

SOME PROPERTIES OF T -FUZZY GENERALIZED SUBGROUPS

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ABSTRACT. In this paper, we deal with Molaei's generalized groups. We define the notion of a fuzzy generalized subgroup with respect to a t -norm (or T -fuzzy generalized subgroup) and give some related properties. Especially, we state and prove the Representation Theorem for these fuzzy generalized subgroups. Next, using the concept of continuity of t -norms we obtain a correspondence between $TF(G)$, the set of all T -fuzzy generalized subgroups of a generalized group G , and the set of all T -fuzzy generalized subgroups of the corresponding quotient generalized group. Subsequently, we study the quotient structure of T -fuzzy generalized subgroups: we define the notion of a T -fuzzy normal generalized subgroup, give some related properties, construct the quotient generalized group, state and prove the homomorphism theorem. Finally, we study the lattice of T -fuzzy generalized subgroups and prove that $TF(G)$ is a Heyting algebra.

1. Introduction

There has been an old problem in mathematical physics: How can one change a Lorentzian Metric to a Rimanian one? In the process of solving this problem many unified theories such as Isotheory have been presented. The theory is based on a method that creates new fields which are called isofields. The notion of a generalized groups introduced by M. R. Molaei, when extending this theory to any manifold, i.e, isomanifold in [10]. This structure was studied more closely by Molaei and others. He studied quotient structure and homomorphism theorems on a generalized group, and also considered them from topological view point. In [6], R. A. Borzooei et al. characterized generalized groups with the order less than 4 and gave some interesting results.

On the other hand, the notion of fuzzy sets has been an interesting subject for mathematicians who have worked on group theory [16] and [17]. In [16], Rosenfeld defined the notion of a fuzzy subgroup. Many authors have worked on fuzzy group theory [11], [12] and [17]. Especially, some authors considered the fuzzy subgroups with respect to a t -norm and gave some results [2], [17] and [13]. In this paper we are interested in studying generalized groups from fuzzy theory point of view with a t -norm.

Key words and phrases: Generalized groups, Fuzzy generalized subgroups, t -norm, Heyting algebra.

This research has been supported in part by Fuzzy Systems Research Center, University of Sistan and Bluchestan, Zahedan, Iran.

2. Preliminaries

Here, we give some definitions and results on generalized groups and t-norms.

Definition 2.1. [10] A nonempty set G endowed with a binary operation “ $*$ ” that satisfies the following properties is called a *generalized group*:

(GG1) For all $x, y, z \in G$, $(x * y) * z = x * (y * z)$.

(GG2) For each $x \in G$, there exists a unique element $e(x) \in G$ such that

$$x * e(x) = e(x) * x = x$$

(GG3) For each $x \in G$, there exists an element $x^{-1} \in G$ such that

$$x * x^{-1} = x^{-1} * x = e(x)$$

A nonempty subset H of a generalized group G is called a *generalized subgroup* if it is a generalized group under the operations of G .

Theorem 2.2. [10] A nonempty subset H of a generalized group G is a generalized subgroup if and only if for all $x, y \in H$, $xy^{-1} \in H$.

Proposition 2.3. [10] Define a binary relation \sim on a generalized group G by $x \sim y$ iff $e(x) = e(y)$. \sim is an equivalence relation on G , and every equivalence class G_a is a (generalized) subgroup of G .

Example 2.4. [10] (i) Let $G = \mathbb{R} \times (\mathbb{R} \setminus \{0\})$. Then G with the operation $(x, y) * (w, z) = (yw, yz)$ is a generalized group in which for all $(x, y) \in G$, $e(x, y) = (x/y, 1)$ and $(x, y)^{-1} = (x/y^2, 1/y)$.

(ii) $M_{2 \times 2}$, the set of all 2×2 matrices with the operation

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & b_{12} \\ b_{21} & a_{22} \end{pmatrix}$$

is a generalized group in which for all $A \in M_{2 \times 2}$, $e(A) = A^{-1} = A$.

(iii) Let $G = \left\{ A = \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix} : a, b \in \mathbb{R}, b \neq 0 \right\}$. Then G together with the natural multiplication of matrices is a generalized group in which for all $A \in G$,

$$e(A) = \begin{pmatrix} 0 & 0 \\ a/b & 1 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} 0 & 0 \\ a/b^2 & 1/b \end{pmatrix}.$$

Proposition 2.5. [10] In any generalized group G the following hold: $\forall x \in G$,

- (i) $e(e(x)) = e(x)$,
- (ii) $e(x^{-1}) = e(x) = (e(x))^{-1}$,
- (iii) x^{-1} is unique,
- (iv) $(x^{-1})^{-1} = x$.

Definition 2.6. A function $T : [0, 1] \times [0, 1] \rightarrow [0, 1]$ satisfying the following properties is called a *t-norm*, if for all $x, y, z \in [0, 1]$,

- (T1) $T(x, 1) = x$,
- (T2) $T(x, y) \leq T(x, z)$, if $y \leq z$,
- (T3) $T(x, y) = T(y, x)$,
- (T4) $T(x, T(y, z)) = T(T(x, y), z)$.

Here are some examples of t-norms:

- $T_D(x, y) = \begin{cases} x \wedge y & ; \quad x \vee y = 1, \\ 0 & ; \quad \text{otherwise.} \end{cases}$ (Drastic product)
- $T_L(x, y) = 0 \vee (x + y - 1)$. (Łukasiewicz)
- $T_P(x, y) = xy$. (Product)
- $T_M(x, y) = x \wedge y$. (Minimum)
- $T_{nM}(x, y) = \begin{cases} 0 & ; \quad x + y \leq 1, \\ x \wedge y & ; \quad \text{otherwise.} \end{cases}$ (Nilpotent minimum)

where $\wedge = \min$ and $\vee = \max$.

Every t-norm T satisfies the inequality $T_D(x, y) \leq T(x, y) \leq T_M(x, y)$.

Definition 2.7. For $n \in \mathbb{N}$, the function $T_n : \prod_{i \in I_n} [0, 1] \rightarrow [0, 1]$ is defined by

$$T_n(a_1, a_2, \dots, a_n) = T(a_i, T_{n-1}(a_1, a_2, \dots, a_{i-1}, a_{i+1}, \dots, a_n)),$$

for all $i \in I_n$ where $T_2 = T$ and $T_1 = id$ (identity).

Definition 2.8. Let $f : X \rightarrow Y$ be a function, μ and ν be fuzzy subsets of X and Y , respectively. The image of μ and the preimage of ν under f are defined as follows:

$$f(\mu)(y) = \begin{cases} \sup_{x \in f^{-1}(y)} \mu(x) & ; \quad f^{-1}(y) \neq \emptyset, \\ 0 & ; \quad \text{otherwise.} \end{cases}$$

$$f^{-1}(\nu)(x) = \nu(f(x)).$$

for all $y \in Y$ and $x \in X$.

A *relatively pseudocomplemented lattice* is an algebra $(L, \wedge, \vee, \rightarrow, 1)$ where $(L, \wedge, \vee, 1)$ is a lattice with the greatest element 1 and the binary operation \rightarrow on L verifies:

$$\forall x, y, z \in L, \quad x \rightarrow y \leq z \Leftrightarrow x \wedge y \leq z.$$

A *Heyting algebra* is a duplicate name for bounded relatively pseudocomplemented lattice. We say that L satisfies the *Join Infinite Distributive identity* (JID) if

$$x \wedge \left(\bigvee_{\alpha \in I} y_\alpha \right) = \bigvee_{\alpha \in I} (x \wedge y_\alpha)$$

for any indexed set I , whenever the arbitrary unions exist.

Remark 2.9. By [4, Theorem 15], a complete lattice is relatively pseudocomplemented (i.e. a Heyting algebra) if and only if it satisfies (JID).

3. T-fuzzy Generalized Subgroups

From now on, in this paper, G denotes a generalized group, T a t-norm and $\text{FS}(G)$ the set of all fuzzy subsets of G , unless otherwise specified.

Definition 3.1. Let $\mu \in \text{FS}(G)$. We say that μ is a *fuzzy generalized subgroup with respect to t-norm T* , or a *T -fuzzy generalized subgroup* if for all $x, y \in G$,

- (TF1) $\mu(xy) \geq T(\mu(x), \mu(y))$,
- (TF2) $\mu(x^{-1}) = \mu(x)$.

From now on, in this paper $TF(G)$ denotes the set of all T -fuzzy generalized subgroups of G

Note 3.2. If G is a group, any fuzzy subset of G satisfying the conditions (TF1) and (TF2) is called a T -fuzzy subgroup, (see [17]).

Example 3.3. (i) Let $G = \{a, b, c\}$ and the multiplication is defined by $xy = x$, for all $x, y \in G$. Then G is a generalized group. Now, define a fuzzy subset μ in G by $\mu(a) = 1/2$, $\mu(b) = 1/4$, and $\mu(c) = 1/8$. Then μ is a T_M -fuzzy generalized subgroup of G .

(ii) Let G be the generalized group defined as in the following table:

*	x_1	x_2	x_3	x_4
x_1	x_1	x_1	x_3	x_3
x_2	x_2	x_2	x_4	x_4
x_3	x_1	x_1	x_3	x_3
x_4	x_2	x_2	x_4	x_4

Let fuzzy subset μ of G be defined by $\mu(x_i) = 1/i$, for $i \in \{1, 2, 3, 4\}$. Then μ is a T_P -fuzzy generalized subgroup of G , which is not a T_M -fuzzy generalized subgroup, because $\mu(x_2 * x_3) = \mu(x_4) = 1/4 \not\geq 1/3 = \mu(x_2) \wedge \mu(x_3)$.

Remark 3.4. It is well-known that any ' T_M -fuzzy generalized subgroup' is called a 'fuzzy generalized subgroup'.

Definition 3.5. [7] For a t -norm T , an element $a \in [0, 1]$ is said to be *idempotent* if $T(a, a) = a$.

Obviously, 0 and 1 are idempotent elements of any t -norm T , which are called the *trivial* idempotent elements of T . For t -norm T_M , the idempotent elements are members of $[0, 1]$ and for nilpotent minimum T_{nM} , the set $\{0\} \cup (0.5, 1]$ consists of all idempotents elements.

Definition 3.6. We say that a T -fuzzy subset μ of a set X is *imaginable* if every element of $Im\mu$ is an idempotent element of T . Obviously, every T_M -fuzzy generalized subgroup is imaginable.

Proposition 3.7. *Every imaginable T -fuzzy generalized subgroup is a fuzzy generalized subgroup.*

Proof. Let μ be a T -fuzzy generalized subgroup of G . It suffices to prove that $\mu(xy) \geq \mu(x) \wedge \mu(y)$. For this, let $x, y \in G$. Since, μ is imaginable, we have:

$$\begin{aligned} \mu(x) \wedge \mu(y) &= T(\mu(x) \wedge \mu(y), \mu(x) \wedge \mu(y)) \\ &\leq T(\mu(x), \mu(y)) \\ &\leq \mu(xy) \end{aligned}$$

Hence, $\mu(xy) \geq \mu(x) \wedge \mu(y)$. □

Lemma 3.8. *Let μ be a fuzzy subset of a set X . Then:*

$$\mu_t = \bigcap_{s \in [0, t)} \mu_s \text{ and } \mu_{t>} = \bigcup_{s \in (t, 1]} \mu_s$$

for all $t \in [0, 1]$, where $\mu_t = \{x \in G : \mu(x) \geq t\}$ and $\mu_{t>} = \{x \in G : \mu(x) > t\}$.

Proof. The proof is easy. □

Theorem 3.9. *The intersection of any family of generalized subgroups of a generalized group is again a generalized subgroup.*

Proof. The proof is straightforward. \square

Theorem 3.10. *For imaginable T -fuzzy subset μ of G , the following statements are equivalent:*

- (i) μ is a T -fuzzy generalized subgroup of G ,
- (ii) every nonempty strong level subset $\mu_{t>}$, $t \in [0, 1]$, is a generalized subgroup of G ,
- (iii) every nonempty level subset μ_t , $t \in [0, 1]$, is a generalized subgroup of G .

Proof. (i) \Rightarrow (ii) By Theorem 2.2, it suffices to show that for any $a, b \in \mu_{t>}$ and $t \in [0, 1]$, $ab^{-1} \in \mu_{t>}$. Now, let $a, b \in \mu_{t>}$. Then, $\mu(a) > t$ and $\mu(b) > t$ and hence $\mu(ab^{-1}) \geq T(\mu(a), \mu(b)) > T(t, t) = t$, that is, $ab^{-1} \in \mu_{t>}$.

(ii) \Rightarrow (iii) Let $a, b \in \mu_t$, for $t \in [0, 1]$. By Lemma 3.8, $a, b \in \mu_{s>}$, for all $s \in [0, t)$, and so by (ii), $ab^{-1} \in \mu_{s>}$. Hence, $ab^{-1} \in \mu_t$ proving that μ_t is a generalized subgroup of G .

(iii) \Rightarrow (i) The proof follows from (ii), and Theorem 3.9 and Lemma 3.8. \square

Lemma 3.11. *Let μ be an imaginable T -fuzzy generalized subgroup of G . Then for all $x \in G$,*

- (i) $\mu(e(x)) \geq \mu(x)$.
- (ii) $\mu(x^n) \geq \mu(x)$, for all $n \in \mathbb{N}$.

Proof. To (i) it suffices to observe that for all $x \in G$, $e(x) = xx^{-1}$, and the proof of (ii) is followed by an inductive procedure on n . \square

Theorem 3.12. *Every T -fuzzy generalized subgroup μ of G , satisfies the following property;*

$$(3.1) \quad \mu(xy^{-1}) \geq T(\mu(x), \mu(y)).$$

for all $x, y \in G$. Conversely, every imaginable T -fuzzy subset μ of G that satisfies (3.1) is a T -fuzzy generalized subgroup of G .

Proof. Obviously, every T -fuzzy generalized subgroup of G satisfies the condition (3.1). Conversely, let μ be an imaginable T -fuzzy subset of G that satisfies condition (3.1) and let $x \in G$. Then,

$$\begin{aligned} \mu(x) &= \mu(e(x)x) = \mu(e(x^{-1})(x^{-1})^{-1}) \geq T(\mu(e(x^{-1})), \mu(x^{-1})) \\ &\geq T(\mu(x^{-1}), \mu(x^{-1})) = \mu(x^{-1}). \end{aligned}$$

By a similar argument, we can prove that $\mu(x^{-1}) \geq \mu(x)$. Thus, $\mu(x^{-1}) = \mu(x)$. Now, let $x, y \in G$. Then,

$$\mu(xy) = \mu(x(y^{-1})^{-1}) \geq T(\mu(x), \mu(y^{-1})) = T(\mu(x), \mu(y))$$

\square

Theorem 3.13. *Every generalized subgroup of G is a level subset of a fuzzy generalized subgroup of G .*

Proof. Let H be a generalized subgroup of G and let the fuzzy subset μ of G be defined by

$$\mu(x) = \begin{cases} s & \text{if } x \in H, \\ t & \text{otherwise,} \end{cases}$$

where $0 \leq t < s \leq 1$. We show that μ is a fuzzy generalized subgroup of G . Let $x, y \in G$ be such that $x, y \in H$. Then $xy \in H$ and so

$$\mu(xy) = s \geq \mu(x) \wedge \mu(y).$$

If $x \in H$ and $y \notin H$ (or $x \notin H$ and $y \in H$), then

$$\mu(xy) = s \geq t = \mu(x) \wedge \mu(y).$$

Now, let $x \in G$. If $x \in H$, then $x^{-1} \in H$ and so $\mu(x^{-1}) = s = \mu(x)$. Otherwise, $x, x^{-1} \notin H$ and so $\mu(x^{-1}) = t = \mu(x)$. \square

Theorem 3.14. Let $\mu \in \text{FS}(G)$ and $\text{Im}\mu = \{s_0, s_1, s_2, \dots, s_n\}$, where $s_i < s_j$, for $i > j$. If there exists a chain

$$H_0 \subseteq H_1 \subseteq \dots \subseteq H_n = G$$

of generalized subgroups of G such that $\mu(H'_k) = s_k$, where $H'_0 = H_0$, $H'_k = H_k \setminus H_{k-1}$, for $k = 1, 2, \dots, n$, then μ is a T -fuzzy generalized subgroup of G .

Proof. Let $x, y \in G$. If $x, y \in H'_k$, for some $k \in \{0, 1, \dots, n\}$, then $x, y \in H_k$ and $x, y \notin H_{k-1}$, which imply that $xy \in H_k$ and $xy \notin H_{k-1}$, because H'_i 's are disjoint. Thus, $\mu(xy) = s_k \geq T(s_k, s_k) = T(\mu(x), \mu(y))$. Otherwise, if $x \in H'_k$ and $y \in H'_l$, for $k \neq l$, without loss of generality, we can assume that $l < k$. Then $\mu(x) = s_k \leq s_l = \mu(y)$ and $H_l \subseteq H_k$ and so $x \in H'_k$, $y \in H_k$ whence, $xy \in H'_k$. This implies that $\mu(xy) = s_k = T(s_k, 1) \geq T(\mu(x), \mu(y))$.

Obviously, for all $x \in G$ there exists $k \in \{0, 1, \dots, n\}$ such that $x \in H'_k$ and so $x^{-1} \in H'_k$. Hence, $\mu(x^{-1}) = \mu(x)$, completes the proof. \square

Theorem 3.15. (*Representation Theorem*) Let $\{H_\alpha : \alpha \in [0, 1]\}$ be a class of generalized subgroups of G . The necessary and sufficient condition for there exists a T -fuzzy generalized subgroup μ of G such that for all $\alpha \in [0, 1]$ we have $\mu_\alpha = H_\alpha$ is that for all $M \subseteq [0, 1]$,

$$(3.2) \quad H_{\bigvee_{\alpha \in M} \alpha} = \bigcap_{\alpha \in M} H_\alpha.$$

Proof. Suppose that (3.2) holds, and define a fuzzy subset μ of G by

$$\mu(x) = \bigvee_{x \in H_\alpha} \alpha, \quad \forall x \in G.$$

We prove that μ is a T -fuzzy generalized subgroup of G . Let $x, y \in G$ and

$$T\left(\bigvee_{x \in H_\beta} \beta, \bigvee_{y \in H_\gamma} \gamma\right) = T(\mu(x), \mu(y)) = t.$$

Then $\bigvee_{x \in H_\beta} \beta \geq t$ and $\bigvee_{y \in H_\gamma} \gamma \geq t$. Now, set $M = \{\alpha \in [0, 1] : x \in H_\alpha\}$. Then,

$$x \in \bigcap_{\beta \in M} H_\beta = H_{\bigvee_{\beta \in M} \beta} \subseteq H_t,$$

and similarly we can deduce that $y \in H_t$. Hence, $xy \in H_t$ and so

$$\mu(xy) = \bigvee_{xy \in H_\alpha} \alpha \geq t = T(\mu(x), \mu(y)).$$

Now, let $x \in G$. Since, H_α is a generalized subgroup of G , then $x \in H_\alpha$ iff $x^{-1} \in H_\alpha$, and hence

$$\mu(x^{-1}) = \bigvee_{x^{-1} \in H_\alpha} \alpha = \bigvee_{x \in H_\alpha} \alpha = \mu(x).$$

Now, we show that $\mu_\alpha = H_\alpha$, for all $\alpha \in [0, 1]$. Let $x \in H_t$, for some $t \in [0, 1]$. Then, $\mu(x) = \bigvee_{x \in H_\alpha} \alpha \geq t$ that is, $x \in \mu_t$ and so $H_t \subseteq \mu_t$. Now, if $x \in \mu_t$, then $\bigvee_{x \in H_\alpha} \alpha = \mu(x) \geq t$ and so for $M = \{\alpha \in [0, 1] : x \in H_\alpha\}$ we have $x \in \bigcap_{\alpha \in M} H_\alpha = H_{\bigvee_{\alpha \in M} \alpha} \subseteq H_t$ that is, $\mu_t \subseteq H_t$. Thus, we proved that $\mu_t = H_t$, for all $t \in [0, 1]$.

Conversely, suppose that there exists a T -fuzzy generalized subgroup μ of G such that for all $t \in [0, 1]$, $\mu_t = H_t$. We prove the validity of (3.2). For this, let $M \subseteq [0, 1]$ and let $x \in H_\beta$, where $\beta = \bigvee_{\alpha \in M} \alpha$. Then, $x \in \mu_\beta$ and so $\mu(x) \geq \alpha$, for all $\alpha \in M$. This implies

that $x \in \mu_\alpha = H_\alpha$, for all $\alpha \in M$, and so $x \in \bigcap_{\alpha \in M} H_\alpha$. Hence,

$$H_{\bigvee_{\alpha \in M} \alpha} \subseteq \bigcap_{\alpha \in M} H_\alpha.$$

The converse of inequality proved similarly. \square

4. Some Results on T -fuzzy Generalized Subgroups

It is easy to verify that the preimage of any T -fuzzy generalized subgroup under a homomorphism is a T -fuzzy generalized subgroup. While, this is not true for the image, in general. But, we expect the continuous t -norms prepare this for us. The next theorems demonstrate the role of continuous t -norms in this situation.

First, we give the definition of a continuous t -norm.

Definition 4.1. A t -norm T is said to be *continuous* if for every $y \in [0, 1]$ and for every convergent sequence $(x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}} \in [0, 1]^{\mathbb{N}}$ we have

$$\lim_{n \rightarrow \infty} T(x_n, y_n) = T\left(\lim_{n \rightarrow \infty} x_n, \lim_{n \rightarrow \infty} y_n\right).$$

Observe that the t -norms T_M, T_P and T_L are continuous and the drastic product, T_D , is not continuous.

Lemma 4.2. Let T be a continuous t -norm on a set X , A and B are nonempty subsets of X and μ be a fuzzy subset of X . Then

$$\sup_{\substack{x \in A \\ y \in B}} T(\mu(x), \mu(y)) = T\left(\sup_{x \in A} \mu(x), \sup_{y \in B} \mu(y)\right).$$

Proof. Obviously,

$$\sup_{\substack{x \in A \\ y \in B}} T(\mu(x), \mu(y)) \leq T\left(\sup_{x \in A} \mu(x), \sup_{y \in B} \mu(y)\right).$$

Now, we prove the inverse of the inequality. We first observe that since T is continuous, for all $\epsilon > 0$ there exists $\delta > 0$ such that for all $x', y' \in [0, 1]$ such that $\sup_{x \in A} \mu(x) - \mu(x') \leq \delta$ and $\sup_{y \in B} \mu(y) - \mu(y') \leq \delta$ imply that

$$T\left(\sup_{x \in A} \mu(x), \sup_{y \in B} \mu(y)\right) \leq T(\mu(x'), \mu(y')) + \epsilon.$$

Now, choose $a \in A$ and $b \in B$ such that $\sup_{x \in A} \mu(x) - \mu(a) \leq \delta$ and $\sup_{y \in B} \mu(y) - \mu(b) \leq \delta$.

Then

$$T\left(\sup_{x \in A} \mu(x), \sup_{y \in B} \mu(y)\right) \leq T(\mu(a), \mu(b)) + \epsilon$$

and since $\epsilon > 0$ is arbitrary,

$$T\left(\sup_{x \in A} \mu(x), \sup_{y \in B} \mu(y)\right) \leq T(\mu(a), \mu(b)),$$

for all $a \in A$ and $b \in B$, and so

$$T\left(\sup_{x \in A} \mu(x), \sup_{y \in B} \mu(y)\right) \leq \sup_{\substack{x \in A \\ y \in B}} T(\mu(x), \mu(y)).$$

□

Theorem 4.3. *Let $f : G \rightarrow H$ be a homomorphism of the generalized groups G and H . If T is a continuous t -norm, then the image of any T -fuzzy generalized subgroup of G is a T -fuzzy generalized subgroup of H .*

Proof. Let $y_1, y_2 \in G$. We first observe that if $f^{-1}(y_1) = f^{-1}(y_2) = \emptyset$, then

$$f(\mu)(y_1 y_2) \geq 0 = T(0, 0) = T\left(\sup_{x_1 \in f^{-1}(y_1)} \mu(x_1), \sup_{x_2 \in f^{-1}(y_2)} \mu(x_2)\right) = T(f(\mu)(y_1), f(\mu)(y_2))$$

Then, suppose that there exist $x_1, x_2 \in G$ such that $y_1 = f(x_1)$, $y_2 = f(x_2)$, and set $x = x_1 x_2$. It is obvious that

$$\{x \in G : x = a_1 a_2, \text{ for some } a_1 \in f^{-1}(y_1) \text{ and } a_2 \in f^{-1}(y_2)\} \subseteq f^{-1}(y_1 y_2).$$

Hence,

$$\begin{aligned} f(\mu)(y_1 y_2) &= \sup_{x \in f^{-1}(y_1 y_2)} \mu(x) \geq \sup_{\substack{x_1 \in f^{-1}(y_1) \\ x_2 \in f^{-1}(y_2)}} \mu(x_1 x_2) \\ &\geq \sup_{\substack{x_1 \in f^{-1}(y_1) \\ x_2 \in f^{-1}(y_2)}} T(\mu(x_1), \mu(x_2)) \\ &\geq T\left(\sup_{x_1 \in f^{-1}(y_1)} \mu(x_1), \sup_{x_2 \in f^{-1}(y_2)} \mu(x_2)\right) \text{ by Lemma 4.2} \\ &= T(f(\mu)(y_1), f(\mu)(y_2)), \end{aligned}$$

completes the proof. □

Now, we give the notion of normal generalized subgroup which will be used in this section. In the next section, we consider this notion, more closely.

Definition 4.4. Let $f : G \rightarrow H$ be a homomorphism of the generalized groups and $f_a = f|_{G_a}$. By $\ker f$ we mean $\bigcup_{a \in G} \ker f_a$.

Definition 4.5. A generalized subgroup N of the generalized group G is said to be *normal* if there exist a generalized group H and a homomorphism $f : G \rightarrow H$ such that for each $a \in G$, $N_a = \ker f_a$ provided that $N_a \neq \emptyset$, where $N_a = N \cap G_a$.

Example 4.6. Let G be the generalized group in Example 2.4(i) and fix the natural numbers m and n . Let $N_i = \{(x, y) \in G : x = iy\}$, for $i \in \{m, n\}$, and $N = N_m \cup N_n$. Now, define a mapping $f : G \rightarrow G$ by $f(x, y) = e(x, y)$. We can see that f is a homomorphism, $N_{(m,1)} = \ker f_{(m,1)}$, $N_{(n,1)} = \ker f_{(n,1)}$, and for all $(x, y) \in G$ with $(x, y) \notin \{(m, 1), (n, 1)\}$ we have $N_{(x,y)} = \emptyset$. Thus, N is a normal generalized subgroup of G .

Note 4.7. By Definition 4.5, it is obvious that for a normal generalized subgroup N of G , N_a (with $a \in G$) is a normal subgroup of the group G_a .

Theorem 4.8. [10] *Let N be a normal generalized subgroup of the generalized group G . Then $G/N = \bigcup_{a \in G} G_a/N_a$ with the multiplication $(xN_a)(yN_b) = xyN_{ab}$ is a generalized group in which $e(xN_a) = e(x)N_a$ and $(xN_a)^{-1} = x^{-1}N_a$.*

Proposition 4.9. *Let T be a continuous t -norm, N a normal generalized subgroup and μ be a T -fuzzy generalized subgroup of G . Then, the fuzzy subset $\bar{\mu} \in \text{FS}(G/N)$ which is defined by $\bar{\mu}(xN_a) = \sup_{w \in xN_a} \mu(w)$ is a T -fuzzy generalized subgroup of G/N .*

Proof. Let $xN_a, yN_b \in G/N$. Then,

$$\begin{aligned} \bar{\mu}(xN_a yN_b) &= \bar{\mu}(xyN_{ab}) = \sup_{w \in xyN_{ab}} \mu(w) \\ &= \sup_{w \in xN_a yN_b} \mu(w) = \sup_{\substack{w_1 \in xN_a \\ w_2 \in yN_b}} \mu(w_1 w_2) \\ &\geq \sup_{\substack{w_1 \in xN_a \\ w_2 \in yN_b}} T(\mu(w_1), \mu(w_2)) \\ &\geq T\left(\sup_{w_1 \in xN_a} \mu(w_1), \sup_{w_2 \in yN_b} \mu(w_2)\right), \text{ by Lemma 4.2} \\ &= T(\bar{\mu}(xN_a), \bar{\mu}(yN_b)). \end{aligned}$$

Also,

$$\bar{\mu}((xN_a)^{-1}) = \bar{\mu}(x^{-1}N_a) = \sup_{w \in x^{-1}N_a} \mu(w) = \sup_{w^{-1} \in xN_a} \mu(w^{-1}) = \sup_{z \in xN_a} \mu(z) = \bar{\mu}(xN_a) \quad \square$$

Proposition 4.10. *Let N be a normal generalized subgroup of G and $\bar{\mu} \in \text{FS}(G/N)$. Then $\mu \in \text{FS}(G)$ defined by $\mu(x) = \bar{\mu}(xN_a)$ is a T -fuzzy generalized subgroup of G .*

Proof. It is easy. □

Theorem 4.11. (*Correspondence Theorem*) *Let T be a continuous t -norm and N be a normal generalized subgroup of G . Then there exists a correspondence bijection between $\text{TF}(G)$ and $\text{TF}(G/N)$.*

Proof. By Propositions 4.9 and 4.10, it is easy to verify that the mapping

$$\begin{aligned} \Phi : \text{TF}(G) &\longrightarrow \text{TF}(G/N) \\ \mu &\longmapsto \bar{\mu} \end{aligned}$$

is a bijection. □

5. T -fuzzy Normal Generalized Subgroups

We start this section with a definition.

Definition 5.1. A T -fuzzy generalized subgroup μ of G is said to be a T -fuzzy normal generalized subgroup, notationally $\mu \triangleleft \text{TF}(G)$, if every nonempty level subset μ_t is a normal generalized subgroup of G .

Example 5.2. Let G be the generalized group in Example 2.4(i), and for $m \in \mathbb{N}$ define a fuzzy subset μ of G by $\mu(x, y) = s_k$ if $x = ky$ (with $k \in \{1, 2, \dots, m\}$) and $\mu(x, y) = 0$, otherwise, where $0 = s_0 < s_1 < s_2 < \dots < s_m < 1$. Then it is easy to verify that, for all $t \in (s_m, 1]$, $\mu_t = \emptyset$, and then there is nothing to do. But for $t \in (s_{k-1}, s_k]$ we have

$$\mu_t = N_k \cup N_{k+1} \cup \dots \cup N_m$$

where $N_k = \{(x, y) \in G : x = ky\}$, and $k \in \{1, 2, \dots, m\}$. As in Example 4.6, for $f_t : G \rightarrow G$ by $f_t(x, y) = e(x, y)$ ($t \in [0, 1]$), we can see that

$$(\mu_t)_{(a,b)} = \begin{cases} \ker(f_t)_{(m,1)} & , (a, b) = (m, 1), \\ \ker(f_t)_{(m-1,1)} & , (a, b) = (m-1, 1), \\ \vdots & \\ \ker(f_t)_{(1,1)} & , (a, b) = (1, 1), \\ \emptyset & , \text{otherwise} \end{cases}$$

which shows that μ is a T -fuzzy normal generalized subgroup of G .

It can be checked easily, that the following theorem holds for a T -fuzzy subgroup as well as fuzzy subgroups.

Theorem 5.3. For any T -fuzzy subgroup μ of the group G , the following conditions hold: $\forall x, y \in G$,

- (1) $\mu(xy) = \mu(yx)$,
- (2) $\mu(xyx^{-1}) = \mu(y)$,
- (3) $\mu(xyx^{-1}) \geq \mu(y)$,
- (4) $\mu(xyx^{-1}) \leq \mu(y)$.

Definition 5.4. A T -fuzzy subgroup μ of the group G that satisfies the equivalent conditions of Theorem 5.3 is said to be a T -fuzzy normal subgroup.

Theorem 5.5. A fuzzy subset μ of the group G is a T -fuzzy (normal) subgroup if and only if every nonempty level subset μ_t is a (normal) subgroup of G , for all $t \in [0, 1]$.

For T -fuzzy (generalized) subgroup μ of the (generalized) group G , let $\mu_* = \{x \in G : \mu(x) = \mu(e)\}$ ($\{x \in G : \mu(x) = \mu(e(x))\}$). By reformulating Theorems 1.3.4 and 1.3.10 of [11] we have the following result.

Theorem 5.6. Let G be a group and μ be a T -fuzzy subgroup of G . Then

- (i) $x\mu = y\mu$ iff $x\mu_* = y\mu_*$.
- (ii) If μ is a T -fuzzy normal subgroup of G , then μ_* is a normal subgroup of G .

Lemma 5.7. For T -fuzzy generalized subgroup μ of G , $a \in G$ and $t \in [0, 1]$ we have $(\mu_a)_t = (\mu_t)_a$. Especially, $(\mu_a)_* = (\mu_*)_a$.

Proof. It is easy. □

Remark 5.8. Obviously, the restriction of any T -fuzzy generalized subgroup μ of G to G_a , notationally μ_a , is a T -fuzzy subgroup of the group G_a , also if μ is normal, then it can be seen that μ_a is normal, too. Now, let $\mu \triangleleft \text{TF}(G)$. By the definition, every nonempty level subset μ_t (with $t \in [0, 1]$) is a normal generalized subgroup of G . So, $G_a/(\mu_t)_a$ is a group and so $G/\mu_t = \bigcup_{a \in G} G_a/(\mu_t)_a$ is a generalized group, by Theorem 4.8. Now, let μ be imaginable. By reformulating [12, Theorem 4.5], with respect to T , it follows that G_a/μ_a , the set of all fuzzy cosets of μ_a , with the multiplication $x\mu_a y\mu_a = xy\mu_a$ is a group. Let $G/\mu = \bigcup_{a \in G} G_a/\mu_a$ and define the binary operation “ \cdot ” in G/μ by $x\mu_a \cdot y\mu_b = xy\mu_{ab}$. We will prove that $(G/\mu, \cdot)$ is a generalized group.

Definition 5.9. A T -fuzzy generalized subgroup μ of G is said to have the *tip* t if for all $x \in G$, $\mu(e(x)) = t$.

Theorem 5.10. Let μ be an imaginable T -fuzzy generalized subgroup of G with tip t . Then the set μ_* is a generalized subgroup of G .

Proof. Let $x, y \in \mu_*$. Then

$$\mu(xy^{-1}) \geq T(\mu(x), \mu(y)) = T(\mu(e(x)), \mu(e(y))) = T(t, t) = t = \mu(e(xy^{-1}))$$

The converse of the inequality holds by Lemma 3.11. \square

Lemma 5.11. Let G be a normal generalized group and μ be an imaginable T -fuzzy generalized subgroup, with tip t , of G . Then

$$(\mu_*)_a(\mu_*)_b \subseteq (\mu_*)_{ab}.$$

Proof. Let $z \in (\mu_*)_a(\mu_*)_b$. Then there exist $x \in (\mu_*)_a$ and $y \in (\mu_*)_b$ such that $z = xy$. But, this means that $x \in G_a$, $\mu(x) = \mu(e(x)) = t$, $y \in G_b$ and $\mu(y) = \mu(e(y)) = t$. Now,

$$\mu(xy) \geq T(\mu(x), \mu(y)) \geq T(t, t) = t = \mu(e(xy))$$

which implies that $xy \in \mu_*$. This together with $xy \in G_a G_b \subseteq G_{ab}$ implies that $xy \in (\mu_*)_{ab}$. \square

Theorem 5.12. Let G be a normal generalized group and μ be an imaginable T -fuzzy normal generalized subgroup with tip t , of G . Then $(G/\mu, \cdot)$ is a generalized group.

Proof. We first prove that \cdot is well-defined. Let $x_1\mu_a = x_2\mu_a$ and $y_1\mu_b = y_2\mu_b$, for $x_1, x_2 \in G_a$ and $y_1, y_2 \in G_b$. By Theorem 5.6(i), $x_1(\mu_a)_* = x_2(\mu_a)_*$ and $y_1(\mu_b)_* = y_2(\mu_b)_*$, and so

$$\begin{aligned} (x_1 y_1)(\mu_{ab})_* &= (x_1 y_1)(\mu_*)_{ab} \quad \text{by Lemma 5.7} \\ &= x_1(\mu_*)_a y_1(\mu_*)_b \quad \text{by Remark 2.9} \\ &= x_1(\mu_a)_* y_1(\mu_b)_* \\ &= x_2(\mu_a)_* y_2(\mu_b)_* \quad \text{by hypothesis} \\ &= x_2(\mu_*)_a y_2(\mu_*)_b \\ &= x_2 y_2(\mu_*)_{ab} \\ &= (x_2 y_2)(\mu_{ab})_*. \end{aligned}$$

This implies that $(x_1 y_1)\mu_{ab} = (x_2 y_2)\mu_{ab}$ proving that \cdot is well-defined. The other properties is easily followed from that of G . \square

Theorem 5.13. [9] If $f : G \longrightarrow H$ is a homomorphism of generalized groups, then $G/\ker f \simeq f(G)$.

Theorem 5.14. *Let μ be an imaginable T -fuzzy normal generalized subgroup, with tip t , of normal generalized group G . Then, $G/\mu \simeq G/\mu_*$.*

Proof. By Theorem 5.6(i) and Lemma 5.7, it is easily verified that the mapping $\Phi : G/\mu \rightarrow G/\mu_*$ by $\Phi(x\mu_a) = x(\mu_*)_a$ is well-defined and one-to-one. Obviously, it is onto. Moreover,

$$\Phi(xy(\mu_a)) = xy(\mu_*)_a = xy(\mu_a)_* = x(\mu_a)_*y(\mu_a)_* = x(\mu_*)_ay(\mu_*)_a = \Phi(x\mu_a)\Phi(y\mu_a)$$

which shows that Φ is a homomorphism. Therefore, Φ is an isomorphism. \square

By Theorems 5.13 and 5.14 we have the following theorem.

Theorem 5.15. (*Homomorphism Theorem*) *Let G be a normal generalized group, μ an imaginable T -fuzzy normal generalized subgroup, with tip t , of G and $f : G \rightarrow H$ be a homomorphism of generalized groups such that $\mu_* = \ker f$. Then, $G/\mu \simeq f(G)$.*

6. Lattice of T -fuzzy Generalized Subgroups

It is easy to verify that the intersection of any family of (T -fuzzy) generalized subgroups of G is again a (T -fuzzy) generalized subgroup of G . Hence, we can speak about the (T -fuzzy) generalized subgroup of G generated by a nonempty subset X of G (of $\text{FS}(G)$), denoted by $\langle X \rangle$, i.e., the intersection of all (T -fuzzy) generalized subgroups of G contain X .

Corollary 6.1. *($\text{FS}(G), \subseteq$) is a complete lattice.*

Notation. For nonempty subset X of G let $X^{-1} = \{x^{-1} : x \in X\}$.

Theorem 6.2. *Let G be a normal generalized group satisfying*

$$(CI) \quad xe(y) = e(y)x, \text{ for } (\emptyset \neq)X \subseteq G \text{ and } \forall x, y \in X \cup X^{-1}.$$

Then,

$$(6.1) \quad \langle X \rangle = \{x_1^{n_1}x_2^{n_2} \cdots x_k^{n_k} : x_i \in X, n_i \in \mathbb{Z}, i = 1, 2, \dots, k\}$$

Proof. Let A be the right hand side of (6.1). Obviously, $X \subseteq A$. Now, let $x, y \in A$. Then there exist natural numbers k and l , integers n_i, m_j with $i \in \{1, 2, \dots, k\}$ and $j \in \{1, 2, \dots, l\}$ and $a_1, a_2, \dots, a_k, b_1, b_2, \dots, b_l \in X$ such that $x = a_1^{n_1}a_2^{n_2} \cdots a_k^{n_k}$ and $y = b_1^{m_1}b_2^{m_2} \cdots b_l^{m_l}$. But then,

$$xy^{-1} = a_1^{n_1}a_2^{n_2} \cdots a_k^{n_k} a_{k+1}^{n_{k+1}} a_{k+2}^{n_{k+2}} \cdots a_{k+l}^{n_{k+l}}$$

where for $j \in \{k+1, k+2, \dots, k+l\}$, $a_j = b_j$ and $n_j = -m_j$, verify that A is a generalized subgroup of G . It is obvious that A is the smallest generalized subgroup of G that contains X . Thus $A = \langle X \rangle$. \square

Now, we have the following lemma.

Lemma 6.3. *Let G be a normal generalized group and μ be an imaginable T -fuzzy generalized subgroup of G such that every nonempty level subset μ_t ($t \in [0, 1]$) has the property (CI). Then,*

$$\langle \mu_s \rangle \langle \mu_t \rangle \subseteq \mu_{T(s,t)}, \quad s, t \in [0, 1].$$

Proof. Let $x = zw$, for $z \in \langle \mu_s \rangle$ and $w \in \langle \mu_t \rangle$. Then there exist natural numbers k, l , integers n_i, m_j with $i \in \{1, 2, \dots, k\}$ and $j \in \{1, 2, \dots, l\}$, and $z_i \in \mu_s, w_j \in \mu_t$, such that $z = z_1^{n_1} z_2^{n_2} \dots z_k^{n_k}$, $w = w_1^{m_1} w_2^{m_2} \dots w_l^{m_l}$. By Lemma 3.11(ii), $\mu(z_i^{n_i}) \geq \mu(z_i) \geq s$, for all $i \in \{1, 2, \dots, k\}$, and $\mu(w_j^{m_j}) \geq \mu(w_j) \geq t$, for all $j \in \{1, 2, \dots, l\}$, and so $\mu(z) \geq T_k(\mu(z_1), \mu(z_2), \dots, \mu(z_k)) \geq s$ and $\mu(w) \geq T_l(\mu(w_1), \mu(w_2), \dots, \mu(w_l)) \geq t$. Thus, $\mu(x) = \mu(zw) \geq T(\mu(z), \mu(w)) \geq T(s, t)$, which means that $x \in \mu_{T(s,t)}$. \square

Theorem 6.4. Let G be a normal generalized group with the property (CI) and $A = \{\mu_i : i \in I\}$ be an indexed family of imaginable T -fuzzy generalized subgroups of G . Then

$$\langle A \rangle (x) = \vee \{k : x \in \langle (\cup_{i \in I} \mu_i)_k \rangle\}.$$

Proof. For $x \in G$ and $i \in I$, let $\mu_i(x) = t_i$ and

$$\lambda(x) = \vee \{k : x \in \langle (\cup_{i \in I} \mu_i)_k \rangle\}.$$

Then

$$x \in (\mu_i)_{t_i} \subseteq \left(\bigcup_{i \in I} \mu_i \right)_{t_i} \subseteq \langle (\cup_{i \in I} \mu_i)_{t_i} \rangle$$

whence $t_i \in \{k : x \in \langle (\cup_{i \in I} \mu_i)_k \rangle\}$, proves that $\lambda(x) \geq \mu_i(x)$, for all $i \in I$. Now, we show that λ is a T -fuzzy generalized subgroup of G . For $x, y \in G$, let $T(\lambda(x), \lambda(y)) = t$ and $\alpha = t - 1/n$, where n is a sufficiently large positive integer. Then

$$\lambda(x) \geq T(\lambda(x), \lambda(y)) > \alpha,$$

and similarly $\lambda(y) > \alpha$ and hence

$$\vee \{k : x \in \langle (\cup_{i \in I} \mu_i)_k \rangle\} = \lambda(x) > \alpha$$

and

$$\vee \{k : y \in \langle (\cup_{i \in I} \mu_i)_k \rangle\} = \lambda(y) > \alpha.$$

Hence, there exist $k_0 > \alpha$ and $k_1 > \alpha$ such that $x \in \langle (\cup_{i \in I} \mu_i)_{k_0} \rangle$ and $y \in \langle (\cup_{i \in I} \mu_i)_{k_1} \rangle$ whence, by Lemma 6.3, $xy \in \langle (\cup_{i \in I} \mu_i)_{T(k_0, k_1)} \rangle$, where $T(k_0, k_1) > \alpha$. So,

$$\lambda(xy) = \vee \{k : xy \in \langle (\cup_{i \in I} \mu_i)_k \rangle\} \geq t = T(\lambda(x), \lambda(y)).$$

The second property, $\lambda(x^{-1}) = \lambda(x)$, follows from the fact that for a T -fuzzy generalized subgroup μ of G , $x \in \langle \mu_k \rangle$ if and only if $x^{-1} \in \langle \mu_k \rangle$.

Now, let σ be a T -fuzzy generalized subgroup of G containing A and let $\lambda(x) = t$. Then $\vee \{k : x \in \langle (\cup_{i \in I} \mu_i)_k \rangle\} > s$, where $s = t - 1/n$ and n is a sufficiently large positive integer, and hence there exists $k_0 > s$ such that $x \in \langle (\cup_{i \in I} \mu_i)_{k_0} \rangle$. But then, $x = a_1^{m_1} a_2^{m_2} \dots a_r^{m_r}$, for $r \in \mathbb{N}$, $m_1, m_2, \dots, m_r \in \mathbb{Z}$ and for $a_1, a_2, \dots, a_r \in \langle (\cup_{i \in I} \mu_i)_{k_0} \rangle$. Thus, $(\cup_{i \in I} \mu_i)(a_j) \geq k_0 > s$, for $j \in \{1, 2, \dots, r\}$, which implies that $\mu_{i_j}(a_j) > s$ with $j \in \{1, 2, \dots, r\}$. Hence,

$$\sigma(x) = \sigma(a_1^{m_1} a_2^{m_2} \dots a_r^{m_r}) \geq T_r(\sigma(a_1), \sigma(a_2), \dots, \sigma(a_r)) > T_r(s, s, \dots, s) = s$$

means that $\sigma(x) \geq t = \lambda(x)$ proving λ is the smallest T -fuzzy generalized subgroup containing A . \square

Theorem 6.5. Let G be a normal generalized group, $\mu, \mu_i \in \text{TF}(G)$, $i \in I$, be T -fuzzy generalized subgroups of G satisfying (CI). Then,

$$\mu \wedge \left(\bigvee_{\alpha \in \Lambda} \mu_\alpha \right) = \bigvee_{\alpha \in \Lambda} (\mu \wedge \mu_\alpha).$$

Proof. Let $x \in G$ be such that $\mu \wedge (\vee \mu_\alpha)(x) = t$, and let $s = t - 1/n$, where n is a sufficiently large positive integer. Then $\mu(x) \geq t$ and

$$\vee \{k : x \in \langle (\cup \mu_\alpha)_k \rangle\} = (\vee \mu_\alpha)(x) > s.$$

Hence, there exists $k_0 > s$ such that $x \in \langle (\cup \mu_\alpha)_{k_0} \rangle$ and so there exist natural number r , integers $n_1, n_2, \dots, n_r, a_1, a_2, \dots, a_r \in (\cup \mu_\alpha)_{k_0}$ such that $x = a_1^{n_1} a_2^{n_2} \dots a_r^{n_r}$. Thus $\cup \mu_\alpha(a_i) \geq k_0 > s$ and so there exists $\lambda \in \Lambda$ such that $\mu_\lambda(a_i) > s$. Hence, $\mu \wedge \mu_\lambda(a_i) > s$ which means that $a_i \in (\cup \mu \wedge \mu_\alpha)_s$ and so $x \in \langle (\cup \mu \wedge \mu_\alpha)_s \rangle$. This implies that $s \in \{k : x \in \langle (\cup \mu \wedge \mu_\alpha)_k \rangle\}$. Thus $\vee_\alpha (\mu \wedge \mu_\alpha)(x) \geq t = \mu \wedge (\vee \mu_\alpha)(x)$. The converse of the inequality is obvious. \square

In virtue of Theorem 6.5 and that the intersection of any family of T -fuzzy generalized subgroups is again a T -fuzzy generalized subgroup we have the following result.

Corollary 6.6. *By the hypothesis of Theorem 6.5, $(TF(G), \subseteq)$ is a Heyting algebra.*

7. Conclusion

We introduced the concept of a fuzzy generalized subgroup with respect to a t -norm, gave some related properties, especially gave some theorems that characterized these fuzzy generalized subgroups, and thus we proposed and proved the representation theorem (section 3). Next, in section 4, we considered the image and preimage of a T -fuzzy generalized subgroup under a homomorphism. Using the notion of the continuity of t -norms, we proved that the preimage of a T -fuzzy generalized subgroup is again a T -fuzzy generalized subgroup, and thus, at the end of the section, we gave a correspondence theorem. Subsequently, in section 5, we defined the concept of a T -fuzzy normal generalized subgroup, using this notion we considered the quotient structure of generalized groups, and so we state and proved some homomorphism theorems. Finally, in section 6, we studied the lattice structure of the set of all T -fuzzy generalized subgroups of a generalized group and proved that this set forms a Heyting algebra. But we would like to reduce the conditions. For example, we would like to know if there exist simpler conditions than (CI) under which we can obtain the same result. These motivate us for the next researches.

Acknowledgements. This paper is partially supported by the Fuzzy Systems and Its Applications Center of Excellence, Shahid Bahonar University of Kerman, Iran.

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