuous, then T is ϕ-conjugate with T_{LK}.

From the above two proposition, we can easily obtain the following proposition.

Proposition 4.4. *Let $I_{g,N}$ be a fuzzy implication defined by (1), and let $G=\{x\in[0,1]|N(x)\geq x\}$. If $I_{g,N}$ and a t-norm T satisfy (TC), then*

(i) $N(x)\geq\widetilde{N}(x)$, $x\in[0,1]$, where $\widetilde{N}(x)=g^{-1}(g(1)-g(x)), x\in[0,1]$.

(ii) $N(N(x))\geq\widetilde{N}(N_T(x))$, $x\in[0,1]$, where $N_T(x)=sup\{y\in[0,1]\mid T(x,y)=0\}$.

(iii) $N(\widetilde{N}(x))\geq\widetilde{N}(N(x))$, $x\in[0,1]$.

(iv) $N(N(x))\geq x, x\in G$.

(v) Let $\overline{G}$ be the complement set of G, i.e., $\overline{G}=\{x\in[0,1]|N(x)<x\}$, then $\overline{G}\neq\{1\}$, and N is a strictly deceasing function on $\overline{G}$.

Proof. Since $I_{g,N}$ and a t-norm T satisfy (TC), then $T(x,I_{g,N}(x,y))\leq y$ for all $x,y\in[0,1]$.

(i) Let $x=1$, then $I_{g,N}(1,y))\leq y, y\in[0,1]$, and so $g^{-1}(g(1)-g(N(y)))\leq y, y\in[0,1]$, thus $N(y)\geq g^{-1}(g(1)-g(y))=\widetilde{N}(y)$, that is, $N(x)\geq\widetilde{N}(x), x\in[0,1]$.

(ii) By Proposition 4.3 (i), we have $N_{I_{g,N}}\leq N_T$, then $g^{-1}(g(1)-g(N(N(x))))\leq N_T(x)$, thus $N(N(x))\geq\widetilde{N}(N_T(x))$.

(iii) By Proposition 4.4 (i), we have $N(N(x))\geq\widetilde{N}(N(x))$, and $N(N(x))\leq N(\widetilde{N}(x))$, then $N(\widetilde{N}(x))\geq\widetilde{N}(N(x)), x\in[0,1]$.

(iv) Suppose there exists a $x_0\in G$, that is $N(x_0)\geq x_0$, such that $N(N(x_0))<x_0$, then

$$I_{g,N}(x_0,N(N(x_0)))=g^{-1}(min(g(1),g(N(x_0)\wedge N(N(x_0)))-g(N(N(x_0)\vee N(N(x_0))))+g(1)))=1,$$

a contradiction to Proposition 4.2.

(v) Suppose $\overline{G}=\{1\}$, then $N(x)=N_{D_2}$, hence, $G=[0,1)$, and $N(N(x))=0$ for $x\in G$, thus $N(N(x))<x$ for $x\in G$, which contradicts $N(N(x))\geq x$ for $x\in G$.

Suppose there exist $x_1,x_2\in\overline{G}$, and $x_1<x_2$, such that $N(x_1)=N(x_2)$, then $N(x_1)<x_1$, $N(x_2)<x_2$. However, $I_{g,N}(x_2,x_1)=g^{-1}(min(g(1),g(N(x_2)\wedge x_1)-g(N(N(x_2)\vee x_1))+g(1)))=1$, a contradiction. □

Remark 4.5. *(i) For the fuzzy negation $N=N_{D_2}$, since $\overline{G}=\{1\}$, hence $I_{g,N_{D_2}}$ does not satisfy (TC) with any t-norm T.*

(ii) For the fuzzy negation $N=N_{D_1}$, obviously, $\overline{G}=(0,1]$, since N_{D_1} does not strictly decreasing on $\overline{G}$, then $I_{g,N_{D_1}}$ does not satisfy (TC) with any t-norm T.

Lemma 4.6. *Let N be a fuzzy negation, then $N=N_{D_1}$ if and only if*

$$N(N(x))=\begin{cases}0, & \text{if } x=0,\\ 1, & \text{if } x>0.\end{cases}$$

Proof. Let $N(N(x))=\begin{cases}0, & \text{if } x=0\\ 1, & \text{if } x>0\end{cases}$. Suppose $N(x)\neq N_{D_1}$, then there exists $x_0\in(0,1)$ such that $N(x_0)>0$. If $N(x_0)=1$, then $N(N(x_0))=N(1)=0$, a contradiction. If $N(x_0)\in(0,1)$, then $N(x)>0$ for all $x\in[0,x_0]$, if $N(x_0)\geq x_0$, then $N(N(x_0))\leq N(x_0)<1$, a contradiction. If $N(x_0)<x_0$, since $N(N(x_0))=1$, then $N(u)=1$ for all $u\in[0,N(x_0)]$, especially $N(u)=1$ for all $u\in(0,N(x_0)]$, thus we obtain $N(N(u))=0$ for all $u\in(0,N(x_0)$, a contradiction.

Let $N(x)=N_{D_1}$, it is easy to obtain $N(N(x))=\begin{cases}0, & \text{if } x=0\\ 1, & \text{if } x>0\end{cases}$. □

Proposition 4.7. *Let $I_{g,N}$ be a fuzzy implication defined by (1), then $I_{g,N}$ does not satisfy (TC) with any positive t-norm T.*

Proof. Suppose that T is a positive t-norm, then $N_T=N_{D_1}$, thus $\widetilde{N}(N_T(x))=\begin{cases}0, & \text{if } x=0\\ 1, & \text{if } x>0\end{cases}$. Since $N(N(x))\geq\widetilde{N}(N_T(x))$, then $N(N(x))=\begin{cases}0, & \text{if } x=0,\\ 1, & \text{if } x>0.\end{cases}$ From Lemma 4.6, we obtain $N=N_{D_1}$, from Remark 4.5 (ii), then $I_{g,N}$ does not satisfy (TC) with any positive t-norm T. □

Remark 4.8. *(i) Since the t-norm T_P is a positive t-norm, then $I_{g,N}$ does not satisfy (TC) with any t-norm $T\geq T_P$.*

(ii) A problem: Let N be a fuzzy negation with $N(x)\geq\widetilde{N}(x)$, $x\in[0,1]$, $N(N(x))\geq\widetilde{N}(N_T(x))$, $x\in[0,1]$, $N(N(x))\geq x, x\in G$, where $\widetilde{N}(x)=g^{-1}(g(1)-g(x))$, $N_T(x)=sup\{y\in[0,1]\mid T(x,y)=0\}$, $G=\{x\in[0,1]|N(x)\geq x\}$. Does there exist a non-positive t-norm T such that $I_{g,N}$ satisfy (TC) with the non-positive t-norm T?

Theorem 4.9. *Let N be a fuzzy negation such that $N(x) \geq \widetilde{N}(x)$, $x \in [0,1]$, $N(N(x)) \geq x, x \in G$, where $G = \{t \in [0,1] | N(t) \geq t\}$. Then the fuzzy implication $I_{g,N}$ and the t-norm T_D satisfy (TC) if and only if N is strictly decreasing on $\overline{G}$, where $\overline{G} = \{t \in [0,1] | t > N(t)\}$.*

Proof. Let $I_{g,N}$ and the t-norm T_D satisfy (TC), from Proposition 4.4 (v), it is easy to obtain that N is strictly decreasing on $\overline{G}$

Conversely, Let $x, y \in [0,1]$, N a be strictly decreasing fuzzy negation on $\overline{G}$. Consider $x > y$.

If $x = 1$, then $T_D(x, I_{g,N}(x,y)) = \widetilde{N}(N(y)) \leq \widetilde{N}(\widetilde{N}(y)) = y$.

If $x \neq 1$ and $x \in G$, then $N(x) \geq x$, note that $x > y$, then $N(x) > y$, thus

$$\begin{aligned} I_{g,N}(x,y) &= g^{-1}(min(g(1), g(N(x) \wedge y) - g(N(N(x) \vee y)) + g(1))) \\ &= g^{-1}(g(y) - g(N(N(x))) + g(1)) \\ &\leq g^{-1}(g(y) - g(x) + g(1)) \\ &< 1, \end{aligned}$$

then $T_D(x, I_{g,N}(x,y)) = 0 \leq y$.

If $x \neq 1$, and $x \in \overline{G}$, $y \in \overline{G}$, then $x > N(x)$, $y \leq N(y)$, obviously, $N(x) < N(y)$. Actually, if $N(x) = N(y)$, then $N(N(x)) = N(N(y))$, from $y \leq N(y)$, we have $N(y) \geq N(N(y))$, thus $x > N(x) = N(y) \geq N(N(y)) = N(N(x))$, which contradicts $N(N(x)) \geq x$. So,

$$\begin{aligned} I_{g,N}(x,y) &= g^{-1}(min(g(1), g(N(x) \wedge y)) - g(N(N(x) \vee y) + g(1))) \\ &= \begin{cases} g^{-1}(g(N(x)) - g(N(y)) + g(1)), & \text{if } y > N(x) \\ g^{-1}(g(y) - g(N(N(x))) + g(1)), & \text{if } y \leq N(x) \end{cases} \\ &< 1, \end{aligned}$$

then $T_D(x, I_{g,N}(x,y)) = 0 \leq y$.

If $x \neq 1$, and $x, y \in \overline{G}$, note that N is strict on $\overline{G}$, then $N(x) < N(y) < y < x$, thus

$$I_{g,N}(x,y) = g^{-1}(min(g(1), g(N(x) \wedge y)) - g(N(N(x) \vee y) + g(1))) = g^{-1}(g(N(x)) - g(N(y)) + g(1)) < 1,$$

then $T_D(x, I_{g,N}(x,y)) = 0 \leq y$. □

Corollary 4.10. *Let N be a strong fuzzy negation, and satisfies $N(x) \geq \widetilde{N}(x)$ for all $x \in [0,1]$, then the fuzzy implication $I_{g,N}$ and the t-norm T_D satisfy (TC).*

Corollary 4.11. *Let N be a strict fuzzy negation, and satisfies $N(x) \geq \widetilde{N}(x)$ for all $x \in [0,1]$, $N(N(x)) \geq x, x \in \{t \in [0,1] | N(t) \geq t\}$, then the fuzzy implication $I_{g,N}$ and the t-norm T_D satisfy (TC).*

Problem 4.12. *Let N be a fuzzy negation with $N(x) \geq \widetilde{N}(x)$, $x \in [0,1]$, $N(N(x)) \geq \widetilde{N}(N_T(x))$, $x \in [0,1]$, $N(N(x)) \geq x$, $x \in G$, and let N be a strictly decreasing fuzzy negation on $\overline{G}$, where $\widetilde{N}(x) = g^{-1}(g(1) - g(x))$, $N_T(x) = sup\{y \in [0,1] \mid T(x,y) = 0\}$, $G = \{x \in [0,1] | N(x) \geq x\}$, $\overline{G} = \{t \in [0,1] | t > N(t)\}$. Does $I_{g,N}$ satisfy (TC) with any non-positive t-norm T?*

Unfortunately, the answer is negative.

Example 4.13. *Let N be a strict fuzzy negation with $N(x) \geq \widetilde{N}(x)$, $x \in [0,1]$, $N(N(x)) \geq \widetilde{N}(N_T(x))$, $x \in [0,1]$, $N(N(x)) \geq x, x \in G$, where $\widetilde{N}(x) = g^{-1}(g(1) - g(x))$, $N_T(x) = sup\{y \in [0,1] \mid T(x,y) = 0\}$, $G = \{x \in [0,1] | N(x) \geq x\}$, and let T be the t-norm T_{nM} defined as*

$$T_{nM}(x,y) = \begin{cases} 0, & \text{if } x + y \leq 1, \\ min(x,y), & \text{otherwise}, \end{cases}$$

then $I_{g,N}$ does not satisfy (TC) with T_{nM}.

Proof. Suppose that $I_{g,N}$ and T_{nM} satisfy (TC). Since T_{nM} is a left-continuous t-norm, then $I_{g,N}(x,y) \leq I_{T_{nM}}(x,y)$ for all $x, y \in [0,1]$, where $I_{T_{nM}}(x,y) = \begin{cases} 1, & \text{if } x \leq y, \\ max(1-x,y), & \text{if } x > y. \end{cases}$ Consider $x > y$, we have

$$I_{g,N}(x,y) \leq max(1-x,y) \text{ for all } x > y,\ x,y \in [0,1], \tag{10}$$

thus

$$min(g(1), g(N(x) \wedge y) - g(N(N(x) \vee y)) + g(1)) \leq max(g(1-x), g(y)). \tag{11}$$

Since $x > y, x, y \in [0,1]$, then

$$N(x) \wedge y < N(N(x) \vee y) \text{ for all } x > y, x, y \in [0,1], \tag{12}$$

thus

$$g(N(x) \wedge y) - g(N(N(x) \vee y)) + g(1) \leq max(g(1-x), g(y)). \tag{13}$$

Let $s = N(s)$, then $s \in (0,1)$, and $x > N(x)$ when $x \in (s,1]$. Let $x_0 \in (s,1) \cap (\frac{1}{2}, 1)$.

Consider the following cases:

Case 1. If $N(x_0) > 1 - x_0$, thus there exists $y_0 \in (0,1)$, such that $N(x_0) > y_0 > 1 - x_0$, obviously, $x_0 > y_0$, and $N(N(x_0)) \geq N(x_0)$. Hence, we obtain $y_0 < N(N(x_0))$. However,

$$I_{g,N}(x_0,y_0) = g^{-1}(g(y_0) + g(1) - g(N(N(x_0)))) > g^{-1}(g(y_0)) = y_0 = max(1-x_0, y_0),$$

which contradicts (10).

Case 2. If $N(x_0) = 1 - x_0$, let $y_0 = N(x_0)$, obviously, $x_0 > y_0$. However,

$$I_{g,N}(x_0,y_0) = g^{-1}(g(y_0) + g(1) - g(x_0)) > g^{-1}(g(y_0)) = y_0 = max(1-x_0, y_0),$$

which contradicts (10).

Case 3. If $N(x_0) < 1 - x_0$, note that $1 - x_0 < x_0$, then there exists $y_0 \in (0,1)$, such that $x_0 > y_0 > 1 - x_0$, from (13), we obtain

$$g(N(x_0)) - g(N(y_0)) + g(1) \leq g(y_0),$$

then

$$g(1) - g(y_0) \leq g(N(y_0)) - g(N(x_0)),$$

note that $y_0 < x_0$, then

$$g(1) - g(x_0) < g(N(y_0)) - g(N(x_0)),$$

fix x_0, let $y_0 \to x_0$, consider the continuity of g and N, then $g(N(y_0)) - g(N(x_0)) \to 0$, but $g(1) - g(x_0)$ is fixed, a contradiction. From above discussion, $I_{g,N}$ does not satisfy (TC) with T_{nM}. □

Theorem 4.14. *Let N be a strict fuzzy negation and T a continuous non-positive t-norm, then the following statements are equivalent:*

(i) $I_{g,N}$ and T satisfy (TC).

(ii) There exists $\varphi \in \Phi$ such that $T = (T_{LK})_\varphi$, $N(N(x)) \geq \widetilde{N}((N_C)_\varphi(x))$ and $I_{g,N}(x,y) \leq (S_{LK})_\varphi((N_C)_\varphi(x), y), x, y \in [0,1]$.

Proof. Since N is a strict fuzzy negation, then $N_{I_{g,N}}(x) = 0 \Leftrightarrow N(N(x)) = 1 \Leftrightarrow x = 1$, i.e., $N_{I_{g,N}}$ is a non-vanishing negation. From Proposition 4.3 (iii), there exists $\varphi \in \Phi$ such that $T = (T_{LK})_\varphi$. The rest of the proof is similarly to the proof of Theorem 7.4.6 in [4]. □

5 The proposed fuzzy implications and the Modus tollens

The most used inference schemes in approximate reasoning are the so-called Modus Ponens for forward inferences and Modus Tollens for backward inferences. In this section, we study the modus tollens property for the proposed implications.

Definition 5.1. [22] *Let I be a fuzzy implication, T a t-norm and N_1 a fuzzy negation. It is said that I satisfies the Modus Tollens property with respect to T and N_1 if $T(N_1(y), I(x,y)) \leq N_1(x)$ for all $x, y \in [0,1]$.*

Proposition 5.2. [22] *Let T be a t-norm, I a fuzzy implication and N_1 a fuzzy negation. If I satisfies the modus tollens with respect to T and N_1, then*
(i) $I(1,y) \leq N_T(N_1(y))$ for all $y \in [0,1]$.
(ii) $N_I(x) \leq N_1(x)$ for all $x \in [0,1]$.
(iii) $N_1(x) = N_1(y)$ for all $(x,y) \in \{(x,y) | x > y, I(x,y) = 1, x, y \in [0,1]\}$.

Corollary 5.3. *Let I be a fuzzy implication, then I satisfies the modus tollens with respect to N_{D_1} and any t-norm T if and only if $N_I = N_{D_1}$.*

Theorem 5.4. *Let T be a positive t-norm, N_1 a fuzzy negation, then the fuzzy implication $I_{g,N}$ satisfies the modus tollens with respect to T and N_1 if and only if*

$$N = N_1, \text{ and } N_1(x) = \begin{cases} 1, & \text{if } x \in B, \\ 0, & \text{if } x \in A, \end{cases}$$

where A and B are a partition of the interval [0,1] such that $A \cap B = \emptyset$, $A \cup B = [0,1]$.

Proof. Let $I_{g,N}$ satisfy the modus tollens with respect to T and N_1. Since T is a positive t-norm, then

$$N_T(x) = \begin{cases} 1, & \text{if } x = 0, \\ 0, & \text{if } x \in (0,1]. \end{cases}$$

From Proposition 5.2 (i), we have

$$I_{g,N}(1,y) \leq \begin{cases} 1, & \text{if } N_1(y) = 0, \\ 0, & \text{if } N_1(y) > 0. \end{cases}$$

Let $A = \{y \in [0,1] | N_1(y) = 0\}$, $B = \{y \in [0,1] | N_1(y) > 0\}$, obviously, $A \cap B = \emptyset$, $A \cup B = [0,1]$ and $A \neq \emptyset$, $B \neq \emptyset$. Consider $y \in B$, then $I_{g,N}(1,y) = 0$, thus, $g^{-1}(g(1) - g(N(y))) = 0$, hence $N(y) = 1$ for all $y \in B$.

From Proposition 5.2 (ii), we have $g^{-1}(g(1) - g(N(N(x)))) \leq N_1(x)$ for all $x \in [0,1]$. Let $x \in B$, then $N_1(x) = 1$. Let $x \in A$, then $N(N(x)) = 1$, thus

$$N_1(x) = \begin{cases} 1, & \text{if } x \in B, \\ 0, & \text{if } x \in A, \end{cases}, \; N(x) = 1, \text{ for all } x \in B, \quad N(N(x)) = 1, \text{ for all } x \in A.$$

Let we consider $x \in A, y \in B$, then

$$T(N_1(y), I_{g,N}(x,y)) = I_{g,N}(x,y), \quad N_1(x) = 0.$$

From the condition that $I_{g,N}$ satisfies the modus tollens with T and N_1, then

$$g(N(x) \wedge y) - g(N(N(x) \vee y)) + g(1) = 0, \text{ for } x \in A, \; y \in B. \tag{14}$$

Fix x, where $x \in A$, from $N(N(x)) = 1$, we have $N(x) < 1$. If $N(x) > 0$, then there exists a y_0, such that $0 < y_0 < N(x)$, then $N(y_0) \geq N(N(x)) = 1$, thus $y_0 \in B$, then from (14), we have

$$g(y_0) - g(N(N(x))) + g(1) = 0.$$

Note that $N(N(x)) = 1$ for $x \in A$, then $y_0 = 0$, a contradiction, then $N(x) = 0$ for $x \in A$, thus, we obtain

$$N(x) = \begin{cases} 1, & \text{if } x \in B \\ 0, & \text{if } x \in A \end{cases} = N_1(x).$$

Conversely, if $N = N_1$, and $N_1(x) = \begin{cases} 1, & \text{if } x \in B \\ 0, & \text{if } x \in A \end{cases}$, it is easy to prove that $I_{g,N}$ satisfies the modus tollens with respect to T and N_1. □

Corollary 5.5. *If $I_{g,N}$ satisfies (OP), then $I_{g,N}$ does not satisfy the modus tollens with respect to any positive T and N_1.*

Problem 5.6. *Let N_1 be a fuzzy negation defined as $N_1(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \in B \end{cases}$, where $A \cap B = \emptyset,\ A \cup B = [0,1]$. And let $N = N_1$, $I_{g,N}$ satisfies the modus tollens with respect to a t-norm T and N_1, does need the t-norm T must be positive?*

The answer is negative.

Proposition 5.7. *Let N_1 be a fuzzy negation defined as*

$$N_1(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \in B, \end{cases}$$

where $A \cap B = \emptyset,\ \ A \cup B = [0,1]$, and let T be a t-norm. Then the fuzzy implication $I_{g,N}$ satisfies the modus tollens with respect to T and N_1 if and only if $N = N_1$.

Proof. Let $I_{g,N}$ satisfy the modus tollens with respect to T and N_1, then for any $x \in B$, $y \in A$, we have

$$T(N_1(y), I_{g,N}(x,y)) \leq N_1(x),$$

i.e.,

$$I_{g,N}(x,y) = g^{-1}(min(g(1), g(N(x) \wedge y) - g(N(N(x) \vee y)) + g(1))) = 0,$$

then

$$g(N(x) \wedge y) + g(1) = g(N(N(x) \vee y)). \tag{15}$$

From (15), we obtain

$$N(N(x) \vee y) = 1, \text{ for any } \ x \in B,\ \ y \in A. \tag{16}$$

$$N(x) \wedge y = 0, \text{ for any } \ x \in B,\ \ y \in A \tag{17}$$

If $A = \{0\}$, then $B = (0,1]$ and $N(N(x)) = 1$ for any $x \in B$, from the proof of Lemma 4.6, we have $N = N_{D_1} = N_1$. If $A \neq \{0\}$, then there exists $y_0 \in A$ such that $y_0 \neq 0$, from (18), we obtain $N(x) = 0$ for any $x \in B$, and from (17), we obtain $N(y) = 1$ for any $y \in A$, thus

$$N(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \in B, \end{cases} = N_1(x).$$

From above discussion, we have $N(x) = N_1(x), x \in [0,1]$.

Conversely, let $N(x) = N_1(x), x \in [0,1]$, consider the following cases:

Case 1. $x \in A$, obviously, $T(N_1(y), I_{g,N}(x,y)) \leq 1 = N_1(x)$.

Case 2. $x \in B$, $y \in B$, obviously, $T(N_1(y), I_{g,N}(x,y)) = 0 = N_1(x)$.

Case 3. $x \in B$, $y \in A$, we have

$$\begin{aligned} T(N_1(y), I_{g,N}(x,y)) &= I_{g,N}(x,y) \\ &= g^{-1}(min(g(1), g(N(x) \wedge y) - g(N(N(x) \vee y)) + g(1))) \\ &= g^{-1}(0 - g(N(y)) + g(1)) \\ &= g^{-1}(-g(1) + g(1)) \\ &= 0 \\ &= N_1(x), \end{aligned}$$

then $T(N_1(y), I_{g,N}(x,y)) \leq N_1(x)$ for all $x, y \in [0,1]$, i.e., the fuzzy implication $I_{g,N}$ satisfies the modus tollens with respect to T and N_1. □

Theorem 5.8. *Let N be a strong fuzzy negation and N_1 a strictly fuzzy negation , and let T_D be a t-norm defined as*

$$T_D(x,y) = \begin{cases} y, & \text{if } x = 1, \\ x, & \text{if } y = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then $I_{g,N}$ satisfies the modus tollens with respect to t-norm T_D and N_1 if and only if $N_1 \geq \widetilde{N}$, where $\widetilde{N}(x) = g^{-1}(g(1) - g(x))$.

Proof. Let $x \leq y$, since $T(N_1(y), I(x,y)) \leq N_1(x)$ is hold for any fuzzy negation N_1 and t-norm T. Hence, it suffices to consider $x > y$.

Since N_1 is a strictly fuzzy negation, then $N_1(y) < 1$ for all $y \in (0,1]$. Since N is strong, then $I_{g,N}(x,y) < 1$ for $x > y, x, y \in [0,1]$. Hence, for $x > y$,

$$T_D(N_1(y), I_{g,N}(x,y)) = \begin{cases} \widetilde{N}(x), & \text{if } y = 0, \\ 0, & \text{otherwise}, \end{cases}$$

thus $N_1(x) \geq \widetilde{N}(x) \Leftrightarrow T_D(N_1(y), I_{g,N}(x,y)) \leq N_1(x), x \in [0,1]$, i.e., $I_{g,N}$ satisfies the modus tollens with respect to t-norm T_D and N_1 if and only if $N_1 \geq \widetilde{N}$. □

Theorem 5.9. *Let N be a strict fuzzy negation, N_1 a fuzzy negation, and let T be a continuous t-norm, then the following statements are equivalent:*

(i) $I_{g,N}$ satisfies modus tollens with respect to T and N_1.

(ii) There exists $\varphi \in \Phi$ such that $T = (T_{LK})_\varphi$, $N(y) \geq \widetilde{N}((N_C)_\varphi(N_1(y)))$ and $I_{g,N}(x,y) \leq (S_{LK})_\varphi((N_C)_\varphi(N_1(y)), N_1(x))$, $x, y \in [0,1]$.

Proof. The proof is similar to the proof of Theorem 4.13. □

6 Conclusions

In this paper, first, we introduced a new method for constructing implications based on fuzzy negation and continuous and strictly monotone function. If the continuous and strictly monotone function is an identity function, then the proposed method can construct implications only from fuzzy negation. Second, some properties of the proposed fuzzy implications are studied, for instance, the laws of contraposition(CP), the exchange principle(EP), T-conditionality(TC), etc, some results are obtained on the condition that the fuzzy negation is strong or continuous. However, there exist problems for the proposed fuzzy implication that need to be solved when the fuzzy negation is not continuous.

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