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Complex fuzzy H_v-subgroups of an H_v-group

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Abstract

The concept of complex fuzzy sets is a generalization of ordinary fuzzy sets. In this paper, we introduce the concept of complex fuzzy subhypergroups (H_v-subgroups) as well as the concept of complex anti-fuzzy subhypergroups (H_v-subgroups). We investigate their properties and their relations with the traditional fuzzy (anti-fuzzy) subhypergroups (H_v-subgroups), and we prove some results in this respect.

Keywords: Complex fuzzy set, complex fuzzy subhypergroup, complex anti-fuzzy subhypergroup.

1 Introduction

Algebraic hyperstructures represent a natural generalization of classical algebraic structures and they were introduced by Marty [5] in 1934 at the eighth Congress of Scandinavian Mathematicians. In classical algebraic structures, the composition of two elements is an element whereas in algebraic hyperstructures, the composition of two elements is a set. Since then, many different hyperstructures (hyperring, hyperalgebra, hyperrepresentation, ...) were widely studied from the theoretical point of view and for their applications to many subjects of pure and applied mathematics: geometry, topology, cryptography and code theory, graphs and hypergraphs, probability theory, binary relations, theory of fuzzy and rough sets, automata theory, economy, etc. (see [1]). The H_v-structures are generalized algebraic hyperstructures where in the axioms of the classical hyperstructures the equality is replaced by the non-empty intersection. They were introduced by Vougiouklis [11], also see [9, 10].

On the other hand, the fuzzy mathematics forms a branch of mathematics related to fuzzy set theory and fuzzy logic. It was introduced in 1965 after the publication of Zadeh (see [12]) as an extension of the classical notion of set, when he proposed the idea of a multi-valued logic, which extends the traditional concept of a bivalent logic, which becomes a particular case of the new theory. The fuzzy set theory is based on the principle called by Zadeh "the principle of incompatibility", that is "the closer a phenomenon is studied, the more indistinct its definition becomes". Fuzzy sets are sets whose elements have degrees of membership. In classical set theory, the membership of elements in a set is assessed in binary terms according to a bivalent condition that an element either belongs or does not belong to the set. By contrast, fuzzy set theory permits the gradual assessment of the membership of elements in a set; this is described with the aid of a membership function valued in the real unit interval $[0, 1]$. Fuzzy sets generalize classical sets, since the indicator functions of classical sets are special cases of the membership functions of fuzzy sets, if the latter only take values 0 or 1. Rosenfield [8] applied this concept to the theory of groups and introduced the concept of a fuzzy subgroup of a group. Since then, a host of mathematicians are engaged in fuzzifying various notions and results of abstract algebra. In [2], Davvaz introduced the concept of fuzzy subhypergroup (H_v-subgroups) of a hypergroup (H_v-group. A short review of the theory of fuzzy algebraic hyperstructures appears in [4].

As an extension of fuzzy sets, Raymot et al. [7, 6] introduced the concept of complex fuzzy sets in which the codomain of membership function is the unit disc of the complex plane. They introduced different fuzzy complex

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operations and relations.

The remainder part of our paper is constructed as follows: after an Introduction, in Section 2 we present some definitions and results about hyperstructures and traditional fuzzy subhyperstructures. In Section 3, we define complex fuzzy H_v-subgroups as well as complex anti-fuzzy H_v-subgroups, investigate their properties and present different examples on them.

2 Hyperstructures and traditional fuzzy subhyperstructures

In this section, we present some definitions and theorems related to hyperstructures and fuzzy subhyperstructures that are used throughout the paper.

Definition 2.1. [3] *Let H be a non-empty set. Then, a mapping $\circ : H \times H \to \mathcal{P}^*(H)$ is called a binary hyperoperation on H, where $\mathcal{P}^*(H)$ is the family of all non-empty subsets of H. The couple $(H, \circ)$ is called a hypergroupoid.*

In the above definition, if A and B are two non-empty subsets of H and $x \in H$, then we define:

$$A \circ B = \bigcup_{\substack{a \in A \\ b \in B}} a \circ b, \ x \circ A = \{x\} \circ A \text{ and } A \circ x = A \circ \{x\}.$$

Definition 2.2. [3] *A hypergroupoid $(H, \circ)$ is called a:*

- *semihypergroup if for every $x, y, z \in H$, we have $x \circ (y \circ z) = (x \circ y) \circ z$;*
- *quasihypergroup if for every $x \in H$, $x \circ H = H = H \circ x$ (This condition is called the reproduction axiom);*
- *hypergroup if it is a semihypergroup and a quasihypergroup;*
- *H_v-group if it is a quasihypergroup and for every $x, y, z \in H$, we have $x \circ (y \circ z) \cap (x \circ y) \circ z \neq \emptyset$.*

Definition 2.3. [3] *Let $(H, \circ)$ be a hypergroup (or H_v-group) and $K \subseteq H$. Then $(K, \circ)$ is a subhypergroup (or H_v-subgroup) of $(H, \circ)$ if for all $a \in K$, we have that $a \circ K = K \circ a = K$.*

Definition 2.4. [12] *A fuzzy set, defined on a universe of discourse U is characterized by a membership function $\mu_A(x)$ that assigns any element a grade of membership in A. The fuzzy set may be represented by the set of ordered pairs $A = \{(x, \mu_A(x)) : x \in U\}$, where $\mu_A(x) \in [0, 1]$.*

Definition 2.5. [4] *Let $(H, \circ)$ be a hypergroup (or H_v-group) and A be a fuzzy subset of H with membership function $\mu_A(x) \in [0, 1]$. Then A is a fuzzy subhypergroup (or H_v-subgroup) of H if the following conditions hold:*

1. $\min\{\mu_A(x), \mu_A(y)\} \leq \inf\{\mu_A(z) : z \in x \circ y\}$ *for all $x, y \in H$;*
2. *For all $x, a \in H$, there exists $y \in H$ such that $x \in a \circ y$ and $\min\{\mu_A(x), \mu_A(a)\} \leq \mu_A(y)$;*
3. *For all $x, a \in H$, there exists $z \in H$ such that $x \in z \circ a$ and $\min\{\mu_A(x), \mu_A(a)\} \leq \mu_A(z)$.*

Lemma 2.6. [2] *Let $(H, \circ)$ be a hypergroup (or H_v-group) and μ be a fuzzy subhypergroup (or H_v-subgroup) of H. Then*

$$\min\{\mu(x_1), \mu(x_2), \cdots, \mu(x_n)\} \leq \inf\{\mu(a) : a \in x_1 \circ (x_2 \circ (\cdots) \circ x_n) \cdots)\}$$

for all $x_1, x_2, \cdots, x_n \in H$.

Definition 2.7. [4] *Let $(H, \circ)$ be a hypergroup (or H_v-group) and A be a fuzzy subset of H with membership function $\mu_A(x)$. Then A is an anti-fuzzy subhypergroup (or H_v-subgroup) of H if the following conditions hold:*

1. $\sup\{\mu_A(z) : z \in x \circ y\} \leq \max\{\mu_A(x), \mu_A(y)\}$ *for all $x, y \in H$;*
2. *For all $x, a \in H$, there exists $y \in H$ such that $x \in a \circ y$ and $\mu_A(y) \leq \max\{\mu_A(x), \mu_A(a)\}$;*
3. *For all $x, a \in H$, there exists $z \in H$ such that $x \in z \circ a$ and $\mu_A(z) \leq \max\{\mu_A(x), \mu_A(a)\}$.*

Lemma 2.8. [4] *Let $(H, \circ)$ be a hypergroup (or H_v-group) and μ be an anti-fuzzy subhypergroup (or H_v-subgroup) of H. Then, for all $x_1, x_2, \cdots, x_n \in H$,*

$$\max\{\mu(x_1), \mu(x_2), \cdots, \mu(x_n)\} \geq \sup\{\mu(a) : a \in x_1 \circ (x_2 \circ (\cdots) \circ x_n) \cdots)\}.$$

Theorem 2.9. [4] *Let $(H, \circ)$ be a hypergroup (or H_v-group) and μ be a fuzzy subset of H. Then μ is a fuzzy subhypergroup (or H_v-subgroup) of H if and only if its complement μ^c is an anti-fuzzy subhypergroup (or H_v-subgroup) of H. Here, $\mu^c(x) = 1 - \mu(x)$ for all $x \in H$.*

3 Complex fuzzy subhyperstructures

In this section, we use the concept of complex fuzzy subsets to define complex fuzzy (anti-fuzzy) subhypergroups. And we investigate their properties.

3.1 Complex fuzzy H_v-subgroups

Definition 3.1. *Let $A = \{(x, \mu_A(x)) : x \in U\}$ be a fuzzy set. Then the set $A_\pi = \{(x, 2\pi\mu_A(x)) : x \in U\}$ is said to be a π-fuzzy set.*

Proposition 3.2. *Let $(H, \circ)$ be a hypergroup (or H_v-group). A π-fuzzy set A_π is a π-fuzzy subhypergroup (or H_v-subgroup) of H if and only if A is a fuzzy subhypergroup (or H_v-subgroup) of H.*

Proof. The proof is straightforward. □

Definition 3.3. *A complex fuzzy set, defined on a universe of discourse U is characterized by a membership function $\mu_A(x)$ that assigns any element, a complex-valued grade of membership in A. The complex fuzzy set may be represented by the set of ordered pairs $A = \{(x, \mu_A(x)) : x \in U\}$, where $\mu_A(x) = r(x)e^{iw(x)}$, $i = \sqrt{-1}$, $r(x) \in [0, 1]$ and $w(x) \in [0, 2\pi]$.*

Remark 3.4. *By setting $w(x) = 0$ in the above definition, we return to the traditional fuzzy set.*

Definition 3.5. [6] *Let $A = \{(x, \mu_A(x)) : x \in U\}$ and $B = \{(x, \mu_B(x)) : x \in U\}$ be two complex fuzzy sets of the same universe U with the membership functions $\mu_A(x) = r_A(x)e^{i\omega_A(x)}$ and $\mu_B(x) = r_B(x)e^{i\omega_B(x)}$, respectively. Then*

- $\mu_{A\cap B}(x) = r_{A\cap B}(x)e^{i\omega_{A\cap B}(x)} = \min\{r_A(x), r_B(x)\}e^{i\min\{\omega_A(x),\omega_B(x)\}}$;
- $\mu_{A\cup B}(x) = r_{A\cup B}(x)e^{i\omega_{A\cup B}(x)} = \max\{r_A(x), r_B(x)\}e^{i\max\{\omega_A(x),\omega_B(x)\}}$;
- $\mu_{A^c}(x) = (1 - r_A(x))e^{i(2\pi-\omega_A(x))}$, *where A^c denotes the complement of A.*

Definition 3.6. *Let $A = \{(x, \mu_A(x)) : x \in H\}$ and $B = \{(x, \mu_B(x)) : x \in H\}$ be complex fuzzy subsets of a non-void set H with membership functions $\mu_A(x) = r_A(x)e^{iw_A(x)}$ and $\mu_B(x) = r_B(x)e^{iw_B(x)}$ respectively. Then*

1. *A complex fuzzy subset A is said to be homogeneous if for all $x, y \in H$, we have*
$$r_A(x) \leq r_A(y) \text{ if and only if } w_A(x) \leq w_A(y).$$
2. *A complex fuzzy subset A is said to be homogeneous with B if for all $x, y \in H$, we have*
$$r_A(x) \leq r_B(y) \text{ if and only if } w_A(x) \leq w_B(y).$$

Notation 3.7. *Let $A = \{(x, \mu_A(x)) : x \in H\}$ and $B = \{(x, \mu_B(x)) : x \in H\}$ be complex fuzzy subsets of a non-void set H with membership functions $\mu_A(x) = r_A(x)e^{iw_A(x)}$ and $\mu_B(x) = r_B(x)e^{iw_B}(x)$ respectively. By $\mu_A(x) \leq \mu_B(x)$, we mean that $r_A(x) \leq r_B(x)$ and $w_A(x) \leq w_B(x)$.*

Throughout this paper, all complex fuzzy sets are considered homogeneous.

Definition 3.8. *Let $(H, \circ)$ be a hypergroup (or H_v-group) and A be a (homogeneous) complex fuzzy subset of H with membership function $\mu_A(x) = r_A(x)e^{iw_A(x)}$. Then A is a complex fuzzy subhypergroup (or H_v-subgroup) of H if the following conditions hold:*

1. $\min\{\mu_A(x), \mu_A(y)\} \leq \inf\{\mu_A(z) : z \in x \circ y\}$ *for all $x, y \in H$;*
2. *For all $x, a \in H$, there exists $y \in H$ such that $x \in a \circ y$ and $\min\{\mu_A(x), \mu_A(a)\} \leq \mu_A(y)$;*
3. *For all $x, a \in H$, there exists $z \in H$ such that $x \in z \circ a$ and $\min\{\mu_A(x), \mu_A(a)\} \leq \mu_A(z)$.*

Example 3.9. *Let $H = \{a, b\}$ and define the hypergroup $(H, \circ)$ by the following table:*

$\circ$	a	b
a	a	H
b	H	b

We define a complex fuzzy subset μ of H as follows: $\mu(a) = 0.5e^{i0}$ and $\mu(b) = 1e^{i\frac{\pi}{2}}$. Then μ is homogeneous complex fuzzy subhypergroup of H.

Theorem 3.10. *Let $(H,\circ)$ be a hypergroup (or H_v-group) and A be a (homogeneous) complex fuzzy subset of H with membership function $\mu_A(x) = r_A(x)e^{iw_A(x)}$. Then A is a complex fuzzy subhypergroup (or H_v-subgroup) of H if and only if r_A is a fuzzy subhypergroup (or H_v-subgroup) of H and w_A is a π-fuzzy subhypergroup (or H_v-subgroup) of H.*

Proof. Suppose that A is a complex fuzzy subhypergroup (or H_v-subgroup) of H. We need to prove that the conditions of Definition 2.5 are satisfied for r_A and w_A. For all $x, y \in H$, we have $\min\{\mu_A(x), \mu_A(y)\} \leq \inf\{\mu_A(z) : z \in x \circ y\}$. The latter and Notation 3.7 imply that $\inf\{r_A(z) : z \in x \circ y\} \geq \min\{r_A(x), r_A(y)\}$ and $\inf\{w_A(z) : z \in x \circ y\} \geq \min\{w_A(x), w_A(y)\}$. Let $a, x \in H$. Then there exist $y, z \in H$ such that $x \in a \circ y$, $x \in z \circ a$ and $\min\{\mu_A(a), \mu_A(x)\} \leq \mu_A(y)$, $\min\{\mu_A(a), \mu_A(x)\} \leq \mu_A(z)$. Notation 3.7 implies that the conditions 2 and 3 of Definition 2.5 are satisfied for both r_A and w_A.

Suppose that r_A is a fuzzy subhypergroup (or H_v-subgroup) of H and w_A is a π-fuzzy subhypergroup (or H_v-subgroup) of H. We need to prove that the conditions of Definition 3.8 are satisfied. For all $x, y \in H$, we have $\inf\{r_A(z) : z \in x \circ y\} \geq \min\{r_A(x), r_A(y)\}$ and $\inf\{w_A(z) : z \in x \circ y\} \geq \min\{w_A(x), w_A(y)\}$. The latter and Notation 3.7 imply that $\min\{\mu_A(x), \mu_A(y)\} \leq \inf\{\mu_A(z) : z \in x \circ y\}$. Let $a, x \in H$. Then there exist $y, z \in H$ such that $x \in a \circ y$, $x \in z \circ a$ and $\min\{r_A(a), r_A(x)\} \leq r_A(y)$, $\min\{r_A(a), r_A(x)\} \leq r_A(z)$, $\min\{w_A(a), w_A(x)\} \leq w_A(y)$, $\min\{w_A(a), w_A(x)\} \leq w_A(z)$. Notation 3.7 implies that the conditions 2 and 3 of Definition 3.8 are satisfied for μ_A. □

Lemma 3.11. *Let $(H,\circ)$ be a hypergroup (or H_v-group) and μ be a (homogeneous) complex fuzzy subhypergroup (or H_v-subgroup) of H. Then, for all $x_1, x_2, \cdots, x_n \in H$,*

$$\min\{\mu(x_1), \mu(x_2), \cdots, \mu(x_n)\} \leq \inf\{\mu(a) : a \in x_1 \circ (x_2 \circ (\cdots) \circ x_n) \cdots)\}.$$

Proof. Let $x_1, x_2, \cdots, x_n \in H$ and $\mu(x) = r(x)e^{iw(x)}$. To prove the lemma, it suffices to show that

$$\min\{r(x_1), r(x_2), \cdots, r(x_n)\} \leq \inf\{r(a) : a \in x_1 \circ (x_2 \circ (\cdots) \circ x_n) \cdots)\}$$

and

$$\min\{w(x_1), w(x_2), \cdots, w(x_n)\} \leq \inf\{w(a) : a \in x_1 \circ (x_2 \circ (\cdots) \circ x_n) \cdots)\}.$$

Since μ is homogeneous, it suffices to show that

$$\min\{r(x_1), r(x_2), \cdots, r(x_n)\} \leq \inf\{r(a) : a \in x_1 \circ (x_2 \circ (\cdots) \circ x_n) \cdots)\}.$$

Theorem 3.10 asserts that r is a fuzzy subhypergroup (or H_v-subgroup) of H. Lemma 2.6 completes the proof. □

Definition 3.12. *Let $A = \{(x, \mu_A(x)) : x \in H\}$ be a (homogeneous) complex fuzzy subset of a non-void set H with membership function $\mu_A(x) = r_A(x)e^{iw_A(x)}$. Define the level subset, μ_t, of H by $\mu_t = \{x \in H : \mu_A(x) \geq t\}$, where $t = se^{i\theta}$, $s \in [0,1]$ and $\theta \in [0, 2\pi]$.*

Remark 3.13. *Let $A = \{(x, \mu_A(x)) : x \in H\}$ be a (homogeneous) complex fuzzy subset of a non-void set H. Then the following are true:*

1. *If $t_1 \leq t_2$ then $\mu_{t_2} \subseteq \mu_{t_1}$.*
2. *$\mu_{0e^{0i}} = H$.*

Theorem 3.14. *Let $(H,\circ)$ be a hypergroup (or H_v-group) and A be a (homogeneous) complex fuzzy subset of H with membership function $\mu_A(x) = r_A(x)e^{iw_A(x)}$. Then A is a complex fuzzy subhypergroup (or H_v-subgroup) of H if and only if for all $t = se^{i\theta}$, $s \in [0,1]$ and $\theta \in [0, 2\pi]$, $\mu_t \neq \emptyset$ is a subhypergroup (or H_v-subgroup) of H.*

Proof. Let A be a complex fuzzy subhypergroup (or H_v-subgroup) of H and $x, y \in \mu_t \neq \emptyset$. Then for all $a \in x \circ y$, we have that $\mu_A(a) \geq \min\{\mu_A(x), \mu_A(y)\} \geq t$. Thus $a \in x \circ y \subseteq \mu_t$. Hence, for every $a \in \mu_t$, we have $a \circ \mu_t \subseteq \mu_t$. Now let $x \in \mu_t$ then by condition 2 of Definition 3.8, there exists $y \in H$ such that $x \in a \circ y$ and $t = \min\{\mu_A(a), \mu_A(x)\} \leq \mu_A(y)$. The latter implies that $y \in \mu_t$. We can use condition 3 of Definition 3.8 to get that $\mu_t \circ a \subseteq \mu_t$.

For the converse, assume that for all $t = se^{i\theta}$, $s \in [0,1]$ and $\theta \in [0, 2\pi]$, $\mu_t \neq \emptyset$ is subhypergroup (or H_v-subgroup) of H. Let $t_0 = s_0e^{i\theta_0} = \min\{\mu_A(x), \mu_A(y)\}$. Then $s_0 = \min\{r_A(x), r_A(y)\}$ and $\theta_0 = \min\{w_A(x), w_A(y)\}$. Since $x, y \in \mu_{t_0}$ and μ_{t_0} is a subhypergroup (or H_v-subgroup) of H, it follows that $x \circ y \subseteq \mu_{t_0}$. Therefore, for every $a \in x \circ y$ we have that $\mu_A(a) \geq t_0 = \min\{\mu_A(x), \mu_A(y)\}$ and thus, condition 1 of Definition 3.8 is verified. We prove now condition 2 and condition 3 is done in a similar manner. For every $a, x \in H$, set $t_1 = s_1e^{i\theta_1} = \min\{\mu_A(x), \mu_A(a)\}$, then $x, a \in \mu_{t_1}$. Having μ_{t_1} a subhypergroup (or H_v-subgroup) of H implies that $a \circ \mu_{t_1} = \mu_{t_1}$. The latter implies that there exists $y \in \mu_{t_1}$ such that $x \in a \circ y$. Therefore, $\mu_A(y) \geq t_1 = \min\{\mu_A(a), \mu_A(x)\}$. □

Corollary 3.15. *Let $(H,\circ)$ be a hypergroup (or H_v-group) and A be a (homogeneous) complex fuzzy subhypergroup (or H_v-subgroup) of H with membership function $\mu_A(x) = r_A(x)e^{iw_A(x)}$. If $0e^{0i} \leq t_1 = s_1e^{i\theta_1} < t_2 = s_2e^{i\theta_2} \leq 1e^{2\pi i}$, then $\mu_{t_1} = \mu_{t_2}$ if and only if there is no $x \in H$ such that $t_1 \leq \mu_A(x) < t_2$.*

Proof. Let $0e^{0i} \leq t_1 = s_1e^{i\theta_1} < t_2 = s_2e^{i\theta_2} \leq 1e^{2\pi i}$ such that $\mu_{t_1} = \mu_{t_2}$. Suppose that there exists $x \in H$ such that $t_1 \leq \mu_A(x) < t_2$. Then $x \in \mu_{t_1} = \mu_{t_2}$. The latter implies that $\mu_A(x) \geq t_2$. Since $0e^{0i} \leq t_1 = s_1e^{i\theta_1} < t_2 = s_2e^{i\theta_2} \leq 1e^{2\pi i}$, it follows by Remark 3.13 that $\mu_{t_2} \subseteq \mu_{t_1}$. To show that $\mu_{t_1} \subseteq \mu_{t_2}$, we let $x \in \mu_{t_1}$. Then $\mu_A(x) \geq t_1$. Since there is no $x \in H$ such that $t_1 \leq \mu_A(x) < t_2$, it follows that $\mu_A(x) \geq t_2$. □

Corollary 3.16. *Let $(H,\circ)$ be a hypergroup (or H_v-group) and A be a (homogeneous) complex fuzzy subhypergroup (or H_v-subgroup) of H with membership function $\mu_A(x) = r_A(x)e^{iw_A(x)}$. If the range of μ_A is the finite set $\{t_1, t_2, \cdots, t_n\}$ then the set $\{\mu_{t_i} : i = 1, 2, \cdots, n\}$ contains all the level subhypergroups (or H_v-subgroups) of H. Moreover, if $t_1 \geq t_2 \geq \cdots \geq t_n$ then all the level subhypergroups (or H_v-subgroups) of H form the chain: $\mu_{t_1} \subseteq \mu_{t_2} \subseteq \cdots \subseteq \mu_{t_n}$.*

Proof. Let $\mu_s \neq \emptyset$ be a level subhypergroup (or H_v-subgroup) of H such that $\mu_s \neq \mu_{t_i}$ for all $1 \leq i \leq n$. Let t_k be closest complex number to s. We have two cases for s: $s < t_k$ and $s > t_k$. We consider only the first case, the second is done in a similar manner. Since the range of μ_A is the finite set $\{t_1, t_2, \cdots, t_n\}$, it follows that there is no $x \in H$ such that $s \leq \mu_A(x) < t_k$. Using Corollary 3.15, we get contradiction. □

Proposition 3.17. *Let $(H,\circ)$ be the biset hypergroup, i.e., $x \circ y = \{x, y\}$ for all $x, y \in H$ and let μ be any homogeneous complex fuzzy subset of H. Then μ is a complex fuzzy subhypergroup of H.*

Proof. Let $t = se^{i\theta}$, $s \in [0,1]$ and $\theta \in [0, 2\pi]$. Then, by Theorem 3.14, it suffices to show that $\mu_t \neq \emptyset$ is a subhypergroup of H. We have that $\mu_t \subseteq a \circ \mu_t$ as for all $x \in \mu_t$, $x \in a \circ x = \{a, x\}$. Moreover, It is clear that $a \circ \mu_t = \mu_t \circ a = \{x \circ a : x \in \mu_t\} = \{x, a\} \subseteq \mu_t$ for all $a \in \mu_t$. □

Proposition 3.18. *Let $(H,\circ)$ be the total hypergroup, i.e., $x \circ y = H$ for all $x, y \in H$ and let μ be any homogeneous complex fuzzy subset of H. Then μ is a complex fuzzy subhypergroup of H if and only if μ is a constant complex function.*

Proof. If μ is a constant complex function then it is clear that μ is a complex fuzzy subhypergroup of H. Let μ be a complex fuzzy subhypergroup of H and suppose for contradiction that μ is not a constant complex function. Then we can find $x, y \in H$, $t = se^{i\theta}$, $s \in [0,1]$ and $\theta \in [0, 2\pi]$ such that $\mu(x) < \mu(y) = t$. It is clear that x is not an element in $\mu_t \ni y$. Since $\mu_t \neq \emptyset$ is a subhypergroup of H, it follows that $H = y \circ y \subseteq \mu_t$. □

Proposition 3.19. *Let $(H,\circ)$ be a hypergroup (or H_v-group) and A be a (homogeneous) complex fuzzy subset of H. Then A is a complex fuzzy subhypergroup (H_v-subgroup) of H if and only if for every $t = se^{i\theta}$, $s \in [0,1]$ and $\theta \in [0, 2\pi]$, the following conditions are satisfied:*

1. *$\mu_t \circ \mu_t \subseteq \mu_t$;*
2. *$a \circ (H - \mu_t) - (H - \mu_t) \subseteq a \circ \mu_t$, for all $a \in \mu_t$;*
3. *$(H - \mu_t) \circ a - (H - \mu_t) \subseteq \mu_t \circ a$, for all $a \in \mu_t$.*

Proof. Let A be a complex fuzzy subhypergroup (H_v-subgroup) of H. Then, by Theorem 3.14, μ_t is a subhypergroup (H_v-subgroup) of H, i.e., $a \circ \mu_t = \mu_t$ for all $a \in \mu_t$. Thus, we get that $\mu_t \circ \mu_t \subseteq \mu_t$. We need to show that $a \circ (H - \mu_t) - (H - \mu_t) \subseteq a \circ \mu_t$. Let $z \in a \circ (H - \mu_t) - (H - \mu_t)$. Then z is not an element in $(H - \mu_t)$. This implies that $z \in \mu_t = a \circ \mu_t$. Condition 3 can be proved in a similar manner. For the converse, suppose that the conditions 1 and 2 hold. Then, by Theorem 3.14, it suffices to show that μ_t is a subhypergroup (H_v-subgroup) of H, i.e., $a \circ \mu_t = \mu_t \circ a = \mu_t$ for all $a \in \mu_t$. Assume that there exists $x \in \mu_t$ such that x is not an element in $a \circ \mu_t$. The reproduction axiom of $(H,\circ)$ asserts that there exists $b \in H$ such that $x \in a \circ b$. We consider the following two cases for b:

- Case $b \in \mu_t$. We get that $x \in a \circ b \subseteq a \circ \mu_t$ which is a contradiction.
- Case b is not an element in μ_t. We get that $b \in H - \mu_t$. And having $x \in a \circ b$ implies that $x \in a \circ (H - \mu_t)$. Since $x \in \mu_t$, it follows that x is not in $H - \mu_t$. Thus, $x \in a \circ (H - \mu_t) - (H - \mu_t) \subseteq a \circ \mu_t$ which is a contradiction.

We can prove that $\mu_t \circ a = \mu_t$, by applying condition 3, in a similar manner. □

Proposition 3.20. *Let $(H, \circ)$ be a hypergroup (or H_v-group). Then every subhypergroup (or H_v-subgroup) of H is a level H_v-subgroup of a fuzzy subhypergroup (H_v-subgroup) of H.*

Proof. Let M be a subhypergroup (or H_v-subgroup) of H. For a fixed complex number $t_0 = se^{i\theta}$, $s \in]0,1], \theta \in]0,2\pi]$, the fuzzy subset μ is defined as follows:

$$\mu(x) = \begin{cases} t_0, & \text{if } x \in M; \\ 0e^{i0}, & \text{otherwise.} \end{cases}$$

We have $M = \mu_{t_0}$ and $\mu_t = \begin{cases} H, & \text{if } t = 0; \\ M, & \text{if } 0 < t \leq t_0; \\ \emptyset, & \text{otherwise.} \end{cases}$ is either the empty set or a subhypergroup (or H_v-subgroup) of H. Then, by Theorem 3.14, we get that μ is a fuzzy subhypergroup (H_v-subgroup) of H. □

Proposition 3.21. *Let $(H, \circ)$ be a hypergroup (or H_v-group) and A be a (homogeneous) complex fuzzy subhypergroup (H_v-subgroup) of H with membership function $\mu_A(x) = r_A(x)e^{iw_A(x)}$. Define $\overline{\mu}$ as follows:*

$$\overline{\mu} = \{x \in H : \mu_A(x) = 1e^{2\pi i}\}.$$

Then $\overline{\mu}$ is empty or subhypergroup (H_v-subgroup) of H.

Proof. Let $x, y \in \overline{\mu} \neq \emptyset$. We show that $a \circ \overline{\mu} = \overline{\mu} = \overline{\mu} \circ a$ for all $a \in \overline{\mu}$. Let $x \in \overline{\mu}$ and $z \in a \circ x$. Having $\mu_A(z) \geq \min\{\mu_A(a), \mu_A(x)\} = 1e^{2\pi i}$ implies that $\mu_A(z) = 1e^{2\pi i}$ and thus $z \in a \circ x \subseteq \overline{\mu}$. For all $a, x \in \overline{\mu}$, there exists $y \in H$ such that $x \in a \circ y$ and $\mu_A(y) \geq \min\{\mu_A(a), \mu_A(x)\} = 1e^{2\pi i}$. The latter implies that $\mu_A(y) = 1e^{2\pi i}$ and thus $y \in \overline{\mu}$. □

Proposition 3.22. *Let $(H, \circ)$ be a hypergroup (or H_v-group) and A be a (homogeneous) complex fuzzy subhypergroup (H_v-subgroup) of H with membership function $\mu_A(x) = r_A(x)e^{iw_A(x)}$. Define the support, $supp(\mu)$, of μ by $supp(\mu) = \{x \in H : \mu_A(x) > 0e^{0i}\}$. Then $supp(\mu)$ is empty or subhypergroup (H_v-subgroup) of H.*

Proof. Let $x, y \in supp(\mu) \neq \emptyset$. We want to show that $a \circ supp(\mu) = supp(\mu) = supp(\mu) \circ a$ for all $a \in supp(\mu)$. Let $x \in supp(\mu)$ and $z \in a \circ x$. Having $\mu_A(z) \geq \min\{\mu_A(a), \mu_A(x)\} > 0e^{0i}$ implies that $\mu_A(z) > 0e^{0i}$ and thus $z \in a \circ x \subseteq supp(\mu)$. For all $a, x \in supp(\mu)$, there exists $y \in H$ such that $x \in a \circ y$ and $\mu_A(y) \geq \min\{\mu_A(a), \mu_A(x)\} > 0e^{0i}$. The latter implies that $\mu_A(y) > 0e^{0i}$ and thus $y \in supp(\mu)$. □

Definition 3.23. *Let $A = \{(x, \mu_A(x) = r_A(x)e^{iw_A(x)}) : x \in H\}$ be a homogeneous complex fuzzy subset of a non-void set H. We define the complement of the complex fuzzy subset A of H as follows:*

$$A^c = \{(x, \mu_{A^c}(x) = (1 - r_A)(x)e^{i(2\pi - w_A(x))}) : x \in H\}.$$

Next, we present some examples where μ and μ^c are complex fuzzy subhypergroups (which in general is not always valid).

Example 3.24. *We consider $(H, \circ)$ defined in Example 3.9 with the complex fuzzy subset μ of H as: $\mu(a) = 0.5e^{i0}$ and $\mu(b) = 1e^{i\frac{\pi}{2}}$. We get $\mu(a) = 0.5e^{i2\pi}$ and $\mu(b) = 0e^{i\frac{3\pi}{2}}$. Then μ and μ^c are homogeneous complex fuzzy subhypergroups of H.*

Example 3.25. *Let $(H, \circ)$ be any hypergroup (H_v-group) with the complex fuzzy subset μ of H as: $\mu(x) = re^{i\theta}$ where $r \in [0,1], \theta \in [0, 2\pi]$ are fixed real numbers. Then μ and μ^c are homogeneous complex fuzzy subhypergroups of H.*

Remark 3.26. *Let $(H, \circ)$ be a hypergroup (or H_v-group) and A be a (homogeneous) complex fuzzy subhypergroup (H_v-subgroup) of H with membership function $\mu_A(x) = r_A(x)e^{iw_A(x)}$. Then A^c is not necessarily a complex fuzzy subhypergroup (or H_v-subgroup) of H.*

We illustrate Remark 3.26 by the following example.

Example 3.27. *Let $H = \{0, 1, 2\}$ and define the H_v-group $(H, +)$ by the following table:*

$+$	0	1	2
0	0	$\{1,2\}$	2
1	$\{1,2\}$	2	0
2	2	0	1

And define a complex fuzzy subset μ of H as: $\mu(0) = 0.2e^{i\pi}$ and $\mu(1) = \mu(2) = 0.1e^{i\frac{\pi}{2}}$. Having

$$\mu_t = \begin{cases} H, & \text{if } t \leq 0.1e^{i\frac{\pi}{2}}; \\ \{0\}, & \text{if } 0.1e^{i\frac{\pi}{2}} < t \leq 0.2e^{i\pi}; \\ \emptyset, & \text{otherwise.} \end{cases}$$

either an empty set or a subhypergroup of H implies that μ is homogeneous complex fuzzy subhypergroup of H.

Since $0.8e^{i\pi} = \mu^c(0) = \mu^c(1+2) < \min\{\mu^c(1), \mu^c(2)\} = 0.9^{i\frac{3\pi}{2}}$, it follows that μ^c is not a complex fuzzy H_v-subgroup of H.

3.2 Complex anti-fuzzy H_v-subgroups

Definition 3.28. *Let $(H, \circ)$ be a hypergroup (or H_v-group) and A be a (homogeneous) complex fuzzy subset of H with membership function $\mu_A(x) = r_A(x)e^{iw_A(x)}$. Then A is a complex anti-fuzzy subhypergroup (or H_v-subgroup) of H if the following conditions hold:*

1. *$\sup\{\mu_A(z) : z \in x \circ y\} \leq \max\{\mu_A(x), \mu_A(y)\}$ for all $x, y \in H$,*
2. *For all $x, a \in H$, there exists $y \in H$ such that $x \in a \circ y$ and $\mu_A(y) \leq \max\{\mu_A(x), \mu_A(a)\}$,*
3. *For all $x, a \in H$, there exists $z \in H$ such that $x \in z \circ a$ and $\mu_A(z) \leq \max\{\mu_A(x), \mu_A(a)\}$.*

Next, we present some examples on complex anti-fuzzy H_v-subgroups.

Example 3.29. *We consider $(H, \circ)$ defined in Example 3.9 with the complex fuzzy subset μ of H as: $\mu(a) = 0.5e^{i0}$ and $\mu(b) = 1e^{i\frac{\pi}{2}}$. We get $\mu(a) = 0.5e^{i2\pi}$ and $\mu(b) = 0e^{i\frac{3\pi}{2}}$. Then μ is a homogeneous complex anti-fuzzy subhypergroup of H.*

Example 3.30. *Let $(H, \circ)$ be any hypergroup (H_v-group) with the complex fuzzy subset μ of H as: $\mu(x) = re^{i\theta}$ where $r \in [0, 1], \theta \in [0, 2\pi]$ are fixed real numbers. Then μ is a homogeneous complex anti-fuzzy subhypergroup of H.*

Proposition 3.31. *Let $(H, \circ)$ be a hypergroup (or H_v-group). A π-fuzzy set A_π is a π-anti-fuzzy subhypergroup (or H_v-subgroup) of H if and only if A is an anti-fuzzy subhypergroup (or H_v-subgroup) of H.*

Proof. The proof is straightforward. □

Theorem 3.32. *Let $(H, \circ)$ be a hypergroup (or H_v-group) and A be a (homogeneous) complex fuzzy subset of H with membership function $\mu_A(x) = r_A(x)e^{iw_A(x)}$. Then A is a complex anti-fuzzy subhypergroup (or H_v-subgroup) of H if and only if r_A is an anti- fuzzy subhypergroup (or H_v-subgroup) of H and w_A is a π-anti-fuzzy subhypergroup (or H_v-subgroup) of H.*

Proof. Suppose that A is a complex anti-fuzzy subhypergroup (or H_v-subgroup) of H. We need to prove that the conditions of Definition 2.7 are satisfied for r_A and w_A. For all $x, y \in H$, we have $\sup\{\mu_A(z) : z \in x \circ y\} \leq \max\{\mu_A(x), \mu_A(y)\}$. The latter and Notation 3.7 imply that $\sup\{r_A(z) : z \in x \circ y\} \leq \max\{r_A(x), r_A(y)\}$ and $\sup\{w_A(z) : z \in x \circ y\} \leq \max\{w_A(x), w_A(y)\}$. Let $a, x \in H$. Then there exist $y, z \in H$ such that $x \in a \circ y$, $x \in z \circ a$ and $\max\{\mu_A(a), \mu_A(x)\} \geq \mu_A(y)$, $\max\{\mu_A(a), \mu_A(x)\} \geq \mu_A(z)$. Notation 3.7 implies that the conditions 2 and 3 of Definition 2.7 are satisfied for both r_A and w_A. Suppose that r_A is an anti-fuzzy subhypergroup (or H_v-subgroup) of H and w_A is a π-anti-fuzzy subhypergroup (or H_v-subgroup) of H. We need to prove that the conditions of Definition 3.28 are satisfied. For all $x, y \in H$, we have $\sup\{r_A(z) : z \in x \circ y\} \leq \max\{r_A(x), r_A(y)\}$ and $\sup\{w_A(z) : z \in x \circ y\} \leq \max\{w_A(x), w_A(y)\}$. The latter and Notation 3.7 imply that $\sup\{\mu_A(z) : z \in x \circ y\} \leq \max\{\mu_A(x), \mu_A(y)\}$. Let $a, x \in H$. Then there exist $y, z \in H$ such that $x \in a \circ y$, $x \in z \circ a$ and $\max\{r_A(a), r_A(x)\} \geq r_A(y)$, $\max\{r_A(a), r_A(x)\} \geq r_A(z)$, $\max\{w_A(a), w_A(x)\} \geq w_A(y)$, $\max\{w_A(a), w_A(x)\} \geq w_A(z)$. Notation 3.7 implies that the conditions 2 and 3 of Definition 3.28 are satisfied for μ_A. □

Lemma 3.33. *Let $(H, \circ)$ be a hypergroup (or H_v-group) and μ be a (homogeneous) complex anti-fuzzy subhypergroup (or H_v-subgroup) of H. Then*

$$\max\{\mu(x_1), \mu(x_2), \cdots, \mu(x_n)\} \geq \sup\{\mu(a) : a \in x_1 \circ (x_2 \circ (\cdots) \circ x_n) \cdots)\}$$

for all $x_1, x_2, \cdots, x_n \in H$.

Proof. Let $x_1, x_2, \cdots, x_n \in H$ and $\mu(x) = r(x)e^{iw(x)}$. To prove the lemma, it suffices to show that

$$\max\{r(x_1), r(x_2), \cdots, r(x_n)\} \geq \sup\{r(a) : a \in x_1 \circ (x_2 \circ (\cdots) \circ x_n) \cdots)\}$$

and

$$\max\{w(x_1), w(x_2), \cdots, w(x_n)\} \geq \sup\{w(a) : a \in x_1 \circ (x_2 \circ (\cdots) \circ x_n) \cdots)\}.$$

Since μ is homogeneous, it suffices to show that

$$\max\{r(x_1), r(x_2), \cdots, r(x_n)\} \geq \sup\{r(a) : a \in x_1 \circ (x_2 \circ (\cdots, x_n) \cdots)\}.$$

Theorem 3.32 asserts that r is an anti-fuzzy subhypergroup (or H_v-subgroup) of H. Lemma 2.8 completes the proof. □

Definition 3.34. *Let $A = \{(x, \mu_A(x)) : x \in H\}$ be a (homogeneous) complex fuzzy subsets of a non-void set H with membership function $\mu_A(x) = r_A(x)e^{iw_A(x)}$. Define the lower subset, μ_t, of H by $\overline{\mu}_t = \{x \in H : \mu_A(x) \leq t\}$, where $t = se^{i\theta}$, $s \in [0,1]$ and $\theta \in [0, 2\pi]$.*

Remark 3.35. *Let $A = \{(x, \mu_A(x)) : x \in H\}$ be a (homogeneous) complex fuzzy subsets of a non-void set H. Then the following are true:*

1. *If $t_1 \leq t_2$ then $\overline{\mu}_{t_1} \subseteq \overline{\mu}_{t_2}$.*
2. *$\overline{\mu}_{1e^{2\pi i}} = H$.*

Theorem 3.36. *Let $(H, \circ)$ be a hypergroup (or H_v-group) and A be a (homogeneous) complex fuzzy subset of H with membership function $\mu_A(x) = r_A(x)e^{iw_A(x)}$. Then A is a complex anti-fuzzy subhypergroup (or H_v-subgroup) of H if and only if for all $t = se^{i\theta}$, $s \in [0,1]$ and $\theta \in [0, 2\pi]$, $\overline{\mu}_t \neq \emptyset$ is a subhypergroup (or H_v-subgroup) of H.*

Proof. The proof is similar to that of Theorem 3.14. □

Corollary 3.37. *Let $(H, \circ)$ be a hypergroup (or H_v-group) and A be a (homogeneous) complex anti-fuzzy subhypergroup (or H_v-subgroup) of H with membership function $\mu_A(x) = r_A(x)e^{iw_A(x)}$. If $0e^{0i} \leq t_1 = s_1e^{i\theta_1} < t_2 = s_2e^{i\theta_2} \leq 1e^{2\pi i}$, then $\overline{\mu}_{t_1} = \overline{\mu}_{t_2}$ if and only if there is no $x \in H$ such that $t_1 \leq \mu_A(x) < t_2$.*

Proof. Let $0e^{0i} \leq t_1 = s_1e^{i\theta_1} < t_2 = s_2e^{i\theta_2} \leq 1e^{2\pi i}$ such that $\overline{\mu}_{t_1} = \overline{\mu}_{t_2}$. Suppose that there exists $x \in H$ such that $t_1 \leq \mu_A(x) < t_2$. Then $x \in \mu_{t_1} = \mu_{t_2}$. The latter implies that $\mu_A(x) \leq t_1$.

Since $0e^{0i} \leq t_1 = s_1e^{i\theta_1} < t_2 = s_2e^{i\theta_2} \leq 1e^{2\pi i}$, it follows by Remark 3.35 that $\overline{\mu}_{t_1} \subseteq \overline{\mu}_{t_2}$. To show that $\overline{\mu}_{t_2} \subseteq \overline{\mu}_{t_1}$, we let $x \in \overline{\mu}_{t_2}$. Then $\mu_A(x) \leq t_2$. Since there is no $x \in H$ such that $t_1 \leq \mu_A(x) < t_2$, it follows that $\mu_A(x) \leq t_1$. □

Corollary 3.38. *Let $(H, \circ)$ be a hypergroup (or H_v-group) and A be a (homogeneous) complex anti-fuzzy subhypergroup (or H_v-subgroup) of H with membership function $\mu_A(x) = r_A(x)e^{iw_A(x)}$. If the range of μ_A is the finite set $\{t_1, t_2, \cdots, t_n\}$ then the set $\{\overline{\mu}_{t_i} : i = 1, 2, \cdots, n\}$ contains all the lower level subhypergroups (or H_v-subgroups) of H. Moreover, if $t_1 \leq t_2 \leq \cdots \leq t_n$ then all the lower level subhypergroups (or H_v-subgroups) of H form the chain: $\overline{\mu}_{t_1} \subseteq \overline{\mu}_{t_2} \subseteq \cdots \subseteq \overline{\mu}_{t_n}$.*

Proof. Let $\overline{\mu}_s \neq \emptyset$ be a lower level subhypergroup (or H_v-subgroup) of H such that $\overline{\mu}_s \neq \overline{\mu}_{t_i}$ for all $1 \leq i \leq n$. Let t_k be closest complex number to s. We have two cases for s: $s < t_k$ and $s > t_k$. We consider only the first case, the second is done in a similar manner. Since the range of μ_A is the finite set $\{t_1, t_2, \cdots, t_n\}$, it follows that there is no $x \in H$ such that $s < \mu_A(x) < t_k$. Using Corollary 3.37, we get contradiction. □

Proposition 3.39. *Let $(H, \circ)$ be the biset hypergroup, i.e., $x \circ y = \{x, y\}$ for all $x, y \in H$ and let μ be any homogeneous complex fuzzy subset of H. Then μ is a complex anti-fuzzy subhypergroup of H.*

Proof. The proof is similar to that of Proposition 3.17. □

Proposition 3.40. *Let $(H, \circ)$ be the total hypergroup, i.e., $x \circ y = H$ for all $x, y \in H$ and let μ be any homogeneous complex fuzzy subset of H. Then μ is a complex anti-fuzzy subhypergroup of H if and only if μ is a constant complex function.*

Proof. The proof is similar to that of Proposition 3.18. □

Proposition 3.41. *Let $(H, \circ)$ be a hypergroup (or H_v-group) and A be a (homogeneous) complex fuzzy subset of H. Then A is a complex anti-fuzzy subhypergroup (H_v-subgroup) of H if and only if for every $t = se^{i\theta}$, $s \in [0,1]$ and $\theta \in [0, 2\pi]$, the following conditions are satisfied:*

1. $\overline{\mu}_t \circ \overline{\mu}_t \subseteq \overline{\mu}_t$;

2. $a \circ (H - \overline{\mu}_t) - (H - \overline{\mu}_t) \subseteq a \circ \overline{\mu}_t$, *for all* $a \in \overline{\mu}_t$;

3. $(H - \overline{\mu}_t) \circ a - (H - \overline{\mu}_t) \subseteq \overline{\mu}_t \circ a$, *for all* $a \in \overline{\mu}_t$.

Proof. Let A be a complex fuzzy subhypergroup (H_v-subgroup) of H. Then, by Theorem 3.36, $\overline{\mu}_t$ is a subhypergroup (H_v-subgroup) of H, i.e., $a \circ \overline{\mu}_t = \overline{\mu}_t$ for all $a \in \overline{\mu}_t$. Thus, we get that $\overline{\mu}_t \circ \overline{\mu}_t \subseteq \overline{\mu}_t$. we need to show that $a \circ (H - \overline{\mu}_t) - (H - \overline{\mu}_t) \subseteq a \circ \overline{\mu}_t$. Let $z \in a \circ (H - \overline{\mu}_t) - (H - \overline{\mu}_t)$. Then z is not an element in $(H - \overline{\mu}_t)$. This implies that $z \in \overline{\mu}_t = a \circ \overline{\mu}_t$. Condition 3 can be proved in a similar manner.

For the converse, suppose that the conditions 1 and 2 hold. Then, by Theorem 3.36, it suffices to show that $\overline{\mu}_t$ is a subhypergroup (H_v-subgroup) of H, i.e., $a \circ \overline{\mu}_t = \overline{\mu}_t \circ a = \overline{\mu}_t$ for all $a \in \overline{\mu}_t$. Assume that there exists $x \in \overline{\mu}_t$ such that x is not an element in $a \circ \overline{\mu}_t$. The reproduction axiom of $(H, \circ)$ asserts that there exists $b \in H$ such that $x \in a \circ b$. We consider the following two cases for b:

- Case $b \in \overline{\mu}_t$. We get that $x \in a \circ b \subseteq a \circ \overline{\mu}_t$ which is a contradiction.
- Case b is not an element in $\overline{\mu}_t$. We get that $b \in H - \overline{\mu}_t$. And having $x \in a \circ b$ implies that $x \in a \circ (H - \overline{\mu}_t)$. Since $x \in \overline{\mu}_t$, it follows that x is not in $H - \overline{\mu}_t$. Thus, $x \in a \circ (H - \overline{\mu}_t) - (H - \overline{\mu}_t) \subseteq a \circ \overline{\mu}_t$ which is a contradiction.

We can prove that $\overline{\mu}_t \circ a = \overline{\mu}_t$, by applying condition 3, in a similar manner. □

Proposition 3.42. *Let $(H, \circ)$ be a hypergroup (or H_v-group). Then every subhypergroup (or H_v-subgroup) of H is a lower level H_v-subgroup of an anti-fuzzy subhypergroup (H_v-subgroup) of H.*

Proof. Let M be a subhypergroup (or H_v-subgroup) of H. For a fixed complex number $t_0 = se^{i\theta}$, $s \in [0, 1[, \theta \in [0, 2\pi[$, the fuzzy subset μ is defined as follows:

$$\mu(x) = \begin{cases} t_0, & \text{if } x \in M; \\ 1e^{2\pi i}, & \text{otherwise.} \end{cases}$$

We have $M = \overline{\mu}_{t_0}$ and $\overline{\mu}_t = \begin{cases} \emptyset, & \text{if } t < t_0; \\ M, & \text{if } t_0 \leq t < 1e^{2\pi i}; \\ H, & \text{if } t = 1e^{2\pi i}. \end{cases}$ is either the empty set a subhypergroup (or H_v-subgroup) of H. Then, by Theorem 3.36, we get that μ is a anti-fuzzy subhypergroup (H_v-subgroup) of H. □

Proposition 3.43. *Let $(H, \circ)$ be a hypergroup (or H_v-group) and A be a (homogeneous) complex anti-fuzzy subhypergroup (H_v-subgroup) of H with membership function $\mu_A(x) = r_A(x)e^{iw_A(x)}$. Define $\overline{\overline{\mu}}$ by $\overline{\overline{\mu}} = \{x \in H : \mu_A(x) = 0e^{0i}\}$. Then $\overline{\overline{\mu}}$ is empty or subhypergroup (H_v-subgroup) of H.*

Proof. Let $x, y \in \overline{\overline{\mu}} \neq \emptyset$. We show that $a \circ \overline{\overline{\mu}} = \overline{\overline{\mu}} = \overline{\overline{\mu}} \circ a$ for all $a \in \overline{\overline{\mu}}$. Let $x \in \overline{\overline{\mu}}$ and $z \in a \circ x$. Having $\mu_A(z) \leq \max\{\mu_A(a), \mu_A(x)\} = 0e^{0i}$ implies that $\mu_A(z) = 0e^{0i}$ and thus $z \in a \circ x \subseteq \overline{\overline{\mu}}$. For all $a, x \in \overline{\overline{\mu}}$, there exists $y \in H$ such that $x \in a \circ y$ and $\mu_A(y) \leq \max\{\mu_A(a), \mu_A(x)\} = 0e^{0i}$. The latter implies that $\mu_A(y) = 0e^{0i}$ and thus $y \in \overline{\overline{\mu}}$. □

Proposition 3.44. *Let $(H, \circ)$ be a hypergroup (or H_v-group) and A be a (homogeneous) complex fuzzy subhypergroup (H_v-subgroup) of H with membership function $\mu_A(x) = r_A(x)e^{iw_A(x)}$. Define the set $\overline{supp}$ as follows:*

$$\overline{supp} = \{x \in H : \mu_A(x) < 1e^{2\pi i}\}.$$

Then supp(μ) is empty or subhypergroup (H_v-subgroup) of H.

Proof. Let $x, y \in \overline{supp} \neq \emptyset$. We want to show that $a \circ \overline{supp} = \overline{supp} = \overline{supp} \circ a$ for all $a \in \overline{supp}$. Let $x \in \overline{supp}$ and $z \in a \circ x$. Having $\mu_A(z) \leq \max\{\mu_A(a), \mu_A(x)\} > 0e^{0i}$ implies that $\mu_A(z) < 1e^{2\pi i}$ and thus $z \in a \circ x \subseteq \overline{supp}$. For all $a, x \in \overline{supp}$, there exists $y \in H$ such that $x \in a \circ y$ and $\mu_A(y) \leq \max\{\mu_A(a), \mu_A(x)\} < 1e^{2\pi i}$. The latter implies that $\mu_A(y) < 1e^{2\pi i}$ and thus $y \in \overline{supp}$. □

Theorem 3.45. *Let $(H, \circ)$ be a hypergroup (or H_v-group) and A be a (homogeneous) complex fuzzy subset of H with membership function $\mu_A(x) = r_A(x)e^{iw_A(x)}$. Then A is a complex fuzzy subhypergroup (or H_v-subgroup) of H if and only if A^c is a complex anti-fuzzy subhypergroup (or H_v-subgroup) of H.*

Proof. The statement A is a complex fuzzy subhypergroup (or H_v-subgroup) of H is equivalent, by Theorem 3.10, to having r_A a fuzzy subhypergroup (or H_v-subgroup) of H and w_A a π-fuzzy subhypergroup (or H_v-subgroup) of H. The latter is equivalent, by Theorem 2.9, to having r_A^c an anti-fuzzy subhypergroup (or H_v-subgroup) of H and w_A^c a π-anti-fuzzy subhypergroup (or H_v-subgroup) of H. Theorem 3.32 completes the proof. □

Corollary 3.46. *Let $(H, \circ)$ be a hypergroup (or H_v-group) and A be a (homogeneous) complex fuzzy subset of H with membership function $\mu_A(x) = r_A(x)e^{iw_A(x)}$. Then A is a complex fuzzy and anti-fuzzy subhypergroup (or H_v-subgroup) of H if and only if A^c is a complex fuzzy and anti-fuzzy subhypergroup (or H_v-subgroup) of H.*

Proof. The proof results from Theorem 3.46. □

Example 3.47. *Let $(H, \circ)$ be the biset hypergroup, i.e., $x \circ y = \{x, y\}$ for all $x, y \in H$ and let μ be any homogeneous complex fuzzy subset of H. Then, by Propositions 3.17 and 3.39, μ and μ^c are complex fuzzy and anti-fuzzy subhypergroup of H.*

Example 3.48. *Let $(H, \circ)$ be any hypergroup (H_v-group) with the complex fuzzy subset μ of H as: $\mu(x) = re^{i\theta}$ where $r \in [0, 1], \theta \in [0, 2\pi]$ are fixed real numbers. Then μ and μ^c are both: homogeneous complex fuzzy and anti-fuzzy subhypergroups of H.*

4 Conclusions

This paper contributed to the study of fuzzy subhyperstructures by introducing the concepts of complex fuzzy (anti-fuzzy) subhyperstructures and investigating their properties.

For future work, we may define the generalized complex fuzzy subhyperstructures and investigate their properties.

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