

Fuzzy convergence structures in the framework of L -convex spaces

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Abstract

In this paper, fuzzy convergence theory in the framework of L -convex spaces is introduced. Firstly, the concept of L -convex remote-neighborhood spaces is introduced and it is shown that the resulting category is isomorphic to that of L -convex spaces. Secondly, by means of L -convex ideals, the notion of L -convergence spaces is introduced and it is proved that the category of L -convex spaces can be embedded in that of L -convergence spaces as a reflective subcategory. Finally, the concepts of convex and preconvex L -convergence spaces are introduced and it is shown that the resulting categories are isomorphic to the categories of L -convex spaces and L -preconvex remote-neighborhood spaces, respectively.

Keywords: Fuzzy convergence structures, fuzzy convex structures, fuzzy remote-neighborhood systems, L -convex ideals.

1 Introduction

As a topology-like spatial structure, convex structures [23] have some characters similar to topologies, but they are essentially different. Concretely, convex structures are used to deal with convex sets and topology mainly deals with open sets. From another aspect, closure operators of convex structures are algebraic, equivalently, directed-unions-preserving, while closure operators of topologies are finite-unions-preserving. Combining with fuzzy set theory, Rosa [19] first introduced the concept of fuzzy convex structures with the unit interval as the truth-value table. Afterwards, Maruyama [8] proposed the concept of L -fuzzy convexity structures by generalizing the truth-value table from the unit interval to a completely distributive lattice L . Adopting the terminology of fuzzy topology, fuzzy convex structures in the sense of Rosa and Maruyama are called L -convex structures nowadays. This kind of fuzzy convex structures has been investigated from different aspects, including topological characterizations [5, 17, 31], interval operators [15] and categorical properties [1, 13, 14]. From a logical aspect, Shi and Xiu proposed a new type of fuzzy convex structures based on a completely distributive lattice M , which is called M -fuzzifying convex structures [21]. In the setting of M -fuzzifying convex structures, many researchers have also made plentiful investigations, such as interval operators [16, 24, 34, 35], topological properties [20, 27, 32, 33] and geometric properties [28, 29, 30]. Recently, Shi and Xiu further proposed the notion of (L, M) -fuzzy convex structures [22] and this kind of fuzzy convex structures also received wide attention from different aspects [11, 12, 25].

Inspired by the relations between fuzzy topologies and fuzzy orders, Li and Shi [7] established the relations between M -fuzzifying convex structures and M -preorders. Wang and Shi [26] made use of fuzzy inclusion orders between fuzzy sets to investigate the relations among fuzzifying interval operators, fuzzifying convex structures and fuzzy preorders. Motivated by fuzzy convergence structures [2, 3, 4, 6, 9, 18] in fuzzy topological spaces, Pang [10] introduced the concept of M -fuzzifying (convex) convergence structures and established its relations with M -fuzzifying convex structures from a categorical viewpoint. By this motivation, a natural question has risen:

Is there a counterpart of convergence structures in the framework of L -convex spaces?

In this paper, we will focus on convergence structures in L -convex spaces. We will show that there really exists such a counterpart of convergence structures in L -convex spaces, which will be called L -convergence structures. Furthermore, we will show that there are close categorical relations between L -convex structures and L -convergence structures.

This paper is organized as follows. In Section 2, we will recall some necessary concepts and notations. In Section 3, we will introduce the notion of L -convex remote-neighborhood systems and study its relations with L -convex structures. In Section 4, we will first propose the concept of L -convex ideals and use it to define L -convergence structures, and then investigate the categorical relations between L -convex spaces and L -convergence spaces. In Section 5, we will propose convex and preconvex L -convergence structures and establish their relations with L -convex spaces and L -preconvex remote-neighborhood spaces, respectively.

2 Preliminaries

Throughout this paper, $(L, ')$ denotes a completely distributive De Morgan algebra. The smallest element and the largest element in L are denoted by \perp and \top , respectively. For $a, b \in L$, we say that a is wedge below b in L , in symbols $a \prec b$, if for every subset $D \subseteq L$, $\bigvee D \geq b$ implies $d \geq a$ for some $d \in D$. Let $\beta(a) = \{b \in L \mid b \prec a\}$. A complete lattice L is completely distributive if and only if $a = \bigvee \beta(a)$ for each $a \in L$.

For a nonempty set X , L^X denotes the set of all L -subsets on X . The smallest element and the largest element in L^X are denoted by $\underline{\perp}$ and $\underline{\top}$, respectively. L^X is also a completely distributive De Morgan algebra when it inherits the structure of the lattice L in a natural way, by defining \bigvee , \bigwedge , \leq and $'$ pointwisely. For each $x \in X$ and $\lambda \in L$, the L -subset x_λ , defined by $x_\lambda(y) = \lambda$ if $y = x$, and $x_\lambda(y) = \perp$ if $y \neq x$, is called a fuzzy point. The set of all fuzzy points in L^X is denoted by $J(L^X)$. For convenience, let $\mathbb{P}(J(L^X))$ denote the powerset of $J(L^X)$. We say $\{A_j\}_{j \in J}$ is a directed subset of L^X , if for each $A_{j_1}, A_{j_2} \in \{A_j\}_{j \in J}$, there exists $A_{j_3} \in \{A_j\}_{j \in J}$ such that $A_{j_1} \leq A_{j_3}$ and $A_{j_2} \leq A_{j_3}$.

We usually use the symbols $\{A_j\}_{j \in J} \stackrel{dir}{\subseteq} \mathcal{A}$ to denote that $\{A_j\}_{j \in J}$ is a directed subset of \mathcal{A} .

Let $\varphi : X \rightarrow Y$ be a mapping. Define $\varphi_L^{\rightarrow} : L^X \rightarrow L^Y$ and $\varphi_L^{\leftarrow} : L^Y \rightarrow L^X$ by $\varphi_L^{\rightarrow}(A)(y) = \bigvee_{\varphi(x)=y} A(x)$ for $A \in L^X$ and $y \in Y$, and $\varphi_L^{\leftarrow}(B) = B \circ \varphi$ for $B \in L^Y$, respectively.

Definition 2.1 ([8, 19]). *An L -convex structure \mathcal{C} on X is a subset of L^X which satisfies:*

- (LCS1) $\underline{\perp}, \underline{\top} \in \mathcal{C}$;
- (LCS2) $\{A_k\}_{k \in K} \subseteq \mathcal{C}$ implies $\bigwedge_{k \in K} A_k \in \mathcal{C}$;
- (LCS3) $\{A_j\}_{j \in J} \stackrel{dir}{\subseteq} \mathcal{C}$ implies $\bigvee_{j \in J} A_j \in \mathcal{C}$.

For an L -convex structure \mathcal{C} on X , the pair (X, \mathcal{C}) is called an L -convex space.

A mapping $\varphi : X \rightarrow Y$ between L -convex spaces (X, \mathcal{C}_X) and (Y, \mathcal{C}_Y) is called L -convexity-preserving (L -CP, in short) provided that for each $B \in L^Y$, $B \in \mathcal{C}_Y$ implies $\varphi_L^{\leftarrow}(B) \in \mathcal{C}_X$.

The category with L -convex spaces as objects and with L -CP mappings as morphisms will be denoted by L -CS.

3 L -convex remote-neighborhood spaces

In this section, we will introduce the concept of L -convex remote-neighborhood spaces and discuss its relations with L -convex spaces in a categorical sense.

Definition 3.1. *An L -convex remote-neighborhood system on X is a set $\mathcal{R} = \{\mathcal{R}_{x_\lambda} \mid x_\lambda \in J(L^X)\}$, where $\mathcal{R}_{x_\lambda} \subseteq L^X$ satisfies:*

- (LCR1) $\underline{\perp} \in \mathcal{R}_{x_\lambda}, \underline{\top} \notin \mathcal{R}_{x_\lambda}$;
- (LCR2) $\forall A \in \mathcal{R}_{x_\lambda}, x_\lambda \leq A'$;
- (LCR3) $\forall A \in \mathcal{R}_{x_\lambda}, B \leq A$ implies $B \in \mathcal{R}_{x_\lambda}$;
- (LCR4) $\forall \{A_j\}_{j \in J} \stackrel{dir}{\subseteq} \mathcal{R}_{x_\lambda}, \bigvee_{j \in J} A_j \in \mathcal{R}_{x_\lambda}$;
- (LCR5) $A \in \mathcal{R}_{x_\lambda}$ if and only if $\exists B \in L^X$ s.t. $x_\lambda \leq B \leq A$ and $\forall y_\mu \prec B, B' \in \mathcal{R}_{y_\mu}$.

For an L -convex remote-neighborhood system \mathcal{R} on X , the pair (X, \mathcal{R}) is called an L -convex remote-neighborhood space.

Definition 3.2. *A mapping $\varphi : (X, \mathcal{R}^X) \rightarrow (Y, \mathcal{R}^Y)$ between L -convex remote-neighborhood spaces is called L -convexity-preserving (L -CP, in short) provided that for each $B \in L^Y$ and $x_\lambda \in J(L^X)$, $B \in \mathcal{R}_{\varphi(x)_\lambda}^Y$ implies $\varphi_L^{\leftarrow}(B) \in \mathcal{R}_{x_\lambda}^X$.*

It is easy to check that L -convex remote-neighborhood spaces as objects and L -CP mappings as morphisms form a category, denoted by $L\text{-CR}$.

Now let us establish the relations between L -convex remote-neighborhood spaces and L -convex spaces.

Proposition 3.3. *Let (X, \mathcal{R}) be an L -convex remote-neighborhood space and define $\mathcal{C}^{\mathcal{R}} \subseteq L^X$ as follows:*

$$\mathcal{C}^{\mathcal{R}} = \{A \in L^X \mid \forall x_\lambda \prec A', A \in \mathcal{R}_{x_\lambda}\}.$$

Then $\mathcal{C}^{\mathcal{R}}$ is an L -convex structure on X .

Proof. (LCS1) It is obvious.

(LCS2) Take each $\{A_k\}_{k \in K} \subseteq \mathcal{C}^{\mathcal{R}}$. Suppose that $x_\lambda \prec (\bigwedge_{k \in K} A_k)'$. That is, $x_\lambda \prec \bigvee_{k \in K} A'_k$. Then there exists $k_0 \in K$ such that $x_\lambda \prec A'_{k_0}$. Since $A_{k_0} \in \mathcal{C}^{\mathcal{R}}$, we have $A_{k_0} \in \mathcal{R}_{x_\lambda}$. Then it follows from (LCR3) that $\bigwedge_{k \in K} A_k \in \mathcal{R}_{x_\lambda}$. This proves $\bigwedge_{k \in K} A_k \in \mathcal{C}^{\mathcal{R}}$.

(LCS3) Take each $\{A_j\}_{j \in J} \stackrel{dir}{\subseteq} \mathcal{C}^{\mathcal{R}}$. Suppose that $x_\lambda \prec (\bigvee_{j \in J} A_j)'$. That is, $x_\lambda \prec \bigwedge_{j \in J} A'_j$. Then it follows that $x_\lambda \prec A'_j$ for each $j \in J$. This implies $A_j \in \mathcal{R}_{x_\lambda}$ for each $j \in J$. Since $\{A_j\}_{j \in J}$ is directed, it follows from (LCR4) that $\bigvee_{j \in J} A_j \in \mathcal{R}_{x_\lambda}$. This shows $\bigvee_{j \in J} A_j \in \mathcal{C}^{\mathcal{R}}$. \square

Proposition 3.4. *If $\varphi : (X, \mathcal{R}^X) \rightarrow (Y, \mathcal{R}^Y)$ is L -CP, then so is $\varphi : (X, \mathcal{C}^{\mathcal{R}^X}) \rightarrow (Y, \mathcal{C}^{\mathcal{R}^Y})$.*

Proof. Since $\varphi : (X, \mathcal{R}^X) \rightarrow (Y, \mathcal{R}^Y)$ is L -CP, it follows that for each $B \in L^Y$ and $x_\lambda \in J(L^X)$, $B \in \mathcal{R}^Y_{\varphi(x)_\lambda}$ implies $\varphi_L^{\leftarrow}(B) \in \mathcal{R}^X_{x_\lambda}$. Then for each $C \in L^Y$, we obtain

$$\begin{aligned} C \in \mathcal{C}^{\mathcal{R}^Y} &\iff \forall y_\mu \prec C', C \in \mathcal{R}^Y_{y_\mu} \\ &\implies \forall \varphi(x)_\lambda \prec C', C \in \mathcal{R}^Y_{\varphi(x)_\lambda} \\ &\implies \forall x_\lambda \prec \varphi_L^{\leftarrow}(C)', \varphi_L^{\leftarrow}(C) \in \mathcal{R}^X_{x_\lambda} \\ &\iff \varphi_L^{\leftarrow}(C) \in \mathcal{C}^{\mathcal{R}^X}. \end{aligned}$$

This proves that $\varphi : (X, \mathcal{C}^{\mathcal{R}^X}) \rightarrow (Y, \mathcal{C}^{\mathcal{R}^Y})$ is L -CP. \square

By Propositions 3.3 and 3.4, we obtain a functor $\mathbb{C}_{\mathbb{R}} : L\text{-CR} \rightarrow L\text{-CS}$ by

$$\mathbb{C}_{\mathbb{R}} : \begin{cases} L\text{-CR} &\rightarrow & L\text{-CS}, \\ (X, \mathcal{R}) &\mapsto & (X, \mathcal{C}^{\mathcal{R}}), \\ \varphi &\mapsto & \varphi. \end{cases}$$

Conversely, we will show that each L -convex space can induce an L -convex remote-neighborhood space.

Proposition 3.5. *Let (X, \mathcal{C}) be an L -convex space. For each $x_\lambda \in J(L^X)$, define $\mathcal{R}^{\mathcal{C}}_{x_\lambda} \subseteq L^X$ by*

$$\mathcal{R}^{\mathcal{C}}_{x_\lambda} = \{A \in L^X \mid \exists B' \in \mathcal{C} \text{ s.t. } x_\lambda \leq B \leq A'\}.$$

Then $\mathcal{R}^{\mathcal{C}} = \{\mathcal{R}^{\mathcal{C}}_{x_\lambda} \mid x_\lambda \in J(L^X)\}$ is an L -convex remote-neighborhood system on X .

Proof. It is enough to show that $\mathcal{R}^{\mathcal{C}}$ satisfies (LCR1)–(LCR5). (LCR1)–(LCR3) are obvious.

(LCR4) Take each $\{A_j\}_{j \in J} \stackrel{dir}{\subseteq} \mathcal{R}^{\mathcal{C}}_{x_\lambda}$. Then for each $j \in J$, there exists $B'_j \in \mathcal{C}$ such that $x_\lambda \leq B_j \leq A'_j$. Let $C = \bigwedge_{j \in J} co(A_j)'$, where $co(A_j) = \bigwedge\{D \in L^X \mid A_j \leq D \in \mathcal{C}\}$. Then it follows from (LCS2) that $co(A_j) \in \mathcal{C}$ for each $j \in J$. By the definition of $co(A_j)$, it is easy to check that $\{co(A_j)\}_{j \in J}$ is directed. By (LCS3), we have $C' = \bigvee_{j \in J} co(A_j) \in \mathcal{C}$. Since $B'_j \in \mathcal{C}$ and $A_j \leq B'_j$, it follows that $co(A_j) \leq B'_j$. That is, $B_j \leq co(A_j)'$. Thus we can obtain

$$x_\lambda \leq \bigwedge_{j \in J} B_j \leq \bigwedge_{j \in J} co(A_j)' = C \leq \bigwedge_{j \in J} A'_j = \left(\bigvee_{j \in J} A_j\right)'.$$

This proves $\bigvee_{j \in J} A_j \in \mathcal{R}^{\mathcal{C}}_{x_\lambda}$.

(LCR5) *Necessity.* Take each $A \in \mathcal{R}^{\mathcal{C}}_{x_\lambda}$. Then there exists $B' \in \mathcal{C}$ such that $x_\lambda \leq B \leq A'$. For each $y_\mu \prec B$, there exists $C (= B)$ such that $C' = B' \in \mathcal{C}$ and $y_\mu \leq B \leq C = (B')'$. This means $B' \in \mathcal{R}^{\mathcal{C}}_{y_\mu}$ for each $y_\mu \prec B$, as desired.

Sufficiency. We first prove the following result.

$$(LCR0) \quad \mathcal{R}_{x_\lambda}^{\mathcal{C}} = \bigcap_{\mu \in \beta(\lambda)} \mathcal{R}_{x_\mu}^{\mathcal{C}}.$$

On one hand, if $A \in \mathcal{R}_{x_\lambda}^{\mathcal{C}}$, then there exists $B' \in \mathcal{C}$ such that $x_\lambda \leq B \leq A'$. Take each $\mu \in \beta(\lambda)$. Then $x_\mu \prec x_\lambda \leq B \leq A'$. This implies $A \in \mathcal{R}_{x_\mu}^{\mathcal{C}}$ for each $\mu \in \beta(\lambda)$.

On the other hand, since $A \in \mathcal{R}_{x_\mu}^{\mathcal{C}}$ for each $\mu \in \beta(\lambda)$, there exists $B'_\mu \in \mathcal{C}$ such that $x_\mu \leq B_\mu \leq A'$. Let $B = \bigvee_{\mu \in \beta(\lambda)} B_\mu$. Then by (LCS2), we have $B' = \bigwedge_{\mu \in \beta(\lambda)} B'_\mu \in \mathcal{C}$. Further,

$$x_\lambda = \bigvee_{\mu \in \beta(\lambda)} x_\mu \leq \bigvee_{\mu \in \beta(\lambda)} B_\mu = B \leq A'.$$

This implies that $A \in \mathcal{R}_{x_\lambda}^{\mathcal{C}}$.

Now we know there exists $B \in L^X$ such that $x_\lambda \leq B \leq A'$ and for each $y_\mu \prec B$, $B' \in \mathcal{R}_{y_\mu}^{\mathcal{C}}$. Then for each $\mu \in \beta(\lambda)$, it follows that $x_\mu \prec x_\lambda \leq B$. This implies $B' \in \mathcal{R}_{x_\mu}^{\mathcal{C}}$. By (LCR0), we know $B' \in \bigcap_{\mu \in \beta(\lambda)} \mathcal{R}_{x_\mu}^{\mathcal{C}} = \mathcal{R}_{x_\lambda}^{\mathcal{C}}$. Then it follows from (LCR3) that $A \in \mathcal{R}_{x_\lambda}^{\mathcal{C}}$. This proves the sufficiency of (LCR5). \square

Proposition 3.6. *If $\varphi : (X, \mathcal{C}_X) \rightarrow (Y, \mathcal{C}_Y)$ is L-CP, then so is $\varphi : (X, \mathcal{R}^{\mathcal{C}_X}) \rightarrow (Y, \mathcal{R}^{\mathcal{C}_Y})$.*

Proof. Since $\varphi : (X, \mathcal{C}_X) \rightarrow (Y, \mathcal{C}_Y)$ is L-CP, it follows that for each $B \in L^Y$, $B \in \mathcal{C}_Y$ implies $\varphi_L^{\leftarrow}(B) \in \mathcal{C}_X$. Then for each $x_\lambda \in J(L^X)$ and $B \in L^Y$, we have

$$\begin{aligned} B \in \mathcal{R}_{\varphi(x)_\lambda}^{\mathcal{C}_Y} &\iff \exists C' \in \mathcal{C}_Y \text{ s.t. } \varphi(x)_\lambda \leq C \leq B' \\ &\implies \exists \varphi_L^{\leftarrow}(C') \in \mathcal{C}_X \text{ s.t. } x_\lambda \leq \varphi_L^{\leftarrow}(C) \leq \varphi_L^{\leftarrow}(B') \\ &\implies \exists A' \in \mathcal{C}_X \text{ s.t. } x_\lambda \leq A \leq \varphi_L^{\leftarrow}(B') \\ &\iff \varphi_L^{\leftarrow}(B) \in \mathcal{R}_{x_\lambda}^{\mathcal{C}_X}. \end{aligned}$$

This shows that $\varphi : (X, \mathcal{R}^{\mathcal{C}_X}) \rightarrow (Y, \mathcal{R}^{\mathcal{C}_Y})$ is L-CP. \square

By Propositions 3.5 and 3.6, we obtain a functor $\mathbb{R}_{\mathcal{C}} : L\text{-CS} \rightarrow L\text{-CR}$ by

$$\mathbb{R}_{\mathcal{C}} : \begin{cases} L\text{-CS} &\rightarrow & L\text{-CR}, \\ (X, \mathcal{C}) &\mapsto & (X, \mathcal{R}^{\mathcal{C}}), \\ \varphi &\mapsto & \varphi. \end{cases}$$

Next we present the main result in this section.

Theorem 3.7. *L-CS is isomorphic to L-CR.*

Proof. It suffices to show that $\mathbb{R}_{\mathcal{C}} \circ \mathbb{C}_{\mathcal{R}} = \mathbb{I}_{L\text{-CR}}$ and $\mathbb{C}_{\mathcal{R}} \circ \mathbb{R}_{\mathcal{C}} = \mathbb{I}_{L\text{-CS}}$. To this end, we need only verify (1) $\mathcal{R}^{\mathcal{C}^{\mathcal{R}}} = \mathcal{R}$ and (2) $\mathcal{C}^{\mathcal{R}^{\mathcal{C}}} = \mathcal{C}$.

For (1), take each $A \in L^X$ and $x_\lambda \in J(L^X)$. Then

$$\begin{aligned} A \in \mathcal{R}_{x_\lambda}^{\mathcal{C}^{\mathcal{R}}} &\iff \exists B' \in \mathcal{C}^{\mathcal{R}} \text{ s.t. } x_\lambda \leq B \leq A' \\ &\iff \exists B \in L^X \text{ s.t. } x_\lambda \leq B \leq A' \text{ and } \forall y_\mu \prec B, B' \in \mathcal{R}_{y_\mu} \\ &\iff A \in \mathcal{R}_{x_\lambda}. \text{ (by (LCR5))} \end{aligned}$$

This shows $\mathcal{R}^{\mathcal{C}^{\mathcal{R}}} = \mathcal{R}$.

For (2), take each $A \in L^X$. Then

$$\begin{aligned} A \in \mathcal{C}^{\mathcal{R}^{\mathcal{C}}} &\iff \forall x_\lambda \prec A', A \in \mathcal{R}_{x_\lambda}^{\mathcal{C}} \\ &\iff \forall x_\lambda \prec A', \exists B' \in \mathcal{C} \text{ s.t. } x_\lambda \leq B \leq A' \\ &\iff A' = \bigvee_{x_\lambda \prec A'} \bigvee_{B' \in \mathcal{C}} \bigvee_{x_\lambda \leq B \leq A'} B \\ &\iff A = \bigwedge_{x_\lambda \prec A'} \bigwedge_{B' \in \mathcal{C}} \bigwedge_{x_\lambda \leq B \leq A'} B' \\ &\iff A \in \mathcal{C}. \end{aligned}$$

This means $\mathcal{C}^{\mathcal{R}^{\mathcal{C}}} = \mathcal{C}$. \square

4 L -convergence spaces

In this section, we will first introduce the concept of L -convex ideals. By means of L -convex ideals, we will propose the counterpart of convergence structures in L -convex spaces, which will be called L -convergence structures. Further, we will investigate the categorical relations between L -convergence spaces and L -convex spaces.

Definition 4.1. A subset $\mathcal{I} \subseteq L^X$ is called an L -convex ideal on X if it satisfies:

- (LCI1) $\perp \in \mathcal{I}$, $\top \notin \mathcal{I}$;
- (LCI2) $A \in \mathcal{I}$ and $B \leq A$ imply $B \in \mathcal{I}$;
- (LCI3) $\forall \{A_j\}_{j \in J} \stackrel{dir}{\subseteq} \mathcal{I}$, $\bigvee_{j \in J} A_j \in \mathcal{I}$.

The family of all L -convex ideals on X is denoted by $\mathcal{I}_L^C(X)$.

Example 4.2. Let X be a nonempty set.

- (1) For each $x_\lambda \in J(L^X)$, we define $[x_\lambda] \subseteq L^X$ by $[x_\lambda] = \{A \in L^X \mid x_\lambda \leq A\}$. Then $[x_\lambda]$ is an L -convex ideal on X .
- (2) For an L -convex structure \mathcal{C} on X and for each $x_\lambda \in J(L^X)$, $\mathcal{R}_{x_\lambda}^{\mathcal{C}}$ is an L -convex ideal on X and $\mathcal{R}_{x_\lambda}^{\mathcal{C}} \subseteq [x_\lambda]$.

Definition 4.3. Let $\mathcal{I} \in \mathcal{I}_L^C(X)$ and let $\varphi : X \rightarrow Y$ be a mapping. Define $\varphi^{\rightarrow}(\mathcal{I}) = \{B \in L^Y \mid \varphi_L^{\leftarrow}(B) \in \mathcal{I}\}$. Then $\varphi^{\rightarrow}(\mathcal{I})$ is an L -convex ideal on Y and is called the image of \mathcal{I} under φ .

Next we will propose the concept of L -convergence structures in the framework of L -convex spaces.

Definition 4.4. An L -convergence structure on X is a mapping $\lim : \mathcal{I}_L^C(X) \rightarrow \mathbb{P}(J(L^X))$ which satisfies:

- (LC1) $x_\lambda \in \lim([x_\lambda])$;
- (LC2) $\mathcal{I}_1 \subseteq \mathcal{I}_2$ implies $\lim(\mathcal{I}_1) \subseteq \lim(\mathcal{I}_2)$;
- (LC3) $\mathcal{R}_{x_\lambda}^{\lim} = \bigcap_{\mu \in \beta(\lambda)} \mathcal{R}_{x_\mu}^{\lim}$, where $\mathcal{R}_{x_\lambda}^{\lim}$ is defined by $\mathcal{R}_{x_\lambda}^{\lim} = \bigcap_{x_\lambda \in \lim(\mathcal{I})} \mathcal{I}$.

For an L -convergence structure \lim on X , the pair (X, \lim) is called an L -convergence space.

Remark 4.5. Formally, L -convergence structures in the framework of L -convex spaces are similar to that in L -topological spaces [18]. However, they are different. On one hand, the tools for defining convergence are different. That is, L -convex ideals are different from L -prefilters in [18]. On the other hand, the definition of L -convergence structures in L -convex spaces includes the condition (LC3), which does not appear in the definition of L -convergence structures in L -topological spaces. Moreover, this condition plays an important role in discussing the categorical relations between L -convex spaces and L -convergence spaces.

Definition 4.6. A mapping $\varphi : (X, \lim_X) \rightarrow (Y, \lim_Y)$ between L -convergence spaces is called L -convexity-preserving (L -CP, in short) provided that for each $\mathcal{I} \in \mathcal{I}_L^C(X)$ and $x_\lambda \in J(L^X)$, $x_\lambda \in \lim_X(\mathcal{I})$ implies $\varphi(x)_\lambda \in \lim_Y(\varphi^{\rightarrow}(\mathcal{I}))$.

It is easy to check that L -convergence spaces as objects and L -CP mappings as morphisms form a category, denoted by $L\text{-Conv}$.

Now let us establish the relations between L -convex spaces and L -convergence spaces from a categorical aspect.

Proposition 4.7. Let (X, \mathcal{C}) be an L -convex space and define $\lim^{\mathcal{C}} : \mathcal{I}_L^C(X) \rightarrow \mathbb{P}(J(L^X))$ by

$$\forall \mathcal{I} \in \mathcal{I}_L^C(X), \quad \lim^{\mathcal{C}}(\mathcal{I}) = \{x_\lambda \in J(L^X) \mid \mathcal{R}_{x_\lambda}^{\mathcal{C}} \subseteq \mathcal{I}\}.$$

Then $\lim^{\mathcal{C}}$ is an L -convergence structure on X .

Proof. It suffices to verify that $\lim^{\mathcal{C}}$ satisfies (LC1)–(LC3). Indeed,

- (LC1) Since $\mathcal{R}_{x_\lambda}^{\mathcal{C}} \subseteq [x_\lambda]$, it follows that $x_\lambda \in \lim^{\mathcal{C}}([x_\lambda])$.
- (LC2) Obviously.
- (LC3) Take each $z_\nu \in J(L^X)$. Then it follows that

$$\mathcal{R}_{z_\nu}^{\lim^{\mathcal{C}}} = \bigcap_{z_\nu \in \lim^{\mathcal{C}}(\mathcal{I})} \mathcal{I} = \bigcap_{\mathcal{R}_{z_\nu}^{\mathcal{C}} \subseteq \mathcal{I}} \mathcal{I} = \mathcal{R}_{z_\nu}^{\mathcal{C}}.$$

Furthermore, by (LCR0), we have

$$\mathcal{R}_{x_\lambda}^{\lim^{\mathcal{C}}} = \mathcal{R}_{x_\lambda}^{\mathcal{C}} = \bigcap_{\mu \in \beta(\lambda)} \mathcal{R}_{x_\mu}^{\mathcal{C}} = \bigcap_{\mu \in \beta(\lambda)} \mathcal{R}_{x_\mu}^{\lim^{\mathcal{C}}},$$

as desired. □

Proposition 4.8. *If $\varphi : (X, \mathcal{C}_X) \rightarrow (Y, \mathcal{C}_Y)$ is L -CP, then so is $\varphi : (X, \lim^{C_X}) \rightarrow (Y, \lim^{C_Y})$.*

Proof. Since $\varphi : (X, \mathcal{C}_X) \rightarrow (Y, \mathcal{C}_Y)$ is L -CP, it follows from Proposition 3.6 that $\varphi : (X, \mathcal{R}^{C_X}) \rightarrow (Y, \mathcal{R}^{C_Y})$ is L -CP. Then for each $x_\lambda \in J(L^X)$ and $B \in L^Y$,

$$B \in \mathcal{R}_{\varphi(x)_\lambda}^{C_Y} \implies \varphi_L^-(B) \in \mathcal{R}_{x_\lambda}^{C_X} \iff B \in \varphi^\Rightarrow(\mathcal{R}_{x_\lambda}^{C_X}).$$

This means that $\mathcal{R}_{\varphi(x)_\lambda}^{C_Y} \subseteq \varphi^\Rightarrow(\mathcal{R}_{x_\lambda}^{C_X})$. Then for each $\mathcal{I} \in \mathcal{I}_L^C(X)$ and $x_\lambda \in J(L^X)$, we have

$$\begin{aligned} x_\lambda \in \lim^{C_X}(\mathcal{I}) &\iff \mathcal{R}_{x_\lambda}^{C_X} \subseteq \mathcal{I} \\ &\implies \varphi^\Rightarrow(\mathcal{R}_{x_\lambda}^{C_X}) \subseteq \varphi^\Rightarrow(\mathcal{I}) \\ &\implies \mathcal{R}_{\varphi(x)_\lambda}^{C_Y} \subseteq \varphi^\Rightarrow(\mathcal{I}) \\ &\iff \varphi(x)_\lambda \in \lim^{C_Y}(\varphi^\Rightarrow(\mathcal{I})). \end{aligned}$$

Therefore, $\varphi : (X, \lim^{C_X}) \rightarrow (Y, \lim^{C_Y})$ is L -CP. □

By Propositions 4.7 and 4.8, we obtain a functor $\mathbb{L}_C : L\text{-CS} \rightarrow L\text{-Conv}$ by

$$\mathbb{L}_C : \begin{cases} L\text{-CS} & \rightarrow & L\text{-Conv}, \\ (X, \mathcal{C}) & \mapsto & (X, \lim^C), \\ \varphi & \mapsto & \varphi. \end{cases}$$

Proposition 4.9. *Let (X, \lim) be an L -convergence space and define $\mathcal{C}^{\lim} \subseteq L^X$ as follows:*

$$\mathcal{C}^{\lim} = \{A \in L^X \mid \forall x_\lambda \prec A', A \in \mathcal{R}_{x_\lambda}^{\lim}\}.$$

Then \mathcal{C}^{\lim} is an L -convex structure on X .

Proof. It suffices to verify that \mathcal{C}^{\lim} satisfies (LCS1)–(LCS3). Indeed,

(LCS1) It is trivial.

(LCS2) For $\{A_k\}_{k \in K} \subseteq \mathcal{C}^{\lim}$, take each $x_\lambda \in J(L^X)$ such that $x_\lambda \prec (\bigwedge_{k \in K} A_k)' = \bigvee_{k \in K} A'_k$. Then there exists k_0 such that $x_\lambda \prec A'_{k_0}$. This implies that $A_{k_0} \in \mathcal{R}_{x_\lambda}^{\lim}$. Since $\mathcal{R}_{x_\lambda}^{\lim} \in \mathcal{I}_L^C(X)$ and $\bigwedge_{k \in K} A_k \leq A_{k_0}$, it follows from (LCI2) that $\bigwedge_{k \in K} A_k \in \mathcal{R}_{x_\lambda}^{\lim}$. This proves that $\bigwedge_{k \in K} A_k \in \mathcal{C}^{\lim}$.

(LCS3) For $\{A_j\}_{j \in J} \stackrel{dir}{\subseteq} \mathcal{C}^{\lim}$, take each $x_\lambda \in J(L^X)$ such that $x_\lambda \prec (\bigvee_{j \in J} A_j)' = \bigwedge_{j \in J} A'_j$. Then $x_\lambda \prec A'_j$ for each $j \in J$. This implies that $A_j \in \mathcal{R}_{x_\lambda}^{\lim}$ for each $j \in J$. By (LCI3), we have $\bigvee_{j \in J} A_j \in \mathcal{R}_{x_\lambda}^{\lim}$. This proves $\bigvee_{j \in J} A_j \in \mathcal{C}^{\lim}$. □

Proposition 4.10. *If $\varphi : (X, \lim_X) \rightarrow (Y, \lim_Y)$ is L -CP, then so is $\varphi : (X, \mathcal{C}^{\lim_X}) \rightarrow (Y, \mathcal{C}^{\lim_Y})$.*

Proof. Since $\varphi : (X, \lim_X) \rightarrow (Y, \lim_Y)$ is L -CP, it follows that

$$\forall \mathcal{I} \in \mathcal{I}_L^C(X), x_\lambda \in J(L^X), x_\lambda \in \lim_X(\mathcal{I}) \implies \varphi(x)_\lambda \in \lim_Y(\varphi^\Rightarrow(\mathcal{I})).$$

This implies that

$$\begin{aligned} \mathcal{R}_{\varphi(x)_\lambda}^{\lim_Y} &= \bigcap_{\varphi(x)_\lambda \in \lim_Y(J)} J \subseteq \bigcap_{\varphi(x)_\lambda \in \lim_Y(\varphi^\Rightarrow(\mathcal{I}))} \varphi^\Rightarrow(\mathcal{I}) \\ &\subseteq \bigcap_{x_\lambda \in \lim_X(\mathcal{I})} \varphi^\Rightarrow(\mathcal{I}) = \varphi^\Rightarrow \left(\bigcap_{x_\lambda \in \lim_X(\mathcal{I})} \mathcal{I} \right) = \varphi^\Rightarrow(\mathcal{R}_{x_\lambda}^{\lim_X}). \end{aligned}$$

Take each $B \in \mathcal{C}^{\lim_Y}$. Then for each $x_\lambda \in J(L^X)$, we have

$$x_\lambda \prec \varphi_L^-(B)' \iff \varphi(x)_\lambda \prec B' \implies B \in \mathcal{R}_{\varphi(x)_\lambda}^{\lim_Y} \subseteq \varphi^\Rightarrow(\mathcal{R}_{x_\lambda}^{\lim_X}) \iff \varphi_L^-(B) \in \mathcal{R}_{x_\lambda}^{\lim_X}.$$

This means that $\varphi_L^-(B) \in \mathcal{C}^{\lim_X}$. Therefore, $\varphi : (X, \mathcal{C}^{\lim_X}) \rightarrow (Y, \mathcal{C}^{\lim_Y})$ is L -CP. □

By Propositions 4.9 and 4.10, we obtain a functor $\mathbb{C}_L : L\text{-Conv} \rightarrow L\text{-CS}$ by

$$\mathbb{C}_L : \begin{cases} L\text{-Conv} & \longrightarrow & L\text{-CS}, \\ (X, \lim) & \longmapsto & (X, \mathcal{C}^{\lim}), \\ \varphi & \longmapsto & \varphi. \end{cases}$$

Theorem 4.11. $(\mathbb{C}_L, \mathbb{L}_C)$ is a Galois correspondence, and \mathbb{C}_L is a left inverse of \mathbb{L}_C .

Proof. It suffices to show that for an L -convergence structure \lim on X and for an L -convex structure \mathcal{C} on X , the following two statements hold:

- (1) $\lim^{\mathcal{C}^{\lim}} \geq \lim$;
- (2) $\mathcal{C}^{\lim^{\mathcal{C}}} = \mathcal{C}$.

For (1), we first prove $\mathcal{R}_{x_\lambda}^{\mathcal{C}^{\lim}} \subseteq \mathcal{R}_{x_\lambda}^{\lim}$. Take each $A \in \mathcal{R}_{x_\lambda}^{\mathcal{C}^{\lim}}$. Then there exists $B' \in \mathcal{C}^{\lim}$ such that $x_\lambda \leq B \leq A'$. That is to say, there exists $B \in L^X$ such that $x_\lambda \leq B \leq A'$ and $B' \in \mathcal{R}_{y_\mu}^{\lim}$ for each $y_\mu \prec B$. Take each $\mu \in \beta(\lambda)$. Then $x_\mu \prec x_\lambda \leq B$. Further we obtain $B' \in \mathcal{R}_{x_\mu}^{\lim}$. Since $A \leq B'$, it follows that $A \in \mathcal{R}_{x_\mu}^{\lim}$. By the arbitrariness of μ , we have $A \in \bigcap_{\mu \in \beta(\lambda)} \mathcal{R}_{x_\mu}^{\lim} = \mathcal{R}_{x_\lambda}^{\lim}$. This means $\mathcal{R}_{x_\lambda}^{\mathcal{C}^{\lim}} \subseteq \mathcal{R}_{x_\lambda}^{\lim}$. Now take each $\mathcal{I} \in \mathcal{I}_L^{\mathcal{C}}(X)$ and $x_\lambda \in J(L^X)$. Then

$$x_\lambda \in \lim(\mathcal{I}) \implies \mathcal{R}_{x_\lambda}^{\lim} \subseteq \mathcal{I} \implies \mathcal{R}_{x_\lambda}^{\mathcal{C}^{\lim}} \subseteq \mathcal{I} \implies x_\lambda \in \lim^{\mathcal{C}^{\lim}}(\mathcal{I}),$$

which shows $\lim^{\mathcal{C}^{\lim}} \geq \lim$.

For (2), by the proof of Proposition 4.7, we first have $\mathcal{R}_{x_\lambda}^{\lim^{\mathcal{C}}} = \mathcal{R}_{x_\lambda}^{\mathcal{C}}$. Then

$$\begin{aligned} A \in \mathcal{C}^{\lim^{\mathcal{C}}} &\iff \forall x_\lambda \prec A', A \in \mathcal{R}_{x_\lambda}^{\lim^{\mathcal{C}}} \\ &\iff \forall x_\lambda \prec A', A \in \mathcal{R}_{x_\lambda}^{\mathcal{C}} \\ &\iff \forall x_\lambda \prec A', \exists B' \in \mathcal{C} \text{ s.t. } x_\lambda \leq B \leq A' \\ &\iff A' = \bigvee_{x_\lambda \prec A'} \bigvee_{B' \in \mathcal{C}} \bigvee_{x_\lambda \leq B \leq A'} B \\ &\iff A = \bigwedge_{x_\lambda \prec A'} \bigwedge_{B' \in \mathcal{C}} \bigwedge_{x_\lambda \leq B \leq A'} B' \\ &\iff A \in \mathcal{C}. \end{aligned}$$

This implies that $\mathcal{C}^{\lim^{\mathcal{C}}} = \mathcal{C}$. □

Corollary 4.12. The category $L\text{-CS}$ can be embedded in $L\text{-Conv}$ as a reflective subcategory.

Now we have shown that L -convex structures can be considered as a subclass of L -convergence structures in a categorical sense. So what kind of L -convergence structures can be equivalent to L -convex structures? We will give an answer to this problem in the next section.

5 Convex and preconvex L -convergence spaces

In this section, we will strengthen the axiomatic conditions of L -convergence spaces and propose the concepts of convex and preconvex L -convergence spaces. Moreover, we will establish their relations with L -convex spaces.

Definition 5.1. An L -convergence structure \lim on X is called *convex* if it satisfies:

$$(LCP) \quad x_\lambda \in \lim(\mathcal{R}_{x_\lambda}^{\lim});$$

$$(LCC) \quad A \in \mathcal{R}_{x_\lambda}^{\lim} \text{ if and only if } \exists B \in L^X \text{ s.t. } x_\lambda \leq B \leq A' \text{ and } \forall y_\mu \prec B, B' \in \mathcal{R}_{y_\mu}^{\lim}.$$

For a convex L -convergence structure \lim on X , the pair (X, \lim) is called a *convex L -convergence space*.

The full subcategory of $L\text{-Conv}$, consisting of convex L -convergence spaces, is denoted by $L\text{-CConv}$.

Next we will show convex L -convergence structures can be used to characterize L -convex structures.

Proposition 5.2. Let (X, \mathcal{C}) be an L -convex space and define $\lim^{\mathcal{C}} : \mathcal{I}_L^{\mathcal{C}}(X) \rightarrow \mathbb{P}(J(L^X))$ by

$$\forall \mathcal{I} \in \mathcal{I}_L^{\mathcal{C}}(X), \quad \lim^{\mathcal{C}}(\mathcal{I}) = \{x_\lambda \in J(L^X) \mid \mathcal{R}_{x_\lambda}^{\mathcal{C}} \subseteq \mathcal{I}\}.$$

Then $\lim^{\mathcal{C}}$ is a convex L -convergence structure on X .

Proof. By Proposition 4.7, it is enough to verify that \lim^C satisfies (LCP) and (LCC). Since $\mathcal{R}_{z_\nu}^{\lim^C} = \mathcal{R}_{z_\nu}^C$ for each $z_\nu \in J(L^X)$ (See Proposition 4.7), we can check (LCP) and (LCC) as follows:

(LCP) By the definition of \lim^C , we have

$$x_\lambda \in \lim^C(\mathcal{R}_{x_\lambda}^C) = \lim^C(\mathcal{R}_{x_\lambda}^{\lim^C}).$$

(LCC) For each $x_\lambda \in J(L^X)$, we have

$$\begin{aligned} A \in \mathcal{R}_{x_\lambda}^{\lim^C} &\iff A \in \mathcal{R}_{x_\lambda}^C \\ &\iff \exists B' \in \mathcal{C} \text{ s.t. } x_\lambda \leq B \leq A' \\ &\iff \exists B' \in \mathcal{C}^{\mathcal{R}^C} \text{ s.t. } x_\lambda \leq B \leq A' \quad (\text{by Theorem 3.7}) \\ &\iff \exists B \in L^X \text{ s.t. } x_\lambda \leq B \leq A' \text{ and } \forall y_\mu \prec B, B' \in \mathcal{R}_{y_\mu}^C \\ &\iff \exists B \in L^X \text{ s.t. } x_\lambda \leq B \leq A' \text{ and } \forall y_\mu \prec B, B' \in \mathcal{R}_{y_\mu}^{\lim^C}, \end{aligned}$$

as desired. □

Proposition 5.3. *If (X, \lim) is a convex L -convergence space, then $\lim^{C^{\lim}} = \lim$.*

Proof. Take each $A \in L^X$ and $x_\lambda \in J(L^X)$. Then

$$\begin{aligned} A \in \mathcal{R}_{x_\lambda}^{C^{\lim}} &\iff \exists B' \in \mathcal{C}^{\lim} \text{ s.t. } x_\lambda \leq B \leq A' \\ &\iff \exists B \in L^X \text{ s.t. } x_\lambda \leq B \leq A' \text{ and } \forall y_\mu \prec B, B' \in \mathcal{R}_{y_\mu}^{\lim} \\ &\iff A \in \mathcal{R}_{x_\lambda}^{\lim}, \quad (\text{by (LCC)}) \end{aligned}$$

which means $\mathcal{R}_{x_\lambda}^{C^{\lim}} = \mathcal{R}_{x_\lambda}^{\lim}$. Hence we have

$$x_\lambda \in \lim^{C^{\lim}}(\mathcal{I}) \iff \mathcal{R}_{x_\lambda}^{C^{\lim}} \subseteq \mathcal{I} \iff \mathcal{R}_{x_\lambda}^{\lim} \subseteq \mathcal{I} \stackrel{(\text{LCP})}{\iff} x_\lambda \in \lim(\mathcal{I}).$$

This shows $\lim^{C^{\lim}} = \lim$. □

By Propositions 5.2–5.3 and Theorem 4.11, we can obtain the following result.

Theorem 5.4. *$L\text{-CConv}$ is isomorphic to $L\text{-CS}$.*

Up to now, we have provided an answer to the problem in the final of Section 4. That is, convex L -convergence structures can be used to characterize L -convex structures from a categorical aspect. Next we will investigate another kind of L -convergence structures, which will be called preconvex L -convergence structures.

Definition 5.5. *An L -convergence structure \lim on X is called preconvex if it satisfies (LCP). For a preconvex L -convergence structure \lim on X , the pair (X, \lim) is called a preconvex L -convergence space.*

The full subcategory of $L\text{-Conv}$, consisting of preconvex L -convergence spaces, is denoted by $L\text{-PConv}$.

Proposition 5.6. *Let \lim be an L -convergence structure on X . Then the following conditions are equivalent.*

- (LCP) $x_\lambda \in \lim(\mathcal{R}_{x_\lambda}^{\lim})$.
- (LCP)* $x_\lambda \in \lim(\mathcal{I})$ iff $\mathcal{R}_{x_\lambda}^{\lim} \subseteq \mathcal{I}$.
- (LCP)* $\lim(\bigcap_{k \in K} \mathcal{I}_k) = \bigcap_{k \in K} \lim(\mathcal{I}_k)$.

Proof. (LCP) \Rightarrow (LCP)*: If $x_\lambda \in \lim(\mathcal{I})$, then $\mathcal{R}_{x_\lambda}^{\lim} = \bigcap_{x_\lambda \in \lim(\mathcal{J})} \mathcal{J} \subseteq \mathcal{I}$. If $\mathcal{R}_{x_\lambda}^{\lim} \subseteq \mathcal{I}$, then it follows from (LCP) and (LC2) that $x_\lambda \in \lim(\mathcal{R}_{x_\lambda}^{\lim}) \subseteq \lim(\mathcal{I})$.

(LCP)* \Rightarrow (LCP)*: By (LC2), it follows that $\lim(\bigcap_{k \in K} \mathcal{I}_k) \subseteq \bigcap_{k \in K} \lim(\mathcal{I}_k)$. Moreover, take each $x_\lambda \in \bigcap_{k \in K} \lim(\mathcal{I}_k)$. Then $x_\lambda \in \lim(\mathcal{I}_k)$ for each $k \in K$. By (LCP)*, $\mathcal{R}_{x_\lambda}^{\lim} \subseteq \mathcal{I}_k$ for each $k \in K$. This implies $\mathcal{R}_{x_\lambda}^{\lim} \subseteq \bigcap_{k \in K} \mathcal{I}_k$. It follows from (LCP)* that $x_\lambda \in \lim(\bigcap_{k \in K} \mathcal{I}_k)$. By the arbitrariness of x_λ , we obtain $\lim(\bigcap_{k \in K} \mathcal{I}_k) \supseteq \bigcap_{k \in K} \lim(\mathcal{I}_k)$, as desired.

(LCP)* \Rightarrow (LCP): By the definition of $\mathcal{R}_{x_\lambda}^{\lim}$, it follows that

$$x_\lambda \in \bigcap_{x_\lambda \in \lim(\mathcal{I})} \lim(\mathcal{I}) \stackrel{(\text{LCP})^*}{=} \lim\left(\bigcap_{x_\lambda \in \lim(\mathcal{I})} \mathcal{I}\right) = \lim(\mathcal{R}_{x_\lambda}^{\lim}).$$

□

Finally, we will find the counterpart of “ L -convex remote-neighborhood systems” corresponding to preconvex L -convergence structures. To this end, we will introduce the following definition by relaxing the axioms of L -convex remote-neighborhood systems, which will be called L -preconvex remote-neighborhood systems.

Definition 5.7. An L -preconvex remote-neighborhood system on X is a set $\mathcal{R} = \{\mathcal{R}_{x_\lambda} \mid x_\lambda \in J(L^X)\}$, where $\mathcal{R}_{x_\lambda} \subseteq L^X$ satisfies (LCR1)–(LCR4) and

$$(LCR0) \quad \mathcal{R}_{x_\lambda} = \bigcap_{\mu \in \beta(\lambda)} \mathcal{R}_{x_\mu}.$$

For an L -preconvex remote-neighborhood system \mathcal{R} on X , the pair (X, \mathcal{R}) is called an L -preconvex remote-neighborhood space.

Definition 5.8. A mapping $\varphi : (X, \mathcal{R}^X) \longrightarrow (Y, \mathcal{R}^Y)$ between L -preconvex remote-neighborhood spaces is called L -convexity-preserving (L -CP, in short) provided that for each $B \in L^Y$ and $x_\lambda \in J(L^X)$, $B \in \mathcal{R}_{\varphi(x)_\lambda}^Y$ implies $\varphi_L^\leftarrow(B) \in \mathcal{R}_{x_\lambda}^X$.

It is easy to check that L -preconvex remote-neighborhood spaces as objects and L -CP mappings as morphisms form a category, denoted by L -PCR.

Next we will show that L -preconvex remote-neighborhood systems and preconvex L -convergence structures are one-to-one corresponding.

Proposition 5.9. Let (X, \mathcal{R}) be an L -preconvex remote-neighborhood space and define $\lim^{\mathcal{R}} : \mathcal{I}_L^C(X) \longrightarrow \mathbb{P}(J(L^X))$ by

$$\forall \mathcal{I} \in \mathcal{I}_L^C(X), \quad \lim^{\mathcal{R}}(\mathcal{I}) = \{x_\lambda \in J(L^X) \mid \mathcal{R}_{x_\lambda} \subseteq \mathcal{I}\}.$$

Then $\lim^{\mathcal{R}}$ is a preconvex L -convergence structure on X .

Proof. It is enough to show that $\lim^{\mathcal{R}}$ satisfies (LC1)–(LC3) and (LCP). (LC1) and (LC2) are obvious.

(LC3) Take each $x_\lambda \in J(L^X)$. Then

$$\mathcal{R}_{x_\lambda}^{\lim^{\mathcal{R}}} = \bigcap_{x_\lambda \in \lim^{\mathcal{R}}(\mathcal{I})} \mathcal{I} = \bigcap_{\mathcal{R}_{x_\lambda} \subseteq \mathcal{I}} \mathcal{I} = \mathcal{R}_{x_\lambda}.$$

By (LCR0), we have

$$\mathcal{R}_{x_\lambda}^{\lim^{\mathcal{R}}} = \mathcal{R}_{x_\lambda} = \bigcap_{\mu \in \beta(\lambda)} \mathcal{R}_{x_\mu} = \bigcap_{\mu \in \beta(\lambda)} \mathcal{R}_{x_\mu}^{\lim^{\mathcal{R}}},$$

as desired.

(LCP) It follows from $\mathcal{R}_{x_\lambda}^{\lim^{\mathcal{R}}} = \mathcal{R}_{x_\lambda}$ that $x_\lambda \in \lim(\mathcal{R}_{x_\lambda}^{\lim^{\mathcal{R}}})$. □

Proposition 5.10. If $\varphi : (X, \mathcal{R}^X) \longrightarrow (Y, \mathcal{R}^Y)$ is L -CP, then so is $\varphi : (X, \lim^{\mathcal{R}^X}) \longrightarrow (Y, \lim^{\mathcal{R}^Y})$.

Proof. Since $\varphi : (X, \mathcal{R}^X) \longrightarrow (Y, \mathcal{R}^Y)$ is L -CP, we know that for each $x_\lambda \in J(L^X)$ and $B \in L^Y$,

$$B \in \mathcal{R}_{\varphi(x)_\lambda}^Y \implies \varphi_L^\leftarrow(B) \in \mathcal{R}_{x_\lambda}^X \iff B \in \varphi^\rightarrow(\mathcal{R}_{x_\lambda}^X).$$

This means that $\mathcal{R}_{\varphi(x)_\lambda}^Y \subseteq \varphi^\rightarrow(\mathcal{R}_{x_\lambda}^X)$. Then for each $\mathcal{I} \in \mathcal{I}_L^C(X)$ and $x_\lambda \in J(L^X)$, we have

$$\begin{aligned} x_\lambda \in \lim^{\mathcal{R}^X}(\mathcal{I}) &\iff \mathcal{R}_{x_\lambda}^X \subseteq \mathcal{I} \\ &\implies \varphi^\rightarrow(\mathcal{R}_{x_\lambda}^X) \subseteq \varphi^\rightarrow(\mathcal{I}) \\ &\implies \mathcal{R}_{\varphi(x)_\lambda}^Y \subseteq \varphi^\rightarrow(\mathcal{I}) \\ &\iff \varphi(x)_\lambda \in \lim^{\mathcal{R}^Y}(\varphi^\rightarrow(\mathcal{I})). \end{aligned}$$

Therefore, $\varphi : (X, \lim^{\mathcal{R}^X}) \longrightarrow (Y, \lim^{\mathcal{R}^Y})$ is L -CP. □

Proposition 5.11. Let (X, \lim) be a preconvex L -convergence space. Then $\mathcal{R}^{\lim} = \{\mathcal{R}_{x_\lambda}^{\lim} \mid x_\lambda \in J(L^X)\}$ is an L -preconvex remote-neighborhood system on X .

Proof. (LCR1) It is obvious.

(LCR2) Take each $A \in \mathcal{R}_{x_\lambda}^{\lim}$. By the definition of $\mathcal{R}_{x_\lambda}^{\lim}$, it follows from $x_\lambda \in \lim([x_\lambda])$ that $A \in [x_\lambda]$. Hence $x_\lambda \leq A'$.

(LCR3) Take each $A, B \in L^X$ such that $A \in \mathcal{R}_{x_\lambda}^{\lim}$ and $B \leq A$. Then it follows that $A \in \mathcal{I}$ for each $x_\lambda \in \lim(\mathcal{I})$. By (LCI2), we have $B \in \mathcal{I}$ for each $x_\lambda \in \lim(\mathcal{I})$. This implies $B \in \mathcal{R}_{x_\lambda}^{\lim}$.

(LCR4) It follows immediately from (LCI3), which is similar to the verifications of (LCR3).

(LCR0) It follows immediately from (LC3). \square

Proposition 5.12. *If $\varphi : (X, \lim_X) \rightarrow (Y, \lim_Y)$ is L -CP, then so is $\varphi : (X, \mathcal{R}^{\lim_X}) \rightarrow (Y, \mathcal{R}^{\lim_Y})$.*

Proof. Since $\varphi : (X, \lim_X) \rightarrow (Y, \lim_Y)$ is L -CP, it follows that

$$\forall \mathcal{I} \in \mathcal{I}_L^C(X), x_\lambda \in J(L^X), x_\lambda \in \lim_X(\mathcal{I}) \implies \varphi(x)_\lambda \in \lim_Y(\varphi^\Rightarrow(\mathcal{I})).$$

By the proof of Proposition 4.10, we know that $\mathcal{R}_{\varphi(x)_\lambda}^{\lim_Y} \subseteq \varphi^\Rightarrow(\mathcal{R}_{x_\lambda}^{\lim_X})$. Now take each $x_\lambda \in J(L^X)$ and $B \in \mathcal{R}_{\varphi(x)_\lambda}^{\lim_Y}$. Then it follows that $B \in \varphi^\Rightarrow(\mathcal{R}_{x_\lambda}^{\lim_X})$. This means $\varphi_L^\leftarrow(B) \in \mathcal{R}_{x_\lambda}^{\lim_X}$. Therefore, $\varphi : (X, \mathcal{R}^{\lim_X}) \rightarrow (Y, \mathcal{R}^{\lim_Y})$ is L -CP. \square

Theorem 5.13. *L -PConv and L -PCR are isomorphic.*

Proof. By Propositions 5.9, 5.10, 5.11 and 5.12, it suffices to show that for an L -preconvex remote-neighborhood system \mathcal{R} on X and a preconvex L -convergence structure \lim on X , the following two statements hold:

(1) $\lim^{\mathcal{R}^{\lim}} = \lim$;

(2) $\mathcal{R}^{\lim^{\mathcal{R}}} = \mathcal{R}$.

For (1), take each $\mathcal{I} \in \mathcal{I}_L^C(X)$ and $x_\lambda \in J(L^X)$. Then

$$x_\lambda \in \lim^{\mathcal{R}^{\lim}}(\mathcal{I}) \iff \mathcal{R}_{x_\lambda}^{\lim} \subseteq \mathcal{I} \stackrel{(LCP)^*}{\iff} x_\lambda \in \lim(\mathcal{I}).$$

This means $\lim^{\mathcal{R}^{\lim}} = \lim$.

For (2), take each $x_\lambda \in J(L^X)$. Then

$$\mathcal{R}_{x_\lambda}^{\lim^{\mathcal{R}}} = \bigcap_{x_\lambda \in \lim^{\mathcal{R}}(\mathcal{I})} \mathcal{I} = \bigcap_{\mathcal{R}_{x_\lambda} \subseteq \mathcal{I}} \mathcal{I} = \mathcal{R}_{x_\lambda}.$$

This implies that $\mathcal{R}^{\lim^{\mathcal{R}}} = \mathcal{R}$. \square

6 Conclusions

In the setting of L -convex spaces, we introduced the concept of L -convergence structures and discussed its relations with L -convex structures from a categorical viewpoint. We showed that there are compatible categorical relations between L -convex spaces and L -convergence spaces. In particular, convex L -convergence spaces are categorically isomorphic to L -convex spaces. Related to the results in this paper, we will consider the following problems as the future research:

- Separation axioms are important spatial properties in convex spaces. Convergence structures provide a new tool for interpreting separation properties of convex spaces. By means of L -convergence structures, we will consider separation properties of L -convex spaces.
- L -convex structures and M -fuzzifying convex structure can be considered as special cases of (L, M) -fuzzy convex structures [22]. That is to say, (L, M) -fuzzy convex structure provide a more general framework for the research of fuzzy convex structures. So we will further consider fuzzy convergence structures in the situation of (L, M) -fuzzy convex structures.

Acknowledgments

The authors would like to express their sincere thanks to the anonymous reviewers and the area editor for their careful reading and constructive comments. This work is supported by the Natural Science Foundation of China (Nos. 11701122, 11871097, 11971448), China Postdoctoral Science Foundation (No. 2017M622563) and Beijing Institute of Technology Research Fund Program for Young Scholars (No. 2020CX04176).

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