Individual ergodic theorem for intuitionistic fuzzy observables using intuitionistic fuzzy state

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Abstract
The classical ergodic theory has been built on σ-algebras. Later the Individual ergodic theorem was studied on more general structures like MV-algebras and quantum structures. The aim of this paper is to formulate the Individual ergodic theorem for intuitionistic fuzzy observables using m-almost everywhere convergence, where m is an intuitionistic fuzzy state. We show the Kolmogorov construction for intuitionistic fuzzy observables, too.

Keywords: The intuitionistic fuzzy event, the intuitionistic fuzzy observable, the intuitionistic fuzzy state, the product, the upper limit, the lower limit, the m-almost everywhere convergence, the m-preserving transformation, the individual ergodic theorem, the Kolmogorov construction.

1 Introduction

In [1, 2] K.T. Atanassov introduced the notion of intuitionistic fuzzy sets. Then in [8] B. Riečan defined the intuitionistic fuzzy state on the family of intuitionistic fuzzy events \( \mathcal{F} = \{(\mu_A, \nu_A) : \mu_A + \nu_A \leq 1_\Omega\} \), where \( \mu_A, \nu_A \) are \( \mathcal{S} \)-measurable functions, \( \mu_A, \nu_A : \Omega \to [0, 1] \), as a mapping \( m \) from the family \( \mathcal{F} \) to the set \( R \) by the formula

\[
m((\mu_A, \nu_A)) = (1 - \alpha) \int_\Omega \mu_A dP + \alpha \left( 1 - \int_\Omega \nu_A dP \right),
\]

where \( P : \mathcal{S} \to [0, 1] \) is a probability measure and \( \alpha \in [0, 1] \).

In paper [4] we defined the upper and the lower limits for sequence of intuitionistic fuzzy observables. We used an intuitionistic fuzzy state \( m \) to define the notion of almost everywhere convergence. We compared two concepts of \( m \)-almost everywhere convergence.

In paper [5] we studied the \( m \)-almost everywhere convergence of sequence of intuitionistic fuzzy observables \( g_n(x_1, \ldots, x_n) : \mathcal{B}(R) \to \mathcal{F} \) given by \( g_n(x_1, \ldots, x_n) = h_n \circ g_n^{-1} \), where \( h_n : \mathcal{B}(R^n) \to \mathcal{F} \) is the joint intuitionistic fuzzy observable of intuitionistic fuzzy observables \( x_1, \ldots, x_n \) and \( g_n : R^n \to R \) is a Borel measurable function. We showed the connection between \( m \)-almost everywhere convergence of this sequence of intuitionistic fuzzy observables and \( P \)-almost everywhere convergence of random variables in classical probability space induced by Kolmogorov construction. This connection is a start point for proving the Individual ergodic theorem for intuitionistic fuzzy observables using \( m \)-almost everywhere convergence.

Recall that the formulation of the Individual ergodic theorem for intuitionistic fuzzy events with product first appeared in the paper [3]. There we used \( P \)-almost everywhere convergence, where \( P \) was a separating intuitionistic fuzzy probability. Since the intuitionistic fuzzy probability \( P \) can be decomposed to two intuitionistic fuzzy states, it is usefull to study \( m \)-almost everywhere convergence, where \( m \) is an intuitionistic fuzzy state. In this paper we formulate the Individual ergodic theorem for intuitionistic fuzzy observables using \( m \)-almost everywhere convergence. We show the Kolmogorov construction for intuitionistic fuzzy observables, too.
Remark that in a whole text we use a notation IF as an abbreviation for intuitionistic fuzzy.

2 IF-events, IF-states, IF-observables and IF-mean value

First we start with definitions of basic notions (see [1, 2, 9]).

Definition 2.1. Let \( \Omega \) be a nonempty set. An IF-set \( A \) on \( \Omega \) is a pair \((\mu_A, \nu_A)\) of mappings \( \mu_A, \nu_A : \Omega \to [0, 1] \) such that \( \mu_A + \nu_A \leq 1_\Omega \).

Definition 2.2. Start with a measurable space \((\Omega, S)\). Hence \( S \) is a \( \sigma \)-algebra of subsets of \( \Omega \). An IF-event is called an IF-set \( A \) = \((\mu_A, \nu_A)\) such that \( \mu_A, \nu_A : \Omega \to [0, 1] \) are \( S \)-measurable.

The family of all IF-events on \((\Omega, S)\) will be denoted by \( \mathcal{F} \), \( \mu_A : \Omega \to [0, 1] \) will call the membership function, \( \nu_A : \Omega \to [0, 1] \) be called the non-membership function.

If \( A = (\mu_A, \nu_A) \in \mathcal{F} \), \( B = (\mu_B, \nu_B) \in \mathcal{F} \), then we define the Lukasiewicz binary operations \( \oplus, \circ \) on \( \mathcal{F} \) by

\[
A \oplus B = ((\mu_A + \mu_B) \wedge 1_\Omega, (\nu_A + \nu_B - 1_\Omega) \vee 0_\Omega)), \quad \text{and} \quad A \circ B = ((\mu_A + \mu_B - 1_\Omega) \vee 0_\Omega, (\nu_A + \nu_B) \wedge 1_\Omega))
\]

and the partial ordering is given by \( A \leq B \) if and only if \( \mu_A \leq \mu_B, \nu_A \geq \nu_B \). In paper we use max-min connectives defined by

\[
A \vee B = (\mu_A \vee \mu_B, \nu_A \wedge \nu_B), \quad \text{and} \quad A \wedge B = (\mu_A \wedge \mu_B, \nu_A \vee \nu_B)
\]

and the de Morgan rules \( (a \vee b)^* = a^* \wedge b^* \), and \( (a \wedge b)^* = a^* \vee b^* \), where \( a^* = 1 - a \).

Example 2.3. Fuzzy set \( f : \Omega \to [0, 1] \) can be regarded as IF-set, if we put \( A = (f, 1_\Omega - f) \). If \( f = \chi_A \), then the corresponding IF-set has the form \( A = (\chi_A, 1_\Omega - \chi_A) = (\chi_A, \chi_A^c) \). In this case \( A \oplus B \) corresponds to the union of sets, \( A \circ B \) to the intersection of sets and \( \leq \) to the set inclusion.

In the IF-probability theory \([10]\) instead of the notion of probability we use the notion of state.

Definition 2.4. Let \( \mathcal{F} \) be the family of all IF-events in \( \Omega \). A mapping \( m : \mathcal{F} \to [0, 1] \) is called an IF-state, if the following conditions are satisfied:

(i) \( m((1_\Omega, 0_\Omega)) = 1 \), \( m((0_\Omega, 1_\Omega)) = 0 \);

(ii) if \( A \circ B = (0_\Omega, 1_\Omega) \) and \( A, B \in \mathcal{F} \), then \( m(A \oplus B) = m(A) + m(B) \);

(iii) if \( A_n \triangleright A \) (i.e. \( \mu_{A_n} \triangleright \mu_A, \nu_{A_n} \triangleright \nu_A \)), then \( m(A_n) \triangleright m(A) \).

Probably the most useful result in the IF-state theory is the following representation theorem \([5]\):

Theorem 2.5. To each IF-state \( m : \mathcal{F} \to [0, 1] \) there exists exactly one probability measure \( P : \mathcal{S} \to [0, 1] \) and exactly one \( \alpha \in [0, 1] \) such that for each \( A = (\mu_A, \nu_A) \in \mathcal{F} \),

\[
m(A) = (1 - \alpha) \int_\Omega \mu_A dP + \alpha \left(1 - \int_\Omega \nu_A dP \right).
\]

Proof. In \([5]\) Theorem. \( \square \)

The third basic notion in the probability theory is the notion of an observable. Let \( \mathcal{F} \) be the family of all intervals in \( \mathbb{R} \) of the form \( [a, b] = \{x \in \mathbb{R} : a \leq x \leq b\} \). Then the \( \sigma \)-algebra \( \sigma(\mathcal{F}) \) is denoted \( \mathcal{B}(\mathbb{R}) \) and it is called the \( \sigma \)-algebra of Borel sets, its elements are called Borel sets.

Definition 2.6. By an IF-observable on \( \mathcal{F} \) we understand each mapping \( x : \mathcal{B}(\mathbb{R}) \to \mathcal{F} \) satisfying the following conditions:

(i) \( x(R) = (1_\Omega, 0_\Omega), x(\emptyset) = (0_\Omega, 1_\Omega) \);

(ii) if \( A \cap B = \emptyset \), then \( x(A) \circ x(B) = (0_\Omega, 1_\Omega) \) and \( x(A \cup B) = x(A) \oplus x(B) \);

(iii) if \( A_n \triangleright A \), then \( x(A_n) \triangleright x(A) \).

If we denote \( x(A) = (x^3(A), 1_\Omega - x^2(A)) \) for each \( A \in \mathcal{B}(\mathbb{R}) \), then \( x^3, x^4 : \mathcal{B}(\mathbb{R}) \to \mathcal{T} \) are observables, where \( \mathcal{T} = \{ f : \Omega \to [0, 1] : f \text{ is } \mathcal{S} \text{-measurable} \} \).
Remark 2.7. Sometimes we need to work with $n$-dimensional IF-observable $x : \mathcal{B}(\mathbb{R}^n) \rightarrow \mathcal{F}$ defined as a mapping with the following conditions:

(i) $x(0^n) = (1_{\Omega}, 0_{\Omega})$, $x(\emptyset) = (0_{\Omega}, 1_{\Omega})$;

(ii) if $A \cap B = \emptyset$, $A, B \in \mathcal{B}(\mathbb{R}^n)$, then $x(A) \odot x(B) = (0_{\Omega}, 1_{\Omega})$ and $x(A \cup B) = x(A) \oplus x(B)$;

(iii) if $A_n \nrightarrow A$, then $x(A_n) \nrightarrow x(A)$ for each $A, A_n \in \mathcal{B}(\mathbb{R}^n)$.

If $n = 1$, then we simply say that $x$ is an IF-observable.

Similarly as in the classical case the following theorem can be proved \cite{7,10}.

Theorem 2.8. Let $x : \mathcal{B}(\mathbb{R}) \rightarrow \mathcal{F}$ be an IF-observable, $\mathbf{m} : \mathcal{F} \rightarrow [0,1]$ be an IF-state. Define the mapping $\mathbf{m}_x : \mathcal{B}(\mathbb{R}) \rightarrow [0,1]$ by the formula $\mathbf{m}_x(C) = \mathbf{m}(x(C))$. Then $\mathbf{m}_x : \mathcal{B}(\mathbb{R}) \rightarrow [0,1]$ is a probability measure.

Proof. In \cite{7} Proposition 3.1. $\Box$

Since now $\mathbf{m}_x : \mathcal{B}(\mathbb{R}) \rightarrow [0,1]$ plays an analogous role as $P_\xi : \mathcal{B}(\mathbb{R}) \rightarrow [0,1]$, we can define IF-expected value $\mathbf{E}(x)$ by the same formula (see \cite{7}).

Definition 2.9. We say that an IF-observable $x$ is an integrable IF-observable, if the integral $\int_\mathbb{R} t \, d\mathbf{m}_x(t)$ exists. In this case we define IF-expected value $\mathbf{E}(x) = \int_\mathbb{R} t \, d\mathbf{m}_x(t)$. If the integral $\int_\mathbb{R} t^2 \, d\mathbf{m}_x(t)$ exists, then we define IF-dispersion $D^2(x)$ by the formula

$$D^2(x) = \int_\mathbb{R} t^2 \, d\mathbf{m}_x(t) - (\mathbf{E}(x))^2 = \int_\mathbb{R} (t - \mathbf{E}(x))^2 \, d\mathbf{m}_x(t).$$

3 Product operation, joint IF-observable and function of several IF-observables

In \cite{5} we introduced the notion of product operation on the family of IF-events $\mathcal{F}$ and showed an example of this operation.

Definition 3.1. We say that a binary operation $\cdot$ on $\mathcal{F}$ is product if it satisfies the following conditions:

(i) $(1_{\Omega}, 0_{\Omega}) \cdot (a_1, a_2) = (a_1, a_2)$ for each $(a_1, a_2) \in \mathcal{F}$;

(ii) the operation $\cdot$ is commutative and associative;

(iii) if $(a_1, a_2) \odot (b_1, b_2) = (0_{\Omega}, 1_{\Omega})$ and $(a_1, a_2), (b_1, b_2) \in \mathcal{F}$, then

$$(c_1, c_2) \cdot ((a_1, a_2) \odot (b_1, b_2)) = ((c_1, c_2) \cdot (a_1, a_2)) \oplus ((c_1, c_2) \cdot (b_1, b_2))$$

and

$$(c_1, c_2) \cdot ((a_1, a_2)) \odot ((c_1, c_2) \cdot (b_1, b_2)) = (0_{\Omega}, 1_{\Omega}), \quad \text{for each } (c_1, c_2) \in \mathcal{F};$$

(iv) if $(a_{1n}, a_{2n}) \nrightarrow (0_{\Omega}, 1_{\Omega})$, $(b_{1n}, b_{2n}) \nrightarrow (0_{\Omega}, 1_{\Omega})$ and $(a_{1n}, a_{2n}), (b_{1n}, b_{2n}) \in \mathcal{F}$, then $(a_{1n}, a_{2n}) \cdot (b_{1n}, b_{2n}) \nrightarrow (0_{\Omega}, 1_{\Omega})$.

In the following theorem is the example of product operation for IF-events.

Theorem 3.2. The operation $\cdot$ defined by $(x_1, y_1) \cdot (x_2, y_2) = (x_1 \cdot x_2, y_1 + y_2 - y_1 \cdot y_2)$ for each $(x_1, y_1), (x_2, y_2) \in \mathcal{F}$ is product operation on $\mathcal{F}$.

Proof. In \cite{5} Theorem 1. $\Box$

In \cite{9} B. Riečan defined the notion of a joint IF-observable and he proved its existence.

Definition 3.3. Let $x, y : \mathcal{B}(\mathbb{R}) \rightarrow \mathcal{F}$ be two IF-observables. The joint IF-observable of the IF-observables $x, y$ is a mapping $h : \mathcal{B}(\mathbb{R}^2) \rightarrow \mathcal{F}$ satisfying the following conditions:

(i) $h(0^2) = (1_{\Omega}, 0_{\Omega})$, $h(\emptyset) = (0_{\Omega}, 1_{\Omega})$;

(ii) if $A, B \in \mathcal{B}(\mathbb{R}^2)$ and $A \cap B = \emptyset$, then $h(A \cup B) = h(A) \oplus h(B)$ and $h(A) \odot h(B) = (0_{\Omega}, 1_{\Omega})$;
(iii) if $A, A_1, \ldots \in \mathcal{B}(R^2)$ and $A_n \not\supset A$, then $h(A_n) \not\supset h(A)$;
(iv) $h(C \times D) = x(C) \cdot y(D)$ for each $C, D \in \mathcal{B}(R)$.

**Theorem 3.4.** For each two IF-observables $x, y : \mathcal{B}(R) \to \mathcal{F}$ there exists their joint IF-observable.

**Proof.** In [9] Theorem 3.3. $\square$

**Remark 3.5.** The joint IF-observable of IF-observables $x, y$ from Definition 3.3 is two-dimensional IF-observable.

If we have several IF-observables and a Borel measurable function, we can define the IF-observable, which is the function of several IF-observables. About this says the following definition.

**Definition 3.6.** Let $x_1, \ldots, x_n : \mathcal{B}(R) \to \mathcal{F}$ be IF-observables, $h_n$ their joint IF-observable and $g_n : R^n \to R$ a Borel measurable function. Then we define the IF-observable $g_n(x_1, \ldots, x_n) : \mathcal{B}(R) \to \mathcal{F}$ by the formula $g_n(x_1, \ldots, x_n)(A) = h_n(g_n^{-1}(A))$, for each $A \in \mathcal{B}(R)$.

## 4 Kolmogorov construction

In this section we introduce the notion of compatibility of intuitionistic fuzzy observables as follows.

**Definition 4.1.** Let $(\mathcal{F}, \cdot)$ be a family of IF-events with product and $x, y : \mathcal{B}(R) \to \mathcal{F}$ be the IF-observables on $\mathcal{F}$. We say that the IF-observables $x, y$ are compatible, if there exists their joint IF-observable $h$.

We can generalize the notion of compatibility for $k$ IF-observables $x_{i_1}, \ldots, x_{i_k}$.

**Definition 4.2.** Let $J \subset N$, $J = \{i_1, \ldots, i_k\}$ and $x_{i_1}, \ldots, x_{i_k}$ be the IF-observables on $\mathcal{F}$. We say that the IF-observables $x_{i_1}, \ldots, x_{i_k}$ are compatible, if there exists a mapping $h_J : \mathcal{B}(R^{|J|}) \to \mathcal{F}$ satisfying the following conditions:

(i) $h_J(R^{|J|}) = (1_{\Omega}, 0_{\Omega})$, $h(\emptyset) = (0_{\Omega}, 1_{\Omega})$
(ii) if $A, B \in \mathcal{B}(R^{|J|})$ and $A \cap B = \emptyset$, then $h_J(A \cup B) = h_J(A) \oplus h_J(B)$ and $h_J(A) \odot h_J(B) = (0_{\Omega}, 1_{\Omega})$;
(iii) if $A, A_1, \ldots \in \mathcal{B}(R^{|J|})$ and $A_n \not\supset A$, then $h_J(A_n) \not\supset h_J(A)$;
(iv) $h_J(A_{i_1} \times \ldots \times A_{i_k}) = x_{i_1}(A_{i_1}) \cdot \ldots \cdot x_{i_k}(A_{i_k})$ for each $A_{i_1}, \ldots, A_{i_k} \in \mathcal{B}(R)$.

By Definition 4.2 to every compatible IF-observables $x_1, \ldots, x_n$ there exists a morphism $h_n : \mathcal{B}(R^n) \to \mathcal{F}$ (i.e. $h_n(R^n) = (1_{\Omega}, 0_{\Omega})$, $h_n$ is additive and continuous) such that $h_n(A_1 \times \ldots \times A_n) = x_1(A_1) \cdot \ldots \cdot x_n(A_n)$, for each $A_1, \ldots, A_n \in \mathcal{B}(R)$.

**Lemma 4.3.** The mapping $h_J : \mathcal{B}(R^{|J|}) \to \mathcal{F}$ from Definition 4.2 satisfies the following conditions:

(v) if $A \in \mathcal{B}(R)$, then $h_J(\{(t_1, \ldots, t_i, \ldots, t_k) \mid (t_1, \ldots, t_i, \ldots, t_k) \in R^{|J|}, t_i \in A\}) = x_i(A)$;
(vi) if $J_1 \subset J_2 \subset N$, then $h_{J_2}(\pi_{J_2, J_1}^{-1}(A)) = h_{J_1}(A)$ for each $A \in \mathcal{B}(R^{|J_1|})$, where $\pi_{J_2, J_1} : \mathcal{B}(R^{|J_2|}) \to \mathcal{B}(R^{|J_1|})$ is the projection.

**Proof.** (v)

$$h_J(\{(t_1, \ldots, t_i, \ldots, t_k) \mid (t_1, \ldots, t_i, \ldots, t_k) \in R^{|J|}, t_i \in A\}) = h_J(R \times \ldots \times R \times A \times R \times \ldots \times R) = x_1(R) \cdot \ldots \times x_{i-1}(R) \cdot x_i(A) \cdot x_{i+1}(R) \cdot \ldots \times x_k(R) = (1_{\Omega}, 0_{\Omega}) \cdot \ldots \cdot (1_{\Omega}, 0_{\Omega}) \cdot x_i(A) \cdot (1_{\Omega}, 0_{\Omega}) \cdot \ldots \cdot (1_{\Omega}, 0_{\Omega}) = (1_{\Omega}, 0_{\Omega}) \cdot x_i(A) \cdot (1_{\Omega}, 0_{\Omega}) = x_i(A)$$

(vi) Let $J_1 \subset J_2 \subset N; A = A_{i_1} \times \ldots \times A_{i_k} \in \mathcal{B}(R^{|J_1|})$.

Then

$$\pi_{J_1, J_2}^{-1}(A) = R \times \ldots \times R \times A_{i_1} \times \ldots \times A_{i_k} \times R \times \ldots \times R \in \mathcal{B}(R^{|J_2|}).$$

Then

$$h_{J_2}(\pi_{J_2, J_1}^{-1}(A)) = x_{i_1}(R) \cdot \ldots \times x_{i_k}(R) \cdot x_{i_1}(A_{i_1}) \cdot \ldots \times x_{i_k}(A_{i_k}) \cdot x_{i_1}(R) \cdot \ldots \times x_{i_k}(R) = (1_{\Omega}, 0_{\Omega}) \cdot \ldots \cdot (1_{\Omega}, 0_{\Omega}) \cdot x_{i_1}(A_{i_1}) \cdot \ldots \times x_{i_k}(A_{i_k}) \cdot (1_{\Omega}, 0_{\Omega}) \cdot \ldots \cdot (1_{\Omega}, 0_{\Omega})$$
Proposition 4.4. Let $m$ be an IF-state on a family of IF-events with product $(\mathcal{F}, \cdot)$. Define $P_n : \mathcal{B}(R^n) \to [0,1]$ by the formula

$$P_n(A) = m(h_n(A)), \quad A \in \mathcal{B}(R).$$

Then $P_n$ is a probability measure such that

$$P_n(\{(t_1, \ldots, t_i, \ldots, t_n) \mid (t_1, \ldots, t_i, \ldots, t_n) \in R^n, t_i \in A\} = m(x_i(A)) = m_{x_i}(A).$$

Proof. The first assertion is clear. Further

$$P_n(\{(t_1, \ldots, t_i, \ldots, t_n) \mid (t_1, \ldots, t_i, \ldots, t_n) \in R^n, t_i \in A\}) = m(h_n(R \times \ldots \times R \times A \times R \times \ldots \times R))$$

$$= m(x_1(R) \cdot \ldots \cdot x_{i-1}(R) \cdot x_i(A) \cdot x_{i+1}(R) \cdot \ldots \cdot x_n(R))$$

$$= m((1_{\Omega}, 0_{\Omega}) \cdot x_i(A) \cdot (1_{\Omega}, 0_{\Omega})) = m(x_i(A)) = m_{x_i}(A).$$

Proposition 4.5. Let $\emptyset \neq J \subset N$, $J$ be finite, $J = \{t_1, \ldots, t_k\}$. Then there exists exactly one probability measure $P_J : \mathcal{B}(R^k) \to [0,1]$ such that

$$P_J(A_1 \times \ldots \times A_k) = m(x_1(A_1) \cdot \ldots \cdot x_k(A_k))$$

for each $A_1, \ldots, A_k \in \mathcal{B}(R)$. 

Proof. Let $I = \{1, \ldots, t_k\} \supset J$, $\pi_{I,J}$ be the projection from $R^t$ to $R^k$. Then

$$\pi_{I,J}^{-1}(A_1 \times \ldots \times A_k) = B_1 \times \ldots \times B_{t_k},$$

where $B_{t_i} = A_i$ $(i = 1, 2, \ldots, k); B_j = R$ if $j \notin J$. Therefore

$$P_{t_k}(\pi_{I,J}^{-1}(A_1 \times \ldots \times A_k)) = P_{t_k}(B_1 \times \ldots \times B_{t_k})$$

$$= m(h_{t_k}(B_1 \times \ldots \times B_{t_k}))$$

$$= m(x_1(B_1) \cdot \ldots \cdot x_{t_k}(B_{t_k}))$$

$$= m(x_1(A_1) \cdot \ldots \cdot x_k(A_k)).$$

Put $P_J = P_{t_k} \circ \pi_{I,J}^{-1} : \mathcal{B}(R^k) \to [0,1]$. Then $P_J$ is a probability measure with the property stated in Proposition 4.5. If $\mu$ is another measure with this property, then $P_J$ coincides with $\mu$ on each rectangles and therefore they coincide on $\mathcal{B}(R^k)$. 

By property (vi) we obtained a family of probability measures $\{P_J \mid \emptyset \neq J \subset N, J \text{ finite}\}$ given by

$$P_J(A) = m(h_J(A)), \quad A \in \mathcal{B}(R).$$

The family satisfies the Kolmogorov consistency condition. E.g., if $J_2 = \{1, 2, 3\}, J_1 = \{1, 3\}$ and $\pi_{J_2,J_1} : R^3 \to R^2$ is the projection (assigning to a triple $(t_1, t_2, t_3)$ a pair $(t_1, t_3)$) then

$$P_{J_2}(\pi_{J_2,J_1}^{-1}(A \times B)) = P_{J_2}(\{(t_1, t_2, t_3) : (t_1, t_3) \in A \times B\})$$

$$= P_{J_2}(A \times R \times B)$$

$$= m(x_1(A) \cdot x_2(R) \cdot x_3(B))$$

$$= m(x_1(A) \cdot x_3(B))$$

$$= P_{J_1}(A \times B),$$

$$P_{J_2} \circ \pi_{J_2,J_1}^{-1} = P_{J_1}. $$
Proposition 4.6. The family \( \{ P_J \mid \emptyset \neq J \subset N, J \text{ finite} \} \) satisfies the Kolmogorov consistency condition, i.e.
\[
P_J(\pi_{J_2,J_1}^{-1}(A)) = P_J(A)
\]
whenever \( J_1 \subset J_2, \ A \in \mathcal{B}(R^{|J_1|}), \) where \( \pi_{J_2,J_1} : R^{|J_2|} \to R^{|J_1|} \) is the projection.

Proof. \( P_J \) and \( P_J \circ \pi_{J_2,J_1}^{-1} \) are two measures on \( \mathcal{B}(R^{|J_1|}) \) coinciding on the family of all rectangles. \( \square \)

At this point we may use the Kolmogorov consistency theorem.

Proposition 4.7. Let \( C \) be the family of all cylinders in \( R^N, \) i.e.
\[
C = \{ \pi_J^{-1}(A) \mid \emptyset \neq J \subset N, J \text{ finite}, A \in \mathcal{B}(R^{|J|}) \}.
\]
Then there exists exactly one probability measure \( P : \sigma(C) \to [0,1] \) such that \( P(\pi_J^{-1}(A)) = P_J(A) \), for each cylinders \( \pi_J^{-1}(A) \). Particularly
\[
P(\{ (t_n)_1^\infty : t_i \in A_i, i = 1,2,\ldots,n \}) = m(h_n(A_1\times\ldots\times A_n)) = m(x_1(A_1)\ldots\cdot x_n(A_n)).
\]

Proof. It follows by the Kolmogorov theorem and Proposition 4.6. \( \square \)

Proposition 4.8. Define the coordinate function \( \xi_n : R^N \to R \) by the formula \( \xi_n((t_i)_1^\infty) = t_n \). Then \( \xi_n \) is a random variable with respect to \( \sigma(C) \) such that \( P_{\xi_n} = m \circ x_n = m_{x_n} \).

Proof. If \( A \in \mathcal{B}(R) \), then \( \xi_n^{-1}(A) = \{ (t_i)_1^\infty ; t_n \in A \} = \pi_{\{n\}}^{-1}(A) \in C \). Moreover
\[
P_{\xi_n}(A) = P(\xi_n^{-1}(A)) = P(\pi_{\{n\}}^{-1}(A)) = m(x_n(A)) = m_{x_n}(A).
\]

Remark 4.9. By the preceding procedure to each sequence \( (x_n)_n \) we can construct a sequence \( (\xi_n)_n \) of a random variables.

5 Lower and upper limits, m-almost everywhere convergence

In \[4\] we defined the notions of lower and upper limits for a sequence of \( IF \)-observables and showed the connection between two kinds of \( m \)-almost everywhere convergence.

Definition 5.1. We shall say that a sequence \( (x_n)_n \) of \( IF \)-observables has \( \limsup \) \( n \to \infty \)
\[
\varpi((\infty,t)) = \bigvee_{p=1}^\infty \bigwedge_{k=1}^\infty x_n\left((-\infty,t-\frac{1}{p})\right)
\]
for every \( t \in R \). We write \( \varpi = \limsup_{n \to \infty} x_n \).

Note that if another \( IF \)-observable \( y \) satisfies the above condition, then \( m \circ y = m \circ \varpi \).

Definition 5.2. A sequence \( (x_n)_n \) of \( IF \)-observables has \( \liminf \) \( n \to \infty \)
\[
\underline{x}((\infty,t)) = \bigwedge_{p=1}^\infty \bigvee_{k=1}^\infty x_n\left((-\infty,t-\frac{1}{p})\right)
\]
for all \( t \in R \). Notation: \( \underline{x} = \liminf_{n \to \infty} x_n \).

Proposition 5.3. A sequence \( (x_n)_n \) of an \( IF \)-observables converges \( m \)-almost everywhere to 0 if and only if
\[
m\left( \bigvee_{p=1}^\infty \bigwedge_{k=1}^\infty x_n\left((-\infty,t-\frac{1}{p})\right) \right) = m\left( \bigwedge_{p=1}^\infty \bigvee_{k=1}^\infty x_n\left((-\infty,t-\frac{1}{p})\right) \right) = m(0_{\varpi}((-\infty,t)))
\]
for every \( t \in R \).

In accordance with Proposition 5.3 we can extend the notion of \( m \)-almost everywhere convergence by the following way.

**Definition 5.4.** A sequence \((x_n)_n\) of an IF-observables converges \( m \)-almost everywhere to an IF-observable \( x \), if

\[
\mathbf{m}\left(\limsup_{n \to \infty} g_n(x_1, \ldots, x_n)((-\infty, t))\right) = \mathbf{m}\left(\liminf_{n \to \infty} g_n(x_1, \ldots, x_n)((-\infty, t))\right)
\]

for every \( t \in \mathbb{R} \).

The next theorem is important for the proof of the Individual ergodic theorem in intuitionistic fuzzy case, where we work with the sequence of several IF-observables induced by the Borel function.

**Theorem 5.5.** Let \((x_n)_n\) be a sequence of IF-observables, \((\xi_n)_n\) be the sequence of corresponding projections, \((g_n)_n\) be a sequence of Borel measurable functions \( g_n : \mathbb{R}^n \to \mathbb{R} \). If the sequence \((g_n(\xi_1, \ldots, \xi_n))_n\) converges \( P \)-almost everywhere, then the sequence \((g_n(x_1, \ldots, x_n))_n\) converges \( m \)-almost everywhere and

\[
\mathbf{m}\left(\limsup_{n \to \infty} g_n(x_1, \ldots, x_n)((-\infty, t))\right) = \mathbf{m}\left(\liminf_{n \to \infty} g_n(x_1, \ldots, x_n)((-\infty, t))\right)
\]

for each \( t \in \mathbb{R} \). Moreover

\[
P\left\{u \in \mathbb{R}^N : \limsup_{n \to \infty} g_n(\xi_1(u), \ldots, \xi_n(u)) < t\right\} = \mathbf{m}\left(\limsup_{n \to \infty} g_n(x_1, \ldots, x_n)((-\infty, t))\right)
\]

for each \( t \in \mathbb{R} \).

Proof. In Theorem 5.1.

\[\square\]

## 6 Individual ergodic theorem

First we recall the classical Individual ergodic theorem. Let \((X, \sigma, P)\) be a probability space, \(T : X \to X\) be a measure preserving transformation (i.e. \( A \in \sigma \) implies \( T^{-1}(A) \in \sigma \) and \( P(T^{-1}(A)) = P(A) \)), \( \xi : X \to \mathbb{R} \) be an integrable random variable. Then there exists an integrable random variable \( \xi^* \) such that the following conditions are satisfied:

(i) \( E(\xi) = E(\xi^*) \),

(ii) \( \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} (\xi \circ T^i) = \xi^* \) \( P \)-almost everywhere,

(iii) \( \xi^* = \xi \circ T \) \( P \)-almost everywhere.

We defined the IF-mean value of an IF-observable and \( m \)-almost everywhere convergence in the previous sections. Now we must define a transformation preserving an IF-state \( m \).

**Definition 6.1.** Let \((\mathcal{F}, \cdot)\) be a family of IF-events with product, \( m \) be an IF-state. Then a mapping \( \tau : \mathcal{F} \to \mathcal{F} \) is said to be a \( m \)-preserving transformation, if the following conditions are satisfied:

(i) \( \tau((1\Omega, 0\Omega)) = (1\Omega, 0\Omega) \);

(ii) if \( A, B \in \mathcal{F} \) and \( A \odot B = (0\Omega, 1\Omega) \), then \( \tau(A) \odot \tau(B) = (0\Omega, 1\Omega) \) and \( \tau(A \oplus B) = \tau(A) \oplus \tau(B) \);

(iii) if \( A_n \not\succ A, A_n, A \in \mathcal{F}, n \in \mathbb{N} \), then \( \tau(A_n) \not\succ \tau(A) \);

(iv) \( m(\tau(A) \cdot \tau(B)) = m(A \cdot B) \) for each \( A, B \in \mathcal{F} \).

The next theorem says about a representation of \( m \)-preserving transformation.
Theorem 6.2. Let \( \tau : \mathcal{F} \to \mathcal{F} \) be a \( \mathbf{m} \)-preserving transformation, where \( \mathbf{m} \) is an IF-state. For each \( \mathbf{A} = (\mu_{\mathbf{A}}, \nu_{\mathbf{A}}) \in \mathcal{F} \) denote
\[
\tau(\mathbf{A}) = (\tau^{\flat}(\mu_{\mathbf{A}}), 1_\Omega - \tau^{\sharp}(1_\Omega - \nu_{\mathbf{A}})).
\]
Then the mappings \( \tau^{\flat}, \tau^{\sharp} : \mathcal{T} \to \mathcal{T} \) are the measure preserving transformations in a tribe \( \mathcal{T} = \{ f : \Omega \to [0, 1] ; f \text{ is } \mathcal{S} - \text{measurable} \} \).

Proof. Let \( \tau : \mathcal{F} \to \mathcal{F} \) be a \( \mathbf{m} \)-preserving transformation. Then from Definition 6.1 we obtain that the mapping \( \tau \) is satisfying the four conditions:

(i) Let \( \tau((1_\Omega, 0_\Omega)) = (1_\Omega, 0_\Omega) \). Then \( (1_\Omega, 0_\Omega) = \tau((1_\Omega, 0_\Omega)) = (\tau^{\flat}(1_\Omega), 1_\Omega - \tau^{\sharp}(1_\Omega)) \) and therefore we have \( 1_\Omega = \tau^{\flat}(1_\Omega), 1_\Omega = \tau^{\sharp}(1_\Omega) \).

(ii) Let \( \mathbf{A}, \mathbf{B} \in \mathcal{F} \) and \( \mathbf{A} \odot \mathbf{B} = (0_\Omega, 1_\Omega) \). Then \( (0_\Omega, 1_\Omega) = (\mu_{\mathbf{A}} \odot \mu_{\mathbf{B}}, \nu_{\mathbf{A}} \odot \nu_{\mathbf{B}}) = ((\mu_{\mathbf{A}} + \mu_{\mathbf{B}} - 1_\Omega) \lor 0_\Omega, (\nu_{\mathbf{A}} + \nu_{\mathbf{B}}) \land 1_\Omega) \).

\[
0_\Omega = (\mu_{\mathbf{A}} + \mu_{\mathbf{B}} - 1_\Omega) \lor 0_\Omega \text{ and } 1_\Omega = (\nu_{\mathbf{A}} + \nu_{\mathbf{B}}) \land 1_\Omega.
\]

Therefore,
\[
\begin{align*}
\mu_{\mathbf{A}} + \mu_{\mathbf{B}} &\leq 1_\Omega & \quad (1) \\
\nu_{\mathbf{A}} + \nu_{\mathbf{B}} &\geq 1_\Omega. & \quad (2)
\end{align*}
\]

By (2) we obtain
\[
(1_\Omega - \nu_{\mathbf{A}}) \odot (1_\Omega - \nu_{\mathbf{B}}) = (1_\Omega - (\nu_{\mathbf{A}} + \nu_{\mathbf{B}})) \lor 0_\Omega = 0_\Omega.
\]

Since \( \tau(\mathbf{A}) \odot \tau(\mathbf{B}) = (0_\Omega, 1_\Omega) \), we get
\[
(0_\Omega, 1_\Omega) = (\tau^{\flat}(\mu_{\mathbf{A}}), 1_\Omega - \tau^{\sharp}(1_\Omega - \nu_{\mathbf{A}})) \odot (\tau^{\flat}(\mu_{\mathbf{B}}), 1_\Omega - \tau^{\sharp}(1_\Omega - \nu_{\mathbf{B}}))
\]
\[
= (\tau^{\flat}(\mu_{\mathbf{A}}) \odot \tau^{\flat}(\mu_{\mathbf{B}}), (1_\Omega - \tau^{\sharp}(1_\Omega - \nu_{\mathbf{A}})) \odot (1_\Omega - \tau^{\sharp}(1_\Omega - \nu_{\mathbf{B}})))
\]
\[
= (\tau^{\flat}(\mu_{\mathbf{A}}) \odot \tau^{\flat}(\mu_{\mathbf{B}}), (1_\Omega - \tau^{\sharp}(1_\Omega - \nu_{\mathbf{A}})) + (1_\Omega - \tau^{\sharp}(1_\Omega - \nu_{\mathbf{B}})) \land 1_\Omega.
\]

Therefore \( \tau^{\flat}(1_\Omega - \nu_{\mathbf{A}}) + \tau^{\sharp}(1_\Omega - \nu_{\mathbf{B}}) \leq 1_\Omega \) and
\[
\tau^{\flat}(1_\Omega - \nu_{\mathbf{A}}) \odot \tau^{\sharp}(1_\Omega - \nu_{\mathbf{B}}) = (\tau^{\flat}(1_\Omega - \nu_{\mathbf{A}}) + \tau^{\sharp}(1_\Omega - \nu_{\mathbf{B}}) - 1_\Omega) \lor 0_\Omega = 0_\Omega.
\]

Finally since \( \tau(\mathbf{A} \oplus \mathbf{B}) = \tau(\mathbf{A}) \oplus \tau(\mathbf{B}) \), then
\[
\tau((\mu_{\mathbf{A}} \odot \mu_{\mathbf{B}}, \nu_{\mathbf{A}} \odot \nu_{\mathbf{B}})) = (\tau^{\flat}(\mu_{\mathbf{A}}), 1_\Omega - \tau^{\sharp}(1_\Omega - \nu_{\mathbf{A}})) \odot (\tau^{\flat}(\mu_{\mathbf{B}}), 1_\Omega - \tau^{\sharp}(1_\Omega - \nu_{\mathbf{B}})),
\]
\[
(\tau^{\flat}(\mu_{\mathbf{A}} \odot \mu_{\mathbf{B}}), 1_\Omega - \tau^{\sharp}(1_\Omega - \nu_{\mathbf{A}}) \odot \nu_{\mathbf{B}})) = (\tau^{\flat}(\mu_{\mathbf{A}}) \odot \tau^{\flat}(\mu_{\mathbf{B}}), (1_\Omega - \tau^{\sharp}(1_\Omega - \nu_{\mathbf{A}})) \odot (1_\Omega - \tau^{\sharp}(1_\Omega - \nu_{\mathbf{B}}))).
\]

Hence \( \tau^{\flat}(\mu_{\mathbf{A}} \odot \mu_{\mathbf{B}}) = \tau^{\flat}(\mu_{\mathbf{A}}) \odot \tau^{\flat}(\mu_{\mathbf{B}}) \), and
\[
1_\Omega - \tau^{\sharp}(1_\Omega - \nu_{\mathbf{A}}) \odot \nu_{\mathbf{B}} = (1_\Omega - \tau^{\sharp}(1_\Omega - \nu_{\mathbf{A}})) \odot (1_\Omega - \tau^{\sharp}(1_\Omega - \nu_{\mathbf{B}}))
\]
and using de Morgan rule: \( f \odot g = 1_\Omega - (1_\Omega - f) \oplus (1_\Omega - g) \) on the second equality we obtain
\[
1_\Omega - \tau^{\sharp}((1_\Omega - \nu_{\mathbf{A}}) \odot (1_\Omega - \nu_{\mathbf{B}})) = 1_\Omega - \tau^{\sharp}(1_\Omega - \nu_{\mathbf{A}}) \oplus \tau^{\sharp}(1_\Omega - \nu_{\mathbf{B}}).
\]

Therefore
\[
\tau^{\sharp}((1_\Omega - \nu_{\mathbf{A}}) \odot (1_\Omega - \nu_{\mathbf{B}})) = \tau^{\sharp}(1_\Omega - \nu_{\mathbf{A}}) \oplus \tau^{\sharp}(1_\Omega - \nu_{\mathbf{B}}).
\]

(iii) If \( \mathbf{A}_n = (\mu_{\mathbf{A}_n}, \nu_{\mathbf{A}_n}) \not\lesssim \mathbf{A} = (\mu_{\mathbf{A}}, \nu_{\mathbf{A}}) \), i.e. \( \mathbf{A}_n \not\lesssim \mu_{\mathbf{A}} \) and \( \nu_{\mathbf{A}_n} \not\gtrsim \nu_{\mathbf{A}} \), then \( \tau(\mathbf{A}_n) \not\lesssim \tau(\mathbf{A}) \).

Hence
\[
\begin{align*}
(\tau^{\flat}(\mu_{\mathbf{A}_n}), 1_\Omega - \tau^{\sharp}(1_\Omega - \nu_{\mathbf{A}_n})) &\not\lesssim (\tau^{\flat}(\mu_{\mathbf{A}}), 1_\Omega - \tau^{\sharp}(1_\Omega - \nu_{\mathbf{A}})) \\
\tau^{\flat}(\mu_{\mathbf{A}_n}) &\not\lesssim \tau^{\flat}(\mu_{\mathbf{A}}) \\
1_\Omega - \tau^{\sharp}(1_\Omega - \nu_{\mathbf{A}_n}) &\not\gtrsim 1_\Omega - \tau^{\sharp}(1_\Omega - \nu_{\mathbf{A}}) \\
\tau^{\sharp}(1_\Omega - \nu_{\mathbf{A}_n}) &\not\lesssim \tau^{\sharp}(1_\Omega - \nu_{\mathbf{A}}).
\end{align*}
\]
Let us return to the Kolmogorov extension process (see Definition 6.1) be an IF-state. Let $\xi$ be an integrable IF-observable and $\tau$ be an $m$-preserving transformation. Then there exists an integrable IF-observable $x^*$ such that

(i) $E(x) = E(x^*)$,

(ii) $\lim \frac{1}{n} \sum_{i=0}^{n-1} (\tau^i \circ x) = x^*$ $m$-almost everywhere.

Proof. Let $x_n = \tau^{n-1} \circ x$ for $n = 1, 2, \ldots$, i.e.

$$x_1 = x, x_2 = \tau \circ x, x_3 = \tau^2 \circ x, \ldots \quad (3)$$

Let us return to the Kolmogorov extension process (see Proposition 4.7). Let us consider the probability space $(R^N, \sigma(C), P)$ such that

$$P(\{(t_i)_{i=1}^\infty : t_1 \in A_1, \ldots, t_n \in A_n\}) = m(x_1(A_1) \cdot \ldots \cdot x_n(A_n)) \quad (4)$$

for each $n \in N$ and $A_i \in B(R)$.

Let $T : R^N \to R^N$ be the shift defined by the formula $T((t_n)_n) = (s_n)_n$, where $s_n = t_{n+1}$ for $n = 1, 2, \ldots$.

Let $A = \{(t_i)_{i=1}^\infty : t_1 \in A_1, \ldots, t_n \in A_n\}$ be the cylinder. In this case

$$T^{-1}(A) = \{(t_i)_{i=1}^\infty : T((t_i)_{i=1}^\infty) \in A\} = \{(t_i)_{i=1}^\infty : t_{i+1} \in A_1, \ldots, t_{n+1} \in A_n\}.$$

Therefore using (4), (3) and (iv) from Definition 6.1 we have

$$P(T^{-1}(A)) = m(x_{i+1}(A_1) \cdot \ldots \cdot x_{n+1}(A_n)) = m(\tau \circ x(A_1) \cdot \ldots \cdot \tau^{n} \circ x(A_n)) = m(\tau(x(A_1)) \cdot \ldots \cdot \tau(x(A_n)))$$

$$= m(\tau(x(A_1)) \cdot \ldots \cdot \tau(x(A_n))) = m(x_1(A_1) \cdot \ldots \cdot x_n(A_n)) = P(A).$$

Hence the mapping $T$ preserves the probability measure $P$, i.e. $P(T^{-1}(A)) = P(A)$.

Since the IF-observable $x = x_1$ is integrable, the first coordinate function $\xi_1$ defined by $\xi_1((t_i)_{i=1}^\infty) = t_1$ is integrable, too (see Proposition 4.8 and Definition 2.9). Therefore by the Individual ergodic theorem there exists an integrable random variable $\xi^*$ such that $E(\xi^*) = E(\xi)$ and

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} (\xi \circ T^i) = \xi^* \quad P - \text{almost everywhere.}$$

Of course $\xi \circ T^i = \xi_{i+1}$, hence

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \xi_j = \xi^* \quad P - \text{almost everywhere.}$$
Put \( g_n(u_1, \ldots, u_n) = \frac{1}{n} \sum_{i=1}^{n} u_i \). By Theorem 5.5 the sequence of IF-observables
\[
\left( g_n(x_1, \ldots, x_n) \right)_n = \left( \frac{1}{n} \sum_{i=1}^{n} x_i \right)_n = \left( \frac{1}{n} \sum_{i=0}^{n-1} \tau^{t} \circ x \right)_n
\]
is convergent \( m \)-almost everywhere to the IF-observable \( x^* = \limsup_{n \to \infty} g_n(x_1, \ldots, x_n) \) and
\[
P(\xi^*(-\infty, t)) = m(x^*((-\infty, t)))
\]
for each \( t \in \mathbb{R} \). Since \( P_{\xi*} = m_{x*} \) and \( P_{\xi_1} = m_{x_1} = m_{x} \), we obtain \( E(x) = E(\xi_1) = E(\xi^*) = E(x^*) \).

\[\square\]

7 Conclusion

The paper is concerned with ergodic theory for family of intuitionistic fuzzy events. We proved the Individual ergodic theorem for intuitionistic fuzzy observables using \( m \)-almost everywhere convergence, where \( m \) is an intuitionistic fuzzy state. Since the intuitionistic fuzzy probability \( P \) can be decomposed to two intuitionistic fuzzy states, then this results can be applied to \( P \)-almost everywhere convergence, too.

References