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Power allocation rules under fuzzy behavior and multicriteria situations

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Abstract

In this paper, we adopt supreme-utilities among fuzzy level (decision) vectors to propose a power allocation rule, its efficient extension and normalization in the framework of multicriteria fuzzy transferable-utility (TU) games. We also provide several axiomatic results to present the rationality for these rules. Based on different viewpoints, we introduce different formulations and dynamic results for the efficient extension and the normalization by applying the reduced game and the excess function respectively.

Keywords: Multicriteria fuzzy TU game, supreme-utility; reduced game, excess function, dynamic result.

1 Introduction

In the framework of transferable-utility (TU) games, the power indexes have been defined to measure the political power of each member of a voting system. A member in a voting system is, e.g., a party in a parliament or a country in a confederation. Each member will have a certain number of votes, and so its power will be different. Results of the power indexes may be found in, e.g., Dubey and Shapley [8], Haller [9], Lehrer [12], van den Brink and van der Laan [5] and so on.

Banzhaf [3] defined a power index in the framework of voting games that was essentially identical to that given by Coleman [7]. This index was later on extended to arbitrary games by Owen [19, 20]. In this paper, we focus on the Banzhaf-Owen index. Briefly speaking, the Banzhaf-Owen index is a rule that gathers each member's marginal contribution from all coalitions in which he/she/it has participated.

The theory of *fuzzy TU games* started with the investigation of Aubin [1, 2] where the notions of a fuzzy TU game and the core of a fuzzy TU game are proposed. Many fuzzy allocation rules have been applied wildly, e.g., Borkotokey and Mesiar [4], Butnariu and Kroupa [6], Hwang [10], Khorram et al. [11], Li and Zhang [13], Liao [14], Liao and Chung [15], Meng and Zhang [17], Tijs et al. [23], and so on.

Consistency is an important property among the axiomatic formulations for allocation rules. Consistency states the independence of a value with respect to fixing some members with their assigned payoffs. It asserts that the recommendation made for any problem should always agree with the recommendation made in the subproblem that appears when the payoffs of some members are settled on. It has been introduced in different ways depending upon how the payoffs of the members that "leave the bargaining" are defined. This property has been investigated in various problems by applying *reduced games* always. In addition to axiomatic formulations for an allocation rule, *dynamic processes* can be described that lead the members to that allocation rule, starting from an arbitrary efficient payoff vector. The foundation of a dynamic theory was laid by Stearns [22].

The above pre-existing results raise one motivation:

- whether the power indexes could be extended under fuzzy behavior and multicriteria situations simultaneously.

The paper is devoted to investigate the motivation. The main results of this paper are as follows.

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1. Different from the framework of fuzzy TU games, we firstly consider the framework of *multicriteria fuzzy TU games* in Section 2. The *fuzzy Banzhaf-Owen index*, the *fuzzy efficient Banzhaf-Owen index* and the *fuzzy normalized Banzhaf-Owen index*, are further proposed by applying supreme-utilities among level (decision) vectors on multicriteria fuzzy TU games.

2. By considering an extended reduced game in the framework of fuzzy TU games, we propose some axiomatic results to analyze the rationality for these indexes in Section 3.

3. In order to establish the dynamic processes, we present alternative formulations for the fuzzy efficient Banzhaf-Owen index and fuzzy normalized Banzhaf-Owen index in terms of the *excess functions* in Section 3. The excess of a coalition could be treated as the *variation* between the productivity and the total payoff of the coalition.

4. In Section 4, we adopt reduction and excess function to show that the fuzzy efficient Banzhaf-Owen index and the fuzzy normalized Banzhaf-Owen index can be reached by members who start from an arbitrary efficient payoff vector. When a player participates in a game, several types of variations or complaints may be occurred from different situations. These dynamic processes are devoted to regulating these variations or complaints to be more balanced among all players.

2 Preliminaries

Let U be the universe of members. For $i \in U$ and $b_i \in (0,1]$, $B_i = [0, b_i]$ could be treated as the level (decision) space of member i and $B_i^+ = (0, b_i]$, where 0 denotes no participation. Let $B^N = \prod_{i\in N} B_i$ be the product set of the level (decision) spaces of all members of N. For all $T \subseteq N$, we define $\theta^T \in B^N$ be the vector with $\theta^T_i = 1$ if $i \in T$, and $\theta^T_i = 0$ if $i \in N \setminus T$. Denote 0_N the zero vector in $\mathbb{R}^N$. For $m \in \mathbb{N}$, let 0_m be the zero vector in $\mathbb{R}^m$ and $\mathbb{N}_m = \{1, 2, \cdots, m\}$.

A **fuzzy TU game**[1] is a triple (N, b, v), where N is a non-empty and finite set of members, $b = (b_i)_{i\in N} \in \prod_{i\in N} B_i^+$ is the vector that presents the highest levels for each member, and $v : B^N \to \mathbb{R}$ is a characteristic mapping with $v(0_N) = 0$ which assigns to each $\alpha = (\alpha_i)_{i\in N} \in B^N$ the worth that the members can gain when each member i participates at level α_i. Given a fuzzy TU game (N, b, v) and $\alpha \in B^N$, we write $A(\alpha) = \{i \in N | \ \alpha_i \neq 0\}$ and α_T to be the restriction of α at T for each $T \subseteq N$. Further, we define $v_*(T) = \sup_{\alpha\in B^N}\{v(\alpha)|A(\alpha) = T\}$ is the **supreme-utility**[2] among all action vector α with $A(\alpha) = T$. A **multicriteria fuzzy TU game** is a triple (N, b, V^m), where $m \in \mathbb{N}$, $V^m = (v^t)_{t\in\mathbb{N}_m}$ and (N, b, v^t) is a fuzzy TU game for all $t \in \mathbb{N}_m$.

Denote the collection of all multicriteria fuzzy TU games by Γ. Let $(N, b, V^m) \in \Gamma$. A **payoff vector** of (N, b, V^m) is a vector $x = (x^t)_{t\in\mathbb{N}_m}$ and $x^t = (x^t_i)_{i\in N} \in \mathbb{R}^N$, where x^t_i denotes the payoff to member i in (N, b, v^t) for all $t \in \mathbb{N}_m$ and for all $i \in N$. A payoff vector x of (N, b, V^m) is **multicriteria efficient** if $\sum_{i\in N} x^t_i = v^t_*\big(N\big)$ for all $t \in \mathbb{N}_m$. The collection of all multicriteria efficient vector of (N, b, V^m) is denoted by $E(N, b, V^m)$. A **solution** is a map σ assigning to each $(N, b, V^m) \in \Gamma$ an element

$$\sigma\big(N, b, V^m\big) = \big(\sigma^t\big(N, b, V^m\big)\big)_{t\in\mathbb{N}_m},$$

where $\sigma^t\big(N, b, V^m\big) = \big(\sigma^t_i\big(N, b, V^m\big)\big)_{i\in N} \in \mathbb{R}^N$ and $\sigma^t_i\big(N, b, V^m\big)$ is the payoff of the member i assigned by σ in $\big(N, b, v^t\big)$.

Next, we provide the fuzzy Banzhaf-Owen index and the fuzzy efficient Banzhaf-Owen index under multicriteria situations.

Definition 2.1. *The* **fuzzy Banzhaf-Owen index (FBOI)**, β, *is defined by*

$$\beta^t_i(N, b, V^m) = \sum_{\substack{S\subseteq N\\ i\in S}} \big[v^t_*(S) - v^t_*(S \setminus \{i\})\big]$$

for all $(N, b, V^m) \in \Gamma$, for all $t \in \mathbb{N}_m$ and for all $i \in N$. Under the solution β, all members receive their **marginal contributions** *related to maximal-utilities in each $S \subseteq N$ respectively.*

In order to present our results, some more properties are needed. A solution σ satisfies **multicriteria efficiency (MEFF)** if for all $(N, b, V^m) \in \Gamma$ and for all $t \in \mathbb{N}_m$, $\sum_{i\in N} \sigma_i(N, b, V^m) = v^t_*(N)$. Property MEFF asserts that all members allocate all the utility completely. It is easy to check that the FBOI violates MEFF. Therefore, we consider an efficient extension and a normalization as follows.

[1]A fuzzy TU game, which is defined by Aubin [1, 2], is a pair (N, v^a), where N is a coalition and v^a is a mapping such that $v^a : [0, 1]^N \longrightarrow \mathbb{R}$ and $v^a(0_N) = 0$. In fact, $(N, v^a) = (N, \theta^N, v)$.

[2]From now on we consider bounded fuzzy TU games, defined as those games (N, b, v) such that, there exists $K_v \in \mathbb{R}$ such that $v(\alpha) \le K_v$ for all $\alpha \in B^N$. We adopt it to ensure that $v_*(T)$ is well-defined.

Definition 2.2.

- *The* **fuzzy efficient Banzhaf-Owen index (FEBOI)**, $\overline{\beta}$, *is defined by*

$$\overline{\beta_i^t}(N,b,V^m) = \beta_i^t(N,b,V^m) + \frac{1}{|N|}\cdot\Big[v_*^t(N) - \sum_{k\in N}\beta_k^t(N,b,V^m)\Big]$$

for all $(N,b,V^m)\in\Gamma$, *for all* $t\in\mathbb{N}_m$ *and for all* $i\in N$.

- *The* **fuzzy normalized Banzhaf-Owen index (FNBOI)**, $\overline{\eta}$, *is defined by*

$$\overline{\eta_i^t}(N,b,V^m) = \frac{v_*^t(N)}{\sum\limits_{k\in N}\beta_k^t(N,b,V^m)}\cdot\beta_i^t(N,b,V^m)$$

for all $(N,b,V^m)\in\Gamma^N$, *for all* $t\in\mathbb{N}_m$ *and for all* $i\in N$, *where* $\Gamma^N=\{(N,b,V^m)\in\Gamma \mid \sum_{k\in N}\beta_k^t(N,b,V^m)\neq 0\}$.

Lemma 2.3. *The FEBOI and the FNBOI satisfy MEFF on* Γ *and* Γ^N *respectively.*

Proof. For all $(N,b,V^m)\in\Gamma$ and for all $t\in\mathbb{N}_m$,

$$\begin{aligned}\sum_{i\in N}\overline{\beta_i^t}(N,b,V^m) &= \sum_{i\in N}\Big[\beta_i^t(N,b,V^m)+\tfrac{1}{|N|}\cdot\Big[v_*^t(N)-\sum_{k\in N}\beta_k^t(N,b,V^m)\Big]\Big]\\ &= \sum_{i\in N}\beta_i^t(N,b,V^m)+\tfrac{|N|}{|N|}\cdot\Big[v_*^t(N)-\sum_{k\in N}\beta_k^t(N,b,V^m)\Big]\\ &= \sum_{i\in N}\beta_i^t(N,b,V^m)+v_*^t(N)-\sum_{k\in N}\beta_k^t(N,b,V^m)\\ &= v_*^t(N).\end{aligned}$$

Thus, the FEBOI satisfies MEFF on Γ. On the other hand, For all $(N,b,V^m)\in\Gamma^N$ and for all $t\in\mathbb{N}_m$,

$$\begin{aligned}\sum_{i\in N}\overline{\eta_i^t}(N,b,V^m) &= \sum_{i\in N}\Big[\tfrac{v_*^t(N)}{\sum_{k\in N}\beta_k^t(N,b,V^m)}\cdot\beta_i^t(N,b,V^m)\Big]\\ &= \tfrac{v_*^t(N)}{\sum_{k\in N}\beta_k^t(N,b,V^m)}\cdot\Big[\sum_{i\in N}\beta_i^t(N,b,V^m)\Big]\\ &= v_*^t(N).\end{aligned}$$

Thus, the FNBOI satisfies MEFF on Γ^N. □

Here we provide a brief application of multicriteria fuzzy TU games in the setting of "management". This kind of problem can be formulated as follows. Let $N=\{1,2,\cdots,n\}$ be a set of all members of a grand management system (N,b,V^m). The function v^t could be treated as an utility function which assigns to each level vector $\alpha=(\alpha_i)_{i\in N}\in B^N$ the worth that the members can obtain when each member i participates at operation strategy $\alpha_i\in B_i$ in the sub-management system (N,b,v^t). Modeled in this way, the grand management system (N,b,V^m) could be considered as a multicriteria fuzzy TU game, with v^t being each characteristic function and B_i being the set of all operation strategies of the member i. In the following sections, we would like to show that the FBOI, the FEBOI and the FNBOI could provide "*optimal allocation mechanisms*" among all members, in the sense that this organization can get payoff from each combination of operation strategies of all members under fuzzy behavior and multicriteria situations.

3 Axiomatic results

In this section, we show that there exists corresponding reduced games that could be adopted to analyze the FBOI, the FEBOI and the FNBOI.

Subsequently, we introduced reduced game and related consistency as follows. Let ψ be a solution, $(N,b,V^m)\in\Gamma$ and $S\subseteq N$. The **1-reduced game** $(S,b_S,V_{S,\psi}^{1,m})$ is defined by $V_{S,\psi}^{1,m}=(v_{S,\psi}^{1,t})_{t\in\mathbb{N}_m}$ and

$$v_{S,\psi}^{1,t}(\alpha)=\begin{cases}\sum\limits_{Q\subseteq N\setminus S}\Big[v_*^t\big(A(\alpha)\cup Q\big)-\sum\limits_{i\in Q}\psi_i^t(N,b,V^m)\Big] & \text{otherwise,}\\ 0 & \text{if } \alpha=0_S.\end{cases}$$

ψ satisfies **1-consistency (1CON)** if $\psi_i^t(S,b_S,V_{S,\psi}^{1,m})=\psi_i^t(N,b,V^m)$ for all $(N,b,V^m)\in\Gamma$, for all $S\subseteq N$ with $|S|=2$, for all $t\in\mathbb{N}_m$ and for all $i\in S$. Further, ψ satisfies **1-standard for games (1SG)** if $\psi(N,b,V^m)=\beta(N,b,V^m)$ for all $(N,b,V^m)\in\Gamma$ with $|N|\le 2$. Next, we characterize the FBOI by applying the properties of 1CON and 1SG.

Lemma 3.1.

1. *The FBOI satisfies 1CON on* Γ.
2. *On* Γ*, the FBOI is the only solution satisfying 1SG and 1CON.*

Proof. To prove result 1, let $(N, b, V^m) \in \Gamma$ and $S \subseteq N$. If $|N| = 1$, then the proof is completed. Assume that $|N| \geq 2$ and $S = \{i, j\}$ for some $i, j \in N$. For all $t \in \mathbb{N}_m$ and for all $i \in S$,

$$\begin{aligned}\beta_i^t(S, b_S, V^{1,m}_{S,\beta}) &= \sum_{\substack{T\subseteq S\\ i\in T}} \big[(v^{1,t}_{S,\beta})_*(T) - (v^{1,t}_{S,\beta})_*(T \setminus \{i\})\big] = \sum_{\substack{T\subseteq S\\ i\in T}} \sum_{Q\subseteq N\setminus S} \big[v^t_*(T \cup Q) - v^t_*(T \setminus \{i\} \cup Q)\big] \\ &= \sum_{\substack{K\subseteq N\\ i\in K}} \big[v(K) - v(K \setminus \{i\})\big] = \beta_i^t(N, b, V^m).\end{aligned}$$

Hence, β satisfies 1CON.

By result 1, β satisfies 1CON on Γ. Clearly, β satisfies 1SG on Γ. To prove uniqueness of result 2, suppose ψ satisfies 1SG and 1CON. Let $(N, b, v) \in \Gamma$. If $|N| \leq 2$, then $\psi(N, b, v) = \beta(N, b, v)$ by 1SG. The case $|N| > 2$: Let $i \in N$ and $t \in \mathbb{N}_m$, and let $S \subset N$ with $|S| = 2$ and $i \in S$. Then,

$$\begin{aligned}\psi_i^t(N, b, V^m) &= \psi_i^t(S, b_S, V^{1,m}_{S,\psi}) && \big(\text{by 1CON}\big)\\ &= \beta_i^t(S, b_S, V^{1,m}_{S,\psi}) && \big(\text{by 1SG}\big)\\ &= \sum_{\substack{T\subseteq S\\ i\in T}} \Big[(v^{1,t}_{S,\psi})_*(T) - (v^{1,t}_{S,\psi})_*(T \setminus \{i\})\Big]\\ &= \sum_{\substack{T\subseteq S\\ i\in T}} \sum_{Q\subseteq N\setminus S} [v^t_*(T \cup Q) - v^t_*\big((T \setminus \{i\}) \cup Q\big)]\\ &= \sum_{\substack{K\subseteq N\\ i\in K}} v^t_*(K) - v^t_*(K \setminus \{i\})\\ &= \beta_i^t(N, b, V^m).\end{aligned}$$

Thus, $\psi(N, b, V^m) = \beta(N, b, V^m)$. □

The following examples are to show that each of the axioms adopted in Lemma 3.1 is logically independent of the remaining axioms.

Example 3.2. *Define a solution* ψ *by for all* $(N, b, V^m) \in \Gamma$*, for all* $t \in \mathbb{N}_m$ *and for all* $i \in N$*,* $\psi_i^t(N, b, V^m) = 0$*. Clearly,* ψ *satisfies 1CON, but it violates 1SG.*

Example 3.3. *Define a solution* ψ *by for all* $(N, b, V^m) \in \Gamma$*, for all* $t \in \mathbb{N}_m$ *and for all* $i \in N$*,*

$$\psi_i^t(N, b, V^m) = \begin{cases} \beta_i^t(N, b, V^m) & \text{if } |N| \leq 2,\\ 0 & \text{otherwise.}\end{cases}$$

On Γ*,* ψ *satisfies 1SG, but it violates 1CON.*

Unfortunately, it is easy to check that the index FEBOI $\overline{\beta}$ violates 1CON. Therefore, we consider the *2-reduced game* as follows. Let ψ be a solution, $(N, b, V^m) \in \Gamma$ and $S \subseteq N$. The **2-reduced game** $(S, b_S, V^{2,m}_{S,\psi})$ is defined by $V^{2,m}_{S,\psi} = (v^{2,t}_{S,\psi})_{t\in\mathbb{N}_m}$ and

$$v^{2,t}_{S,\psi}(\alpha) = \begin{cases} 0 & \text{if } \alpha = 0_S,\\ v^t_*(N) - \sum\limits_{i\in N\setminus S} \psi_i^t(N, b, V^m) & \text{if } A(\alpha) = S,\\ \sum\limits_{Q\subseteq N\setminus S} \big[v^t_*\big(A(\alpha) \cup Q\big) - \sum\limits_{i\in Q} \psi_i^t(N, b, V^m)\big] & \text{otherwise.}\end{cases}$$

ψ satisfies **2-consistency (2CON)** if $\psi_i^t(S, b_S, V^{2,m}_{S,\psi}) = \psi_i^t(N, b, V^m)$ for all $(N, b, V^m) \in \Gamma$, for all $S \subseteq N$ with $|S| = 2$, for all $t \in \mathbb{N}_m$ and for all $i \in S$. Further, ψ satisfies **2-standard for games (2SG)** if $\psi(N, b, V^m) = \overline{\beta}(N, b, V^m)$ for all $(N, b, V^m) \in \Gamma$ with $|N| \leq 2$.

It is easy to check that the index FNBOI $\overline{\eta}$ violates 1CON. It is also easy to see that $\sum_{k\in S} \beta_k^t(N, b, V^m) = 0$ for some $(N, b, V^m) \in \Gamma$, for some $t \in \mathbb{N}_m$ and for some $S \subseteq N$, i.e., $\overline{\eta}(S, b_S, v^{2,m}_{S,\psi})$ doesn't exist for some $(N, b, V^m) \in \Gamma$, for some $t \in \mathbb{N}_m$ and for some $S \subseteq N$. Therefore, we consider the *resilient 2-consistency* as follows. A solution ψ satisfies **resilient 2-consistency (R2CON)** if $(S, b_S, v^{2,m}_{S,\psi})$ and $\psi(S, b_S, v^{2,m}_{S,\psi})$ exist for some $(N, b, V^m) \in \Gamma$, for some $t \in \mathbb{N}_m$ and for some $S \subseteq N$ with $|S| = 2$, it holds that $\psi^t_i(S, b_S, V^{2,m}_{S,\psi}) = \psi^t_i(N, b, V^m)$ for all $i \in S$. Further, ψ satisfies **normalized-standard for games (NSG)** if $\psi(N, b, V^m) = \overline{\beta}(N, b, V^m)$ for all $(N, b, V^m) \in \Gamma$ with $|N| \leq 2$. Next, we characterize the FEBOI and the FNBOI by applying the properties of 2CON and R2CON.

In order to establish consistency of the FEBOI, it will be useful to present alternative formulation for the FEBOI in terms of *excess*. Let $(N, b, V^m) \in \Gamma$, $S \subseteq N$ and x be a payoff vector in (N, b, V^m). Define that $x^t(S) = \sum_{i \in S} x^t_i$ for all $t \in \mathbb{N}_m$. The **excess** of a coalition $S \subseteq N$ at x is the real number

$$e(S, V^m, x) = (e(S, v^t, x^t))_{t \in \mathbb{N}_m} \text{ and } e(S, v^t, x^t) = v^t_*(S) - x^t(S). \tag{1}$$

The value $e(S, v^t, x^t)$ can be treated as the **complaint** of coalition S when all members receive their payoffs from x^t in (N, b, v^t).

Lemma 3.4.

1. *Let $(N, b, V^m) \in \Gamma$ and $x \in E(N, b, V^m)$. Then for all $i, j \in N$ and $t \in \mathbb{N}_m$,*

$$\sum_{S \subseteq N \setminus \{i,j\}} e(S \cup \{i\}, v^t, \tfrac{x^t}{2^{|N|-1}}) = \sum_{S \subseteq N \setminus \{i,j\}} e(S \cup \{j\}, v^t, \tfrac{x^t}{2^{|N|-1}}) \Longleftrightarrow x = \overline{\beta}(N, b, V^m).$$

2. *Let $(N, b, V^m) \in \Gamma^N$ and $x \in E(N, b, V^m)$. Then for all $i, j \in N$ and $t \in \mathbb{N}_m$,*

$$\sum_{S \subseteq N \setminus \{i,j\}} e(S \cup \{i\}, v^t, \tfrac{x^t}{b^t \cdot 2^{|N|-1}}) = \sum_{S \subseteq N \setminus \{i,j\}} e(S \cup \{j\}, v^t, \tfrac{x^t}{b^t \cdot 2^{|N|-1}}) \Longleftrightarrow x = \overline{\eta}(N, b, V^m).$$

where $b^t = \frac{v^t(N)}{\sum\limits_{k \in N} \beta^t_k(N, b, V^m)}$

Proof. To prove result 1, let $(N, b, V^m) \in \Gamma$ and $x \in E(N, b, V^m)$. For all $t \in \mathbb{N}_m$ and for all $i, j \in N$,

$$\begin{aligned}
&\sum_{S \subseteq N \setminus \{i,j\}} e(S \cup \{i\}, v^t, \tfrac{x^t}{2^{|N|-1}}) = \sum_{S \subseteq N \setminus \{i,j\}} e(S \cup \{j\}, v^t, \tfrac{x^t}{2^{|N|-1}}) \\
\Longleftrightarrow &\sum_{S \subseteq N \setminus \{i,j\}} \Big[v^t_*(S \cup \{i\}) - \tfrac{x^t(S \cup \{i\})}{2^{|N|-1}}\Big] = \sum_{S \subseteq N \setminus \{i,j\}} \Big[v^t_*(S \cup \{j\}) - \tfrac{x^t(S \cup \{j\})}{2^{|N|-1}}\Big] \\
\Longleftrightarrow &\Big[\sum_{S \subseteq N \setminus \{i,j\}} v^t_*(S \cup \{i\})\Big] - \tfrac{x^t_i}{2} = \Big[\sum_{S \subseteq N \setminus \{i,j\}} v^t_*(S \cup \{j\})\Big] - \tfrac{x^t_j}{2} \\
\Longleftrightarrow &\, x^t_i - x^t_j = 2 \cdot \sum_{S \subseteq N \setminus \{i,j\}} \Big[v^t_*(S \cup \{i\}) - v^t_*(S \cup \{j\})\Big].
\end{aligned} \tag{2}$$

By definition of $\overline{\beta}$,

$$\overline{\beta^t_i}(N, b, V^m) - \overline{\beta^t_j}(N, b, V^m) \quad = 2 \cdot \sum_{S \subseteq N \setminus \{i,j\}} \Big[v^t_*(S \cup \{i\}) - v^t_*(S \cup \{j\})\Big] \tag{3}$$

By equations (2) and (3), for all $i, j \in N$, $x^t_i - x^t_j = \overline{\beta^t_i}(N, b, V^m) - \overline{\beta^t_j}(N, b, V^m)$. Hence,

$$\sum_{j \in N} \Big[x^t_i - x^t_j\Big] = \sum_{j \in N} \Big[\overline{\beta^t_i}(N, b, V^m) - \overline{\beta^t_j}(N, b, V^m)\Big].$$

That is, $|N| \cdot x^t_i - \sum_{j \in N} x^t_j = |N| \cdot \overline{\beta^t_i}(N, b, V^m) - \sum_{j \in N} \overline{\beta^t_j}(N, b, V^m)$. Since $x \in E(N, b, V^m)$ and $\overline{\beta}$ satisfies MEFF, $|N| \cdot x^t_i - v^t_*(N) = |N| \cdot \overline{\beta^t_i}(N, b, V^m) - v^t_*(N)$. Therefore, $x^t_i = \overline{\beta^t_i}(N, b, V^m)$ for all $t \in \mathbb{N}_m$ and for all $i \in N$, i.e., $x = \overline{\beta}(N, b, V^m)$.

To prove result 2, let $(N, b, V^m) \in \Gamma^N$ and $x \in E(N, b, V^m)$. For all $t \in \mathbb{N}_m$ and for all $i, j \in N$,

$$\begin{aligned}
&\sum_{S \subseteq N \setminus \{i,j\}} e(S \cup \{i\}, v^t, \tfrac{x^t}{b^t \cdot 2^{|N|-1}}) = \sum_{S \subseteq N \setminus \{i,j\}} e(S \cup \{j\}, v^t, \tfrac{x^t}{b^t \cdot 2^{|N|-1}}) \\
\Longleftrightarrow &\sum_{S \subseteq N \setminus \{i,j\}} [v^t(S \cup \{i\}) - \tfrac{x^t(S \cup \{i\})}{b^t \cdot 2^{|N|-1}}] = \sum_{S \subseteq N \setminus \{i,j\}} [v^t(S \cup \{j\}) - \tfrac{x^t(S \cup \{j\})}{b \cdot 2^{|N|-1}}] \\
\Longleftrightarrow &[\sum_{S \subseteq N \setminus \{i,j\}} v^t(S \cup \{i\})] - \tfrac{x^t_i}{2b^t} = [\sum_{S \subseteq N \setminus \{i,j\}} v^t(S \cup \{j\})] - \tfrac{x^t_j}{2b^t} \\
\Longleftrightarrow &\, x^t_i - x^t_j = 2b^t \cdot \sum_{S \subseteq N \setminus \{i,j\}} [v^t(S \cup \{i\}) - v^t(S \cup \{j\})].
\end{aligned} \tag{4}$$

On the other hand, by definitions of $\overline{\eta}$, β,

$$\begin{aligned} & \overline{\eta_i^t}(N,b,V^m) - \overline{\eta_j^t}(N,b,V^m) \\ = \;& b^t \cdot [\beta_i^t(N,b,V^m) - \beta_j^t(N,b,V^m)] \\ = \;& b^t \cdot \big[[\sum_{S\subseteq N\setminus\{i\}} (v^t(S\cup\{i\}) - v^t(S)] - [\sum_{S\subseteq N\setminus\{j\}} (v^t(S\cup\{j\}) - v^t(S)]\big] \\ = \;& b^t \cdot \big[[\sum_{S\subseteq N\setminus\{i,j\}} (v^t(S\cup\{i,j\}) - v^t(S\cup\{j\})) + \sum_{S\subseteq N\setminus\{i,j\}} (v^t(S\cup\{i\}) - v^t(S))] \\ & - [\sum_{S\subseteq N\setminus\{i,j\}} v^t(S\cup\{i,j\}) - v^t(S\cup\{i\}) + \sum_{S\subseteq N\setminus\{i,j\}} (v^t(S\cup\{j\}) - v^t(S))]\big] \\ = \;& b^t \cdot \sum_{S\subseteq N\setminus\{i,j\}} 2\cdot[v^t(S\cup\{i\}) - v^t(S\cup\{j\})]. \end{aligned} \tag{5}$$

By equations (4) and (5), for all pairs $\{i,j\}\subseteq N$, $x_i^t - x_j^t = \overline{\eta_i^t}(N,b,V^m) - \overline{\eta_j^t}(N,b,V^m)$. Hence,

$$\sum_{j\neq i}[x_i^t - x_j^t] = \sum_{j\neq i}[\overline{\eta_i^t}(N,b,V^m) - \overline{\eta_j^t}(N,b,V^m)].$$

That is, $(|N|-1)x_i^t - \sum_{j\neq i} x_j^t = (|N|-1)\overline{\eta_i^t}(N,b,V^m) - \sum_{j\neq i}\overline{\eta_j^t}(N,b,V^m)$. By $x\in E(N,b,V^m)$ and $\overline{\eta}$ is MEFF, $|N|x_i^t - v^t(N) = |N|\overline{\eta_i^t}(N,b,V^m) - v^t(N)$. So, $x = \overline{\eta}$. □

Proposition 3.5. *It is easy to check that*

$$\sum_{S\subseteq N\setminus\{i,j\}} e(S\setminus\{i\}, V^m, \beta(N,b,V^m)) = \sum_{S\subseteq N\setminus\{i,j\}} e(S\setminus\{j\}, V^m, \beta(N,b,V^m))$$

for all $(N,b,V^m)\in\Gamma$ and for all $i,j\in N$.

Theorem 3.6.

1. *The FEBOI satisfies 2CON on Γ.*
2. *If ψ satisfies 2SG and 2CON, then it also satisfies MEFF.*
3. *On Γ, the FEBOI is the only solution satisfying 2SG and 2CON.*

Proof. To verify result 1, let $(N,b,V^m)\in\Gamma$ and $S\subseteq N$. If $|N|=1$, then the proof is completed. Assume that $|N|\geq 2$, $x = \overline{\beta}(N,b,V^m)$ and $S=\{i,j\}$ for some $i,j\in N$. For all $t\in\mathbb{N}_m$ and for all $l\in S$,

$$\begin{aligned} \sum_{T\subseteq S\setminus\{i,j\}} e\big(T\cup\{l\}, v_{S,x}^{2,t}, \tfrac{x_S^t}{2^{|S|-1}}\big) =\;& e\big(\{l\}, v_{S,x}^{2,t}, \tfrac{x_S^t}{2}\big) \\ =\;& v_{S,x}^{2,t}(\{l\}) - \tfrac{x_l^t}{2} \\ =\;& \Big(\sum_{Q\subseteq N\setminus S}\big[v_*^t(\{l\}\cup Q) - \sum_{k\in Q} x_k^t\big]\Big) - \tfrac{x_l^t}{2} \\ =\;& \sum_{Q\subseteq N\setminus S}\Big[v_*^t(\{l\}\cup Q) - \sum_{k\in Q} x_k^t - \tfrac{x_l^t}{2\cdot 2^{|N\setminus S|}}\Big] \\ =\;& \sum_{Q\subseteq N\setminus S}\Big[v_*^t(\{l\}\cup Q) - \sum_{k\in Q} x_k^t - \tfrac{x_l^t}{2^{|N|-1}}\Big] \\ =\;& \sum_{Q\subseteq N\setminus S}\Big[v_*^t(\{l\}\cup Q) - \sum_{k\in Q} x_k^t + \sum_{k\in Q}\tfrac{x_k^t}{2^{|N|-1}} - \sum_{k\in Q}\tfrac{x_k^t}{2^{|N|-1}} - \tfrac{x_l^t}{2^{|N|-1}}\Big] \\ =\;& \sum_{Q\subseteq N\setminus S}\Big[v_*^t(\{l\}\cup Q) - \sum_{k\in Q}\big(1-\tfrac{1}{2^{|N|-1}}\big)x_k^t - \sum_{k\in Q\cup\{l\}}\tfrac{x_k^t}{2^{|N|-1}}\Big] \\ =\;& \sum_{Q\subseteq N\setminus S}\Big[e\big(\{l\}\cup Q, v^t, \tfrac{x}{2^{|N|-1}}\big) - \sum_{k\in Q}\big(1-\tfrac{1}{2^{|N|-1}}\big)x_k^t\Big] \\ =\;& \sum_{Q\subseteq N\setminus S} e\big(\{l\}\cup Q, v^t, \tfrac{x}{2^{|N|-1}}\big) - \sum_{Q\subseteq N\setminus S}\sum_{k\in Q}\big(1-\tfrac{1}{2^{|N|-1}}\big)x_k^t. \end{aligned} \tag{6}$$

Since $\overline{\beta}$ satisfies MEFF, $x_S^t\in E(S,b_S,v_{S,x}^{2,t})$ by definition of 2-reduced game. In addition, by equation (6) and Lemma 3.2,

$$\begin{aligned} e\big(\{i\}, v_{S,x}^{2,t}, x_S^t\big) =\;& \sum_{Q\subseteq N\setminus S} e\big(\{i\}\cup Q, v^t, \tfrac{x}{2^{|N|-1}}\big) - \sum_{Q\subseteq N\setminus S}\sum_{k\in Q}\big(1-\tfrac{1}{2^{|N|-1}}\big)x_k^t \quad \text{(by equation (6))} \\ =\;& \sum_{Q\subseteq N\setminus S} e\big(\{j\}\cup Q, v^t, \tfrac{x}{2^{|N|-1}}\big) - \sum_{Q\subseteq N\setminus S}\sum_{k\in Q}\big(1-\tfrac{1}{2^{|N|-1}}\big)x_k^t \quad \text{(by Lemma 3.2)} \\ =\;& e\big(\{j\}, v_{S,x}^{2,t}, x_S^t\big). \quad \text{(by equation (6))} \end{aligned}$$

So, $x_S = \overline{\beta}(S, b_S, V^{2,m}_{S,\overline{\beta}})$. That is, $\overline{\beta}$ satisfies 2CON.

To prove result 2, suppose ψ satisfies 2SG and 2CON. Let $(N, b, V^m) \in \Gamma$ and $t \in \mathbb{N}_m$. If $|N| \leq 2$, it is trivial that ψ satisfies MEFF by 2SG. The case $|N| > 2$: Assume, on the contrary, that there exists $(N, b, V^m) \in \Gamma$ such that $\sum_{i\in N} \psi^t_i(N, b, V^m) \neq v^t_*(N)$. This means that there exist $i \in N$ and $j \in N$ such that $[v^t_*(N) - \sum_{k\in N\setminus\{i,j\}} \psi^t_k(N, b, V^m)] \neq [\psi^t_i(N, b, V^m) + \psi^t_j(N, b, V^m)]$. By 2CON and ψ satisfies MEFF for two-person games, this contradicts

$$\psi^t_i(N, b, V^m) + \psi^t_j(N, b, V^m) = \psi^t_i(\{i, j\}, b_{\{i,j\}}, v^{2,t}_{\{i,j\},\psi}) + \psi^t_j(\{i, j\}, b_{\{i,j\}}, v^{2,t}_{\{i,j\},\psi}) = v^t_*(N) - \sum_{k\in N\setminus\{i,j\}} \psi^t_k(N, b, V^m).$$

Hence ψ satisfies MEFF.

To prove result 3, $\overline{\beta}$ satisfies 2CON by result 1. Clearly, $\overline{\beta}$ satisfies 2SG. To prove uniqueness, suppose ψ satisfies 2SG and 2CON, hence by result 2, ψ also satisfies MEFF. Let $(N, b, V^m) \in \Gamma$. If $|N| \leq 2$, it is trivial that $\psi(N, b, V^m) = \overline{\beta}(N, b, V^m)$ by 2SG. The case $|N| > 2$: Let $i \in N$, $t \in \mathbb{N}_m$ and $S = \{i, j\}$ for some $j \in N \setminus \{i\}$. Then

$$\begin{aligned} \psi^t_i(N, b, V^m) - \overline{\beta^t_i}(N, b, V^m) = \; & \psi^t_i(S, b_S, V^{2,m}_{S,\psi}) - \overline{\beta^t_i}(S, b_S, V^{2,m}_{S,\overline{\beta}}) \quad \big(\text{by 2CON of } \psi, \overline{\beta}\big) \\ = \; & \overline{\beta^t_i}(S, b_S, V^{2,m}_{S,\psi}) - \overline{\beta^t_i}(S, b_S, V^{2,m}_{S,\overline{\beta}}) \quad \big(\text{by 2SG of } \psi, \overline{\beta}\big) \\ = \; & \tfrac{1}{2} \cdot \Big[(v^{2,t}_{S,\psi})_*(S) + (v^{2,t}_{S,\psi})_*(\{i\}) - (v^{2,t}_{S,\psi})_*(\{j\})\Big] \\ & -\tfrac{1}{2} \cdot \Big[(v^{2,t}_{S,\overline{\beta}})_*(S) + (v^{2,t}_{S,\overline{\beta}})_*(\{i\}) - (v^{2,t}_{S,\overline{\beta}})_*(\{j\})\Big]. \end{aligned} \tag{7}$$

By definitions of $v^{2,t}_{S,\psi}$ and $v^{2,t}_{S,\overline{\beta}}$,

$$(v^{2,t}_{S,\psi})_*(\{i\}) - (v^{2,t}_{S,\psi})_*(\{j\}) = \tfrac{1}{2^{|N\setminus S|}} \cdot \sum_{Q\subseteq N\setminus S} \Big[v^t_*(\{i\} \cup Q) - v^t_*(\{j\} \cup Q)\Big] = (v^{2,t}_{S,\overline{\beta}})_*(\{i\}) - (v^{2,t}_{S,\overline{\beta}})_*(\{j\}). \tag{8}$$

By equation (8), equation (7) becomes

$$\begin{aligned} \psi^t_i(N, b, V^m) - \overline{\beta^t_i}(N, b, V^m) &= \tfrac{1}{2} \cdot \Big((v^{2,t}_{S,\psi})_*(S) - (v^{2,t}_{S,\overline{\beta^t}})_*(S)\Big) \\ &= \tfrac{1}{2} \cdot \Big(\psi^t_i(N, b, V^m) + \psi^t_j(N, b, V^m) - \overline{\beta^t_i}(N, b, V^m) + \overline{\beta^t_j}(N, b, V^m)\Big). \end{aligned}$$

That is, $\psi^t_i(N, b, V^m) - \psi^t_j(N, b, V^m) = \overline{\beta^t_i}(N, b, V^m) - \overline{\beta^t_j}(N, b, V^m)$ for all $i, j \in N$. By MEFF of ψ and $\overline{\beta}$,

$$\begin{aligned} \sum_{j\in N} \big[\psi^t_i(N, b, V^m) - \psi^t_j(N, b, V^m)\big] = \sum_{j\in N} \big[\overline{\beta^t_i}(N, b, V^m) - \overline{\beta^t_j}(N, b, V^m)\big] &\Longleftrightarrow |N| \cdot \psi^t_i(N, b, V^m) - v^t_*(N) \\ &= |N| \cdot \overline{\beta^t_i}(N, b, V^m) - v^t_*(N). \end{aligned}$$

Thus, $\psi^t_i(N, b, V^m) = \overline{\beta^t_i}(N, b, V^m)$ for all $i \in N$, i.e., $\psi(N, b, V^m) = \overline{\beta}(N, b, V^m)$. □

Theorem 3.7.

1. *The FNBOI satisfies R2CON on Γ^N.*
2. *If ψ satisfies NSG and R2CON on Γ^N, then it also satisfies MEFF on Γ^N.*
3. *On Γ^N, the FNBOI is the only solution satisfying NSG and R2CON.*

Proof. The proof of this theorem are similar to Theorem 3.6. Hence, we omit it. □

The following examples are to show that each of the axioms adopted in Theorems 3.6 and 3.7 is logically independent of the remaining axioms.

Example 3.8. *Define a solution ψ by for all $(N, b, V^m) \in \Gamma$, for all $t \in \mathbb{N}_m$ and for all $i \in N$, $\psi^t_i(N, b, V^m) = 0$. Clearly, ψ satisfies 2CON and R2CON, but it violates 2SG and NSG.*

Example 3.9. *Define a solution ψ by for all $(N, b, V^m) \in \Gamma$, for all $t \in \mathbb{N}_m$ and for all $i \in N$,*

$$\psi^t_i(N, b, V^m) = \begin{cases} \overline{\beta^t_i}(N, b, V^m) & \text{if } |N| \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

On Γ, ψ satisfies 2SG, but it violates 2CON.

Example 3.10. *Define a solution ψ by for all $(N,b,V^m)\in\Gamma$, for all $t\in\mathbb{N}_m$ and for all $i\in N$,*

$$\psi_i^t(N,b,V^m)=\begin{cases}\overline{\eta_i^t}(N,b,V^m) & \text{if } |N|\le 2,\\ 0 & \text{otherwise.}\end{cases}$$

On Γ^N, ψ satisfies NSG, but it violates R2CON.

4 Dynamic processes

In this section, we adopt excess function and reduction to propose dynamic processes for the FEBOI and the FNBOI.

In order to establish the dynamic processes of the FEBOI and the FNBOI, we firstly define correction functions by means of excess functions. The correction functions are based on the idea that, each member shortens the complaint relating to his own and others' non-participation, and adopts these regulations to correct the original payoff.

Definition 4.1.

- *Let $(N,b,V^m)\in\Gamma$. The* **1-correction function** *$f^1=(f^{1,t})_{t\in\mathbb{N}_m}$ is defined as $f^{1,t}=(f_i^{1,t})_{i\in N}$ and $f_i^{1,t}: E(N,b,V^m)\to\mathbb{R}$ with*

$$f_i^{1,t}(x)=x_i^t+w\sum_{j\in N\setminus\{i\}}\sum_{Q\subseteq N\setminus\{i,j\}}\Big[e(Q\setminus\{j\},v^t,\frac{x^t}{2^{|N|-1}})-e(Q\setminus\{i\},v^t,\frac{x^t}{2^{|N|-1}})\Big],$$

Define $[x]^0=x,\ [x]^1=f([x]^0),\cdots,\ [x]^q=f([x]^{q-1})$ for all $q\in\mathbb{N}$.

- *Let $(N,b,V^m)\in\Gamma^N$. The* **2-correction function** *$f^2=(f^{2,t})_{t\in\mathbb{N}_m}$ is defined as $f^{2,t}=(f_i^{2,t})_{i\in N}$ and $f_i^{2,t}: E(N,b,V^m)\to\mathbb{R}$ with*

$$f_i^{2,t}(x)=x_i^t+w\sum_{j\in N\setminus\{i\}}\sum_{Q\subseteq N\setminus\{i,j\}}\Big[e(Q\setminus\{j\},v^t,\frac{x^t}{b\cdot 2^{|N|-1}})-e(Q\setminus\{i\},v^t,\frac{x^t}{b\cdot 2^{|N|-1}})\Big],$$

Define $[z]^0=x,\ [z]^1=f([x]^0),\cdots,\ [z]^q=f([z]^{q-1})$ for all $q\in\mathbb{N}$,

where $w\in\{r\in\mathbb{R}|r>0\}$ reflects the assumption that member i does not ask for full correction (when $w=1$) but only (usually) a fraction of it.

Lemma 4.2.

1. *$f^1(x)\in E(N,b,V^m)$ for all $(N,b,V^m)\in\Gamma$ and for all $x\in E(N,b,V^m)$.*
2. *$f^2(x)\in E(N,b,V^m)$ for all $(N,b,V^m)\in\Gamma^N$ and for all $x\in E(N,b,V^m)$.*

Proof. To prove result 1, let $(N,b,V^m)\in\Gamma$, $t\in\mathbb{N}_m$, $i,j\in N$ and $x\in E(N,b,V^m)$. Similar to equation (2),

$$\begin{aligned}&\sum_{j\in N\setminus\{i\}}\sum_{Q\subseteq N\setminus\{i,j\}}\Big[e(Q\setminus\{j\},v^t,\tfrac{x^t}{2^{|N|-1}})-e(Q\setminus\{i\},v^t,\tfrac{x^t}{2^{|N|-1}})\Big]\\ =&\sum_{j\in N\setminus\{i\}}\Big[\Big(\sum_{Q\subseteq N\setminus\{i,j\}}\big[v^t(Q\setminus\{j\})-v^t(Q\setminus\{i\})\big]\Big)-\tfrac{x_i^t}{2}+\tfrac{x_j^t}{2}\Big].\end{aligned}\tag{9}$$

By definition of $\overline{\beta}$,

$$\overline{\beta_i^t}(N,b,V^m)-\overline{\beta_j^t}(N,b,V^m)=2\sum_{Q\subseteq N\setminus\{i,j\}}\big[v^t(Q\setminus\{j\})-v^t(Q\setminus\{i\})\big].\tag{10}$$

By equations (9) and (10),

$$\begin{aligned}&\sum_{j\in N\setminus\{i\}}\sum_{Q\subseteq N\setminus\{i,j\}}\Big[e(Q\setminus\{j\},v^t,\tfrac{x^t}{2^{|N|-1}})-e(Q\setminus\{i\},v^t,\tfrac{x^t}{2^{|N|-1}})\Big]\\ =&\ \tfrac{1}{2}\cdot\sum_{j\in N\setminus\{i\}}\Big(\overline{\beta_i^t}(N,b,V^m)-\overline{\beta_j^t}(N,b,V^m)-x_i^t+x_j^t\Big)\\ =&\ \tfrac{1}{2}\cdot\Big((|N|-1)\big(\overline{\beta_i^t}(N,b,V^m)-x_i^t\big)-\sum_{j\in N\setminus\{i\}}\overline{\beta_j^t}(N,b,V^m)+\sum_{j\in N\setminus\{i\}}x_j^t\Big)\\ =&\ \tfrac{1}{2}\cdot\Big(|N|\big(\overline{\beta_i^t}(N,b,V^m)-x_i^t\big)-v_*^t(N)+v_*^t(N)\Big)\\ =&\ \tfrac{|N|}{2}\cdot\Big(\overline{\beta_i^t}(N,b,V^m)-x_i^t\Big).\end{aligned}\tag{11}$$

Moreover,

$$\begin{aligned}\sum_{i\in N}\sum_{j\in N\setminus\{i\}}\sum_{Q\subseteq N\setminus\{i,j\}}\Big[e(Q\setminus\{j\},v^t,x^t)-e(Q\setminus\{i\},v^t,x^t)\Big] &= \sum_{i\in N}\tfrac{|N|}{2}\cdot\Big(\overline{\beta_i^t}(N,b,V^m)-x_i^t\Big)\\ &= \tfrac{|N|}{2}\cdot\Big(\sum_{i\in N}\overline{\beta_i^t}(N,b,V^m)-\sum_{i\in N}x_i^t\Big)\\ &= \tfrac{|N|}{2}\cdot\big(v_*^t(N)-v_*^t(N)\big)\\ &= 0.\end{aligned} \tag{12}$$

So we have that

$$\begin{aligned}\sum_{i\in N}f_i^{1,t}(x) &= \sum_{i\in N}\Big[x_i^t+w\sum_{j\in N\setminus\{i\}}\sum_{Q\subseteq N\setminus\{i,j\}}\Big[e(Q\setminus\{j\},v^t,x^t)-e(Q\setminus\{i\},v^t,x^t)\Big]\Big]\\ &= \sum_{i\in N}x_i^t+w\sum_{i\in N}\sum_{j\in N\setminus\{i\}}\sum_{Q\subseteq N\setminus\{i,j\}}\Big[e(Q\setminus\{j\},v^t,x^t)-e(Q\setminus\{i\},v^t,x^t)\Big]\\ &= v_*^t(N)+0 \quad \big(\text{by equation (12) and } x\in E(N,b,V^m)\big)\\ &= v_*^t(N).\end{aligned}$$

Hence, $f^1(x)\in E(N,b,V^m)$ if $x\in E(N,b,V^m)$. The proof of result 2 is similar. Hence, we omit it. □

Theorem 4.3.

1. *Let $(N,b,V^m)\in\Gamma$. If $0<t<\frac{2}{|N|}$, then $\{[x]^q\}_{q=1}^{\infty}$ converges geometrically to $\overline{\beta}(N,b,V^m)$ for each $x\in E(N,b,V^m)$.*
2. *Let $(N,b,V^m)\in\Gamma^N$. If $0<t<\frac{2}{|N|}$, then $\{[z]^q\}_{q=1}^{\infty}$ converges geometrically to $\overline{\eta}(N,b,V^m)$ for each $x\in E(N,b,V^m)$.*

Proof. To prove result 1, let $(N,b,V^m)\in\Gamma$, $t\in\mathbb{N}_m$, $i\in N$ and $x\in E(N,b,V^m)$. By equation (11) and definition of f^1,

$$f_i^{1,t}(x)-x_i^t = w\sum_{j\in N\setminus\{i\}}\sum_{Q\subseteq N\setminus\{i,j\}}\Big[e(Q\setminus\{j\},v^t,x^t)-e(Q\setminus\{i\},v^t,x^t)\Big] = w\cdot\tfrac{|N|}{2}\cdot\Big(\overline{\beta_i^t}(N,b,V^m)-x_i^t\Big).$$

Hence,

$$\begin{aligned}\overline{\beta_i^t}(N,b,V^m)-f_i^{1,t}(x) &= \overline{\beta_i^t}(N,b,V^m)-x_i^t+x_i^t-f_i^{1,t}(x)\\ &= \overline{\beta_i^t}(N,b,V^m)-x_i^t-w\cdot\tfrac{|N|}{2}\cdot(\overline{\beta_i^t}(N,b,V^m)-x_i^t)\\ &= \Big(1-w\cdot\tfrac{|N|}{2}\Big)\Big[\overline{\beta_i^t}(N,b,V^m)-x_i^t\Big].\end{aligned}$$

So, for all $q\in\mathbb{N}$,

$$\overline{\beta}(N,b,V^m)-[x]^q=\Big(1-w\cdot\frac{|N|}{2}\Big)^q\Big[\overline{\beta}(N,b,V^m)-x\Big].$$

If $0<w<\frac{4}{|N|}$, then $-1<\Big(1-w\cdot\frac{|N|}{2}\Big)<1$ and $\{[x]^q\}_{q=1}^{\infty}$ converges geometrically to $\overline{\beta}(N,b,V^m)$. The proof of result 2 is similar. Hence, we omit it. □

Inspired by Maschler and Owen [16], we will find a dynamic process under reductions.

Definition 4.4. *Let ψ be a solution, $(N,b,V^m)\in\Gamma$, $S\subseteq N$ and $x\in E(N,b,V^m)$. The (x,ψ)-**reduced game** $(S,b_S,V^{r,m}_{\psi,S,x})$ is given by $V^{r,m}_{\psi,S,x}=(v^{r,t}_{\psi,S,x})_{t\in\mathbb{N}_m}$ and for all $T\subseteq S$,*

$$v^{r,t}_{\psi,S,x}(\alpha)=\begin{cases} v_*^t(N)-\sum_{i\in N\setminus S}x_i^t & A(\alpha)=S,\\ v^{2,t}_{S,\psi}(\alpha) & \text{otherwise.}\end{cases}$$

Inspired by Maschler and Owen [16], we also define different correction function as follow.

- Let $(N,b,V^m)\in\Gamma$. The **1-R-correction function** to be $g^1=(g^{1,t})_{t\in\mathbb{N}_m}$, where $g^{1,t}=(g_i^{1,t})_{i\in N}$ and $g_i^{1,t}:E(N,b,V^m)\to\mathbb{R}$ is define by

$$g_i^{1,t}(x)=x_i^t+w\sum_{k\in N\setminus\{i\}}\Big(\overline{\beta_i^t}(\{i,k\},v^t_{\overline{\beta},\{i,k\},x})-x_i^t\Big).$$

Define $[\xi]^0=x,\ [\xi]^1=g^1([\xi]^0),\cdots,\ [\xi]^q=g^1([\xi]^{q-1})$ for all $q\in\mathbb{N}$.

- Let $(N,b,V^m)\in\Gamma$. The **2-R-correction function** to be $g^2=(g^{2,t})_{t\in\mathbb{N}_m}$, where $g^{2,t}=(g_i^{2,t})_{i\in N}$ and $g_i^{2,t}:E(N,b,V^m)\to\mathbb{R}$ is define by

$$g_i^{2,t}(x)=x_i^t+w\sum_{k\in N\setminus\{i\}}\Big(\overline{\eta_i^t}\big(\{i,k\},v^t_{\overline{\eta},\{i,k\},x}\big)-x_i^t\Big).$$

Define $[\theta]^0=x,\ [\theta]^1=g^2([\theta]^0),\cdots,\ [\theta]^q=g^2([\theta]^{q-1})$ for all $q\in\mathbb{N}$.

Lemma 4.5.

1. *$g^1\big(x\big)\in E(N,b,V^m)$ for all $(N,b,V^m)\in\Gamma$ and for all $x\in E(N,b,V^m)$.*
2. *$g^2\big(x\big)\in E(N,b,V^m)$ for all $(N,b,V^m)\in\Gamma^N$ and for all $x\in E(N,b,V^m)$.*

Proof. To prove result 1, let $(N,b,V^m)\in\Gamma$, $t\in\mathbb{N}_m$, $i,k\in N$ and $x\in E(N,b,V^m)$. Let $S=\{i,k\}$, by MEFF of $\overline{\beta}$ and Definition 4.4,

$$\overline{\beta_i^t}(S,b_S,V^{r,m}_{\overline{\beta},S,x})+\overline{\beta_k^t}(S,b_S,V^{r,m}_{\overline{\beta},S,x})=x_i^t+x_k^t.$$

By 2CON and 2SG of $\overline{\beta}$,

$$\begin{aligned}\overline{\beta_i^t}(S,b_S,V^{r,m}_{\overline{\beta},S,x})-\overline{\beta_k^t}(S,b_S,V^{r,m}_{\overline{\beta},S,x})&=(v^{r,t}_{\overline{\beta},S,x})_*(\{i\})-(v^{r,t}_{\overline{\beta},S,x})_*(\{k\})=(v^{2,t}_{S,\overline{\beta}})_*(\{i\})-(v^{2,t}_{S,\overline{\beta}})_*(\{k\})\\&=\overline{\beta_i^t}(S,b_S,V^{2,m}_{S,\overline{\beta}})-\overline{\beta_k^t}(S,b_S,V^{2,m}_{S,\overline{\beta}})=\overline{\beta_i^t}(N,b,V^m)-\overline{\beta_k^t}(N,b,V^m).\end{aligned}$$

Therefore,

$$2\cdot\Big[\overline{\beta_i^t}(S,b_S,V^{r,m}_{\overline{\beta},S,x})-x_i^t\Big]=\overline{\beta_i^t}(N,b,V^m)-\overline{\beta_k^t}(N,b,V^m)-x_i^t+x_k^t.\tag{13}$$

By definition of g^1 and equation (13),

$$\begin{aligned}g_i^{1,t}(x)&=x_i^t+\tfrac{w}{2}\cdot\Big[\sum_{k\in N\setminus\{i\}}\overline{\beta_i^t}(N,b,V^m)-\sum_{k\in N\setminus\{i\}}x_i^t\\&\qquad-\sum_{k\in N\setminus\{i\}}\overline{\beta_k^t}(N,b,V^m)+\sum_{k\in N\setminus\{i\}}x_k^t\Big]\\&=x_i^t+\tfrac{w}{2}\cdot\Big[\sum_{k\in N\setminus\{i\}}\overline{\beta_i^t}(N,b,V^m)-\big(|N|-1\big)x_i^t\\&\qquad-\sum_{k\in N\setminus\{i\}}\overline{\beta_k^t}(N,b,v)+\big(v_*^t(N)-x_i^t\big)\Big]\\&=x_i^t+\tfrac{w}{2}\cdot\Big[\big(|N|-1\big)\overline{\beta_i^t}(N,b,V^m)-\big(|N|-1\big)x_i^t\\&\qquad-\big(v_*^t(N)-\overline{\beta_i^t}(N,b,V^m)\big)+\big(v_*^t(N)-x_i^t\big)\Big]\\&=x_i^t+\tfrac{|N|\cdot w}{2}\cdot\Big[\overline{\beta_i^t}(N,b,V^m)-x_i^t\Big].\end{aligned}\tag{14}$$

So we have that

$$\begin{aligned}\sum_{i\in N}g_i^{1,t}(x)&=\sum_{i\in N}\Big[x_i^t+\tfrac{|N|\cdot w}{2}\cdot\big[\overline{\beta_i^t}(N,b,V^m)-x_i^t\big]\Big]=\sum_{i\in N}x_i^t+\tfrac{|N|\cdot w}{2}\cdot\big[\sum_{i\in N}\overline{\beta_i^t}(N,b,V^m)-\sum_{i\in N}x_i^t\big]\\&=v_*^t(N)+\tfrac{|N|\cdot w}{2}\cdot\big[v_*^t(N)-v_*^t(N)\big]=v_*^t(N).\end{aligned}$$

Thus, $g^1\big(x\big)\in E(N,b,V^m)$ for all $x\in E(N,b,V^m)$. The proof of result 2 is similar. Hence, we omit it. □

Theorem 4.6.

1. *Let $(N,b,V^m)\in\Gamma$. If $0<\alpha<\frac{4}{|N|}$, then $\{[\xi]^q\}_{q=1}^\infty$ converges to $\overline{\beta}(N,b,V^m)$ for each $x\in E(N,b,V^m)$.*
2. *Let $(N,b,V^m)\in\Gamma^N$. If $0<\alpha<\frac{4}{|N|}$, then $\{[\theta]^q\}_{q=1}^\infty$ converges to $\overline{\eta}(N,b,V^m)$ for each $x\in E(N,b,V^m)$.*

Proof. To prove result 1, let $(N,b,V^m)\in\Gamma$, $t\in\mathbb{N}_m$ and $x\in E(N,b,V^m)$. By equation (14), $g_i^{1,t}(x)=x_i^t+\frac{|N|\cdot w}{2}\cdot\Big[\overline{\beta_i^t}(N,b,V^m)-x_i^t\Big]$ for all $i\in N$. Therefore,

$$\big(1-\frac{|N|\cdot w}{2}\big)\cdot\big[\overline{\beta_i^t}(N,b,V^m)-x_i^t\big]=\big[\overline{\beta_i^t}(N,b,V^m)-g_i^{1,t}(x)\big]$$

So, for all $q \in \mathbb{N}$,

$$\overline{\beta}(N,b,V^m) - [\eta]^q = \Big(1 - \frac{|N| \cdot w}{2}\Big)^q \Big[\overline{\beta}(N,b,V^m) - x\Big].$$

If $0 < w < \frac{4}{|N|}$, then $-1 < \Big(1 - \frac{|N| \cdot w}{2}\Big) < 1$ and $\{[\eta]^q\}_{q=1}^{\infty}$ converges to $\overline{\beta}(N,b,v)$ for all $(N,b,V^m) \in \Gamma$, for all $t \in \mathbb{N}_m$ and for all $i \in N$. The proof of result 2 is similar. Hence, we omit it. □

5 Conclusions

In different fields, from sciences to industry, engineering and the social sciences, managers face an increasing need to focus on multiple goals efficiently under operational processes. Related situations include analyzing distribution tradeoffs, selecting optimal decision or process designs, or any other condition where you need an efficient allocation rule with tradeoffs between two or more goals. In many cases these real world efficient situations could be formulated as multicriteria mathematical models. Allocation rules of such situations requires appropriate techniques to provide optimal results that - unlike traditional notions or methods - take several properties of the goals into account. On the other hand, each member is permitted to participate with infinite various activity levels (or decisions, strategies) in many situations. Thus, we would like to provide game-theoretical foundation of optimal allocation rules to analyze these conditions by considering fuzzy behavior and multicriteria situations simultaneously.

In this paper, we propose the fuzzy Banzhaf-Owen index, its efficient extension and normalization in the framework of multicriteria fuzzy TU games. Based on reduction, several axiomatic results for these indexes are proposed to present the rationality for these indexes. Based on different viewpoints, we also introduce alternative formulations and related dynamic processes for the fuzzy efficient Banzhaf-Owen index and the normalized Banzhaf-Owen index by applying the reductions and the excess functions respectively. One should compare our main results with related pre-existing results:

- The fuzzy Banzhaf-Owen index, the fuzzy efficient Banzhaf-Owen index and the fuzzy normalized Banzhaf-Owen index are introduced initially in the framework of multicriteria fuzzy TU games.
- The idea of our correction functions in Definitions 4.1, 4.4 and related dynamic processes are based on that of Maschler and Owen's [16] dynamic results for the Shapley value [21]. The major difference is that our correction functions in Definition 4.1 are based on the "excess function", and Maschler and Owen's [16] correction function is based on the "reduced games".

These mentioned above raise several motivations:

- Whether there exist other axiomatic results and dynamic processes for the indexes proposed in this paper.
- Whether there exist some more allocation rules and related results in the framework of multicriteria fuzzy TU games.

To our knowledge, these issues are still open questions.

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