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Product operation and joint interval valued observable

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Abstract

The aim of this paper is to define the product operation on a family of interval valued events and the notion of joint interval valued observable. We show the connection between product operations for interval valued events and intuitionistic fuzzy events, too. We display the relation between joint interval valued observable and joint intuitionistic fuzzy observable. We define a function of several interval valued observables with help of a Borel function and a joint interval valued observable, too.

Keywords: The interval valued event, the interval valued observable, the product operation, the joint interval valued observable, the joint intuitionistic fuzzy observable, the isomorphism, the function of several interval valued observables.

1 Introduction

In papers [5, 7] B. Riečan, P. Král and A. Michalíková studied a connection between the family of intuitionistic fuzzy events introduced by K.T. Atanassov in [1, 2]

$$\mathcal{F} = \{(\mu_A, \nu_A)\ ; \forall x \in \Omega \ \ \mu_A(x) + \nu_A(x) \leq 1 \text{ and } \mu_A, \nu_A : \Omega \to [0,1] \text{ are } \mathcal{S}-\text{measurable functions}\}$$

with the operations and relation

$$\begin{aligned} \mathbf{A} \leq \mathbf{B} &\iff \mu_A \leq \mu_B, \nu_A \geq \nu_B, \\ \mathbf{A} \oplus \mathbf{B} &= \big((\mu_A + \mu_B) \wedge 1_\Omega, (\nu_A + \nu_B - 1_\Omega) \vee 0_\Omega\big), \\ \mathbf{A} \odot \mathbf{B} &= \big((\mu_A + \mu_B - 1_\Omega) \vee 0_\Omega, (\nu_A + \nu_B) \wedge 1_\Omega)\big). \end{aligned}$$

and the family of interval valued events introduced by L.A. Zadeh in [9]

$$\mathcal{K} = \{(\pi_C, \rho_C)\ ; \forall x \in \Omega \ \ \pi_C(x) \leq \rho_C(x) \text{ and } \pi_C, \rho_C : \Omega \to [0,1] \text{ are } \mathcal{S}-\text{measurable functions}\}$$

with the operations and relation

$$\begin{aligned} \mathbf{C} \preceq \mathbf{D} &\Leftrightarrow \pi_C \leq \pi_D, \rho_C \leq \rho_D \\ \mathbf{C} \widehat{\oplus} \mathbf{D} &= \big((\pi_C + \pi_D) \wedge 1_\Omega, (\rho_C + \rho_D) \wedge 1_\Omega\big) \\ \mathbf{C} \widehat{\odot} \mathbf{D} &= \big((\pi_C + \pi_D - 1_\Omega) \vee 0_\Omega, (\rho_C + \rho_D - 1_\Omega) \vee 0_\Omega\big). \end{aligned}$$

They showed that these two systems are isomorphic by the mapping $\psi : \mathcal{F} \to \mathcal{K}$ given by $\psi\big((\mu_A, \nu_A)\big) = (\mu_A, 1_\Omega - \nu_A)$ for each $\mathbf{A} = (\mu_A, \nu_A) \in \mathcal{F}$. Therefore the following relations hold

$$\psi(\mathbf{A} \oplus \mathbf{B}) = \psi(\mathbf{A}) \widehat{\oplus} \psi(\mathbf{B}), \tag{1}$$

$$\psi(\mathbf{A} \odot \mathbf{B}) = \psi(\mathbf{A}) \widehat{\odot} \psi(\mathbf{B}), \tag{2}$$

$$\mathbf{A} \leq \mathbf{B} \Leftrightarrow \psi(\mathbf{A}) \preceq \psi(\mathbf{B}), \tag{3}$$

$$\mathbf{A}_n \nearrow \mathbf{A} \Leftrightarrow \psi(\mathbf{A}_n) \nearrow \psi(\mathbf{A}), \tag{4}$$

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for each $\mathbf{A}_n, \mathbf{A}, \mathbf{B} \in \mathcal{F}$. They illustrated the connection between intuitionistic fuzzy state $\mathbf{m} : \mathcal{F} \to [0,1]$ and interval valued state $k : \mathcal{K} \to [0,1]$ and that was $\mathbf{m} = k \circ \psi$. Remember, that the mapping ψ was already introduced in a paper [3] by K.T. Atanassov and G. Gargov in 1989.

In paper [4] we defined the notion of interval valued observable $z : \mathcal{B}(R) \to \mathcal{K}$ and we displayed the connection to the intuitionistic fuzzy observable $x : \mathcal{B}(R) \to \mathcal{F}$, which was $z = \psi \circ x$.

In this paper we define the product operation on a family of interval valued events and the notion of joint interval valued observable. We show the connection between product operations for interval valued events and intuitionistic fuzzy events, too. We display the relation between joint interval valued observable and joint intuitionistic fuzzy observable. We define a function of several interval valued observables with help of a Borel function, too.

Remark that in a whole text we use a notation IF as an abbreviation for intuitionistic fuzzy and a notation IV as an abbreviation for interval valued.

2 Interval valued events and interval valued observables

First we start with definitions of basic notions (see [5, 7]).

Definition 2.1. *Let Ω be a nonempty set. An interval valued set (IV-set) $\mathbf{C}$ on Ω is a pair (π_C, ρ_C) of mappings $\pi_C, \rho_C : \Omega \to [0,1]$ such that $\pi_C \leq \rho_C$.*

Definition 2.2. *Start with a measurable space $(\Omega, \mathcal{S})$. Hence $\mathcal{S}$ is a σ-algebra of subsets of Ω. An interval valued event (IV-event) is called an IV-set $\mathbf{C} = (\pi_C, \rho_C)$ such that $\pi_C, \rho_C : \Omega \to [0,1]$ are $\mathcal{S}$-measurable. The family of all IV-events on $(\Omega, \mathcal{S})$ will be denoted by $\mathcal{K}$.*

If $\mathbf{C} = (\pi_C, \rho_C) \in \mathcal{K}$, $\mathbf{D} = (\pi_D, \rho_D) \in \mathcal{K}$, then we define the Lukasiewicz binary operations $\widehat{\oplus}, \widehat{\odot}$ on $\mathcal{K}$ by

$$\begin{aligned} \mathbf{C} \widehat{\oplus} \mathbf{D} &= \big((\pi_C + \pi_D) \wedge 1_\Omega, (\rho_C + \rho_D) \wedge 1_\Omega \big) \\ \mathbf{C} \widehat{\odot} \mathbf{D} &= \big((\pi_C + \pi_D - 1_\Omega) \vee 0_\Omega, (\rho_C + \rho_D - 1_\Omega) \vee 0_\Omega \big) \end{aligned}$$

and the partial ordering is given by $\mathbf{C} \preceq \mathbf{D} \Leftrightarrow \pi_C \leq \pi_D, \rho_C \leq \rho_D$. The continuity is given by

$$\begin{aligned} \mathbf{C} \nearrow \mathbf{D} &\Leftrightarrow \pi_C \nearrow \pi_D, \rho_C \nearrow \rho_D, \\ \mathbf{C} \searrow \mathbf{D} &\Leftrightarrow \pi_C \searrow \pi_D, \rho_C \searrow \rho_D. \end{aligned}$$

The next basic notion in the probability theory is the notion of an observable. Let $\mathcal{J}$ be the family of all intervals in R of the form $[a,b) = \{x \in R : a \leq x < b\}$. Then the σ-algebra $\sigma(\mathcal{J})$ is denoted $\mathcal{B}(R)$ and it is called the σ-algebra of Borel sets, its elements are called Borel sets. Now we start with definition of basic notions (see [4]).

Definition 2.3. *By an interval valued observable (IV-observable) on $\mathcal{K}$ we understand each mapping $z : \mathcal{B}(R) \to \mathcal{K}$ satisfying the following conditions:*

(i) $z(R) = (1_\Omega, 1_\Omega)$, $z(\emptyset) = (0_\Omega, 0_\Omega)$;

(ii) if $A \cap B = \emptyset$, then $z(A) \widehat{\odot} z(B) = (0_\Omega, 0_\Omega)$ and $z(A \cup B) = z(A) \widehat{\oplus} z(B)$;

(iii) if $A_n \nearrow A$, then $z(A_n) \nearrow z(A)$.

Remark 2.4. *If we denote $z(A) = \big(z^\flat(A), z^\sharp(A)\big)$ for each $A \in \mathcal{B}(R)$, then $z^\flat, z^\sharp : \mathcal{B}(R) \to \mathcal{T}$ are observables, where $\mathcal{T} = \{f : \Omega \to [0,1]; f \text{ is } \mathcal{S} - \text{measurable}\}$.*

Remark 2.5. *Sometimes we need to work with n-dimensional IV-observable $z : \mathcal{B}(R^n) \to \mathcal{K}$ defined as a mapping with the following conditions:*

(i) $z(R^n) = (1_\Omega, 1_\Omega)$, $z(\emptyset) = (0_\Omega, 0_\Omega)$;

(ii) if $A \cap B = \emptyset$, $A, B \in \mathcal{B}(R^n)$, then $z(A) \widehat{\odot} z(B) = (0_\Omega, 0_\Omega)$ and $z(A \cup B) = z(A) \widehat{\oplus} z(B)$;

(iii) if $A_n \nearrow A$, then $z(A_n) \nearrow z(A)$ for each $A, A_n \in \mathcal{B}(R^n)$.

If $n = 1$, then we simply say that z is an IV-observable.

Between IV-observable and IF-observable is the connection. About this says the following proposition (see [4]).

Proposition 2.6. *Let $\psi : \mathcal{F} \to \mathcal{K}$, $\psi((u, v)) = (u, 1_\Omega - v)$. If $x : \mathcal{B}(R) \to \mathcal{F}$ is an IF-observable and $z = \psi \circ x : \mathcal{B}(R) \to \mathcal{K}$, then z is an IV-observable.*

Recall that by **intuitionistic fuzzy observable** (IF-observable) on $\mathcal{F}$ we understand each mapping $x : \mathcal{B}(R) \to \mathcal{F}$ satisfying the following conditions (see [8]):

(i) $x(R) = (1_\Omega, 0_\Omega)$, $x(\emptyset) = (0_\Omega, 1_\Omega)$;

(ii) if $A \cap B = \emptyset$, then $x(A) \odot x(B) = (0_\Omega, 1_\Omega)$ and $x(A \cup B) = x(A) \oplus x(B)$;

(iii) if $A_n \nearrow A$, then $x(A_n) \nearrow x(A)$.

If we denote $x(A) = \big(x^\flat(A), 1 - x^\sharp(A)\big)$ for each $A \in \mathcal{B}(R)$, then $x^\flat, x^\sharp : \mathcal{B}(R) \to \mathcal{T}$ are observables, where $\mathcal{T} = \{f : \Omega \to [0,1]; f \textit{ is } \mathcal{S} - \textit{measurable}\}$.

3 Product operation

In this section we define the product operation on a family of interval valued events and we show its connection to the product operation on the family of intuitionistic fuzzy events.

Definition 3.1. *We say that a binary operation $\widehat{\cdot}$ on $\mathcal{K}$ is product if it satisfying the following conditions:*

(i) $(1_\Omega, 1_\Omega)\widehat{\cdot}(\pi_C, \rho_C) = (\pi_C, \rho_C)$ *for each* $(\pi_C, \rho_C) \in \mathcal{K}$*;*

(ii) the operation $\widehat{\cdot}$ is commutative and associative;

(iii) if $(\pi_C, \rho_C)\widehat{\odot}(\pi_D, \rho_D) = (0_\Omega, 0_\Omega)$ *and* $(\pi_C, \rho_C), (\pi_D, \rho_D) \in \mathcal{K}$*, then*

$$(\pi_E, \rho_E)\widehat{\cdot}\big((\pi_C, \rho_C)\widehat{\oplus}(\pi_D, \rho_D)\big) = \big((\pi_E, \rho_E)\widehat{\cdot}(\pi_C, \rho_C)\big)\widehat{\oplus}\big((\pi_E, \rho_E)\widehat{\cdot}(\pi_D, \rho_D)\big)$$

and

$$\big((\pi_E, \rho_E)\widehat{\cdot}(\pi_C, \rho_C)\big)\widehat{\odot}\big((\pi_E, \rho_E)\widehat{\cdot}(\pi_D, \rho_D)\big) = (0_\Omega, 0_\Omega),$$

for each $(\pi_E, \rho_E) \in \mathcal{K}$*;*

(iv) if $(\pi_{C_n}, \rho_{C_n}) \searrow (0_\Omega, 0_\Omega)$, $(\pi_{D_n}, \rho_{D_n}) \searrow (0_\Omega, 0_\Omega)$ *and* $(\pi_{C_n}, \rho_{C_n}), (\pi_{D_n}, \rho_{D_n}) \in \mathcal{K}$*, then* $(\pi_{C_n}, \rho_{C_n})\widehat{\cdot}(\pi_{D_n}, \rho_{D_n}) \searrow (0_\Omega, 0_\Omega)$.

Now we show an example of product operation on the family of interval valued events $\mathcal{K}$.

Theorem 3.2. *The operation $\widehat{\cdot}$ defined by $(\pi_C, \rho_C)\widehat{\cdot}(\pi_D, \rho_D) = (\pi_C \cdot \pi_D, \rho_C \cdot \rho_D)$ for each $(\pi_C, \rho_C), (\pi_D, \rho_D) \in \mathcal{K}$ is product operation on $\mathcal{K}$.*

Proof. (i) Let (π_C, ρ_C) be an element of family the $\mathcal{K}$. Then $(1_\Omega, 1_\Omega)\widehat{\cdot}(\pi_C, \rho_C) = (1_\Omega \cdot \pi_C, 1_\Omega \cdot \rho_C) = (\pi_C, \rho_C)$.
(ii) The operation $\widehat{\cdot}$ is commutative and associative.
(iii) Let $(\pi_C, \rho_C)\widehat{\odot}(\pi_D, \rho_D) = (0_\Omega, 0_\Omega)$. Then $\pi_C + \pi_D \le 1_\Omega$, $\rho_C + \rho_D \le 1_\Omega$ and

$$\begin{aligned}(\pi_E, \rho_E)\widehat{\cdot}\big((\pi_C, \rho_C)\widehat{\oplus}(\pi_D, \rho_D)\big) &= (\pi_E, \rho_E)\widehat{\cdot}\big((\pi_C + \pi_D) \wedge 1_\Omega, (\rho_C + \rho_D) \wedge 1_\Omega\big)\\ &= (\pi_E, \rho_E)\widehat{\cdot}(\pi_C + \pi_D, \rho_C + \rho_D)\\ &= \big(\pi_E \cdot (\pi_C + \pi_D), \rho_E \cdot (\rho_C + \rho_D)\big)\end{aligned}$$

and

$$\begin{aligned}\big((\pi_E, \rho_E)\widehat{\cdot}(\pi_C, \rho_C)\big)\widehat{\oplus}\big((\pi_E, \rho_E)\widehat{\cdot}(\pi_D, \rho_D)\big) &= (\pi_E \cdot \pi_C, \rho_E \cdot \rho_C)\widehat{\oplus}(\pi_E \cdot \pi_D, \rho_E \cdot \rho_D)\\ &= \big((\pi_E \cdot \pi_C + \pi_E \cdot \pi_D) \wedge 1_\Omega, (\rho_E \cdot \rho_C + \rho_E \cdot \rho_D) \wedge 1_\Omega\big)\\ &= \big((\pi_E \cdot (\pi_C + \pi_D)) \wedge 1_\Omega, (\rho_E \cdot (\rho_C + \rho_D)) \wedge 1_\Omega\big)\\ &= \big(\pi_E \cdot (\pi_C + \pi_D), \rho_E \cdot (\rho_C + \rho_D)\big).\end{aligned}$$

Hence $(\pi_E,\rho_E)\widehat{\cdot}\big((\pi_C,\rho_C)\widehat{\oplus}(\pi_D,\rho_D)\big)=\big((\pi_E,\rho_E)\widehat{\cdot}(\pi_C,\rho_C)\big)\widehat{\oplus}\big((\pi_E,\rho_E)\widehat{\cdot}(\pi_D,\rho_D)\big)$. Moreover

$$\begin{aligned}\big((\pi_E,\rho_E)\widehat{\cdot}(\pi_C,\rho_C)\big)\widehat{\odot}\big((\pi_E,\rho_E)\widehat{\cdot}(\pi_D,\rho_D)\big) &= (\pi_E\cdot\pi_C,\rho_E\cdot\rho_C)\widehat{\odot}(\pi_E\cdot\pi_D,\rho_E\cdot\rho_D)\\ &= \big((\pi_E\cdot\pi_C+\pi_E\cdot\pi_D-1_\Omega)\vee 0_\Omega,(\rho_E\cdot\rho_C+\rho_E\cdot\rho_D-1_\Omega)\vee 0_\Omega\big)\\ &= (0_\Omega,0_\Omega).\end{aligned}$$

(iv) Let $(\pi_{C_n},\rho_{C_n})\searrow(0_\Omega,0_\Omega)$, $(\pi_{D_n},\rho_{D_n})\searrow(0_\Omega,0_\Omega)$. Since $\pi_{C_n}\searrow 0$, $\pi_{D_n}\searrow 0$ and $\rho_{C_n}\searrow 0$, $\rho_{D_n}\searrow 0$, then

$$(\pi_{C_n},\rho_{C_n})\widehat{\cdot}(\pi_{D_n},\rho_{D_n})=(\pi_{C_n}\cdot\pi_{D_n},\rho_{C_n}\cdot\rho_{D_n})\searrow(0_\Omega,0_\Omega).$$

□

Now we explain the connection between product operations on the family of interval valued events $\mathcal{K}$ and the family of intuitionistic fuzzy events $\mathcal{F}$.

Theorem 3.3. *If $\cdot$ is a product operation on the family of intuitionistic events $\mathcal{F}$ defined by*

$$(\mu_A,\nu_A)\cdot(\mu_B,\nu_B)=(\mu_A\cdot\mu_B,\nu_A+\nu_B-\nu_A\cdot\nu_B)=\big(\mu_A\cdot\mu_B,1_\Omega-(1_\Omega-\nu_A)\cdot(1_\Omega-\nu_B)\big)$$

for each $\mathbf{A}=(\mu_A,\nu_A),\mathbf{B}=(\mu_B,\nu_B)\in\mathcal{F}$ and $\widehat{\cdot}$ is a product operation on a family of interval valued events $\mathcal{K}$ defined by $(\pi_C,\rho_C)\widehat{\cdot}(\pi_D,\rho_D)=(\pi_C\cdot\pi_D,\rho_C\cdot\rho_D)$, for each $\mathbf{C}=(\pi_C,\rho_C),\mathbf{D}=(\pi_D,\rho_D)\in\mathcal{K}$ and $\psi:\mathcal{F}\to\mathcal{K}$ is a function given by $\psi\big((u,v)\big)=(u,1-v)$, then $\psi(\mathbf{A}\cdot\mathbf{B})=\psi(\mathbf{A})\widehat{\cdot}\psi(\mathbf{B})$, for each $\mathbf{A},\mathbf{B}\in\mathcal{F}$.

Proof. Let $\mathbf{A}=(\mu_A,\nu_A),\mathbf{B}=(\mu_B,\nu_B)\in\mathcal{F}$. Then

$$\psi(\mathbf{A}\cdot\mathbf{B})=\psi\big((\mu_A\cdot\mu_B,1_\Omega-(1_\Omega-\nu_A)\cdot(1_\Omega-\nu_B))\big)=\big(\mu_A\cdot\mu_B,(1_\Omega-\nu_A)\cdot(1_\Omega-\nu_B)\big)$$

and on the other hand

$$\psi(\mathbf{A})\widehat{\cdot}\psi(\mathbf{B})=(\mu_A,1_\Omega-\nu_A)\widehat{\cdot}(\mu_B,1_\Omega-\nu_B)=\big(\mu_A\cdot\mu_B,(1_\Omega-\nu_A)\cdot(1_\Omega-\nu_B)\big)$$

Therefore, $\psi(\mathbf{A}\cdot\mathbf{B})=\psi(\mathbf{A})\widehat{\cdot}\psi(\mathbf{B})$, for each $\mathbf{A},\mathbf{B}\in\mathcal{F}$. □

The next important notion is the notion of joint interval valued observable.

Definition 3.4. *Let $z_1,z_2:\mathcal{B}(R)\to\mathcal{K}$ be two IV-observables. The joint interval valued observable (joint IV-observable) of the IV-observables z_1,z_2 is a mapping $\widehat{h}:\mathcal{B}(R^2)\to\mathcal{K}$ satisfying the following conditions:*

(i) $\widehat{h}(R^2)=(1_\Omega,1_\Omega)$, $\widehat{h}(\emptyset)=(0_\Omega,0_\Omega)$;

(ii) *if $A,B\in\mathcal{B}(R^2)$ and $A\cap B=\emptyset$, then $\widehat{h}(A\cup B)=\widehat{h}(A)\widehat{\oplus}\widehat{h}(B)$ and $\widehat{h}(A)\widehat{\odot}\widehat{h}(B)=(0_\Omega,0_\Omega)$;*

(iii) *if $A,A_1,\ldots\in\mathcal{B}(R^2)$ and $A_n\nearrow A$, then $\widehat{h}(A_n)\nearrow\widehat{h}(A)$;*

(iv) *$\widehat{h}(C\times D)=z_1(C)\widehat{\cdot}z_2(D)$ for each $C,D\in\mathcal{B}(R)$.*

In the following proposition we show the connection between the joint interval valued observable and the intuitionistic fuzzy observable. Recall that by **joint intuitionistic fuzzy observable** (joint IF-observable) we understand each mapping $h:\mathcal{B}(R^2)\to\mathcal{F}$ satisfying the following conditions (see [6, 8]):

(i) $h(R^2)=(1_\Omega,0_\Omega)$, $h(\emptyset)=(0_\Omega,1_\Omega)$;

(ii) if $A,B\in\mathcal{B}(R^2)$ and $A\cap B=\emptyset$, then $h(A\cup B)=h(A)\oplus h(B)$ and $h(A)\odot h(B)=(0_\Omega,1_\Omega)$;

(iii) if $A,A_1,\ldots\in\mathcal{B}(R^2)$ and $A_n\nearrow A$, then $h(A_n)\nearrow h(A)$;

(iv) $h(C\times D)=x(C)\cdot y(D)$ for each $C,D\in\mathcal{B}(R)$.

Proposition 3.5. *Let $\psi:\mathcal{F}\to\mathcal{K}$, $\psi((u,v))=(u,1_\Omega-v)$. If $h:\mathcal{B}(R^2)\to\mathcal{F}$ be a joint IF-observable of IF-observables $x_1,x_2:\mathcal{B}(R)\to\mathcal{F}$ and $\widehat{h}=\psi\circ h:\mathcal{B}(R^2)\to\mathcal{K}$, then $\widehat{h}$ is the joint IV-observable of IV-observables $z_1,z_2:\mathcal{B}(R)\to\mathcal{K}$, where $z_1=\psi\circ x_1$, $z_2=\psi\circ x_2$.*

Proof. Let $\psi((u,v)) = (u, 1_\Omega - v)$ and $h : \mathcal{B}(R^2) \to \mathcal{F}$ be a joint IF-observable of IF-observables $x_1, x_2 : \mathcal{B}(R) \to \mathcal{F}$. Put $\widehat{h} = \psi \circ h$. Then

$$\begin{aligned}\widehat{h}(R^2) &= \psi\big(h(R^2)\big) = \psi\big((1_\Omega, 0_\Omega)\big) = (1_\Omega, 1_\Omega - 0_\Omega) = (1_\Omega, 1_\Omega),\\ \widehat{h}(\emptyset) &= \psi\big(h(\emptyset)\big) = \psi\big((0_\Omega, 1_\Omega)\big) = (0_\Omega, 1_\Omega - 1_\Omega) = (0_\Omega, 0_\Omega).\end{aligned}$$

Let $A, B \in R^2$. If $A \cap B = \emptyset$, then $h(A) \odot h(B) = (0_\Omega, 1_\Omega)$ and using (2) we have

$$\widehat{h}(A)\widehat{\odot}\widehat{h}(B) = \psi\big(h(A)\big)\widehat{\odot}\psi\big(h(B)\big) = \psi\big(h(A) \odot h(B)\big) = \psi\big((0_\Omega, 1_\Omega)\big) = (0_\Omega, 0_\Omega).$$

Moreover using (1) we obtain

$$\widehat{h}(A \cup B) = \psi\big(h(A \cup B)\big) = \psi\big(h(A) \oplus h(B)\big) = \psi\big(h(A)\big)\widehat{\oplus}\psi\big(h(B)\big) = \widehat{h}(A)\widehat{\oplus}\widehat{h}(B).$$

Let $A_n, A \in \mathcal{B}(R^2)$, $A_n \nearrow A$. Then $h(A_n) \nearrow h(A)$ and by (4)

$$\widehat{h}(A_n) = \psi(h(A_n)) \nearrow \psi(h(A)) = \widehat{h}(A).$$

Finally let $h(C \times D) = x_1(C) \cdot x_2(D)$ for each $C, D \in \mathcal{B}(R)$. Then by Theorem 3.3 we have

$$\widehat{h}(C \times D) = \psi\big(h(C \times D)\big) = \psi\big(x_1(C) \cdot x_2(D)\big) = \psi\big(x_1(C)\big)\widehat{\cdot}\psi\big(x_2(D)\big) = z_1(C)\widehat{\cdot}z_2(D),$$

where $z_1 = \psi \circ x_1$, $z_2 = \psi \circ x_2$ are the IV-observables by Proposition 2.6. □

Theorem 3.6. *To each two IV-observables $z_1, z_2 : \mathcal{B}(\mathbf{R}) \to \mathcal{K}$ there exists their joint IV-observable.*

Proof. Let $z_1, z_2 : \mathcal{B}(\mathbf{R}) \to \mathcal{K}$ be two IV-observables, $\psi((u,v)) = (u, 1_\Omega - v)$. Then using Proposition 2.6 there exists IF-observables $x_1, x_2 : \mathcal{B}(\mathbf{R}) \to \mathcal{F}$ given by $x_1 = \psi^{-1} \circ z_1$, $x_2 = \psi^{-1} \circ z_2$. Therefore, by Theorem 3.3 in [8] there exists the joint IF-observable $h : \mathcal{B}(R^2) \to \mathcal{F}$ of IF-observables x_1, x_2. Put $\widehat{h} = \psi \circ h$. Then $\widehat{h} : \mathcal{B}(R^2) \to \mathcal{K}$ is the joint IV-observable of IV-observables z_1, z_2 by Proposition 3.5. □

If we have several IV-observables and a Borel measurable function, we can define the IV-observable, which is the function of several IV-observables. About this says the following definition.

Definition 3.7. *Let $z_1, \dots, z_n : \mathcal{B}(R) \to \mathcal{K}$ be IV-observables, $\widehat{h}_n$ their joint IV-observable and $g_n : R^n \to R$ a Borel measurable function. Then we define the IV-observable $\widehat{y}_n = g_n(z_1, \dots, z_n) : \mathcal{B}(R) \to \mathcal{K}$ by the formula*

$$\widehat{y}_n = g_n(z_1, \dots, z_n)(A) = \widehat{h}_n\big(g_n^{-1}(A)\big).$$

for each $A \in \mathcal{B}(R)$.

Example 3.8. *Let $z_1, \dots, z_n : \mathcal{B}(R) \to \mathcal{K}$ be IV-observables and $\widehat{h}_n : \mathcal{B}(R^n) \to \mathcal{K}$ be their joint IV-observable. Then*

1. *the IV-observable $\widehat{y}_n = \frac{\sqrt{n}}{\sigma}\Big(\frac{1}{n}\sum\limits_{i=1}^{n} z_i - a\Big)$ is defined by the equality $\widehat{y}_n = \widehat{h}_n \circ g_n^{-1}$, where $g_n(u_1, \dots, u_n) = \frac{\sqrt{n}}{\sigma}\Big(\frac{1}{n}\sum\limits_{i=1}^{n} u_i - a\Big)$;*
2. *the IV-observable $\widehat{y}_n = \frac{1}{n}\sum\limits_{i=1}^{n} z_i$ is defined by the equality $\widehat{y}_n = \widehat{h}_n \circ g_n^{-1}$, where $g_n(u_1, \dots, u_n) = \frac{1}{n}\sum\limits_{i=1}^{n} u_i$;*
3. *the IV-observable $\widehat{y}_n = \frac{1}{n}\sum\limits_{i=1}^{n} (z_i - \widehat{E}(z_i))$ is defined by the equality $\widehat{y}_n = \widehat{h}_n \circ g_n^{-1}$, where $g_n(u_1, \dots, u_n) = \frac{1}{n}\sum\limits_{i=1}^{n} (u_i - \widehat{E}(z_i))$;*
4. *the IV-observable $\widehat{y}_n = \frac{1}{a_n}\big(\max(z_1, \dots, z_n) - b_n\big)$ is defined by the equality $\widehat{y}_n = \widehat{h}_n \circ g_n^{-1}$, where $g_n(u_1, \dots, u_n) = \frac{1}{a_n}\big(\max(u_1, \dots, u_n) - b_n\big)$.*

Between a function of several IV-observables $\widehat{y}_n = g_n(z_1, \ldots, z_n)$ and a function of several IF-observables $y_n = g_n(x_1, \ldots, x_n)$ exists a connection. Recall that by **a function of several intuitionistic fuzzy observables** we understand the IF-observable defined by

$$y_n = g_n(x_1, \ldots, x_n)(A) = h_n\big(g_n^{-1}(A)\big).$$

for each $A \in \mathcal{B}(R)$, where h_n is a joint IF-oservable of IF-observables $x_1, \ldots, x_n$.

Proposition 3.9. *Let $\psi : \mathcal{F} \to \mathcal{K}$, $\psi((u, v)) = (u, 1_\Omega - v)$. If $y_n = g_n(x_1, \ldots, x_n) : \mathcal{B}(R) \to \mathcal{F}$ is a function of several IF-observables $x_1, \ldots, x_n$ and $\widehat{y}_n = \psi \circ y_n : \mathcal{B}(R) \to \mathcal{K}$, then $\widehat{y}_n = g_n(z_1, \ldots, z_n)$ is a function of several IV-observables $z_1, \ldots, z_n$, where $z_i = \psi \circ x_i$, $i = 1, \ldots, n$.*

Proof. It follows from Proposition 3.5 and Definition 3.7. Really

$$\widehat{y}_n(A) = \psi\big(y_n(A)\big) = \psi\big(h_n(g_n^{-1}(A))\big) = \widehat{h}_n\big(g_n^{-1}(A)\big),$$

for each $A \in \mathcal{B}(R)$. □

4 Conclusion

In this paper we defined the product operation on family of interval valued events. We illustrated the connection between joint intuitionistic fuzzy observable $h : \mathcal{B}(R^2) \to \mathcal{F}$ and joint interval valued observable $\widehat{h} : \mathcal{B}(R^2) \to \mathcal{K}$ and that is $\widehat{h} = \psi \circ h$. The notion of joint interval valued observable is the starting point for definition of independence. We introduced the notion of a function of several interval valued observables, too.

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