Iranian Journal of Fuzzy Systems

Volume 17, Number 5, (2020), pp. 29-41

University of Sistan and Baluchestan

Arrow theorems in the fuzzy setting

A. Raventós-Pujol[1], M. J. Campión[2] and E. Induráin[3]

[1,2] *Inarbe (Institute for Advanced Research in Business and Economics) and Departamento de Estadística, Informáticay Matemáticas. Universidad Pública de Navarra. 31006 Pamplona, Spain.*

[3] *InaMat (Institute for Advanced Materials) and Departamento de Estadística, Informática y Matemáticas. Universidad Pública de Navarra. 31006 Pamplona, Spain.*

armajac.raventos@unavarra.es, mjesus.campion@unavarra.es, steiner@unavarra,es

Abstract

Throughout this paper, our main idea is to analyze the Arrovian approach in a fuzzy context, paying attention to different extensions of the classical Arrow's model arising in mathematical Social Choice to aggregate preferences that the agents define on a set of alternatives. There is a wide set of extensions. Some of them give rise to an impossibility theorem as in the Arrovian classical model. But others lead to possibility results. We explore the main grounds that lead to impossibility or possibility. In this analysis, representative examples arise. One of them corresponds to an impossibility result, and the other ones allow the aggregation of fuzzy individual preferences to an individual one. We introduce new techniques for the proofs, specially for the one that leads to impossibility.

Keywords: Mathematical social choice, fuzzy preferences, fusion of individual preferences, social aggregation rules, arrovian models in the fuzzy setting.

1 Introduction

The well-known Arrow's impossibility theorem arises in mathematical Social Choice (see e.g. [2, 15]). It clearly states that under an apparently mild set of conditions that our common sense would naturally impose to any aggregation rule that fuses individual preferences into a social one, no rule is possible. The rules are necessary in order to retrieve the main features of each of the individual components, so that the social rule should reflect in a sense the characteristics of the individual opinions. However, due to the impossibility result stated by Arrow's theorem, we see that any rule to aggregate rankings will forcedly have some kind of bad behavior or undesired properties, as regards the classical Arrovian model, since it will not accomplish some of the conditions that the model would expect a priori to be satisfied by the rules (if any). However, all this happens in the crisp setting, where the preferences are dichotomic, so that, if we consider that an element $x \in X$ is related to another element $y \in X$, it happens that the relationship is either void ($= 0$) or total ($= 1$).

Unlike the crisp classical approach, in several generalizations of the Arrovian model that deal with fuzzy preferences to formalize uncertainty, there is still room for some possibility results and, consequently, existence of suitable aggregation rules. But, in this literature about possible extensions to the fuzzy setting of the classical Arrovian model, one may encounter impossibility theorems as well as possibility theorems, depending on the extension made and its terms and conditions imposed (see e.g. [14, 20]). Therefore, we may wonder why in some situations a rule is still possible, whereas in other ones an impossibility theorem is found. In the present manuscript, working with the corresponding Arrovian models, in Section 4 we introduce some possibility results, whereas in Section 5 an impossibility theorem arises. In Section 6 it is our aim to explain the mathematical grounds that could lead to either possibility or impossibility issues.

Corresponding Author: E. Induráin

Received: May 2019; Revised: March 2020; Accepted: April 2020.

2 Preliminaries

2.1 Arrow's impossibility result and some of its variants

Let X stand for a nonempty set.

Definition 2.1. *A preorder $\precsim$ on X (see [6]) is a binary relation on X which is reflexive and transitive. An antisymmetric preorder is said to be an order. A total preorder $\precsim$ is a preorder such that if $x, y \in X$ then $(x \precsim y) \vee (y \precsim x)$ holds. This last property is usually called completeness. If $\precsim$ is a preorder on X, then as usual we denote the associated asymmetric relation by $\succ$ and the associated equivalence relation by $\sim$. These are defined by $x \succ y \Leftrightarrow (x \succsim y) \wedge \neg(y \succsim x)$ and $x \sim y \Leftrightarrow (x \succsim y) \wedge (y \succsim x)$.*

Definition 2.2. *A total preorder $\succsim$ is usually called a (crisp) preference on X.*

Suppose now that A is a finite set of alternatives with at least three elements.

Let us assume that a finite number of $n \geq 3$ agents define on A their individual preferences. Thus, each individual will be represented by a number $i \in N_n = \{1, \ldots, n\}$, and her/his preference will be a total preorder $\succsim_i$ defined on A.

Definition 2.3. *A n-profile $\mathcal{P}$ of preferences (see [14]) is then a n-tuple $(\succsim_1, \ldots, \succsim_n)$ such that $\succsim_i$ stands for the preference of the agent $i \in N_n$. A nonempty subset of A is said to be an agenda. If $\mathcal{A}$ denotes the set of all agendas, a choice function is a map $C : \mathcal{A} \to \mathcal{A}$ such that for every $a \in \mathcal{A}$ we have that $C(a) \subseteq a$. A social choice rule f is a map that assigns a choice function $C_{\mathcal{P}}$ to a profile $\mathcal{P}$.*

Definition 2.4. *The Arrovian model (see [15]) consists of a finite set A of at least three alternatives, a finite set of agents, whose cardinality n must be at least three, and a set of choice rules satisfying the following conditions:*

(i) Standard domain constraint: Each rule f acts on all possible profiles of preferences on A, and given a profile $\mathcal{P}$, f assigns to $\mathcal{P}$ a choice rule $C_{\mathcal{P}}$ that has in its domain all nonempty agendas.

(ii) Strong Pareto condition: For every profile $\mathcal{P}$ and every pair of alternatives $x, y \in A$ such that for every $i \in N_n$ we have that $x \succsim_i y$, and there is also at least one element $j \in N_n$ for which $x \succ_j y$, it holds that, for every agenda $a \in \mathcal{A}$, if x belongs to a, then y does not belong to $C_{\mathcal{P}}(a)$.

(iii) Independence of irrelevant alternatives: For any agenda a and each pair of profiles $\mathcal{P}$ and $\mathcal{P}'$ whose restriction to the agenda a coincide, the associated choice rules $C_{\mathcal{P}}$ and $C_{\mathcal{P}'}$ should accomplish that $C_{\mathcal{P}}(a) = C_{\mathcal{P}'}(a)$.

(iv) Transitive explanation: For each profile $\mathcal{P}$ there exists a total preorder $\succsim_{\mathcal{P},f}$ such that the corresponding choice rule $C_{\mathcal{P}}$ that f assigns to $\mathcal{P}$ satisfies that $C_{\mathcal{P}}(a) = \{x \in a : x \succsim_{\mathcal{P},f} y \text{ for all } y \in a\}$, for any agenda a.

(v) Non-dictatorship: There is no $i \in N_n$ such that for any profile $\mathcal{P}$, every agenda a and $x, y \in A$ it holds that $x \succ_i y$ implies that if $x \in a$ then y does not lie in $C_{\mathcal{P}}(a)$, where $C_{\mathcal{P}}$ is the choice rule that f assigns to $\mathcal{P}$.

Theorem 2.5. *(Arrow's impossibility theorem): The Arrovian model is empty. That is, the five conditions that appear in the model are incompatible.*

Proof. Many proofs are known. It was originally issued by K.J. Arrow in [2]). □

We also include here the Fishburn's variation on Arrow's impossibility theorem ([12]). We will use it then in the proof of a key result, namely Theorem 5.15.

Theorem 2.6. *Let A stand for a finite set whose cardinality is at least three. Let $n \geq 3 \in \mathbb{N}$. Let $\mathcal{A}$ stand for the set of n-profiles of preferences on A. Denote by $\mathcal{O}$ the family of the asymmetric parts of all the total preorders defined on A. Then there is no map $F : \mathcal{O}^n \longrightarrow \mathcal{O}$ satisfying the following properties:*

(i) Unanimity: For every $x, y \in A$ and every profile $\mathcal{P} = (\succsim_1, \ldots, \succsim_n) \in \mathcal{A}$ such that $x \succ_i y$ holds for every $i \in N_n$, it holds that $xF(\mathcal{P})y$,

(ii) Independence of irrelevant alternatives (Fishburn's version): For every $x, y \in A$ and $\mathcal{P} = (\succsim_1, \ldots, \succsim_n); \mathcal{P}' = (\succsim'_1, \ldots, \succsim'_n) \in \mathcal{A}$ if the restriction to $\{x, y\}$ of $\succ_i$ and $\succ'_i$ coincide for every $i \in N_n$, then $xF(\mathcal{P})y = xF(\mathcal{P}')y$.

(iii) Non-dictatorship: There is no $k \in N_n$ such that for every $x, y \in A$ and $\mathcal{P} = (\succsim_1, \ldots, \succsim_n) \in \mathcal{A}$ it holds that $x \succ_k y \Rightarrow xF(\mathcal{P})y$.

Proof. See the main result in [12]. □

2.2 Fuzzy sets and fuzzy preferences

A common feature, that nowadays is typical in many models, is the consideration of comparisons or suitable binary relations that are *graded*. For instance, this is done in order to describe an intensity in the relationship between two given elements. Or this approach can also be encountered when confronting uncertainty. In these cases, two elements could be related at any level between 0 and 1. Of course, now the relation becomes *fuzzy*.

Definition 2.7. *Given a nonempty set U, called universe, a fuzzy subset X of U is defined (see [22]) by means of a map $\mu_X : U \to [0,1]$. This map is said to be the membership function of the fuzzy set X. For any $\alpha \in [0,1]$ we define the α-cut of X as the crisp subset of the universe U given by $X_\alpha = \{t \in U : \mu_X(t) \geq \alpha\}$.*

The support of X is the crisp subset $Supp(X) = \{t \in U : \mu_X(t) \neq 0\} \subseteq U$, whereas the kernel of X is the crisp subset $Ker(X) = \{t \in U : \mu_X(t) = 1\} \subseteq U$. The fuzzy set X is said to be normal if its kernel is nonempty.

Also, a fuzzy binary relation defined on a universe U is established through a function $\mathcal{F} : U \times U \to [0,1]$. So $\mathcal{F}$ is a fuzzy subset of the Cartesian product $U \times U$. Again, for any $\alpha \in [0,1]$ it is defined the α-cut of $\mathcal{F}$ as the crisp subset of $U \times U$ defined by $\mathcal{F}_\alpha = \{(x,y) \in U \times U : \mathcal{F}(x,y) \geq \alpha\}$.

Remark 2.8. *Henceforward, any universe U considered throughout the present manuscript is assumed to have at least three pairwise different elements.*

Definition 2.9. *Let $\mathcal{F}$ and $\mathcal{F}'$ be two fuzzy binary relations on the universe U. We say that $\mathcal{F}$ is equivalent to $\mathcal{F}'$, and we denote this by $\mathcal{F} \approx \mathcal{F}'$ if the supports of $\mathcal{F}$ and $\mathcal{F}'$ coincide and, in addition, there exists an order isomorphism $g : \mathcal{F}(U \times U) \longrightarrow \mathcal{F}'(U \times U)$ such that for every $\alpha \in \mathcal{F}(U \times U)$ the α-cut $\mathcal{F}_\alpha$ coincides with the $g(\alpha)$-cut $\mathcal{F}'_{g(\alpha)}$.*

Remark 2.10. *It is straightforward to see that $\approx$ is actually an equivalence relation. Moreover, when the universe U is finite this definition is equivalent to Definition 4.11 in [14]. Definition 2.9 essentially means that if for every fuzzy binary relation $\mathcal{F}$ we elaborate a ranking on the Cartesian product $U \times U$, so that given a pair $(x,y) \in U \times U$, the number $\mathcal{F}(x,y) \in [0,1]$ is understood as the score assigned to that couple (x,y), then two fuzzy binary relations $\mathcal{F}$ and $\mathcal{F}'$ are equivalent when the scores corresponding to $\mathcal{F}$ define on $U \times U$ exactly the same ranking as the scores corresponding to $\mathcal{F}'$. This is formalized in Proposition 2.11 below.*

Proposition 2.11. *Let $\mathcal{F}$ and $\mathcal{F}'$ be two fuzzy binary relations on a universe U. Then $\mathcal{F} \approx \mathcal{F}'$ holds if and only if $Supp(\mathcal{F}) = Supp(\mathcal{F}')$ and, in addition, for all $\overline{z} = (a,b)$; $\overline{z}' = (a',b') \in U \times U$ we have that $\mathcal{F}(\overline{z}) < \mathcal{F}(\overline{z}') \Leftrightarrow \mathcal{F}'(\overline{z}) < \mathcal{F}'(\overline{z}')$.*

Proof. First observe that the condition on the respective supports is common in both sides of the equivalence implication. Now suppose that $\mathcal{F} \approx \mathcal{F}'$. Then there exists an order isomorphism $g : \mathcal{F}(U \times U) \longrightarrow \mathcal{F}'(U \times U)$ accomplishing that $\mathcal{F}_\alpha = \mathcal{F}'_{g(\alpha)}$ holds for every $\alpha \in \mathcal{F}(U \times U)$. If $\mathcal{F}(\overline{z}) < \mathcal{F}(\overline{z}')$, then $\overline{z} \notin \mathcal{F}_{\mathcal{F}(\overline{z}')} = \mathcal{F}'_{g(\mathcal{F}(\overline{z}'))}$, but $\overline{z}' \in \mathcal{F}_{\mathcal{F}(\overline{z}')} = \mathcal{F}'_{g(\mathcal{F}(\overline{z}'))}$, so $\mathcal{F}'(\overline{z}) < \mathcal{F}'(\overline{z}')$. We can proceed equivalently with g^{-1} to obtain $\mathcal{F}'(\overline{z}) < \mathcal{F}'(\overline{z}') \Rightarrow \mathcal{F}(\overline{z}) < \mathcal{F}(\overline{z}')$.

To prove the converse implication we define $g(t)$ (with $t \in \mathcal{F}(U \times U)$) as the unique value $s \in \mathcal{F}'(U \times U)$ such that $\mathcal{F}_t = \mathcal{F}'_s$. Such a value exists because there is a $\overline{z}_0 \in U \times U$ with $\mathcal{F}(\overline{z}_0) = t$, so $\mathcal{F}_t = \{\overline{z} \in U \times U : \mathcal{F}(\overline{z}_o) \leq \mathcal{F}(\overline{z})\} = \{\overline{z} \in U \times U : \mathcal{F}'(\overline{z}_0) \leq \mathcal{F}'(\overline{z})\} = \mathcal{F}'_{\mathcal{F}'(\overline{z}_0)}$. Notice that, if there exists another value $s' = \mathcal{F}'(\overline{z}_1) \in \mathcal{F}'(U \times U)$ with $\mathcal{F}_t = \mathcal{F}'_{s'}$, then, as $\overline{z}_i \in \mathcal{F}'_{\mathcal{F}'(\overline{z}_i)}$ for $i = 0,1$, from $\mathcal{F}'_{\mathcal{F}'(\overline{z}_0)} = \mathcal{F}_t = \mathcal{F}'_{\mathcal{F}'(\overline{z}_1)}$ we would obtain that $\overline{z}_1 \in \mathcal{F}'_s$ and also $\overline{z}_0 \in \mathcal{F}'_{s'}$, so concluding that $s \leq \mathcal{F}'(\overline{z}_1) = s' \leq \mathcal{F}'(\overline{z}_0) = s$. Moreover, if $t < t'$ then $\mathcal{F}_{t'} \subsetneq \mathcal{F}_t$, so that $\mathcal{F}'_{s'} \subsetneq \mathcal{F}'_s$. Hence $s < s'$ and g is increasing, preserving the usual order $\leq$ of the real line. Bearing all this in mind we can define analogously an order-preserving map $h : \mathcal{F}'(U \times U) \longrightarrow \mathcal{F}(U \times U)$ such that g and h are inverses one another. □

Definition 2.12. *(See e.g. [17]) A triangular norm (t-norm) is a function $T : [0,1] \times [0,1] \longrightarrow [0,1]$ satisfying the following properties:*

(i) Boundary conditions: $T(x,0) = T(0,x) = 0$, and $T(x,1) = T(1,x) = x$, for every $x \in [0,1]$,

(ii) Monotonicity: T is non-decreasing with respect to each variable: $T(x_1,x_2) \leq T(y_1,y_2)$ if $(x_1 \leq y_1) \wedge (x_2 \leq y_2)$.

(iii) T is commutative: $T(x,y) = T(y,x)$ holds for every $x,y \in [0,1]$,

(iv) T is associative: $T(x,T(y,z)) = T(T(x,y),z)$ holds for any $x,y,z \in [0,1]$.

Definition 2.13. *(See e.g. [14]) Let $\mathcal{F}$ be a fuzzy relation on a universe U. Let T be a t-norm. We say that $\mathcal{F}$ is:*

(i) Reflexive if, for all $x \in U$, $\mathcal{F}(x,x) = 1$ holds.

(ii) Symmetric if $\mathcal{F}(x,y) = \mathcal{F}(y,x)$ $(x,y \in U)$.

(iii) Asymmetric if $\mathcal{F}(x,y) > 0 \Rightarrow \mathcal{F}(y,x) = 0$ *holds, for every* $x, y \in U$.

(iv) Connected if $\mathcal{F}(x,y) + \mathcal{F}(y,x) \geq 1$ *holds, for every* $x, y \in U$.

(v) Complete (or total) if $\max\{\mathcal{F}(x,y), \mathcal{F}(y,x)\} = 1$ *holds, for all* $x, y \in U$.

(vi) Weakly transitive if $[\mathcal{F}(x,y) \geq \mathcal{F}(y,x)] \wedge [\mathcal{F}(y,z) \geq \mathcal{F}(z,y)] \Rightarrow \mathcal{F}(x,z) \geq \mathcal{F}(z,x)$ *holds, for every* $x, y, z \in U$.

(vii) T*-transitive if* $\mathcal{F}(x,z) \geq T(\mathcal{F}(x,y), \mathcal{F}(y,z))$ *holds, for any* $x, y, z \in U$.

Remark 2.14. *In the classical Arrovian crisp model, the preferences of an individual agent* i *are implemented through a total preorder* $\precsim_i$. *But a total preorder has associated an asymmetric part* $\succ_i$ *and a symmetric part* $\sim_i$. *This will allow us to study those binary relations, namely* $\precsim_i$, $\succ_i$ *and* $\sim_i$ *separately.*

When looking for extensions to the fuzzy setting we could work with triplets (P_i, R_i, I_i) *of fuzzy binary relations on a universe* U*, such that, in a sense,* R_i *plays the role of a total preorder* $\precsim_i$*, whereas* P_i *(respectively* I_i*) plays the role of* $\succ_i$ *(respectively, of* $\sim_i$*). Here, each of these relations* P_i, R_i *and* I_i*, even being obviously related one another, are considered as items to be handled one by one.*

Definition 2.15. *Given a universe* U*, a fuzzy preference* Λ *on* U *is a triplet* (P, R, I) *of fuzzy binary relations on* U *that satisfies the following conditions:*

(FP1) $P(x,y) > 0 \Rightarrow P(y,x) = 0$ *for every* $x, y \in U$ *(*P *is asymmetric),*
(FP2) $I(x,y) = I(y,x)$ *for all* $x, y \in U$ *(*I *is symmetric),*
(FP3) $P(x,y) \leq R(x,y)$ *for every* $x, y \in U$,
(FP4) $R(x,y) > R(y,x) \Leftrightarrow P(x,y) > 0$ *for all* $x, y \in U$,
(FP5) $P(x,y) = 0 \Rightarrow R(x,y) = I(x,y) \quad (x, y \in U)$,
(FP6) $[I(x,y) \leq I(z,w)] \wedge [P(x,y) \leq P(z,w)] \Rightarrow [R(x,y) \leq R(z,w)]$ *holds for every* $x, y, z, w \in U$.

We call to I *(respectively, to* P*,* R*) the indifference component (respectively, the strict preference, the weak preference component) associated to* Λ.

Definition 2.16. *Given a fuzzy preference* $\Lambda = (P, R, I)$ *on a universe* U*, and a triangular norm* T*, we say that* Λ *is:*

(i) Reflexive if its component R *is reflexive.*

(ii) Connected if R *is connected.*

(iii) Complete if R *if complete.*

(iv) Weakly transitive if R *is weakly transitive.*

(v) T*- transitive if* R *is* T*-transitive.*

(vi) Partially T*-transitive if* P *is* T*-transitive.*

Definition 2.17. *Given two fuzzy preferences* $\Lambda = (P, R, I)$ *and* $\Lambda' = (P', R', I')$ *on* X*. We say that both preferences are equivalent, and we denote it by (*$\Lambda \approx \Lambda'$*), if their second components in the corresponding triplets are equivalent in the sense of Definition 2.9 (that is, when* $R \approx R'$*).*

3 Arrovian models with fuzzy preferences

Let us analyze now different extensions to a fuzzy approach of the classical crisp Arrovian model. To do so, we will consider fuzzy preferences imposed to a universe U. Then we will generalize to this new kind of preferences the definitions of the conditions involved in the model considered in subsection 2.1. Thus, in the previous sections we have established an extension of preferences from a crisp environment to a fuzzy one. Now it is the turn to define how fuzzy preferences can be aggregated, and then analyze the main properties of those aggregations.

Definition 3.1. *(See e.g.* [14]*) Let* U *denote a universe. Fix an integer number* $n \geq 3$. *Let* $\mathcal{P}$ *denote a nonempty set of fuzzy preferences defined on* U. *A* n*-aggregation rule (FPAR) for fuzzy preferences that belong to* $\mathcal{P}$ *is a function* $f : \mathcal{P}^n \to \mathcal{P}$. *An element* $(\Lambda_1, \ldots, \Lambda_n) \in \mathcal{P}^n$ *is called a fuzzy profile of preferences. We will denote it, for short, as* (Λ_i)*, understanding that* $i \in N_n = \{1, \ldots, n\}$. *Moreover* Λ_i *is a triplet of fuzzy binary relations on* U*, namely* (P_i, R_i, I_i).

Notation 3.2. *Given a* n*-aggregation rule* f*, for fuzzy preferences that belong to a set* $\mathcal{P}$*, and a fuzzy profile* (Λ_i)*, the image* $f[(\Lambda_i)]$ *is a new fuzzy preference that belongs to* $\mathcal{P}$. *We will denote it by* $[P_f((\Lambda_i)), R_f((\Lambda_i)), I_f((\Lambda_i))]$. *Given two alternatives* x, y *in the universe* U*, the restriction of a fuzzy preference* Λ *to the subset* $\{x, y\}$ *will be denoted as* $\Lambda_{\rceil\{x,y\}}$.

3.1 Extending the condition of independence of irrelevant alternatives to the fuzzy setting

In the classical crisp Arrovian model, the condition of independence of irrelevant alternatives, already introduced in Definition 2.4 above, has been subject to criticism. It is well known that if we drop such restriction from the classical Arrow's setting, it is still possible to find suitable aggregation rules for crisp individual preferences: an example is the well-known Borda's rule (see e.g. [15], pp. 72 and ff.).

In the fuzzy setting, various non-equivalent ways to extend the condition of independence of irrelevant alternatives have already been introduced (see *[14]*). The choice of some particular extension of this condition will be then the responsible of the existence of some impossibility theorem in fuzzy Arrovian models.

Definition 3.3. [14] *Let $f : \mathcal{P}^n \to \mathcal{P}$ stand for a FPAR on a nonempty set of fuzzy preferences $\mathcal{P}$ defined on a universe U. Let (Λ_i) and (Λ'_i) be two profiles that belong to $\mathcal{P}$. We say that f satisfies the condition of independence of the first type- of irrelevant alternatives if for every $x, y \in U$ we have that $\Lambda_{i\rceil\{x,y\}} = \Lambda'_{i\rceil\{x,y\}}$ for any $i \in N_n$ implies $f((\Lambda_i))_{\rceil\{x,y\}} = f((\Lambda'_i))_{\rceil\{x,y\}}$.*

Using the concept of equivalent fuzzy preferences ($\approx$), already introduced in Definition 2.17, we define a new extension to the fuzzy setting of the independence of irrelevant alternatives condition, as follows.

Definition 3.4. [14] *Let $f : \mathcal{P}^n \to \mathcal{P}$ be a FPAR on a nonempty set of fuzzy preferences $\mathcal{P}$ defined on a universe U. Let (Λ_i) and (Λ'_i) be two profiles that belong to $\mathcal{P}$. We say that f satisfies the condition of independence –of the second type– of irrelevant alternatives if for all $x, y \in X$ we have that $\Lambda_{i\rceil\{x,y\}} \approx \Lambda'_{i\rceil\{x,y\}}$ for any $i \in N_n$ implies $f((\Lambda_i))_{\rceil\{x,y\}} \approx f((\Lambda'_i))_{\rceil\{x,y\}}$.*

Remark 3.5. *The just introduced new notion of independence -of the second type- of irrelevant alternatives had already been introduced in* [20] *for the particular case in which we deal with a finite universe U.*

Definition 3.4 makes reference to pairs (x, y) of alternatives in the universe U. For this reason, it may be helpful to have at hand some characterization of the equivalence of fuzzy preferences when they act on two alternatives. This is made in the spirit of Proposition 2.11, in the particular case of reflexive fuzzy preferences.

Proposition 3.6. *Let $\Lambda = (P, R, I))$ and $\Lambda' = (P', R', I')$ be two reflexive fuzzy preferences on a universe U. Then for every $x, y \in U$ it holds that:*

$$\Lambda_{\rceil\{x,y\}} \approx \Lambda'_{\rceil\{x,y\}} \quad \Leftrightarrow \quad \{\ [R(a,b) > R(b,a) \Leftrightarrow R'(a,b) > R'(b,a)\}] \ \wedge$$

$$\wedge \ \ [R(a,b) = 0 \Leftrightarrow R'(a,b) = 0] \ \ \wedge \ \ [R(a,b) = 1 \Leftrightarrow R'(a,b) = 1] \ \ (a, b \in \{x, y\})\}.$$

Proof. First suppose that $\Lambda_{\rceil\{x,y\}} \approx \Lambda'_{\rceil\{x,y\}}$. By Proposition 2.11 the first and second conditions follow immediately. Moreover, $R(a, b) = 1$, then $R(a, b) \geq R(x, x)$ and, again by Proposition 2.11, it follows that $R'(a, b) \geq R'(x, x) = 1$. Finally, the fact $R'(a, b) = 1 \Rightarrow R(a, b)$ is proved in an analogous way. For the converse, notice that the second condition guarantees the equality of the corresponding supports, whereas the first one carries the equivalence $R(\overline{z}) < R(\overline{z}') \Leftrightarrow R'(\overline{z}) < R'(\overline{z}')$ for any $\overline{z}, \overline{z}' \in \{(x, y), (y, x)\}$. Finally, when $\overline{z}' \in \{(x, x), (y, y)\}$ the corresponding equivalence is granted by the third condition, while if $\overline{z} \in \{(x, x), (y, y)\}$ it never happens that $R(\overline{z}) < R(\overline{z}')$ nor $R'(\overline{z}) < R'(\overline{z}')$. □

Example 3.7. *Definition 3.3 and Definition 3.4 are different in nature:*

Consider the universe $U = \{a, b, c\}$ and a set of fuzzy preferences that only consists of two elements, that we call $\Lambda = (P, R, I)$ and $\Lambda' = (P', R', I')$, where $P(a, a) = P(b, b) = P(c, c) = P(b, a) = P(c, a) = P(c, b) = R(b, a) = R(c, a) = R(c, b) = I(a, b) = I(a, c) = I(b, c) = I(b, a) = I(c, a) = I(c, b) = 0; P(a, b) = R(a, b) = 0.2; P(a, c) = R(a, c) = 0.6; P(b, c) = R(b, c) = 0.9;\ I(a, a) = I(b, b) = I(c, c) = R(a, a) = R(b, b) = R(c, c) = 1,$ whereas $P'(a, a) = P'(b, b) = P'(c, c) = P'(b, a) = P'(c, a) = P'(c, b) = R'(b, a) = R'(c, a) = R'(c, b) = I'(a, b) = I'(a, c) = I'(b, c) = I'(b, a) = I'(c, a) = I'(c, b) = 0; P'(a, b) = R'(a, b) = 0.5; P'(a, c) = R'(a, c) = 0.6; P'(b, c) = R'(b, c) = 0.7; I'(a, a) = I'(b, b) = I'(c, c) = R'(a, a) = R'(b, b) = R'(c, c) = 1$.

Given a profile (Λ_i) with $i = 1, \ldots, n$ and either $\Lambda_i = \Lambda$ or else $\Lambda_i = \Lambda'$ $\ (i \in N_n)$, we consider the following n-aggregation rule f: $P_{f((\Lambda_i))}(a, a) = P_{f((\Lambda_i))}(b, b) = P_{f((\Lambda_i))}(c, c) = P_{f((\Lambda_i))}(b, a) = P_{f((\Lambda_i))}(c, a) = P_{f((\Lambda_i))}(c, b) = R_{f((\Lambda_i))}(b, a) = R_{f((\Lambda_i))}(c, a) = R_{f((\Lambda_i))}(c, b) = I_{f((\Lambda_i))}(a, b) = I_{f((\Lambda_i))}(a, c) = I_{f((\Lambda_i))}(b, c) = I_{f((\Lambda_i))}(b, a) = I_{f((\Lambda_i))}(c, a) = I_{f((\Lambda_i))}(c, b) = 0$, and $P_{f((\Lambda_i))}(x, y) = R_{f((\Lambda_i))}(x,\ y) = 0$ if $(x, y) \in \{(a, b), (a, c), (b, c)\} \wedge max_{i \in N_n} P_i(x, y) \leq 0.5$; $P_{f((\Lambda_i))}(x, y) = R_{f((\Lambda_i))}(x, y) = 1$ if $(x, y) \in \{(a, b), (a, c), (b, c)\} \wedge max_{i \in N_n} P_i(x, y) > 0.5$. It is straightforward to see that f satisfies the independence -of the first type- or irrelevant alternatives condition. However, if we consider the profiles $(\Lambda, \Lambda, \Lambda)$ and $(\Lambda, \Lambda, \Lambda')$, that are obviously equivalent in the sense of Definition 2.17, we see that

$R_{f((\Lambda,\Lambda,\Lambda))}(a,b)=0$ while $R_{f((\Lambda,\Lambda,\Lambda'))}(a,b)=1$, so that $f((\Lambda,\Lambda,\Lambda))$ fails to be equivalent to $f((\Lambda,\Lambda,\Lambda'))$. Hence f does not satisfy the condition of independence -of the second type- of irrelevant alternatives.

Now consider the rule g such that $g((\Lambda_i))$ acts on a pair $(x,y)\in U\times U$ as follows: $P_{g((\Lambda_i))}(a,a)=P_{g((\Lambda_i))}(b,b)=P_{g((\Lambda_i))}(c,c)=P_{g((\Lambda_i))}(b,a)=P_{g((\Lambda_i))}(c,a)=P_{g((\Lambda_i))}(c,b)=R_{g((\Lambda_i))}(b,a)=R_{g((\Lambda_i))}(c,a)=R_{g((\Lambda_i))}(c,b)=I_{g((\Lambda_i))}(a,b)=I_{g((\Lambda_i))}(a,c)=I_{g((\Lambda_i))}(b,c)=I_{g((\Lambda_i))}(b,a)=I_{g((\Lambda_i))}(c,a)=I_{g((\Lambda_i))}(c,b)=0$, $P_{g((\Lambda_i))}(a,a)=P_{g((\Lambda_i))}(b,b)=P_{g((\Lambda_i))}(c,c)=P_{g((\Lambda_i))}(b,a)=P_{g((\Lambda_i))}(c,a)=P_{g((\Lambda_i))}(c,b)=I_{g((\Lambda_i))}(a,b)=I_{g((\Lambda_i))}(a,c)=I_{g((\Lambda_i))}(b,c)=0$, and $P_{g((\Lambda_i))}(a,b)=R_{g((\Lambda_i))}(a,b)=P_{g((\Lambda_i))}(a,c)=R_{g((\Lambda_i))}(a,c)=P_{g((\Lambda_i))}(b,c)=R_{g(\Lambda_i))}(b,c)=0.9$ if Λ is one of the coordinates of the profile (Λ_i) ; $P_{g((\Lambda_i))}(a,b)=R_{g((\Lambda_i))}(a,b)=P_{g((\Lambda_i))}(a,c)=R_{g((\Lambda_i))}(a,c)=P_{g((\Lambda_i))}(b,c)=R_{g((\Lambda_i))}(b,c)=0.7$ if Λ does not appear as one of the coordinates of the profile (Λ_i). A direct check shows that g satisfy the independence -of the second type- of irrelevant alternatives. Nevertheless, the profiles $(\Lambda,\Lambda,\Lambda)$ and $(\Lambda',\Lambda',\Lambda')$ coincide on $\{a,c\}$ but $R_{f((\Lambda,\Lambda,\Lambda))}(a,c)=0.7$ whilst $R_{f((\Lambda,\Lambda,\Lambda'))}(a,c)=0.9$, so that g does not satisfy the condition of independence -of the first type- of irrelevant alternatives.

3.2 Extension to the fuzzy setting of the Paretian and non-dictatorship conditions

Concerning the extensions of some *Paretian* condition, we may observe that this type of conditions represents the following idea: if all the individuals prefer one alternative x to another one y, this must be kept by the aggregation rule, so that the final fused (social) preference should also prefer x to y.

Definition 3.8. [14] *Let $f:\mathcal{P}^n\to\mathcal{P}$ stand for a n-aggregation rule for fuzzy preferences. We say that f is weakly Paretian if for every profile (Λ_i) and any alternatives $x,y\in U$ the following condition holds: $P_i(x,y)>0$ for every $i\in N_n\Rightarrow P_f((\Lambda_i))(x,y)>0$.*

Also, we say that f is Paretian if for every profile (Λ_i) and any alternatives $x,y\in U$ it holds that $P_f((\Lambda_i))(x,y)\geq \min_{i\in N}\{P_i(x,y)\}$.

Now we pass to consider some extension of the concept of dictatorship. The number of extensions used is scarce. The following one is perhaps the most common.

Definition 3.9. [14] *Let $f:\mathcal{P}^n\to\mathcal{P}$ stand for a n-aggregation rule for fuzzy preferences. We say that f is dictatorial if there exists $k\in N_n$, called dictator, such that for every profile (Λ_i) and any alternatives $x,y\in U$ we have that $P_k(x,y)>0\Rightarrow P_f((\Lambda_i))(x,y)>0$.*

4 Aggregation rules for fuzzy preferences: Possibility results

Once we have introduced different extensions to the fuzzy approach of the conditions that arise in the classical Arrovian model, we analyze a combination of them that leads to some possibility results. These make use of the following requirements: independence of irrelevant alternatives, the Paretian condition and transitivity as regards the triangular norm defined by the minimum.

Definition 4.1. *Let U denote a universe. Let T be a t-norm on $[0,1]$. A fuzzy preference $\Lambda=(P,R,I)$ on U is said to be T-fair if it satisfies reflexibility, T-transitivity and connectedness. When the triangular norm considered is the minimum ($T=\min$), we will use the nomenclature "fair" instead of "min-fair". In other words, a fuzzy preference Λ on U is said to be fair if it satisfies reflexibility and connectedness, and also $R(x,z)\geq\min(R(x,y),R(y,z))$ holds, for any $x,y,z\in U$. We will denote by $\mathcal{L}$ the set of fair fuzzy preferences on U.*

Remark 4.2. *Unless otherwise stated from now on the only t-norm T considered will be the* minimum. *That is* $T(x,y)=\min\{x,y\}\quad(x,y\in[0,1])$.

Theorem 4.3. *There exists a n-aggregation rule for fair fuzzy preferences $f:\mathcal{L}^n\longrightarrow\mathcal{L}$ satisfying the conditions of independence -of the first type- of irrelevant alternatives, Pareto and non-dictatorship.*

Proof. We define the following aggregation function f: Given a profile $(\Lambda_i)\in\mathcal{L}^n$, and two alternatives $x,y\in U$, set $I_f((\Lambda_i))(x,y)=1$ if $x=y$ and $I_f((\Lambda_i))(x,y)=\beta$ if $x\neq y$, and also $P_f((\Lambda_i))(x,y)=1$ if $P_i(x,y)>0$ holds for every $i\in N_n$, and $P_f((\Lambda_i))(x,y)=0$ otherwise, with $\frac{1}{2}\leq\beta<1$.

Observe that, from the definition of the concept of a fuzzy preference (see Definition 2.15 above), the component R_f is determined from I_f and P_f as: $R_f((\Lambda_i))(x,y)=1$ if $x=y$; $R_f((\Lambda_i))(x,y)=1$ if $P_i(x,y)>0$ holds for every $i\in N_n$, and finally $R_f((\Lambda_i))(x,y)=\beta$ otherwise.

Notice that $f((\Lambda_i))$ is a fuzzy preference since it accomplishes all the conditions of Definition 2.15. In addition, it is straightforward to see that f satisfies independence -of the first type- of irrelevant alternatives as well as the Paretian and non-dictatorship conditions. Moreover, $f((\Lambda_i))$ is also connected, because $\beta \geq \frac{1}{2}$.

To conclude, let us prove now $f((\Lambda_i))$ is transitive as regards the minimum triangular norm: Suppose that $f((\Lambda_i))$ fails to satisfy min-transitivity. If this happens, then there exist three pairwise different elements $x, y, z \in U$ such that $R_f((\Lambda_i))(x,z) < \min\{R_f((\Lambda_i))(x,y),\ R_f((\Lambda_i))(y,z)\}$. But this may only happen when $R_f((\Lambda_i))(x,z) = \beta$ and $R_f((\Lambda_i))(x,y) = R_f((\Lambda_i))(y,z) = 1$. Consequently, there exists $k \in N_n$ such that $P_k(x,z) = 0$ whereas $P_k(x,y),\ P_k(y,z) > 0$. Thus we get $R_k(x,z) \leq R_k(z,x)$, $R_k(x,y) > R_k(y,x)$ and $R_k(y,z) > R_k(z,y)$. Taking into account the transitive inequality, we have that $R_k(x,z) \geq \min\{R_k(x,y), R_k(y,z)\}$, we analyze two cases, namely $R_k(x,y) \leq R_k(y,z)$ and $R_k(y,z) < R_k(x,y)$:

In the first case, from $R_k(x,y) \leq R_k(y,z)$ and $R_k(y,x) < R_k(x,y)$ we obtain $R_k(y,x) < R_k(y,z)$. Using the inequality $R_k(y,x) \geq \min\{R_k(y,z),\ R_k(z,x)\}$ it follows that $R_k(y,z) > R_k(y,x) \geq \min\{R_k(y,z),\ R_k(z,x)\}$, so that a fortiori $R_k(y,x) \geq R_k(z,x)$. Therefore, we arrive at $R_k(z,x) \leq R_k(y,x) < R_k(x,y) \leq R_k(x,z) \leq R_k(z,x)$, which is a contradiction.

In the second case, from $R_k(y,z) < R_k(x,y)$ jointly with $R_k(x,z) \geq \min\{R_k(x,y),\ R_k(y,z)\}$ we get $R_k(x,z) \geq R_k(y,z)$. In addition, we also have that $R_k(x,y) > R_k(y,z) > R_k(z,y)$, so that, in particular, it follows that $R_k(x,y) > R_k(z,y)$. Thus we arrive at $R_k(x,y) > R_k(z,y) \geq \min\{R_k(z,x), R_k(x,y)\}$. Therefore $R_k(z,y) \geq R_k(z,x)$. Hence $R_k(z,x) \leq R_k(z,y) < R_k(y,z) \leq R_k(x,z)$, so that $R_k(z,x) < R_k(x,z)$ which contradicts the fact $R_k(x,z) \leq R_k(z,x)$. □

Theorem 4.4. *There exists a n-aggregation rule for fair fuzzy preferences $f : \mathcal{L}^n \longrightarrow \mathcal{L}$ satisfying the conditions of independence -of the second type- of irrelevant alternatives, Pareto and non-dictatorship.*

Proof. We take the aggregation function f defined in theorem 4.3. We only need to prove that it accomplishes independence -of the second type- of irrelevant alternatives. Let $(\Lambda_i), (\Lambda'_i) \in \mathcal{L}^n$ be two profiles and $x, y \in U$ two alternatives such that $\Lambda_{\rceil\{x,y\}} \approx \Lambda'_{\rceil\{x,y\}}$. By Proposition 3.6 and *FP4* $[P_i(a,b) > 0 \Leftrightarrow P'_i(z,b) > 0]$ holds for every $a, b \in \{x,y\}$ and $i \in N_n$. By the definition of f, it follows that $[P_f(a,b) > 0 \Leftrightarrow P'_f(a,b) > 0]$ holds for $a, b \in \{x,y\}$. Using a similar argument we also get that $[R_f(a,b) = 1 \Leftrightarrow R'_f(a,b) = 1]$ holds for $a, b \in \{x,y\}$. Finally R_f is always positive. With the previous three facts we can use now Proposition 3.6 to conclude that $f((\Lambda_i))_{\rceil\{x,y\}} \approx f((\Lambda'_i))_{\rceil\{x,y\}}$. □

5 Aggregation rules for fuzzy preferences: An impossibility result

The acceptance of T-transitivities has not been unanimous in the fuzzy community. It has generated controversy and criticism. For instance, A. Billot exposed his discrepancies relative to the case in which fuzzy aggregation rules should be applied to handle some economic contexts (see [5]). This is also mainly related, too, with requirements on independence of irrelevant alternatives, that could actually be incompatible with some T-transitivities. Bearing this in mind, Billot introduced the concept of weak transitivity instead of any kind of T-transitivity, trying to solve this problem. Besides, Billot proposed (see [5]) a special kind of the *independence of irrelevant alternatives* condition, that could be in synthony with weak transitivity. The corresponding definition was refined then by Mordeson et alia in [20] and also by Giblisco et alia in [14]. This is the so-called independence -of the second type- of irrelevant alternatives, already introduced in this article in Definition 3.4 above.

Thus, we can consider a new Arrovian model in the fuzzy setting, inspired in the new definitions introduced by Billot in [5], and characterized by the imposition of the conditions of weak transitivity, independence -of the second type- of irrelevant alternatives, and completeness to the aggregation rules for fuzzy preferences that we will deal with. Nevertheless, unlike the main results in the previous Section 4, in this case we will arrive to an impossibility theorem (see Theorem 5.15 below).

Definition 5.1. *Let U denote a universe. A fuzzy preference Λ on U is said to be* neat *if it satisfies reflexivity, completeness and weak transitivity. We will denote by $\mathcal{N}$ the set of neat fuzzy preferences on U.*

To start with our analysis, we introduce the following lemma.

Lemma 5.2. *Let $\Lambda = (P, R, I)$ and $\Lambda' = (P', R', I')$ denote two neat fuzzy preferences on the universe U. Let $x, y \in U$ be two alternatives. Then $\Lambda_{\rceil\{x,y\}} \approx \Lambda'_{\rceil\{x,y\}} \Leftrightarrow \{\ [P(x,y) > 0 \Leftrightarrow P'(x,y) > 0] \wedge [P(y,x) > 0 \Leftrightarrow P'(y,x) > 0] \wedge [I(x,y) = 0 \Leftrightarrow I'(x,y) = 0\ \}$.*

Proof. It is an straightforward exercise to use Proposition 3.6 in order to prove the equivalence between the sets of conditions $\{[R(a,b) > R(b,a) \Leftrightarrow R'(a,b) > R'(b,a); a,b \in \{x,y\}],\ [R(a,b) = 0 \Leftrightarrow R'(a,b) = 0; a,b \in \{x,y\}],\ [R(a,b) = 1 \Leftrightarrow R'(a,b) = 1; a,b \in \{x,y\}]\}$ and $\{[P(x,y) > 0 \Leftrightarrow P'(x,y) > 0],\ [P(y,x) > 0 \Leftrightarrow P'(y,x) > 0],\ [I(x,y) = 0 \Leftrightarrow I'(x,y) = 0]\}$, under the additional hypothesis of completeness for neat fuzzy preferences. □

The result stated on the last lemma only pays attention to the fact of the values $P(x,y), P'(x,y), P(y,x)$ and $P'(y,x)$ being strictly positive, no matter which is its value in $(0,1)$. This suggest us to study a particular kind of fuzzy preferences, in which the possible values belong to the set $\{0, \frac{1}{2}, 1\}$.

Definition 5.3. *A neat fuzzy preference $\Lambda = (P,R,I)$ on a universe U is said to be* naive *provided that $I(U \times U), R(U \times U) \subseteq \{0, \frac{1}{2}, 1\}$ and $P(U \times U) \subseteq \{0,1\}$. We will denote by $\mathcal{N}_0$ the set of naive fuzzy preferences on U. Notice that $\mathcal{N}_0 \subseteq \mathcal{N}$.*

Definition 5.4. *Given a universe U, the map $\eta : \mathcal{N} \longrightarrow \mathcal{N}_0$, defined for all $(P,R,I) \in \mathcal{N}$ as $\eta(P,R,I) = (P_0, R_0, I_0)$ with $I_0(x,y) = I(x,y)$ if $I(x,y) \in \{0,1\}$; $I_0(x,y) = \frac{1}{2}$ otherwise $(x,y \in U)$, and $P_0(x,y) = 1$ if $P(x,y) > 0$; $P_0(x,y) = 0$ otherwise $(x,y \in U)$, is said to be the* standard projection *of the set $\mathcal{N}$ of neat fuzzy preferences onto its subset $\mathcal{N}_0$ of naive preferences. Notice here that the unique possible definition of R_0 in order to accomplish the conditions in Definition 2.15 is: $R_0(x,y) = 1$ if $P(x,y) > 0$; $R_0(x,y) = I_0(x,y)$ if $P(x,y) = 0$ $(x,y \in U)$.*

Proposition 5.5.

(i) Let $\Lambda \in \mathcal{N}$ be a neat fuzzy preference on a universe U. Then for all $x,y \in U$, it holds that $\Lambda_{\rceil\{x,y\}} \approx \eta(\Lambda)_{\rceil\{x,y\}}$.

(ii) Let Λ and Λ' be two naive preferences, in $\mathcal{N}_0$. Let $x,y \in U$. The following property holds: If $\Lambda_{\rceil\{x,y\}} \approx \Lambda'_{\rceil\{x,y\}}$ then $\Lambda_{\rceil\{x,y\}} = \Lambda'_{\rceil\{x,y\}}$.

Proof. These are direct consequences of Lemma 5.2 and Definition 5.4. □

Definition 5.6. *Given a n-aggregation rule for neat fuzzy preferences on a universe U, $f : \mathcal{N}^n \longrightarrow \mathcal{N}$, its associated map $\tilde{f} : (\mathcal{N}_0)^n \longrightarrow \mathcal{N}_0$, defined by $\tilde{f} = \eta \circ f_{|\mathcal{N}_0}$, is a n-aggregation rule for naive fuzzy preferences, called the* standard reduction *of f.*

Proposition 5.7. *Let $f : \mathcal{N}^n \to \mathcal{N}$ be a n-aggregation rule for neat fuzzy preferences on a universe U. Let $\tilde{f}$ be its standard reduction. If f satisfies the condition of independence -of the second type- of irrelevant alternatives (respectively, the weak Paretian condition), then $\tilde{f}$ also satisfies it.*

Proof. This follows immediately from Definitions 3.8 and 5.6 and Lemma 5.2 □

The following Proposition 5.8 allows us to study the standard reduction $\tilde{f}$, instead of the whole rule f, in order to analyze the dictatorship condition.

Proposition 5.8. *Let $f : \mathcal{N}^n \longrightarrow \mathcal{N}$ n-aggregation rule for neat fuzzy preferences on a universe U. Assume that f satisfies the condition of independence -of the second type- of irrelevant alternatives. Let $\tilde{f} : (\mathcal{N}_0)^n \longrightarrow \mathcal{N}_0$ be its standard reduction. Then the FPAR f is dictatorial if, and only if, its reduction $\tilde{f}$ is dictatorial. Besides, both have the same dictator.*

Proof. Suppose that k is a dictator of f and let $(\Lambda_i) \in (\mathcal{N}_0)^n$ be such that $P_k(x,y) > 0$ for some $x,y \in U$. Then by part (i) of Proposition 5.5 it follows that $\tilde{f}((\Lambda_i))_{\rceil\{x,y\}} \approx f((\Lambda_i))_{\rceil\{x,y\}}$. Moreover, $P_f((\Lambda_i))(x,y) > 0$ because k is a dictator, and by Lemma 5.2 we get $P_{\tilde{f}}((\Lambda_i))(x,y) > 0$. Conversely, if k is a dictator of $\tilde{f}$ and $(\Lambda_i) \in \mathcal{N}^n$ accomplishes that $P_k(x,y) > 0$, then by the condition of independence -of the second type- of irrelevant alternatives on the map f, and Proposition 5.5, it follows that $\tilde{f}((\eta(\Lambda_i)))_{\rceil\{x,y\}} \approx \eta(f((\Lambda_i)))_{\rceil\{x,y\}} \approx f((\Lambda_i))_{\rceil\{x,y\}}$. Hence we finally obtain that $P_f((\Lambda_i))(x,y) > 0$. □

Definition 5.9. *Starting from a naive fuzzy preference $\Lambda = (P,R,I) \in \mathcal{N}_0$, we can define now a (crisp) total preorder on the universe U, as follows: Given $x,y \in U$ we declare that $x \succsim_\Lambda y$ if, by definition, $P(y,x) = 0$. Considering the asymmetric part $\succ_\Lambda$ of $\succsim_\Lambda$ we may also observe that $x \succ_\Lambda y$ holds if and only if $P(x,y) > 0$ $(x,y) \in U \times U$. The total preorder $\succsim_\Lambda$ is said to be the* standard total preorder *associated to the naive fuzzy preference Λ. We denote by p the map from $\mathcal{N}_0$ to $\mathcal{O}$ (set of asymmetric parts of total preorders on U) given by $p(\Lambda) = \succ_\Lambda$.*

Remark 5.10. *Looking for a converse construction to that in Definition 5.9, we may try to assign some naive fuzzy preference to a given total preorder $\succsim$ defined on a universe U. However, in this case it is not immediate to define a suitable preference $\Lambda_{\succsim} = (P, R, I) \in \mathcal{N}_0$. However, we can establish some conditions that we would like to be satisfied by any adequate $\Lambda_{\succsim}$ (if any), for any alternatives $x, y \in U$, namely: (i) $x \succ y \Leftrightarrow P(x, y) > 0$, (ii) If $(x \succsim y) \wedge (y \succsim x)$ holds then $I(x, y) = 1$, (iii) If $x \succ y$, then $I(x, y) < 1$. The main problem appears here when the role of 0 and $\frac{1}{2}$ with respect to I has to be distinguished someway. In other words, if $x \succ y$, do we establish $I(x, y) = 0$ or else $I(x, y) = \frac{1}{2}$?*

Definition 5.11. *Given a universe U, an* assignation *is a symmetric function $G : U \times U \longrightarrow \{0, \frac{1}{2}\}$.*

Proposition 5.12. *Each assignation G defines a rule i_G that associates naive fuzzy preferences to total preorders.*

Proof. Consider the rule i_G, which sends a total preorder $\succsim$ on U into the preference $(P, R, I) \in \mathcal{N}_0$, defined as follows: For every $x, y \in U$, set $I(x, y) = 1$ if $(x \succsim y) \wedge (y \succsim x)$; $I(x, y) = G(x, y)$ otherwise, and also $P(x, y) = 1$ if $x \succ y$; $P(x, y) = 0$ otherwise. Notice in addition that, by a direct application of the preference constrains stated in Definition 2.15, the unique possibility to define here the component R is $R(x, y) = 1$ if $x \succ y$; $R(x, y) = 1$ if $(x \succsim y) \wedge (y \succsim x)$; $R(x, y) = G(x, y)$ otherwise. It is straightforward to see now that $i_G(\succsim)$ is actually a naive fuzzy preference. □

As in Theorem 2.6, denote by $\mathcal{O}$ the set of asymmetric parts of total preorders defined on a universe U. Our main objective is to define suitable maps $\{H : \mathcal{O}^n \longrightarrow \mathcal{O}\}$ from a given n-aggregation rule for naive fuzzy preferences $h : (\mathcal{N}_0)^n \longrightarrow \mathcal{N}_0$ defined on a universe U. To do so, if an assignation G_i is chosen for each individual $i \in N_n = \{1, \ldots, n\}$, we build a crisp n-aggregation function $h_{\mathcal{G}} : \mathcal{O}^n \longrightarrow \mathcal{O}$ for asymmetric parts of total preorders defined on U as follows: $h_{\mathcal{G}} := p \circ h \circ (i_{G_1} \times \cdots i_{G_n})$. We will denote $i_{G_1} \times \cdots \times i_{G_n}$ by $i_{\mathcal{G}}$. So, for each possible n-tuple $(G_1, \ldots, G_n)$ of assignations we obtain a crisp aggregation function for asymmetric parts of total preorders.

Proposition 5.13. *Let $\mathcal{G} = (G_i)_{i \in N}$ be a n-tuple of assignations and $h : (\mathcal{N}_0)^n \longrightarrow \mathcal{N}_0$ a n-aggregation rule for naive fuzzy preferences on a universe U. Assume that h satisfies the weak Paretian condition as well as the independence -of the second type- of irrelevant alternatives. Then the crisp aggregation function $h_{\mathcal{G}}$ is dictatorial.*

Proof. We are going to check that $h_{\mathcal{G}}$ satisfies the conditions in the statement of Theorem 2.6. Hence $h_{\mathcal{G}}$ is, a fortiori, a dictatorial rule.

By hypothesis, U has at least three pairwise different elements. Moreover, the map $h_{\mathcal{G}}$ is defined over all possible profiles in $\mathcal{O}^n$.

The condition of independence of irrelevant alternatives follows immediately, since for every pair $x, y \in U$ and elements $\succ, \succ' \in \mathcal{O}$ such that $\succ_{\rceil\{x,y\}} = \succ'_{\rceil\{x,y\}}$, we have that $i_{G_j}(\succ)_{\rceil\{x,y\}} = i_{G_j}(\succ')_{\rceil\{x,y\}}$ for all $j \in N_n$.

Suppose that for a profile $(\succ_i) \in \mathcal{O}^n$ and two alternatives $x, y \in U$, we have that $x \succ_i y$ holds for any $i \in N_n$. Then it follows that $P_{(i_{G_i}(\succ_i))}(x, y) = 1$. By the weak Paretian property of $\tilde{f}$, we have now that $P_{((\tilde{f} \circ i_{\mathcal{G}})((\succ_i)))}(x, y) > 0$. Finally, by definition of p, we may conclude that $x h_{\mathcal{G}}((\succ_i)) y$. Consequently, $h_{\mathcal{G}}$ also satisfies the unanimity property. □

Proposition 5.13 guarantees that for each n-tuple $\mathcal{G}$ of assignations, $h_{\mathcal{G}}$ is dictatorial, but nothing assures that all possible functions $h_{\mathcal{G}}$ have the same dictator. A priori, it could depend on the n-tuple $\mathcal{G}$. We denote the dictator of $h_{\mathcal{G}}$ by $D(\mathcal{G}, h)$.

Proposition 5.14. *Let $\mathcal{G}$ and $\mathcal{G}'$ be two n-tuples of assignations. Let $h : (\mathcal{N}_0)^n \longrightarrow \mathcal{N}_0$ be a n-aggregation rule for naive fuzzy preferences on the universe U. Assume that h satisfies the weak Paretian condition as well as the independence -of the second type- of irrelevant alternatives. Then the dictators of $h_{\mathcal{G}}$ and $h_{\mathcal{G}'}$ coincide (that is, $D(\mathcal{G}, h) = D(\mathcal{G}', h)$).*

Proof. By Proposition 5.13, both $h_{\mathcal{G}}$ and $h_{\mathcal{G}'}$ are dictatorial rules. We denote the dictator of $h_{\mathcal{G}}$ by k, and we will prove that k is also the dictator of $h_{\mathcal{G}'}$.

To do so, let us consider the case in which there exists a pair $w, v \in U$ such that $G_i(w, v) = G'_i(w, v)$ for all $i \in N_n$. We chose a profile $(\succ_i) \in \mathcal{O}^n$ such that $w \succ_k v$ and $v \succ_j w$ if $j \neq k$. If we prove now that $w h_{\mathcal{G}'}((\succ_i)) v$, k would be the unique possible dictator of $h_{\mathcal{G}'}$, and since $h_{\mathcal{G}'}$ is indeed dictatorial, k will be its dictator. But $w h_{\mathcal{G}'}((\succ_i)) v$ is true because $i_{\mathcal{G}\rceil\{w,v\}} = i_{\mathcal{G}'\rceil\{w,v\}}$ and h satisfies the independence -of the second type- of irrelevant alternatives condition. Finally, for arbitrary $\mathcal{G}$ and $\mathcal{G}'$, consider a pair $w, v \in X$ and define $\mathcal{G}''$ as $G''_i(w, v) = G_i(w, v)$ and $G''_i(x, y) = G'_i(x, y)$ if $\{x, y\} \neq \{w, v\}$. It is plain that $\mathcal{G}$ and $\mathcal{G}''$ coincide on $\{v, w\}$ so, by the previous argument, the dictator of $h_{\mathcal{G}''}$ is k. In addition $\mathcal{G}''$ and $\mathcal{G}'$ coincide on every pair of elements in U, except maybe on $\{w, v\}$. So the dictator of $h_{\mathcal{G}'}$ is also k, and we are done. □

Now we are ready to prove the main result in this Section 5.

Theorem 5.15. *Let $f : \mathcal{N}^n \longrightarrow \mathcal{N}$ denote a weakly Paretian n-aggregation rule for neat fuzzy preferences on a universe U. Assume also that f satisfies the independence -of the second type- of irrelevant alternatives. Then f is dictatorial.*

Proof. Let f be a fuzzy n-aggregation function satisfying the hypotheses of the statement. By Proposition 5.8 it is enough to see that $\tilde{f}$ is dictatorial. Besides, by Proposition 5.7 we have that $\tilde{f}$ is weak Paretian and satisfies the independence -of the second type- of irrelevant alternatives and the completeness condition. By Proposition 5.14 all the functions $\tilde{f}_{\mathcal{G}}$ have the same dictator (where $\mathcal{G}$ stands here for a n-tuple of assignations). We denote this dictator by k and we will see now that $\tilde{f}$ also has k as a dictator. To do so, let $x, y \in U$ and let (Λ_i) be a profile whose k-th strict preference $P_k(x,y)$ is strictly positive. We are going to prove that $P_{\tilde{f}((\Lambda_i))}(x,y) > 0$. In order to prove it, consider a n-tuple of assignations $\mathcal{G}$ such that $G_i(x,y) = \frac{1}{2}$ if $I_i(x,y) > 0$; $G_i(x,y) = 0$ if $I_i(x,y) = 0$. Define the crisp profile $(\succ_i) \in \mathcal{O}^n$ as $\succ_j = p(\Lambda_j)$ for all $j \in N_n$. It is easy to see now that $i_{G_j}(\succ_j)_{\rceil\{x,y\}} \approx \Lambda_{j\rceil\{x,y\}}$ and $x \succ_k y$. Finally, $x\tilde{f}_{\mathcal{G}}((\succ_i))y$ because k is the dictator of $\tilde{f}_{\mathcal{G}}$ and $x \succ_k y$. Using the definition of p we get that $P_{\tilde{f}((i_{G_i}(\succ_i)))}(x,y) > 0$. By the independence -of the second type- of irrelevant alternatives, and $i_{G_j}(\succ_j)_{\rceil\{x,y\}} \approx \Lambda_{j\rceil\{x,y\}}$, we see now that $P_{\tilde{f}((\Lambda_i))}(x,y) > 0$. So k is the dictator of $\tilde{f}$. □

6 Discussion

Despite being true that several other results concerning possibility and/or impossibility results in some Arrovian fuzzy setting have already appeared in the specialized literature (see e.g. [3, 4, 8, 9, 11, 13, 14, 18, 19, 20]), our approach throughout the present manuscript introduces some novelties.

Concerning the *concepts*, as far as we know, our definition of a fuzzy preference (see Definition 2.15 in Section 2) is new. The notion of a fuzzy preference has been considered in the literature in several different manners: On the one hand, we could consider a (fuzzy) preference $\mathcal{F}_i$ of an agent i as a fuzzy binary relation –but just *one*– defined on a universe U, and satisfying certain suitable set of properties (see e.g. [1]) that could remind us the idea of a total preorder (i.e.. transitivity and completeness). On the other hand, and maybe much more often encountered in this literature, given a fuzzy binary relation $\mathcal{F}_i$ that an agent i has defined on a universe U, we could try to *decompose* it into two or three parts, that could play the aforementioned corresponding roles of P_i, R_i and I_i, so that R_i plays the role of a total preorder $\precsim_i$, whereas P_i (respectively I_i) plays the role of $\succ_i$ (respectively, of $\sim_i$). There are several studies in the literature about different kinds of decompositions of fuzzy binary relations (see e.g. [7, 21] or Chapter 3 in [14]). A typical problem that arises now is that the possible decompositions may or may not be unique, strongly depending on the criteria established to do those decompositions (see [7, 14, 21]).

Trying to reconcile these ideas with the notion of a fuzzy preference that we have introduced in the paper in Definition 2.15, we may now define the concept of a decomposition rule for fuzzy binary relations, as follows:

Definition 6.1. *Let $\mathcal{B}$ denote the set of all fuzzy binary relations on a given universe U. A decomposition rule is a map $\phi : \mathcal{B} \to \mathcal{B} \times \mathcal{B}$ such that for every $R \in \mathcal{B}$, if $\phi(R) = (I_R, P_R)$, then $\Lambda_R = (P_R, R, I_R)$ is a fuzzy preference in the sense of Definition 2.15. Once ϕ has been fixed, and given $R \in \mathcal{B}$, the components I_R and P_R of $\phi(R)$ are respectively said to be the symmetric and the asymmetric component of R as regards the rule ϕ.*

Example 6.2. *Given a universe U, we can define a decomposition rule $\phi_\vee : \mathcal{B} \to \mathcal{B} \times \mathcal{B}$ in the following way: Given $R \in \mathcal{B}$, and $x, y \in U$, we declare that $\phi_\vee(R) = (I_R, P_R)$ such that $I_R(x,y) = \min\{R(x,y), R(y,x)\}$ and $P_R(x,y) = R(x,y)$ provided that $R(x,y) > R(y,x)$, and $P_R(x,y) = 0$ otherwise.*

What is crucial now is to realize that, even if the fuzzy preferences (P_R, R, I_R) are a particular case of the ones introduced in Definition 2.15, *the corresponding Arrovian models and possibility or impossibility results are totally independent.* None of the possibility and/or impossibility results got in the case of fuzzy preferences as in Definition 2.15 is a consequence or a particular case of the corresponding one (if any) in the case of fuzzy preferences defined through a decomposition. Let us see why: Given a universe U, call $\mathcal{X}$ to the set of fuzzy preferences considered as in Definition 2.15, on U. Let $\mathcal{Y}$ stand for the set of fuzzy preferences defined on U by means of a decomposition. It is plain that $\mathcal{Y}$ is a subset of $\mathcal{X}$. Suppose now that we have proved some possibility theorem in an Arrovian context that uses n-aggregation rules from $\mathcal{X}^n$ into $\mathcal{X}$. Obviously we may derive the existence of a possibility theorem for n-aggregation rules from $\mathcal{Y}^n \subset \mathcal{X}^n$ into $\mathcal{X}$, but the existing n-aggregation rule got through an existence theorem in that direction takes values on $\mathcal{X}$, and it is by no means guaranteed that it actually takes values in the smaller codomain $\mathcal{Y} \subset \mathcal{X}$. In other words: we cannot state, a priori, a parallel possibility theorem for n-rules that go from $\mathcal{Y}^n$ to $\mathcal{Y}$. Furthermore, if we have proved an impossibility theorem for n-aggregation rules that go from $\mathcal{X}^n$ to $\mathcal{X}$, it might still happen that n-aggregation rules from from $\mathcal{Y}^n$ to $\mathcal{Y}$ actually exist. That is, we cannot state, a priori, a parallel impossibility result for n-rules from $\mathcal{Y}^n$ to $\mathcal{Y}$, either. With similar arguments, we can see that the existence of a possibility (respectively, an impossibility) theorem for n-rules from $\mathcal{Y}^n$ to $\mathcal{Y}$ does not imply the existence of a parallel possibility (respectively, impossibility) result for rules from $\mathcal{X}^n$ into $\mathcal{X}$.

Despite the contexts not being equivalent, it may still happen that, in some *particular* situation, a theorem for n-aggregation rules from $\mathcal{X}^n$ to $\mathcal{X}$, has a parallel result of possibility or impossibility, but now for n-rules that go from $\mathcal{Y}^n$ to $\mathcal{Y}$. Thus, we may observe that several authors have used some kinds of decompositions of fuzzy preferences in the literature to deal with Arrovian models in the fuzzy setting. The corresponding definitions of decomposition they use are similar to Definition 6.1 above. Thus, Proposition 3.9 in [11] (who works with the $\phi_\vee$ decomposition of Example 6.2), and Theorem 4.22 in [14], would be the corresponding versions for rules from $\mathcal{Y}^n$ to $\mathcal{Y}$ of our Theorem 4.3 and Theorem 5.15, well understood, as aforementioned, that those results are indeed independent one another of the parallel ones established for n-aggregation rules from $\mathcal{X}^n$ into $\mathcal{X}$. In addition, the *techniques* used to prove them are also quite different from the ones we have used.

Concerning the *techniques* introduced through this manuscript, a special attention could be paid to Section 5, where we encounter several key results that are previous to the main Theorem 5.15. Along those results we deal with a fuzzy Arrovian model and, to handle it, the main underlying idea is to go back from the fuzzy setting to the classical crisp one, so building there some associated Arrovian (crisp) model that gives rise to an impossibility result. Then this impossibility result can be then interpreted in terms of the original fuzzy Arrovian model considered at the beginning. As far as we know, these ideas of leaning on some associated crisp Arrovian model to derive possibility or impossibility (as it is the case in Theorem 5.15) theorems in the fuzzy approach constitute a clear novelty in this specialized theory. As a matter of fact, most of the proofs of impossibility Arrovian theorems, both in the classical crisp setting and in the fuzzy approach, lean on concepts as *decisive coalitions* -not developed here in the present paper- (see e.g. [2, 14, 15]). Roughly speaking, a decisive coalition which only consists of an individual actually corresponds to a dictator. Because the number of individuals or agents considered, as well as the universe U, is finite by hypothesis, the usual techniques to derive impossibility results in an Arrovian contexts use some combinatorial arguments to prove that a given decisive coalition with more than two elements strictly contains another decisive coalition, with at least one element less. Therefore, we would finally arrive, in a finite number of steps, to a decisive coalition with a single individual. So, we would prove that the rule is dictatorial. *The methods of proof introduced here do not make any use of the concept of a coalition.*

Apart from the fact that the consideration of fuzzy preferences may give rise sometimes to possibility results in an Arrovian model, it is important at this stage to analyze the *mathematical grounds* that lie in the basis of a possibility and/or impossibility result. That is, why some Arrovian models lead to a possibility result, as Theorem 4.3, and others give rise to a result of impossibility of aggregation as Theorem 5.15? At this point, we may pay attention to situations in the crisp setting where some possibility result appears. This actually happens in contexts where an infinite number of individuals or agents is involved in the decision process, as stated by Kirman and Sondermann in [16] (see also [12]). There is a topological approach to this kind of questions that is based on concepts as nets, filters and ultrafilters (see [10, 16]). Basically, and roughly speaking, one may define some suitable topology on the sets of agents, preferences or alternatives, such that a decisive coalition acts as a filter. The role of a dictator is now played by an ultrafilter, that is proved to exist. However, it may happen that an existing ultrafilter still has infinitely many elements. In other words, there is no dictator, understanding a dictator as a singleton that only has one element. In the Economics and Social Choice literature, the set of elements that constitutes that ultrafilter is popularly known as "the invisible hand" or the invisible dictator. In the finite crisp classical Arrow's model a finite set of agents defines individual preferences on a finite set of alternatives. When there are at least three agents and three alternatives, this give rise to dictatorship. It can be proved, in topological terms, that the corresponding ultrafilter of decisive coalitions collapses to a point. Indeed, the existence of a dictator can be proved through combinatorial methods, instead of topological ones, as in [15]. However, unlike the case where only a finite of agents and alternatives appear, when we deal with infinitely many agents, the corresponding ultrafilter may be much bigger, non degenerating to a point and giving rise to the so-called invisible hand. Paying attention to this fact, we may understand better why in the fuzzy Arrovian approach we may encounter some possibility results. Even working with a finite number of agents and a finite number of alternatives, the rank of a fuzzy binary relation (as the P, R, I that appear in a fuzzy preference in the sense of Definition 2.15) is now $[0, 1]$, which is an *infinite* set of numbers. In spite of this not being a particular object of analysis in the present manuscript, we point out that this detail could give rise to the definition of suitable topologies on agents, preferences and alternatives that, because of the sets where they are defined could have infinitely many elements, sometimes give rise to non-degenerate ultrafilters. In the case of impossibility theorems, as Theorem 5.15 above, one important reason for the impossibility to appear is that the corresponding famililly of fuzzy preferences (namely, the neat ones) can be handled through another smaller family (the naive ones) where the corresponding fuzzy binary relations only take a *finite* set of values, namely $\{0, \frac{1}{2}, 1\}$. This leads to a situation that can be handled as in the crisp Arrovian model.

Concerning *open problems* that appear now in this theory, we may consider the following ones, that give us suggestions for further research in next future:

(i) Try to prove in parallel the analogous results to Theorem 4.3, Theorem 4.4 and Theorem 5.15, but working now with fuzzy preferences got through a decomposition, instead of with general fuzzy preferences. Despite some of those analogous results having been already got in the literature (see e.g. [11] and Theorem 4.22 in [14]) it is also appealing to analyze them again using alternative techniques.

(ii) Try to prove the analogous results to Theorem 4.3, Theorem 4.4, but for T-fair fuzzy preferences on a universe U, that is, considering T-transitivity instead of transitivity with respect to the minimum t-norm.

(iii) Try to get similar possibility and/or impossibility results, but using other alternative definitions of the extensions to the fuzzy setting of the classical requirements of the Arrovian model. Is Theorem 5.15 still valid if we consider the independence -of the first type- of irrelevant alternatives instead of that of the second type? Do Theorem 4.3 and/or Theorem 4.4 remain valid if we consider a different kind of transitivity for the fuzzy binary relations?

7 Conclusion

The extension to the fuzzy setting of classical Arrovian models arising in Social Choice may give rise to some possibility results as well as to new impossibility theorems. Therefore, it is important to analyze why sometimes we arrive at impossibility whereas, under other constraints, a possibility result sees the light, unlike the classical crisp Arrovian approach. In that analysis, it is crucial to pay an special attention to definitions: most of the classical concepts encountered in the crisp Arrovian model have several non-equivalent extensions to the fuzzy setting. Depending on the definitions chosen, possibility and/or impossibility results are achieved, so that it is interesting to see how the choice of a particular definition may provoke possibility or impossibility. This has been the objective of the present manuscript. Also, new techniques consisting in the (re)-interpretation of some fuzzy Arrovian model through an associated crisp one have been introduced as a by-product.

Acknowledgement

Thanks are given to the Managing Editor R. A. Borzooei and an anonymous referee for their helpful comments.

References

[1] L. Agud, R. G. Catalán, S. Díaz, E. Induráin, S. Montes, *Numerical representability of fuzzy total preorders*, International Journal of Computational Intelligence Systems (IJCIS), **5**(6) (2012), 996-1009.

[2] K. J. Arrow, *Social choice and individual values*, J. Wiley and sons, New York, 1951.

[3] A. Banerjee, *Fuzzy preferences and arrow-type problems in social choice*, Social Choice and Welfare, **11**(2) (1994), 121-130.

[4] C. R. Barret, P. K. Pattanaik, M. Salles, *On the structure of fuzzy social welfare functions*, Fuzzy Sets and Systems, **19**(1) (1986), 1-10.

[5] A. Billot, *Economic theory of fuzzy equilibria: An axiomatic analysis*, Springer Verlag, Heidelberg, 2012.

[6] D. S. Bridges, G. B. Mehta, *Representation of preference orderings*, Springer-Verlag Berlin Heidelberg, 1995.

[7] S. Díaz, B. De Baets, S. Montes, *Transitivity and negative transitivity in the fuzzy setting*, Advances in Intelligent and Soft Computing, **107** (2011), 91-100.

[8] D. Dubois, J. L. Koning, *Social choice axioms for fuzzy set aggregation*, Fuzzy Sets and Systems, **43** (1991), 257-274.

[9] C. Duddy, A. Piggins, *On some oligarchy results when social preference is fuzzy*, Social Choice and Welfare, **51**(4) (2018), 717-735.

[10] J. Dugundji, *Topology*, Allyn and Bacon, Boston, 1966.

[11] B. Dutta, *Fuzzy preferences and social choice*, Mathematical Social Sciences, **13**(3) (1987), 215-229.

[12] P. C. Fishburn, *Arrow's impossibility theorem: Concise proof and infinite voters*, Journal of Economic Theory, **2**(1) (1970), 103-106.

[13] L. W. Fung, K. S. Fu, *An axiomatic approach to rational decision making in a fuzzy environment*, in: L.A. Zadeh, F. King-Sum, K. Tanaka, M. Shimura (eds.), Fuzzy Sets and their Applications to Cognitive and Decision Processes, Academic Press, New York, 1975.

[14] M. B. Gibilisco, A. M. Gowen, K. E. Albert, J. N. Mordeson, M. J. Wierman, T. D. Clark: *Fuzzy social choice theory*, Studies in Fuzziness and Soft Computing, **315**, Springer International Publishing Switzerland, Cham, 2014.

[15] J. S. Kelly, *Social choice theory: An introduction*, Springer-Verlag Berlin Heidelberg, 1988.

[16] A. P. Kirman, D. Sondermann, *Arrow's theorem, many agents, and invisible dictators*, Journal of Economic Theory, **5** (1972), 267-277.

[17] E. P. Klement, R. Mesiar, E. Pap, *Triangular norms*, Kluwer Academic Publishers, Dordrecht, 2000.

[18] F. J. Montero, *A note on Fung-Fuś theorem*, Fuzzy Sets and Systems, **17** (1985), 259-269.

[19] F. J. Montero, *Social welfare functions in a fuzzy environment*, Kybernetes, **16** (1987), 241-245.

[20] J. N. Mordeson, M. B. Gibilisco, T. D. Clark, *Independence of irrelevant alternatives and fuzzy Arrow's theorem*, New Mathematics and Natural Computation, **8**(2) (2012), 219-237.

[21] S. Ovchinnikov, *Structure of fuzzy binary relations*, Fuzzy Sets and Systems, **6**(2) (1981), 169-195.

[22] L. A. Zadeh, *Fuzzy sets*, Information and Control, **8** (1965), 338-353.