Iranian Journal of Fuzzy Systems

Volume 17, Number 4, (2020), pp. 121-138

Alternative approaches to obtain t-norms and t-conorms on bounded lattices

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Abstract

Triangular norms in the study of probabilistic metric spaces as a special kind of associative functions defined on the unit interval. These functions have found applications in many areas since then. In this study, we present new methods for constructing triangular norms and triangular conorms on an arbitrary bounded lattice under some constraints. Also, we give some illustrative examples for the clarity. Finally, we show that our construction methods can be generalized by induction to a modified ordinal sum for triangular norms and triangular conorms on an arbitrary bounded lattice, respectively.

Keywords: Bounded lattice, triangular norm, triangular conorm, ordinal sum.

1 Introduction and motivation

1.1 A brief review on the development of triangular norms and triangular conorms

The triangular norms (t-norms for short) with 1 as neutral element and triangular conorms (t-conorms for short) with 0 as neutral element were introduced by Schweizer and Sklar in [16]. These operators are extensively used in many applications in fuzzy set theory, fuzzy logics, multicriteria decision support and several branches of information sciences.

The notion of ordinal sum of semigroups in Cliffords sense [4] was further developed by Mostert and Shields [13] and later used for introducing new t-norms and t-conorms on the unit interval $[0, 1]$, see [11]. Note that there is a minor difference in ordinal sum construction for triangular norms (based on min operator) with those for triangular conorms (based on max operator). Since Goguen' s [9] generalization of the classical fuzzy sets (with membership values from $[0, 1]$) to L-fuzzy sets (with membership values from a bounded lattice L), there is a growing interest in t-norms and t-conorms on bounded lattices, in particular in ordinal sum constructions. Saminger [15] focused on ordinal sums of t-norms acting on some particular bounded lattice which is not necessarily a chain or an ordinal sum of lattices. Also, she provided necessary and sufficient conditions for an ordinal sum operation yielding again a t-norm on some bounded lattice whereas the operation is determined by an arbitrary selection of subintervals as carriers for arbitrary summand t-norms. Medina [12] presented several necessary and sufficient conditions for ensuring whether an ordinal sum on a bounded lattice of arbitrary t-norm is a t-norm.

Ertuğrul, Karaçal, Mesiar [8] and Çaylı [5] showed a modification of ordinal sums of t-norms and t-conorms resulting in t-norms and t-conorms on particularly bounded lattice. Also, they presented a new method for constructing t-norms and t-conorms on special bounded lattices L by using the existence of t-norms on a sublattice $[a, 1]$ and t-conorms on a sublattice $[0, a]$, respectively, where $a \in L \setminus \{0, 1\}$.

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Received: March 2019; Revised: November 2019; Accepted: February 2020.

1.2 The motivation

Recently, the topic related to the construction of t-norms and t-conorms on bounded lattices by means of ordinal sums have been extensively studied.

In this paper, we present ordinal sum construction of t-norms and t-conorms on an arbitrary bounded lattice satisfying some constraints for a fixed element $a \in L \setminus \{0,1\}$, by using the existence of t-norms on the sublattice $[0,a]$ and of t-conorms on the sublattice $[a,1]$, respectively.

This paper is organized as follows. In Section 2, some basic notions are shortly presented. In Section 3, we present ordinal sum construction of t-norms and t-conorms on an arbitrary bounded lattice under some constraints, respectively. Also, we give some illustrative examples. Then, we provide some examples to illustrate that our new construction approaches presented in this paper are different from the approaches proposed by Ertuğrul, Karaçal, Mesiar [8] and Çaylı [5]. In Section 4, we present our modified ordinal sum constructions in its full generality. And we provide some illustrative examples. Finally, some concluding remarks are added.

2 Preliminaries

A bounded lattice $(L, \leq)$ is a lattice which has the top and bottom elements, which are written as 1 and 0, respectively, that is, there exist two elements $1, 0 \in L$ such that $0 \leq x \leq 1$, for all $x \in L$.

Definition 2.1. [3, 6] *Given a bounded lattice $(L, \leq, 0, 1)$ and $a, b \in L$, if a and b are incomparable, in this case, we use the notation $a \parallel b$. We denote the set of elements which are incomparable with a by I_a. So $I_a = \{x \in L \mid x \parallel a\}$.*

Definition 2.2. [3, 7] *Given a bounded lattice $(L, \leq, 0, 1)$ and $a, b \in L$, $a \leq b$, a subinterval $[a, b]$ of L is defined by $[a, b] = \{x \in L \mid a \leq x \leq b\}$. Similarly, $[a, b) = \{x \in L \mid a \leq x < b\}$, $(a, b] = \{x \in L \mid a < x \leq b\}$ and $(a, b) = \{x \in L \mid a < x < b\}$.*

Definition 2.3. [2, 15, 10] *Let $(L, \leq, 0, 1)$ be a bounded lattice. A triangular norm T (briefly t-norm) is a binary operation on L which is commutative, associative, increasing with respect to both variables and it satisfies $T(x, 1) = x$ for all $x \in L$.*

Definition 2.4. [1, 15] *Let $(L, \leq, 0, 1)$ be a bounded lattice. A triangular conorm S (briefly t-conorm) is a binary operation on L which is commutative, associative, increasing with respect to both variables and it satisfies $S(x, 0) = x$ for all $x \in L$.*

3 A new method for ordinal sum construction of t-norms and t-conorms on bounded lattices

In this section, we give methods to construct t-norms and t-conorms on an arbitrary bounded lattice L with some constraints. We have stated in Section 1, by means of ordinal sums, both Ertuğrul, Karaçal, Mesiar [8] and Çaylı [5] constructed t-norms and t-conorms on an arbitrary bounded lattice L, where $a \in L \setminus \{0,1\}$, V is a t-norm on $[a,1]$ and W is a t-conorm on $[0,a]$.

Next, we constructing ordinal sums of t-norms and t-conorms on an arbitrary bounded lattice L under some constraints in Theorem 3.8 and Theorem 3.15, respectively, where $a \in L \setminus \{0,1\}$, V is t-norm on $[0,a]$ and W is t-conorm on $[a,1]$, respectively. Also, we give some illustrative examples for clarity.

First we give some illustrative examples to discuss the literature. Then, we list the construction approach for t-norms and t-conorms presented by [8] and [5].

The following definition of an ordinal sum of t-norms defined on subintervals of a bounded lattice $(L, \leq, 0, 1)$ has been extracted from [15], which generalizes the methods given in [11] on subintervals of $[0,1]$.

Definition 3.1. [15] *Let $(L, \leq, 0, 1)$ be a bounded lattice and fix some subinterval $[a, b]$ of L. Let $T^{[a,b]}$ be a t-norm on $[a, b]$. Then $T : L^2 \to L$ defined by*

$$T(x,y) = \begin{cases} T^{[a,b]}(x,y) & \text{if } (x,y) \in [a,b]^2 \ , \\ x \wedge y & \text{otherwise.} \end{cases} \tag{1}$$

is an ordinal sum $(< a, b, T^{[a,b]} >)$ of $T^{[a,b]}$ on L.

Definition 3.2. [15] *Let $(L, \leq, 0, 1)$ be a bounded lattice and fix some subinterval $[a, b]$ of L. Let $S^{[a,b]}$ be a t-conorm on $[a, b]$. Then $S : L^2 \to L$ defined by*

$$S(x,y) = \begin{cases} S^{[a,b]}(x,y) & if \ (x,y) \in [a,b]^2 \ , \\ x \vee y & otherwise. \end{cases} \tag{2}$$

is an ordinal sum $(< a, b, S^{[a,b]} >)$ of $S^{[a,b]}$ on L.

However, the operation T (resp. S) given by Formula (1) (resp. Formula (2)) need not be a t-norm (resp. t-conorm), in general. Observe that condition ensuring that T (resp. S) given by (1) ((2)) is a t-norm (t-conorm) on L are given in Saminger's paper [15]. If L is a chain, then this T (S) is a t-norm (t-conorm) for any $[a, b] \subseteq L$.

Example 3.3. *Consider the lattice $(L_1 = \{0_{L_1}, d, a, b, c, 1_{L_1}\}, \leq, 0_{L_1}, 1_{L_1})$ given in Figure 1 and define the t-norm $V : [0_{L_1}, a]^2 \to [0_{L_1}, a]$ as follows.*

$$V(x,y) = \begin{cases} x \wedge y & if \ \ a \in \{x,y\} \ , \\ 0_{L_1} & otherwise. \end{cases}$$

Then, using Formula (1), the operation T on L_1 defined by Table 1 is not a t-norm.

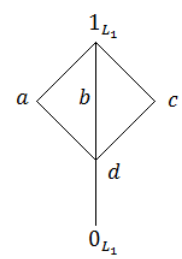

Figure 1: The lattice L_1

Table 1: The operation T on L_1

T	0_{L_1}	d	a	b	c	1_{L_1}
0_{L_1}	0_{L_1}	0_{L_1}	0_{L_1}	0_{L_1}	0_{L_1}	0_{L_1}
d	0_{L_1}	0_{L_1}	d	d	d	d
a	0_{L_1}	d	a	d	d	a
b	0_{L_1}	d	d	b	d	b
c	0_{L_1}	d	d	d	c	c
1_{L_1}	0_{L_1}	d	a	b	c	1_{L_1}

Indeed, the operation T does not satisfy associativity, because $T(a, T(c, d)) = T(a, d) = d > 0_{L_1} = T(d, d) = T(T(a, c), d)$. We obtain that T is not a t-norm on L_1.

Example 3.4. *Consider the lattice $(L_2 = \{0_{L_2}, a, k, t, s, 1_{L_2}\}, \leq, 0_{L_2}, 1_{L_2})$ given in Figure 2 and define the t-conorm $W : [a, 1_{L_2}]^2 \to [a, 1_{L_2}]$ as follows.*

$$W(x,y) = \begin{cases} x \vee y & if \ \ a \in \{x,y\} \ , \\ 1_{L_2} & otherwise. \end{cases}$$

Then, using Formula (2), the operation S on L_2 defined by Table 2 is not a t-conorm.

Indeed, the operation S does not satisfy associativity, because $S(a, S(t, s)) = S(a, s) = s < 1_{L_2} = S(s, s) = S(S(a, t), s)$. We obtain that S is not a t-conorm on L.

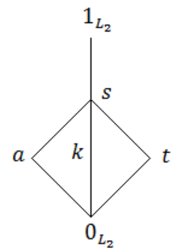

Figure 2: The lattice L_2

Table 2: The operation S on L_2

S	0_{L_2}	a	k	t	s	1_{L_2}
0_{L_2}	0_{L_2}	a	k	t	s	1_{L_2}
a	a	a	s	s	s	1_{L_2}
k	k	s	k	s	s	1_{L_2}
t	t	s	s	t	s	1_{L_2}
s	s	s	s	s	1_{L_2}	1_{L_2}
1_{L_2}	1_{L_2}	1_{L_2}	1_{L_2}	1_{L_2}	1_{L_2}	1_{L_2}

Theorem 3.5. [8] *Let $(L, \leq, 0, 1)$ be a bounded lattice and $a \in L \setminus \{0, 1\}$. If V is a t-norm on $[a, 1]$ and W is a t-conorm on $[0, a]$, then the functions $T^* : L^2 \to L$ and $S^* : L^2 \to L$ are, respectively, a t-norm and a t-conorm on L, where*

$$T^*(x,y) = \begin{cases} x \wedge y & \text{if } x = 1\ y = 1\ , \\ V(x,y) & \text{if } x, y \in [a, 1)\ , \\ x \wedge y \wedge a & \text{otherwise.} \end{cases}$$

$$S^*(x,y) = \begin{cases} x \vee y & \text{if } x = 0\ y = 0\ , \\ W(x,y) & \text{if } x, y \in (0, a]\ , \\ x \vee y \vee a & \text{otherwise.} \end{cases}$$

Remark 3.6. [8] *Observe that the t-norm T^* and t-conorm S^* considered in Theorem 3.5 can be described alternatively as*

$$T^*(x,y) = \begin{cases} V(x,y) & \text{if } (x,y) \in [a,1)^2\ , \\ y \wedge a & \text{if } x \in [a,1), y \parallel a\ , \\ x \wedge a & \text{if } y \in [a,1), x \parallel a\ , \\ x \wedge y \wedge a & \text{if } x \parallel a, y \parallel a\ , \\ x \wedge y & \text{otherwise.} \end{cases}$$

$$S^*(x,y) = \begin{cases} W(x,y) & \text{if } (x,y) \in (0,a]^2\ , \\ y \vee a & \text{if } x \in (0,a], y \parallel a\ , \\ x \vee a & \text{if } y \in (0,a], x \parallel a\ , \\ x \vee y \vee a & \text{if } x \parallel a, y \parallel a\ , \\ x \vee y & \text{otherwise.} \end{cases}$$

Theorem 3.7. [5] *Let $(L, \leq, 0, 1)$ be a bounded lattice and $a \in L \setminus \{0, 1\}$. If V is a t-norm on $[a, 1]$ and W is a t-conorm on $[0, a]$, then the functions $T^{**} : L^2 \to L$ and $S^{**} : L^2 \to L$ are, respectively, a t-norm and a t-conorm on L, where*

$$T^{**}(x,y) = \begin{cases} V(x,y) & \text{if } (x,y) \in [a,1)^2\ , \\ x \wedge y & 1 \in \{x,y\}, \\ 0 & \text{otherwise.} \end{cases}$$

$$S^{**}(x,y)=\begin{cases} W(x,y) & \textit{if } (x,y)\in(0,a]^2\ , \\ x\vee y & 0\in\{x,y\}, \\ 1 & \textit{otherwise.}\end{cases}$$

Now, we present ordinal sum construction of t-norms on an arbitrary bounded lattice L with some properties related to an element $a \in L \setminus \{0,1\}$.

Theorem 3.8. *Let $(L,\leq,0,1)$ be a bounded lattice, $a\in L\setminus\{0,1\}$ and*
i) If $x\in I_a$ and $y\in(0,a]$ then $x\parallel y$,
ii) If $x\in I_a$ and $y\in(a,1]$ then $x<y$.
Then the function $T: L^2\to L$ defined as follows is a t-norm on L, where V is a t-norm on $[0,a]$.

$$T(x,y)=\begin{cases} V(x,y) & \textit{if } (x,y)\in[0,a]^2\ , \\ y & \textit{if } (x,y)\in(a,1]\times I_a\ , \\ x & \textit{if } (x,y)\in I_a\times(a,1]\ , \\ 0 & \textit{if } (x,y)\in[0,a]\times I_a\cup I_a\times[0,a]\cup I_a\times I_a\ , \\ x\wedge y & \textit{otherwise.}\end{cases}$$

Proof. We have $T(x,y)=\begin{cases} x\wedge 1 & \text{if } x\notin I_a \\ x & \text{if } x\in I_a\end{cases}$. So, the fact that $1\in L$ is a neutral element of T. It is easy to see commutativity of T.

i) Monotonicity: We prove that if $x\leq y$, then $T(x,z)\leq T(y,z)$ for all $z\in L$. The proof can be split into all possible cases.

If $z=1$, then we have that $T(x,z)=T(x,1)=x\leq y=T(y,1)=T(y,z)$ for all $x,y\in L$.
If $x=0$, then we have that $T(0,z)=0\leq T(y,z)$ for all $y,z\in L$.
If $z=0$, then we have that $T(x,0)=0=T(y,0)$ for all $x,y\in L$.
If $y=0$, then it must be $x=0$. And we have $T(0,z)=0=T(0,z)$ for all $z\in L$.

1. $x\in(0,a]$

1.1 $y\in(0,a]$

1.1.1. $z\in(0,a]$
$$T(x,z)=V(x,z)\leq V(y,z)=T(y,z)$$

1.1.2. $z\in(a,1)$
$$T(x,z)=x\wedge z\leq y\wedge z=T(y,z)$$

1.1.3. $z\in I_a$
$$T(x,z)=0=T(y,z)$$

1.2. $y\in(a,1]$

1.2.1. $z\in(0,a]$
$$T(x,z)=V(x,z)\leq z=y\wedge z=T(y,z)$$

1.2.2. $z\in(a,1)$
$$T(x,z)=x\wedge z\leq y\wedge z=T(y,z)$$

1.2.3. $z\in I_a$
$$T(x,z)=0\leq z=T(y,z)$$

1.3. $y\in I_a$. Since $x\in(0,a]$ and $y\in I_a$, then it holds $x\parallel y$. So, it can not be the case $y\in I_a$.

2. $x\in(a,1)$. Then, it must be the case that $y\in(a,1]$.

2.1 $y\in(a,1]$

2.1.1. $z\in(0,a]$ or $z\in I_a$
$$T(x,z)=z=T(y,z)$$

2.1.2. $z\in(a,1)$
$$T(x,z)=x\wedge z\leq y\wedge z=T(y,z)$$

3. $x \in I_a$. Then, it must be the case that $y \in I_a$ or $y \in (a, 1]$.

3.1 $y \in I_a$

3.1.1. $z \in (0, a]$ or $z \in I_a$

$$T(x, z) = 0 = T(y, z)$$

3.1.2. $z \in (a, 1]$

$$T(x, z) = x \le y = T(y, z)$$

3.2. $y \in (a, 1]$

3.2.1. $z \in (0, a]$

$$T(x, z) = 0 \le z = T(y, z)$$

3.2.2. $z \in (a, 1]$. It can be obtained from the constraint of Theorem 3.8 that $x < z$ for $x \in I_a$ and $z \in (a, 1]$.

$$T(x, z) = x < y \wedge z = T(y, z)$$

3.2.3. $z \in I_a$

$$T(x, z) = 0 \le z = T(y, z)$$

4. $x = 1$. Then, since $y = 1$, $T(x, z) = T(y, z)$.

ii) Associativity: We need to prove that $T(x, T(y, z)) = T(T(x, y), z)$ for all $x, y, z \in L$. If at least one of x, y, z in L is 1, then it is obvious. So, the proof is split into all possible cases.

1. $x \in [0, a]$

1.1 $y \in [0, a]$

1.1.1. $z \in [0, a]$

$$T(x, T(y, z)) = T(x, V(y, z)) = V(x, V(y, z)) = V(V(x, y), z) = T(T(x, y), z)$$

1.1.2. $z \in (a, 1)$

$$T(x, T(y, z)) = T(x, y) = V(x, y) = T(V(x, y), z) = T(T(x, y), z)$$

1.1.3. $z \in I_a$

$$T(x, T(y, z)) = T(x, 0) = 0 = T(V(x, y), z) = T(T(x, y), z)$$

1.2. $y \in (a, 1)$

1.2.1. $z \in [0, a]$

$$T(x, T(y, z)) = T(x, z) = T(T(x, y), z)$$

1.2.2. $z \in (a, 1)$

If $y \wedge z > a$, then

$$T(x, T(y, z)) = T(x, y \wedge z) = x = T(x, z) = T(T(x, y), z)$$

If $y \wedge z = a$, then

$$T(x, T(y, z)) = T(x, y \wedge z) = T(x, a) = V(x, a) = x = T(x, z) = T(T(x, y), z)$$

1.2.3. $z \in I_a$

$$T(x, T(y, z)) = T(x, z) = T(T(x, y), z)$$

1.3. $y \in I_a$

1.3.1. $z \in [0, a]$ or $z \in I_a$

$$T(x, T(y, z)) = T(x, 0) = 0 = T(0, z) = T(T(x, y), z)$$

1.3.2. $z \in (a, 1)$

$$T(x, T(y, z)) = T(x, y) = 0 = T(0, z) = T(T(x, y), z)$$

2. $x \in (a, 1)$

2.1 $y \in [0, a]$

2.1.1. $z \in [0, a]$
$$T(x, T(y, z)) = T(x, V(y, z)) = V(y, z) = T(y, z) = T(T(x, y), z)$$

2.1.2. $z \in (a, 1)$
$$T(x, T(y, z)) = T(x, y) = y = T(y, z) = T(T(x, y), z)$$

2.1.3. $z \in I_a$
$$T(x, T(y, z)) = T(x, 0) = 0 = T(y, z) = T(T(x, y), z)$$

2.2. $y \in (a, 1)$

2.2.1. $z \in [0, a]$
If $x \wedge y > a$, then
$$T(x, T(y, z)) = T(x, z) = z = T(x \wedge y, z) = T(T(x, y), z)$$
If $x \wedge y = a$, then
$$T(x, T(y, z)) = T(x, z) = z = V(a, z) = T(a, z) = T(x \wedge y, z) = T(T(x, y), z)$$

2.2.2. $z \in (a, 1)$
Let $y \wedge z > a$. If $x \wedge y > a$, then
$$T(x, T(y, z)) = T(x, y \wedge z) = x \wedge y \wedge z = T(x \wedge y, z) = T(T(x, y), z)$$
If $x \wedge y = a$, then
$$T(x, T(y, z)) = T(x, y \wedge z) = x \wedge y \wedge z = a \wedge z = T(a, z) = T(x \wedge y, z) = T(T(x, y), z)$$
Let $y \wedge z = a$. If $x \wedge y > a$, then
$$T(x, T(y, z)) = T(x, y \wedge z) = T(x, a) = x \wedge a = x \wedge y \wedge z = T(x \wedge y, z) = T(T(x, y), z)$$
If $x \wedge y = a$, then
$$T(x, T(y, z)) = T(x, y \wedge z) = T(x, a) = x \wedge a = a = a \wedge z = T(a, z) = T(x \wedge y, z) = T(T(x, y), z)$$

2.2.3. $z \in I_a$
If $x \wedge y > a$, then
$$T(x, T(y, z)) = T(x, z) = z = T(x \wedge y, z) = T(T(x, y), z)$$
According to the our constraint, since $z \in I_a$ and $x \in (a, 1)$ and $y \in (a, 1)$, then it must be $z < y$ and $z < x$. So, we obtain $z \leq x \wedge y$. So, it can not be $x \wedge y = a$ since $z \in I_a$.

2.3. $y \in I_a$

2.3.1. $z \in [0, a]$ or $z \in I_a$
$$T(x, T(y, z)) = T(x, 0) = 0 = T(y, z) = T(T(x, y), z)$$

2.3.2. $z \in (a, 1)$
$$T(x, T(y, z)) = T(x, y) = y = T(y, z) = T(T(x, y), z)$$

3. $x \in I_a$

3.1 $y \in [0, a]$

3.1.1. $z \in [0, a]$
$$T(x, T(y, z)) = T(x, V(y, z)) = 0 = T(0, z) = T(T(x, y), z)$$

3.1.2. $z \in (a, 1)$
$$T(x, T(y, z)) = T(x, y) = 0 = T(0, z) = T(T(x, y), z)$$

3.1.3. $z \in I_a$
$$T(x, T(y, z)) = T(x, 0) = 0 = T(0, z) = T(T(x, y), z)$$

3.2. $y \in (a, 1)$

3.2.1. $z \in [0, a]$

$$T(x, T(y, z)) = T(x, z) = T(T(x, y), z)$$

3.2.2. $z \in (a, 1)$

If $y \wedge z > a$, then

$$T(x, T(y, z)) = T(x, y \wedge z) = x = T(x, z) = T(T(x, y), z)$$

Since $x \in I_a$ and $y \in (a, 1)$ and $z \in (a, 1)$, then it must be $x < y$ and $x < z$ from our constraint. Thus, we obtain $x \leq y \wedge z$. So, it can not be $y \wedge z = a$ since $x \in I_a$.

3.2.3. $z \in I_a$

$$T(x, T(y, z)) = T(x, z) = T(T(x, y), z)$$

3.3. $y \in I_a$

3.3.1. $z \in [0, a]$ or $z \in I_a$

$$T(x, T(y, z)) = T(x, 0) = 0 = T(0, z) = T(T(x, y), z)$$

3.3.2. $z \in (a, 1)$

$$T(x, T(y, z)) = T(x, y) = 0 = T(0, z) = T(T(x, y), z)$$

So, we have the fact that T is a t-norm on L. □

Corollary 3.9. *If we take $V = T_\wedge$ on $[0, a]$ given in Theorem 3.8, then we obtain the following t-norm on L.*

$$T(x, y) = \begin{cases} y & \text{if } (x, y) \in (a, 1] \times I_a \ , \\ x & \text{if } (x, y) \in I_a \times (a, 1] \ , \\ 0 & \text{if } (x, y) \in [0, a] \times I_a \cup I_a \times [0, a] \cup I_a \times I_a \ , \\ x \wedge y & \text{otherwise.} \end{cases}$$

In the following, we provide two lattices which satisfy and do not satisfy the constraints of Theorem 3.8, respectively.

Example 3.10. *The lattice $(L_3 = \{0_{L_3}, n, m, k, a, p, r, s, 1_{L_3}\}, \leq, 0_{L_3}, 1_{L_3})$ in Figure 3 satisfies the constraints of Theorem 3.8 (for element $a \in L_3$) That is, $x \parallel y$ for all $x \in I_a$, $y \in (0_{L_3}, a]$, and $x < y$ for all $x \in I_a$ and $y \in (a, 1_{L_3}]$. Consider the t-norm $V : [0_{L_3}, a]^2 \to [0_{L_3}, a]$ as follows:*

$$V(x, y) = \begin{cases} x \wedge y & \text{if } a \in \{x, y\} \ , \\ 0_{L_3} & \text{otherwise.} \end{cases}$$

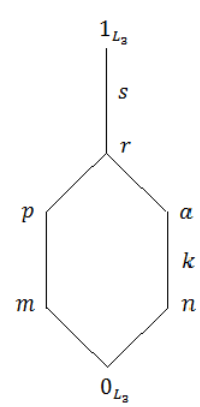

Figure 3: The lattice L_3

Then, by using Theorem 3.8, the function T on L_3 defined by Table 3 is a t-norm.

Table 3: The t-norm T on L_3

T	0_{L_3}	n	m	k	a	p	r	s	1_{L_3}
0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}
n	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	n	0_{L_3}	n	n	n
m	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	m	m	m
k	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	k	0_{L_3}	k	k	k
a	0_{L_3}	n	0_{L_3}	k	a	0_{L_3}	a	a	a
p	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	p	p	p
r	0_{L_3}	n	m	k	a	p	r	r	r
s	0_{L_3}	n	m	k	a	p	r	s	s
1_{L_3}	0_{L_3}	n	m	k	a	p	r	s	1_{L_3}

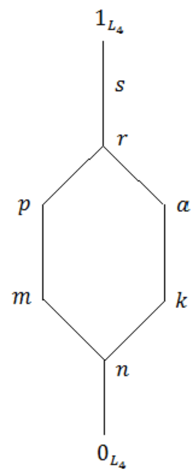

Figure 4: The lattice L_4

Table 4: The function T on L_4

T	0_{L_4}	n	m	k	a	p	r	s	1_{L_4}
0_{L_4}	0_{L_4}	0_{L_4}	0_{L_4}	0_{L_4}	0_{L_4}	0_{L_4}	0_{L_4}	0_{L_4}	0_{L_4}
n	0_{L_4}	n	0_{L_4}	n	n	0_{L_4}	n	n	n
m	0_{L_4}	0_{L_4}	0_{L_4}	0_{L_4}	0_{L_4}	0_{L_4}	m	m	m
k	0_{L_4}	n	0_{L_4}	k	k	0_{L_4}	k	k	k
a	0_{L_4}	n	0_{L_4}	k	a	0_{L_4}	a	a	a
p	0_{L_4}	0_{L_4}	0_{L_4}	0_{L_4}	0_{L_4}	0_{L_4}	p	p	p
r	0_{L_4}	n	m	k	a	p	r	r	r
s	0_{L_4}	n	m	k	a	p	r	s	s
1_{L_4}	0_{L_4}	n	m	k	a	p	r	s	1_{L_4}

Example 3.11. *The lattice* $(L_4 = \{0_{L_4}, n, m, k, a, p, r, s, 1_{L_4}\}, \leq, 0_{L_4}, 1_{L_4})$ *in Figure 4 does not satisfy (for* $a \in L_4$*) one of the constraints of Theorem 3.8. That is, there is* $n \in L_4$ *such that* $n < m$ *for* $m \in I_a$ *and* $n \in (0_{L_4}, a)$*. Consider the t-norm* $V : [0_{L_4}, a]^2 \to [0_{L_4}, a]$*,* $V(x, y) = x \wedge y$*.*

The function T *on* L_4 *is not a t-norm. Indeed, it does not satisfy monotonicity. Clearly,* $n < m$ *and* $T(n, k) = n \nleq 0_{L_4} = T(m, k)$*.*

Example 3.12. *The lattice* $(L_5 = \{0_{L_5}, n, m, k, a, p, r, s, 1_{L_5}\}, \leq, 0_{L_5}, 1_{L_5})$ *in Figure 5 does not satisfy one of the constraints of Theorem 3.8. Namely, there is the element* $r \in L_5$ *such that* $r \parallel p$ *for* $p \in I_a$ *and* $r \in (a, 1_{L_5})$*. Consider* *the t-norm* $V : [0_{L_5}, a]^2 \to [0_{L_5}, a]$ *as follows:*

$$V(x,y) = \begin{cases} x \wedge y & \text{if } a \in \{x,y\} , \\ 0_{L_5} & \text{otherwise.} \end{cases}$$

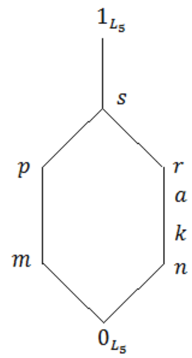

Figure 5: The lattice L_5

The function T *on* L_5 *is not a t-norm. It does not satisfy monotonicity. Clearly* $p < s$ *and* $T(r,p) = p \nleq r = T(r,s)$.

Table 5: The function T on L_5

T	0_{L_5}	n	m	k	a	p	r	s	1_{L_5}
0_{L_5}	0_{L_5}	0_{L_5}	0_{L_5}	0_{L_5}	0_{L_5}	0_{L_5}	0_{L_5}	0_{L_5}	0_{L_5}
n	0_{L_5}	0_{L_5}	0_{L_5}	0_{L_5}	n	0_{L_5}	n	n	n
m	0_{L_5}	0_{L_5}	0_{L_5}	0_{L_5}	0_{L_5}	0_{L_5}	m	m	m
k	0_{L_5}	0_{L_5}	0_{L_5}	0_{L_5}	k	0_{L_5}	k	k	k
a	0_{L_5}	n	0_{L_5}	k	a	0_{L_5}	a	a	a
p	0_{L_5}	0_{L_5}	0_{L_5}	0_{L_5}	0_{L_5}	0_{L_5}	p	p	p
r	0_{L_5}	n	m	k	a	p	r	r	r
s	0_{L_5}	n	m	k	a	p	r	s	s
1_{L_5}	0_{L_5}	n	m	k	a	p	r	s	1_{L_5}

Remark 3.13. *The t-norm defined in Theorem 3.8 is different from the t-norms defined in Theorem 3.5 and Theorem 3.7 proposal by [8] and [5], respectively. In general* $T \neq T^*$ *and* $T \neq T^{**}$. *We show these arguments by an example as follows.*

Example 3.14. *Consider the lattice* $(L_3 = \{0_{L_3}, n, m, k, a, p, r, s, 1_{L_3}\}, \leq, 0_{L_3}, 1_{L_3})$ *in Figure 3 and consider the t-norm* $V : [a, 1_{L_3}]^2 \to [a, 1_{L_3}]$ *as follows:*

$$V(x,y) = \begin{cases} x \wedge y & \text{if } 1_{L_3} \in \{x,y\} , \\ a & \text{otherwise.} \end{cases}$$

Using the construction approaches in Theorem 3.5 and Theorem 3.7, we define the t-norms T^* *and* T^{**} *by Table 6 and Table 7, respectively, and consider the t-norm* T *defined by Table 3. Then,*
- $T \neq T^*$ *since* $T(r,m) = m \neq 0_{L_3} = T^*(r,m)$.
- $T \neq T^{**}$ *since* $T(p,r) = p \neq 0_{L_3} = T^{**}(p,r)$.

Next, we present ordinal sum construction of t-conorms on arbitrary bounded lattice L with some properties related to an element $a \in L \setminus \{0,1\}$. We omit the proof of the next Theorem due to its similarity to the proof of Theorem 3.8.

Table 6: The t-norm T^* on L_3

T^*	0_{L_3}	n	m	k	a	p	r	s	1_{L_3}
0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}
n	0_{L_3}	n	0_{L_3}	n	n	0_{L_3}	n	n	n
m	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	m
k	0_{L_3}	n	0_{L_3}	k	k	0_{L_3}	k	k	k
a	0_{L_3}	n	0_{L_3}	k	a	0_{L_3}	a	a	a
p	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	p
r	0_{L_3}	n	0_{L_3}	k	a	0_{L_3}	a	a	r
s	0_{L_3}	n	0_{L_3}	k	a	0_{L_3}	a	a	s
1_{L_3}	0_{L_3}	n	m	k	a	p	r	s	1_{L_3}

Table 7: The t-norm T^{**} on L_3

T^{**}	0_{L_3}	n	m	k	a	p	r	s	1_{L_3}
0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}
n	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	n
m	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	m
k	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	k
a	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	a	0_{L_3}	a	a	a
p	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	p
r	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	a	0_{L_3}	a	a	r
s	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	a	0_{L_3}	a	a	s
1_{L_3}	0_{L_3}	n	m	k	a	p	r	s	1_{L_3}

Theorem 3.15. *Let $(L, \leq, 0, 1)$ be a bounded lattice, $a \in L \setminus \{0, 1\}$ and*
i) If $x \in I_a$ and $y \in [a, 1)$ then $x \parallel y$,
ii) If $x \in I_a$ and $y \in [0, a)$ then $y < x$.
Then the function $S : L^2 \to L$ defined as follows is a t-conorm on L, where W is a t-conorm on $[a, 1]$.

$$S(x,y) = \begin{cases} W(x,y) & \text{if } (x,y) \in [a,1]^2 \ , \\ y & \text{if } (x,y) \in [0,a) \times I_a \ , \\ x & \text{if } (x,y) \in I_a \times [0,a) \ , \\ 1 & \text{if } (x,y) \in [a,1] \times I_a \cup I_a \times [a,1] \cup I_a \times I_a \ , \\ x \vee y & \text{otherwise.} \end{cases}$$

Corollary 3.16. *If we take $W = S_\vee$ on $[a, 1]$ given in Theorem 3.15, then we obtain the following t-conorm on L.*

$$S(x,y) = \begin{cases} y & \text{if } (x,y) \in [0,a) \times I_a \ , \\ x & \text{if } (x,y) \in I_a \times [0,a) \ , \\ 1 & \text{if } (x,y) \in [a,1] \times I_a \cup I_a \times [a,1] \cup I_a \times I_a \ , \\ x \vee y & \text{otherwise.} \end{cases}$$

In the following, we provide two lattices which satisfy and do not satisfy the constraints of Theorem 3.15, respectively.

Example 3.17. *The lattice $(L_6 = \{0_{L_6}, b, c, d, a, e, t, q, 1_{L_6}\}, \leq, 0_{L_6}, 1_{L_6})$ in Figure 6 satisfies the constraints of Theorem 3.15. That is, $x \parallel y$ for all $x \in I_a$, $y \in [a, 1_{L_6})$, and $x > y$ for all $x \in I_a$ and $y \in [0_{L_6}, a)$. Consider the t-conorm $W : [a, 1_{L_6}]^2 \to [a, 1_{L_6}]$ as follows:*

$$W(x,y) = \begin{cases} x \vee y & \text{if } a \in \{x,y\} \ , \\ 1_{L_6} & \text{otherwise.} \end{cases}$$

Then, by using Theorem 3.15, the function S on L_6 defined by Table 8 is a t-conorm.

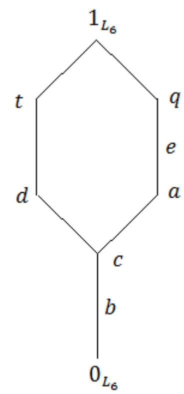

Figure 6: The lattice L_6

Table 8: The t-conorm S on L_6

S	0_{L_6}	b	c	d	a	e	t	q	1_{L_6}
0_{L_6}	0_{L_6}	b	c	d	a	e	t	q	1_{L_6}
b	b	b	c	d	a	e	t	q	1_{L_6}
c	c	c	c	d	a	e	t	q	1_{L_6}
d	d	d	d	1_{L_6}	1_{L_6}	1_{L_6}	1_{L_6}	1_{L_6}	1_{L_6}
a	a	a	a	1_{L_6}	a	e	1_{L_6}	q	1_{L_6}
e	e	e	e	1_{L_6}	e	1_{L_6}	1_{L_6}	1_{L_6}	1_{L_6}
t	t	t	t	1_{L_6}	1_{L_6}	1_{L_6}	1_{L_6}	1_{L_6}	1_{L_6}
q	q	q	q	1_{L_6}	q	1_{L_6}	1_{L_6}	1_{L_6}	1_{L_6}
1_{L_6}	1_{L_6}	1_{L_6}	1_{L_6}	1_{L_6}	1_{L_6}	1_{L_6}	1_{L_6}	1_{L_6}	1_{L_6}

Example 3.18. *The lattice* $(L_7 = \{0_{L_7}, b, c, d, a, e, t, q, 1_{L_7}\}, \leq, 0_{L_7}, 1_{L_7})$ *in Figure 7 does not satisfy one of the constraints of Theorem 3.15. Indeed, there is an element* $t \in L_7$ *such that* $t < q$ *for* $t \in I_a$ *and* $q \in (a, 1_{L_7})$*. Consider the t-conorm* $W : [a, 1_{L_7}]^2 \to [a, 1_{L_7}]$, $W(x, y) = x \vee y$.

The function S *on* L_7 *is not a t-conorm. It does not satisfy monotonicity. Clearly* $t < q$, $S(t, e) = 1_{L_7} \nleq q = S(q, e)$.

Table 9: The function S on L_7

S	0_{L_7}	b	c	d	a	e	t	q	1_{L_7}
0_{L_7}	0_{L_7}	b	c	d	a	e	t	q	1_{L_7}
b	b	b	c	d	a	e	t	q	1_{L_7}
c	c	c	c	d	a	e	t	q	1_{L_7}
d	d	d	d	1_{L_7}	1_{L_7}	1_{L_7}	1_{L_7}	1_{L_7}	1_{L_7}
a	a	a	a	1_{L_7}	a	e	1_{L_7}	q	1_{L_7}
e	e	e	e	1_{L_7}	e	e	1_{L_7}	q	1_{L_7}
t	t	t	t	1_{L_7}	1_{L_7}	1_{L_7}	1_{L_7}	1_{L_7}	1_{L_7}
q	q	q	q	1_{L_7}	q	q	1_{L_7}	q	1_{L_7}
1_{L_7}	1_{L_7}	1_{L_7}	1_{L_7}	1_{L_7}	1_{L_7}	1_{L_7}	1_{L_7}	1_{L_7}	1_{L_7}

Example 3.19. *The lattice* $(L_8 = \{0_{L_8}, b, c, d, a, e, t, q, 1_{L_8}\}, \leq, 0_{L_8}, 1_{L_8})$ *in Figure 8 does not satisfy one of the constraints of Theorem 3.15. Namely, there is an element* $c \in L_8$ *such that* $c \parallel d$ *for* $d \in I_a$ *and* $c \in (0_{L_8}, a)$*. Consider the t-conorm* $W : [a, 1_{L_8}]^2 \to [a, 1_{L_8}]$ *as follows:*

$$W(x, y) = \begin{cases} x \vee y & \textit{if } a \in \{x, y\} \ , \\ 1_{L_8} & \textit{otherwise.} \end{cases}$$

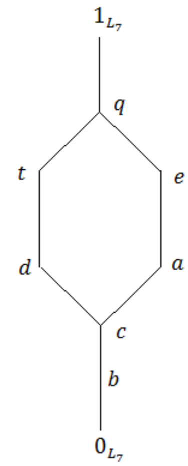

Figure 7: The lattice L_7

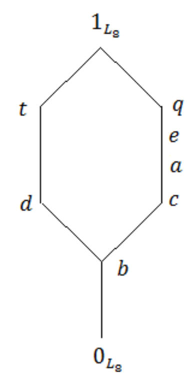

Figure 8: The lattice L_8

The function S on L_8 is not a t-conorm. It does not satisfy monotonicity. Clearly $b < d$, $S(b, c) = c \nleq d = S(d, c)$.

Table 10: The function S on L_8

S	0_{L_8}	b	c	d	a	e	t	q	1_{L_8}
0_{L_8}	0_{L_8}	b	c	d	a	e	t	q	1_{L_8}
b	b	b	c	d	a	e	t	q	1_{L_8}
c	c	c	c	d	a	e	t	q	1_{L_8}
d	d	d	d	1_{L_8}	1_{L_8}	1_{L_8}	1_{L_8}	1_{L_8}	1_{L_8}
a	a	a	a	1_{L_8}	a	e	1_{L_8}	q	1_{L_8}
e	e	e	e	1_{L_8}	e	1_{L_8}	1_{L_8}	1_{L_8}	1_{L_8}
t	t	t	t	1_{L_8}	1_{L_8}	1_{L_8}	1_{L_8}	1_{L_8}	1_{L_8}
q	q	q	q	1_{L_8}	q	1_{L_8}	1_{L_8}	1_{L_8}	1_{L_8}
1_{L_8}	1_{L_8}	1_{L_8}	1_{L_8}	1_{L_8}	1_{L_8}	1_{L_8}	1_{L_8}	1_{L_8}	1_{L_8}

Remark 3.20. *The t-conorm defined in Theorem 3.15 is different from the t-conorms defined in Theorem 3.5 and*

Theorem 3.7 proposal by [8] and [5], respectively. In general $S \neq S^*$ and $S \neq S^{**}$. We show these arguments by an example as follows.

Example 3.21. *Consider the lattice $(L_6 = \{0_{L_6}, b, c, d, a, e, t, q, 1_{L_6}\}, \leq, 0_{L_6}, 1_{L_6})$ in Figure 6 and the t-conorm $W : [0_{L_6}, a]^2 \to [0_{L_6}, a]$ as follows:*

$$W(x,y) = \begin{cases} x \vee y & \text{if } 0_{L_6} \in \{x,y\} \ , \\ a & \text{otherwise.} \end{cases}$$

Using the construction approaches in Theorem 3.5 and Theorem 3.7, we define the t-conorms S^* and S^{**} by Table 11 and Table 12, respectively and consider the t-conorm S defined by Table 8. Then,

- *$S \neq S^*$ since $S(c,d) = d \neq 1_{L_6} = S^*(c,d)$.*
- *$S \neq S^{**}$ since $S(e,a) = e \neq 1_{L_6} = S^{**}(e,a)$.*

Table 11: The t-conorm S^* on L_6

S^*	0_{L_6}	b	c	d	a	e	t	q	1_{L_6}
0_{L_6}	0_{L_6}	b	c	d	a	e	t	q	1_{L_6}
b	b	a	a	1_{L_6}	a	e	1_{L_6}	q	1_{L_6}
c	c	a	a	1_{L_6}	a	e	1_{L_6}	q	1_{L_6}
d	d	1_{L_6}	1_{L_6}	1_{L_6}	1_{L_6}	1_{L_6}	1_{L_6}	1_{L_6}	1_{L_6}
a	a	a	a	1_{L_6}	a	e	1_{L_6}	q	1_{L_6}
e	e	e	e	1_{L_6}	e	e	1_{L_6}	q	1_{L_6}
t	t	1_{L_6}	1_{L_6}	1_{L_6}	1_{L_6}	1_{L_6}	1_{L_6}	1_{L_6}	1_{L_6}
q	q	q	q	1_{L_6}	q	q	1_{L_6}	q	1_{L_6}
1_{L_6}	1_{L_6}	1_{L_6}	1_{L_6}	1_{L_6}	1_{L_6}	1_{L_6}	1_{L_6}	1_{L_6}	1_{L_6}

Table 12: The t-conorm S^{**} on L_6

S^{**}	0_{L_6}	b	c	d	a	e	t	q	1_{L_6}
0_{L_6}	0_{L_6}	b	c	d	a	e	t	q	1_{L_6}
b	b	a	a	1_{L_6}	a	1_{L_6}	1_{L_6}	1_{L_6}	1_{L_6}
c	c	a	a	1_{L_6}	a	1_{L_6}	1_{L_6}	1_{L_6}	1_{L_6}
d	d	1_{L_6}	1_{L_6}	1_{L_6}	1_{L_6}	1_{L_6}	1_{L_6}	1_{L_6}	1_{L_6}
a	a	a	a	1_{L_6}	a	1_{L_6}	1_{L_6}	1_{L_6}	1_{L_6}
e	e	1_{L_6}	1_{L_6}	1_{L_6}	1_{L_6}	1_{L_6}	1_{L_6}	1_{L_6}	1_{L_6}
t	t	1_{L_6}	1_{L_6}	1_{L_6}	1_{L_6}	1_{L_6}	1_{L_6}	1_{L_6}	1_{L_6}
q	q	1_{L_6}	1_{L_6}	1_{L_6}	1_{L_6}	1_{L_6}	1_{L_6}	1_{L_6}	1_{L_6}
1_{L_6}	1_{L_6}	1_{L_6}	1_{L_6}	1_{L_6}	1_{L_6}	1_{L_6}	1_{L_6}	1_{L_6}	1_{L_6}

4 Modified ordinal sum construction of t-norms and t-conorms on bounded lattices

In [5, 8], we know that new t-norms and t-conorms on bounded lattices can be obtained by means of recursion. In this section, based on the approach of constructing t-norms and t-conorms proposed in Section 3, we introduce a new ordinal sum construction of t-norms and t-conorms on an arbitrary bounded lattice L by means of recursion.

Theorem 4.1. *Let $(L, \leq, 0, 1)$ be a bounded lattice and $\{a_0, a_1, a_2, \cdots, a_n\}$ be a finite chain in L such that $0 = a_0 < a_1 < a_2 < \cdots < a_n = 1$. Let $x \parallel y$ for all $x \in I_{a_i}$ and $y \in (0, a_i]$, and $x < y$ for all $x \in I_{a_i}$ and $y \in (a_i, 1]$, and $V : [0, a_1]^2 \to [0, a_1]$ be a t-norm. Then, the function $T_n : L^2 \to L$ defined as follows is a t-norm, where $T_1 = V$ and for*

$i \in \{2, \cdots, n\}$, *the function* $T_i : [0, a_i]^2 \to [0, a_i]$ *is given by*

$$T_i(x,y) = \begin{cases} T_{i-1}(x,y) & \text{if } (x,y) \in [0,a_{i-1}]^2 \ , \\ y & \text{if } (x,y) \in (a_{i-1}, a_i] \times I_{a_{i-1}} \ , \\ x & \text{if } (x,y) \in I_{a_{i-1}} \times (a_{i-1}, a_i] \ , \\ 0 & \text{if } (x,y) \in [0,a_{i-1}] \times I_{a_{i-1}} \cup I_{a_{i-1}} \times [0,a_{i-1}] \cup I_{a_{i-1}} \times I_{a_{i-1}} \ , \\ x \wedge y & \text{otherwise.} \end{cases} \tag{3}$$

Proof. The proof follows easily from Theorem 3.8 by induction and therefore it is omitted. □

It should be pointed out that if L is a chain then the Formula (3), can be reformulated into

$$T_i(x,y) = \begin{cases} T_{i-1}(x,y) & \text{if } (x,y) \in [0,a_{i-1}]^2 \ , \\ x \wedge y & \text{otherwise.} \end{cases}$$

Example 4.2. *Consider the lattice* $(L_9 = \{0_{L_9}, a_1, a_2, k, t, a_3, a_4, 1_{L_9}\}, \leq, 0_{L_9}, 1_{L_9})$ *described in Figure 9 with the finite chain* $0_{L_9} < a_1 < a_2 < a_3 < a_4 < 1_{L_9}$ *in* L_9 *and define the t-norm* $V : [0_{L_9}, a_1]^2 \to [0_{L_9}, a_1]$ *by* $V = T_\wedge$*. By using Theorem 4.1, where* $V = T_1$*, t-norms* $T_2 : [0_{L_9}, a_2]^2 \to [0_{L_9}, a_2]$*,* $T_3 : [0_{L_9}, a_3]^2 \to [0_{L_9}, a_3]$*,* $T_4 : [0_{L_9}, a_4]^2 \to [0_{L_9}, a_4]$*,* $T_5 : {L_9}^2 \to L_9$ *are defined in Tables 13-16.*

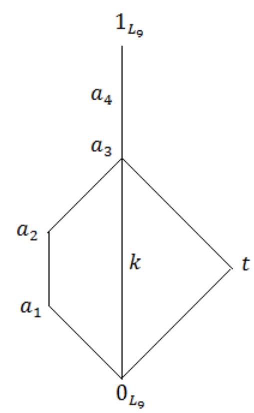

Figure 9: The lattice L_9

Table 13: The t-norm T_2 on L_9

T_2	0_{L_9}	a_1	a_2
0_{L_9}	0_{L_9}	0_{L_9}	0_{L_9}
a_1	0_{L_9}	a_1	a_1
a_2	0_{L_9}	a_1	a_2

Theorem 4.3. *Let* $(L, \leq, 0, 1)$ *be a bounded lattice and* $\{a_0, a_1, a_2, ..., a_n\}$ *be a finite chain in* L *such that* $1 = a_0 > a_1 > a_2 > ... > a_n = 0$*. Let* $x \parallel y$ *for all* $x \in I_{a_i}$ *and* $y \in [a_i, 1)$*, and* $x > y$ *for all* $x \in I_{a_i}$ *and* $y \in [0, a_i)$*, and* $W : [a_1, 1]^2 \to [a_1, 1]$ *be a t-conorm. Then, the function* $S_n : L^2 \to L$ *defined recursively as follows is a t-conorm, where* $S_1 = W$ *and for* $i \in \{2, ..., n\}$*, the function* $S_i : [a_i, 1]^2 \to [a_i, 1]$ *is given by*

$$S_i(x,y) = \begin{cases} S_{i-1}(x,y) & \text{if } (x,y) \in [a_{i-1},1]^2 \ , \\ y & \text{if } (x,y) \in [a_i, a_{i-1}) \times I_{a_{i-1}} \ , \\ x & \text{if } (x,y) \in I_{a_{i-1}} \times [a_i, a_{i-1}) \ , \\ 1 & \text{if } (x,y) \in [a_{i-1},1] \times I_{a_{i-1}} \cup I_{a_{i-1}} \times [a_{i-1},1] \cup I_{a_{i-1}} \times I_{a_{i-1}} \ , \\ x \vee y & \text{otherwise.} \end{cases} \tag{4}$$

Table 14: The t-norm T_3 on L_9

T_3	0_{L_9}	a_1	a_2	k	t	a_3
0_{L_9}	0_{L_9}	0_{L_9}	0_{L_9}	0_{L_9}	0_{L_9}	0_{L_9}
a_1	0_{L_9}	a_1	a_1	0_{L_9}	0_{L_9}	a_1
a_2	0_{L_9}	a_1	a_2	0_{L_9}	0_{L_9}	a_2
k	0_{L_9}	0_{L_9}	0_{L_9}	0_{L_9}	0_{L_9}	k
t	0_{L_9}	0_{L_9}	0_{L_9}	0_{L_9}	0_{L_9}	t
a_3	0_{L_9}	a_1	a_2	k	t	a_3

Table 15: The t-norm T_4 on L_9

T_4	0_{L_9}	a_1	a_2	k	t	a_3	a_4
0_{L_9}	0_{L_9}	0_{L_9}	0_{L_9}	0_{L_9}	0_{L_9}	0_{L_9}	0_{L_9}
a_1	0_{L_9}	a_1	a_1	0_{L_9}	0_{L_9}	a_1	a_1
a_2	0_{L_9}	a_1	a_2	0_{L_9}	0_{L_9}	a_2	a_2
k	0_{L_9}	0_{L_9}	0_{L_9}	0_{L_9}	0_{L_9}	k	k
t	0_{L_9}	0_{L_9}	0_{L_9}	0_{L_9}	0_{L_9}	t	t
a_3	0_{L_9}	a_1	a_2	k	t	a_3	a_3
a_4	0_{L_9}	a_1	a_2	k	t	a_3	a_4

Table 16: The t-norm T_5 on L_9

T_5	0_{L_9}	a_1	a_2	k	t	a_3	a_4	1_{L_9}
0_{L_9}	0_{L_9}	0_{L_9}	0_{L_9}	0_{L_9}	0_{L_9}	0_{L_9}	0_{L_9}	0_{L_9}
a_1	0_{L_9}	a_1	a_1	0_{L_9}	0_{L_9}	a_1	a_1	a_1
a_2	0_{L_9}	a_1	a_2	0_{L_9}	0_{L_9}	a_2	a_2	a_2
k	0_{L_9}	0_{L_9}	0_{L_9}	0_{L_9}	0_{L_9}	k	k	k
t	0_{L_9}	0_{L_9}	0_{L_9}	0_{L_9}	0_{L_9}	t	t	t
a_3	0_{L_9}	a_1	a_2	k	t	a_3	a_3	a_3
a_4	0_{L_9}	a_1	a_2	k	t	a_3	a_4	a_4
1_{L_9}	0_{L_9}	a_1	a_2	k	t	a_3	a_4	1_{L_9}

It should be pointed out that if L is a chain then the Formula (4), can be reformulated into

$$S_i(x,y) = \begin{cases} S_{i-1}(x,y) & \text{if } (x,y) \in [a_{i-1},1]^2 \ , \\ x \vee y & \text{otherwise.} \end{cases}$$

Example 4.4. *Consider the lattice* $(L_{10} = \{0_{L_{10}}, a_4, a_3, a_2, a_1, b, c, 1_{L_{10}}\}, \leq, 0_{L_{10}}, 1_{L_{10}})$ *described in Figure 10 with the finite chain* $1_{L_{10}} > a_1 > a_2 > a_3 > a_4 > 0_{L_{10}}$ *in* L_{10} *and define the t-conorm* $W : [a_1, 1_{L_{10}}]^2 \to [a_1, 1_{L_{10}}]$ *by* $W = S_\vee$. *By using Theorem 4.3, where* $W = S_1$, *t-conorms* $S_2 : [a_2, 1_{L_{10}}]^2 \to [a_2, 1_{L_{10}}]$, $S_3 : [a_3, 1_{L_{10}}]^2 \to [a_3, 1_{L_{10}}]$, $S_4 : [a_4, 1_{L_{10}}]^2 \to [a_4, 1_{L_{10}}]$, $S_5 : L_{10}^2 \to L_{10}$ *are defined in Tables 17-20.*

Table 17: The t-conorm S_2 on L_{10}

S_2	a_2	a_1	$1_{L_{10}}$
a_2	a_2	a_1	$1_{L_{10}}$
a_1	a_1	a_1	$1_{L_{10}}$
$1_{L_{10}}$	$1_{L_{10}}$	$1_{L_{10}}$	$1_{L_{10}}$

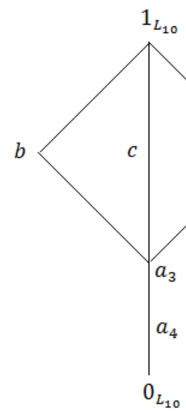

Figure 10: The lattice L_{10}

Table 18: The t-conorm S_3 on L_{10}

S_3	a_3	a_2	a_1	b	c	$1_{L_{10}}$
a_3	a_3	a_2	a_1	b	c	$1_{L_{10}}$
a_2	a_2	a_2	a_1	$1_{L_{10}}$	$1_{L_{10}}$	$1_{L_{10}}$
a_1	a_1	a_1	a_1	$1_{L_{10}}$	$1_{L_{10}}$	$1_{L_{10}}$
b	b	$1_{L_{10}}$	$1_{L_{10}}$	$1_{L_{10}}$	$1_{L_{10}}$	$1_{L_{10}}$
c	c	$1_{L_{10}}$	$1_{L_{10}}$	$1_{L_{10}}$	$1_{L_{10}}$	$1_{L_{10}}$
$1_{L_{10}}$	$1_{L_{10}}$	$1_{L_{10}}$	$1_{L_{10}}$	$1_{L_{10}}$	$1_{L_{10}}$	$1_{L_{10}}$

Table 19: The t-conorm S_4 on L_{10}

S_4	a_4	a_3	a_2	a_1	b	c	$1_{L_{10}}$
a_4	a_4	a_3	a_2	a_1	b	c	$1_{L_{10}}$
a_3	a_3	a_3	a_2	a_1	b	c	$1_{L_{10}}$
a_2	a_2	a_2	a_2	a_1	$1_{L_{10}}$	$1_{L_{10}}$	$1_{L_{10}}$
a_1	a_1	a_1	a_1	a_1	$1_{L_{10}}$	$1_{L_{10}}$	$1_{L_{10}}$
b	b	b	$1_{L_{10}}$	$1_{L_{10}}$	$1_{L_{10}}$	$1_{L_{10}}$	$1_{L_{10}}$
c	c	c	$1_{L_{10}}$	$1_{L_{10}}$	$1_{L_{10}}$	$1_{L_{10}}$	$1_{L_{10}}$
$1_{L_{10}}$	$1_{L_{10}}$	$1_{L_{10}}$	$1_{L_{10}}$	$1_{L_{10}}$	$1_{L_{10}}$	$1_{L_{10}}$	$1_{L_{10}}$

Table 20: The t-conorm S_5 on L_{10}

S_5	$0_{L_{10}}$	a_4	a_3	a_2	a_1	b	c	$1_{L_{10}}$
$0_{L_{10}}$	$0_{L_{10}}$	a_4	a_3	a_2	a_1	b	c	$1_{L_{10}}$
a_4	a_4	a_4	a_3	a_2	a_1	b	c	$1_{L_{10}}$
a_3	a_3	a_3	a_3	a_2	a_1	b	c	$1_{L_{10}}$
a_2	a_2	a_2	a_2	a_2	a_1	$1_{L_{10}}$	$1_{L_{10}}$	$1_{L_{10}}$
a_1	a_1	a_1	a_1	a_1	a_1	$1_{L_{10}}$	$1_{L_{10}}$	$1_{L_{10}}$
b	b	b	b	$1_{L_{10}}$	$1_{L_{10}}$	$1_{L_{10}}$	$1_{L_{10}}$	$1_{L_{10}}$
c	c	c	c	$1_{L_{10}}$	$1_{L_{10}}$	$1_{L_{10}}$	$1_{L_{10}}$	$1_{L_{10}}$
$1_{L_{10}}$	$1_{L_{10}}$	$1_{L_{10}}$	$1_{L_{10}}$	$1_{L_{10}}$	$1_{L_{10}}$	$1_{L_{10}}$	$1_{L_{10}}$	$1_{L_{10}}$

5 Concluding remarks

We have investigated and introduced construction method for building t-norms and t-conorms on an arbitrary bounded lattice with some constraints. First, we have presented t-norm T and t-conorm S on any bounded lattices under some constraints, in Theorem 3.8 and Theorem 3.15, respectively. In order to better understand the introduced t-norm T and t-conorm S, we have given some illustrative examples. Second, we have shown that our construction methods for t-norms T and t-conorms S can be generalized by induction to a modified ordinal sum for t-norms and t-conorms on relevant bounded lattice in Theorem 4.1 and Theorem 4.3, respectively. Again, we have given some illustrative examples.

Acknowledgement

This work was done when the first author was visiting researcher at the Department of Mathematics and Descriptive Geometry, Faculty of Civil Engineering, Slovak University of Technology, Bratislava. The work of the second author on this paper was supported by the grants VEGA 1/0006/19 and by the project of Grant Agency of the Czech Republic (GAR) no.18-06915S.

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