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Solving fuzzy stochastic multi-objective programming problems based on a fuzzy inequality

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Abstract

Probabilistic or stochastic programming is a framework for modeling optimization problems that involve uncertainty. In this paper, we focus on multi-objective linear programming problems in which the coefficients of constraints and the right hand side vector are fuzzy random variables. There are several methods in the literature that convert this problem to a stochastic or fuzzy problem. By using a special type of fuzzy inequality, we transform the problem into a convenient stochastic problem. Then some known methods are applied to obtain the optimal solution. Finally, the equivalent multi-objective problem is solved by an interactive approach. A numerical example is provided to illustrate the procedure.

Keywords: Multi-objective programming, stochastic programming, fuzzy programming, interactive algorithm.

1 Introduction

According to the complexity of many programming problems, we may need to optimize two or more objective functions simultaneously in one problem. In the literature many papers have been devoted to solving these kind of problems. Among them, at first, using of interactive algorithms was considered by Benayoun [4], which is basically some sequential computations and decision phases that terminate in a compromise solution. The obtained solution is satisfactory for the decision maker (DM).

Multi-objective programming (MLP) has important applications in many areas of engineering and management. In applications, one of the major works faced by experts and decision makers (DM) is to determine the values of parameters in MLP model. Since real world problems are very complex, experts and DM frequently do not precisely know the values of those parameters. Therefore, it may be more appropriate to consider the knowledge of experts or DM about the parameters as fuzzy data. In addition, real decision problems usually have parameters that are made of uncertain information. In the existing methods, uncertainty reasoning is mostly based on fuzziness and randomness relates to the stochastic variability of all possible outcomes of a situation and can be perfectly and mathematically described by probability theory with random variable. Fuzziness, on the other hand, stems from the imprecision of subjective human knowledge and exists objectively with a variety of manifestations in numbers of situations such as data capture and process, blurred boundaries of the parameters, expertise applications, and lack of precise knowledge. Following the development of these methods, fuzzy and stochastic programming problems have been proposed which are considered as two powerful tools for solving uncertain optimization models. As long as fuzziness is in some sense complementary to randomness in real decision making models, fuzzy random variables were introduced and fuzzy stochastic programming problems were formulated. Several authors have proposed different approaches for solving such problems e.g. [9, 10, 8, 11, 12, 13, 14, 16, 17, 18, 20, 26, 27, 28, 30]. Zimmermann [30] presented an implementation

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of fuzzy linear programming approach for dealing with problems containing several objectives. Safi et. al. [23] showed the difficulties of the Zimmermann method and proposed an improved algorithm. They also proposed a geometric approach that coincide to the Zimmermann method [22] and then illustrated the difficulties of Zimmermann method by their geometric approach [24]. Rommelfanger [20] utilized the extreme points of α - levels to transform a linear optimization problems with fuzzy parameters in their objective functions into a deterministic optimization system. Moreover, Ramik and Rommelfanger [18] defined a new inequality relation and applied this inequality in the constraints of fuzzy programming problems.

In the field of stochastic programming, one of the major approaches to solve problems with stochastic constraints is the chance constrained programming technique introduced by Charnes and Cooper[6]. The chance-constrained method is one of the major approaches to solving optimization problems under various uncertainties. It is a formulation of an optimization problem that ensures that the probability of meeting a certain constraint is above a certain level. Multi-objective stochastic programming problems with random variable coefficients in objective functions and/or constraints have been studied by Sakawa, Kato, Katagiri, and Nishizaki [13, 14, 26, 27, 28]. They considered several stochastic models such as, an expectation optimization model, a probability maximization model, a variance minimization model, and a fractile criterion optimization model. Using these models, the stochastic programming problems are transformed into deterministic ones. Caballero et al. [5] introduced some different solution concepts and their relations for these problems. Other authors as Mohan, and Nguyen[16] and Rommelfanger [19] have also considered programming problems with the two kinds of uncertainties and presented an interactive solution process for solving such problems.

In the field of fuzzy random programming, Luhandjula[15] utilized a semi-infinite method for solving the problem. Also Katagiri and Sakawa [11] constructed an interactive algorithm using the level sets of fuzzy random coefficients. Same authors proposed an interactive algorithm based both on a stochastic programming technique and on a possibilistic programming technique in [12]. Arjmandzadeh et. al. solved random interval linear programming problem, [2, 3, 1, 21]. They used goal programming, neural network and a generalized variance method for solving the problems.

In this paper, we consider a multi-objective linear programming problem in which the coefficients of technological matrix and the right hand side vector are fuzzy random variables. We aim to describe an approach for solving such a problem with converting the original problem into a stochastic problem by means of a fuzzy inequality relation.
The sections of this paper are organized as follows: Section 2 reviews some basic concepts of fuzzy numbers, fuzzy random variables and structure of multi-objective fuzzy stochastic programming. In section 3, we utilize the chance constrained approach and the fuzzy inequality to transform the constraints from fuzzy stochastic to deterministic form. Section 4 is devoted to introduce an interactive algorithm for finding a satisfying solution for the DM. To improve the comprehension of the proposed method, an illustrative numerical example is presented in the last section.

2 Preliminaries

In this section, some properties of fuzzy numbers and the definition of fuzzy random variables are recalled. Moreover, the fuzzy stochastic programming problem with fuzzy random coefficients in its constraints will be formulated.
Fuzzy numbers are often used to represent uncertainty in information in decision making problems. In the literature, different kinds of fuzzy numbers are used. We consider triangular case for describing fuzzy numbers denoted by $\tilde{A} = (a, \alpha, \beta)$ with the following membership function:

$$\mu_{\tilde{A}}(x) = \begin{cases} 1 + \dfrac{x-a}{\alpha}, & a-\alpha \leq x \leq a, \\ \\ 1 + \dfrac{a-x}{\beta}, & a < x \leq a+\beta, \\ 0, & otherwise. \end{cases}$$

We may consider the following three special cases for triangular fuzzy numbers:
$\tilde{A} = (a, 0, \beta)$ with membership function $\mu_{\tilde{A}}(x) = \begin{cases} 1, & x \leq a, \\ 1 + \frac{a-x}{\beta}, & a < x \leq a+\beta, \\ 0, & x > a+\beta, \end{cases}$
$\tilde{B} = (b, \alpha, 0)$ with membership function $\mu_{\tilde{B}}(x) = \begin{cases} 1, & x \geq b, \\ 1 + \frac{x-b}{\alpha}, & b-\alpha \leq x < b, \\ 0, & x < b-\alpha, \end{cases}$
$\tilde{C} = (c, 0, 0)$ with membership function $\mu_{\tilde{C}}(c) = 1$ and $\mu_{\tilde{C}}(x) = 0$ for all $x \neq c$.

α-cut or α-level set, for an $\alpha \in (0,1]$, of a fuzzy number $\tilde{A}$ is a crisp set that is defined by:

$$\tilde{A}_\alpha = \{x \in \mathbb{R} : \mu_{\tilde{A}}(x) \geq \alpha\}, \quad 0 < \alpha \leq 1,$$
$$\tilde{A}_0 = cl\{x \in \mathbb{R} : \mu_{\tilde{A}}(x) > 0\},$$

where $cl\{.\}$ denotes the closure of A. Every α- cut of a fuzzy number $\tilde{A}$ is a closed interval $\tilde{A}_\alpha = [\tilde{A}^L_\alpha, \tilde{A}^U_\alpha]$, where

$$\tilde{A}^L_\alpha = \inf \tilde{A}_\alpha, \quad \text{and} \quad \tilde{A}^U_\alpha = \sup \tilde{A}_\alpha.$$

In mathematical problems, fuzzy numbers, just like real numbers, are subjected to binary operations, so the definition of some basic operators for fuzzy numbers in fuzzy arithmetic is mandatory. Let $\tilde{A} = (a, \alpha, \beta)$ and $\tilde{B} = (b, \gamma, \theta)$ be triangular fuzzy numbers. According to Zadeh's extension principle [29] some of the results of fuzzy arithmetic on these numbers are as follows:

$$\begin{cases} \textit{scalar multiplication} : \begin{cases} \lambda > 0, \lambda \in \mathbb{R} : \lambda.\tilde{A} = (\lambda a, \lambda\alpha, \lambda\beta), \\ \lambda < 0, \lambda \in \mathbb{R} : \lambda.\tilde{A} = (\lambda a, -\lambda\beta, -\lambda\alpha), \end{cases} \\ \\ \textit{addition} : \tilde{A} + \tilde{B} = (a + b, \alpha + \gamma, \beta + \theta), \\ \\ \textit{subtraction} : \tilde{A} - \tilde{B} = (a - b, \alpha + \theta, \beta + \gamma). \end{cases}$$

Now we need the definition of fuzzy random variables that are often used in the literature. In this paper, we focus on the definition given by Katagiri [11].

Definition 2.1. *Let (Ω, A, P) be a probability space, where Ω is a sample space, A is a σ- field and P is a probability measure. Let $F(\mathbb{R})$ be the set of all fuzzy numbers and B be the Borel σ- field of $\mathbb{R}$. Then, a map $\tilde{\bar{C}} : \Omega \to F(\mathbb{R})$ is called a fuzzy random variable if it holds that*

$$\{(\omega, \tau) \in \Omega \times \mathbb{R} : \tau \in \tilde{\bar{C}}_\alpha(\omega)\} \in A \times B, \quad \forall \alpha \in [0, 1],$$

where $\tilde{\bar{C}}_\alpha(\omega) = [\tilde{\bar{C}}^L_\alpha(\omega), \tilde{\bar{C}}^U_\alpha(\omega)]$ is an α- level set of the fuzzy number $\tilde{\bar{C}}(\omega)$ for $\omega \in \Omega$.

In fact, a fuzzy random variable is a measurable function from the sample space into the set of fuzzy numbers. According to Definition 2.1, under the occurrence of each elementary event ω in the sample space Ω, $\tilde{\bar{C}}(\omega)$ is a fuzzy number. Also, the closed interval $[\tilde{\bar{C}}^L_\alpha(\omega), \tilde{\bar{C}}^U_\alpha(\omega)]$ is a random interval that $\tilde{\bar{C}}^L_\alpha$ and $\tilde{\bar{C}}^U_\alpha$ are random variables. The overview of this definition shows that a fuzzy random variable is actually a random variable with values that are not real, but fuzzy numbers.

2.1 Multi-objective programming problem with fuzzy random variable parameters in the constraints

Consider the following form of multi-objective fuzzy stochastic programming problem:

$$\begin{array}{ll} \min & c_i^T x, \quad i = 1, \ldots, k, \\ s.t. & \widetilde{a}_i x \leq \widetilde{b}_i, \quad i = 1, \ldots, m, \\ & x \geq 0, \end{array} \tag{1}$$

where $\widetilde{a}_i = (\widetilde{a}_{i1}, \ldots, \widetilde{a}_{in})$, $i = 1, \ldots, m$ are n-dimensional row vectors of fuzzy random variables, $\widetilde{b}_i$, $i = 1, \ldots, m$ are fuzzy random variables, $x \in \mathbb{R}^n$ and $c_i^T \in \mathbb{R}^n$ $i = 1, \ldots, k$. The symbols '_' and '$\sim$ ' mean randomness and fuzziness, respectively. Suppose that under occurence of each elementary event ω, $\tilde{\bar{a}}_{ij}(\omega)$ and $\tilde{\bar{b}}_i(\omega)$ $i = 1, \ldots, m$, $j = 1, \ldots, n$ are triangular fuzzy numbers in the following froms:

$$\begin{array}{l} \tilde{\bar{b}}_i(\omega) = (\bar{b}_i(\omega), 0, \beta_i), \quad i = 1, \ldots, m, \\ \tilde{\bar{a}}_{ij}(\omega) = (\bar{a}_{ij}(\omega), \alpha_{ij}, \delta_{ij}), \quad i = 1, \ldots, m, \quad j = 1, \ldots, n, \end{array} \tag{2}$$

where for $i = 1, 2, ..., m,\ j = 1, 2, ..., n$ $\bar{b}_i(\omega)$ and $\bar{a}_{ij}(\omega)$ are random variables, β_i, α_{ij} and δ_{ij} are positive real numbers.

According to Zadeh's extension principle, $\tilde{\bar{a}}_i(\omega)x$, $i = 1, \ldots, m$ are triangular fuzzy numbers $(\bar{a}_i(\omega)x, \alpha_i x, \delta_i x)$ where

$$\bar{a}_i(\omega)x = \sum_{j=1}^{n} \bar{a}_{ij}(\omega)x_j, \quad \delta_i x = \sum_{j=1}^{n} \delta_{ij}x_j, \qquad x \geq 0. \tag{3}$$

Therefore, their membership functions are as follows:

$$\mu_{\tilde{\bar{a}}_i(\omega)x}(\tau) = \begin{cases} 1 + \dfrac{\tau - \bar{a}_i(\omega)x}{\alpha_i x}, & \bar{a}_i(\omega)x - \alpha_i x \leq \tau < \bar{a}_i(\omega)x, \\ 1 + \dfrac{\bar{a}_i(\omega)x - \tau}{\delta_i x}, & \bar{a}_i(\omega)x \leq \tau \leq \bar{a}_i(\omega)x + \delta_i x, \\ 0, & otherwise. \end{cases} \tag{4}$$

3 Optimization model based on a fuzzy inequality

In general, when fuzziness and randomness occur in parameters in a problem simultaneously, then traditional methods for solving ordinary mathematical programming problems cannot be applied directly. To deal with this situation, the most convenient approaches are by combining available methods for solving fuzzy programming problems and stochastic programming problems. The main difference between the developed approaches is just the way in which these algorithms are mixed. As an example for these approaches, Katagiri, et al. [12] incorporated a probability maximization model with possibilistic programming approach. In another work, Katagiri and Sakawa [11] introduced a new Pareto optimal solution concept using the level sets of fuzzy random coefficients in the objective functions.

In the current paper, we look at the problem from a fuzzy programming point of view.

According to the definition of a fuzzy stochastic variable, for each event w, the constraints coefficients, and the right hand vector parameters are fuzzy numbers. Hence, for each possible event w, we have a fuzzy multi-objective programming problem to be solved. In doing so, one of the possible approaches is by applying a fuzzy inequality relation to transform the constraints into deterministic ones.

Using the fuzzy inequality relation proposed by Rommelfanger [20], the constraints of the problem (1) are transformed as follows:

$$\tilde{\bar{a}}_i(\omega)x \leq \tilde{\bar{b}}_i(\omega) \Longrightarrow \begin{cases} \bar{a}_i(\omega)x + \delta_i x \leq \bar{b}_i(\omega) + \beta_i, \\ \mu_i(\bar{a}_i(\omega)x) \longrightarrow \max, \qquad i = 1, ..., m. \end{cases} \tag{5}$$

It is worth mentioning that all of the m added objective functions are continuous and strictly decreasing. The membership function for the latest relation is also as follows:

$$\mu_i(\bar{a}_i(\omega)x) = \begin{cases} 1, & \bar{a}_i(\omega)x \leq \bar{b}_i(\omega), \\ 1 + \dfrac{\bar{b}_i(\omega) - \bar{a}_i(\omega)x}{\beta_i}, & \bar{b}_i(\omega) < \bar{a}_i(\omega)x \leq \bar{b}_i(\omega) + \beta_i, \\ 0, & \bar{b}_i(\omega) + \beta_i < \bar{a}_i(\omega)x. \end{cases} \tag{6}$$

This inequality is a general definition of inequality relations in fuzzy optimization models. In fact, a given variable x satisfies the inequality when the triangular fuzzy number, $\tilde{\bar{a}}_i(\omega)x$, is equal to or less than the triangular fuzzy number $\tilde{\bar{b}}_i(\omega)$. Using of this inequality we may also take the advantage that the possible surplus $\bar{a}_i(\omega)x - \bar{b}_i(\omega)$ affects the decision process, directly.

By substituting the inequality (5) in the problem (1) for each elementary event ω, we have:

$$\begin{array}{ll} \min & c_i^T x, \quad i = 1, \ldots, k, \\ \max & \mu_i(\bar{a}_i(\omega)x), \quad i = 1, \ldots, m, \\ s.t. & \bar{a}_i(\omega)x + \delta_i x \leq \bar{b}_i(\omega) + \beta_i, \quad i = 1, \ldots, m, \\ & x \geq 0. \end{array} \tag{7}$$

Problem (7) is a stochastic programming problem. We can transform the constraints of problem (7) into deterministic ones, based on chance constrained conditions [6]. In this manner, the constraints need to be satisfied with a certain probability level. In the next step, the constraints of the problem(7) are replaced by the chance constrained conditions with satisfying levels θ_i, $i = 1, \ldots, m$:

$$Pr[\bar{a}_i(\omega)x + \delta_i x \leq \bar{b}_i(\omega) + \beta_i] \geq \theta_i, \quad i = 1, \ldots, m. \tag{8}$$

Therefore we have $n + 1$ different random variables, $\bar{b}_i(\omega), \bar{a}_{i,j}(\omega)$, $j = 1, 2, ..., n$, in $i - th$ constraint, $i = 1, 2, ..., m$. As mentioned in [27, 28] we can consider these variables have the same distribution with different parameters. Thus, we

can describe all of them by a random variable $t_i(\omega)$ with suitable difference in a constant and a suitable coefficient, i.e: $\bar{b}_i(\omega) = b_i^1 + t_i(\omega)b_i^2, \bar{a}_{ij}(\omega) = a_{ij}^1 + t_i(\omega)a_{ij}^2,\ j = 1, 2, ..., m$. The values of $b_i^1, b_i^2 a_{ij}^1, a_{ij}^2$ depend on the probability distribution function and its parameters.
As an example, let X and Y be two Gaussian random variables with (μ_X, σ_X^2) and (μ_Y, σ_Y^2) as the pair of mean and variance, respectively. We can describe Y as $a^1 + a^2 X$ where

$$a^1 = \mu_Y - \frac{\sigma_Y}{\sigma_X}\mu_X, \quad a^2 = \frac{\sigma_Y}{\sigma_X}. \tag{9}$$

Also supposing $a_i^2 x - b_i^2 > 0$, $i = 1, \ldots, m$ for any $x \in X$, we have:

$$\begin{aligned}
&P[(a_i^1 x + t_i(\omega)a_i^2 x) + \delta_i x \leq b_i^1 + t_i(\omega)b_i^2 + \beta_i] \geq \theta_i \\
&\Longleftrightarrow P[t_i(\omega).(a_i^2 x - b_i^2) \leq b_i^1 + \beta_i - \delta_i x - a_i^1 x] \geq \theta_i \\
&\Longleftrightarrow P[t_i(\omega) \leq \frac{b_i^1 + \beta_i - \delta_i x - a_i^1 x}{a_i^2 x - b_i^2}] \geq \theta_i \\
&\Longleftrightarrow F_i(\frac{b_i^1 + \beta_i - \delta_i x - a_i^1 x}{a_i^2 x - b_i^2}) \geq \theta_i \\
&\Longleftrightarrow b_i^1 + \beta_i - \delta_i x - a_i^1 x \geq F_i^{-1}(\theta_i).(a_i^2 x - b_i^2)
\end{aligned} \tag{10}$$

where $F_i^{-1}(\theta_i) = inf\{y|F_i(y) = \theta_i\}$ is pseudo inverse of F_i. Now, problem (7) changes as follows:

$$\begin{array}{ll}
\min & c_i^T x, \quad i = 1, \ldots, k, \\
\max & \mu_i(\bar{a}_i(\omega)x), \quad i = 1, \ldots, m, \\
s.t. & b_i^1 + \beta_i - \delta_i x - a_i^1 x \geq F_i^{-1}(\theta_i).(a_i^2 x - b_i^2), \quad i = 1, \ldots, m, \\
& x \geq 0,
\end{array} \tag{11}$$

However, due to the added objective functions, the recent problem still remains a stochastic programming problem. $\mu_i(\bar{a}_i(\omega)x)$ are stochastic. At this time, by introducing the auxiliary variable h, this problem can be transformed into the following problem:

$$\begin{array}{ll}
\min & c_i^T x, \quad i = 1, \ldots, k, \\
\max & h, \\
s.t. & h \leq \mu_i(\bar{a}_i(\omega)x), \quad i = 1, \ldots, m, \\
& b_i^1 + \beta_i - \delta_i x - a_i^1 x \geq F_i^{-1}(\theta_i)a_i^2 x - b_i^2, \quad i = 1, \ldots, m, \\
& x \geq 0, \quad 0 \leq h \leq 1,
\end{array} \tag{12}$$

In a similar way, suppose that the stochastic constraints in (12) have the probability levels $\hat{\theta}_i$. In this case we have:

$$\begin{aligned}
&P[\mu_i(\bar{a}_i(\omega)x) \geq h] \geq \hat{\theta}_i \\
&\Longleftrightarrow P[1 + \frac{\bar{b}_i(\omega) - \bar{a}_i(\omega)x}{\beta_i} \geq h] \geq \hat{\theta}_i \\
&\Longleftrightarrow P[t_i(\omega).(a_i^2 x - b_i^2) \leq \beta_i - \beta_i h + b_i^1 - a_i^1 x] \geq \hat{\theta}_i \\
&\Longleftrightarrow P[t_i(\omega) \leq \frac{\beta_i - \beta_i h + b_i^1 - a_i^1 x}{a_i^2 x - b_i^2}] \geq \hat{\theta}_i \\
&\Longleftrightarrow F_i(\frac{\beta_i - \beta_i h + b_i^1 - a_i^1 x}{a_i^2 x - b_i^2}) \geq \hat{\theta}_i \\
&\Longleftrightarrow \beta_i - \beta_i h + b_i^1 - a_i^1 x \geq F_i^{-1}(\hat{\theta}_i).(a_i^2 x - b_i^2)
\end{aligned} \tag{13}$$

and then problem (12) changes to:

$$\begin{array}{ll}
\min & c_i^T x, \quad i = 1, \ldots, k, \\
\min & -h, \\
s.t. & b_i^1 + \beta_i - \delta_i x - a_i^1 x \geq F_i^{-1}(\hat{\theta}_i).(a_i^2 x - b_i^2), \quad i = 1, \ldots, m, \\
& \beta_i - \beta_i h + b_i^1 - a_i^1 x \geq F_i^{-1}(\hat{\theta}_i).(a_i^2 x - b_i^2), \quad i = 1, \ldots, m, \\
& x \geq 0, \quad 0 \leq h \leq 1.
\end{array} \tag{14}$$

Problem (14) is a deterministic multi-objective linear programming problem. In order to achieve a satisfying solution for the DM, we continue to apply the interactive fuzzy method proposed by Sakawa [25]. First, according to

Zimmermann method [30], we consider the following membership functions for the objective functions:

$$\mu_i(c_i^T x) = \begin{cases} 0, & c_i^T x > z_i^0, \\ \dfrac{c_i^T x - z_i^0}{z_i^1 - z_i^0}, & z_i^0 \geq c_i^T x \geq z_i^1, \quad i = 1, 2, ..., k \\ 1, & z_i^1 > c_i^T x, \end{cases} \tag{15}$$

where $z_i^1 = \min_{x \in \hat{X}} c_i^T x$ and $z_i^0 = \max_{j \neq i} \{c_i^T x^{oj}\}$ (x^{oj} is the optimal solution of $\min_{x \in \hat{X}} c_j^T x$.).

$$\mu_{k+1}(-h) = \begin{cases} 0, & -h > z_{k+1}^0, \\ \dfrac{-h - z_{k+1}^0}{z_{k+1}^1 - z_{k+1}^0}, & z_{k+1}^1 \geq -h \geq z_{k+1}^0, \\ 1, & -h < z_{k+1}^1, \end{cases} \tag{16}$$

where $z_{k+1}^1 = \min_{x \in \hat{X}} h$ and $z_{k+1}^0 = \max_{x \in \hat{X}} \{h\}$.
Now, considering reference point $\bar{\mu} = (\bar{\mu}_1, \bar{\mu}_2, ..., \bar{\mu}_{k+1})$ chosen by the DM, problem (14) can be rewritten as:

$$\begin{array}{ll} \min & \max(\bar{\mu}_1 - \mu_1(c_1^T x), \ldots, \bar{\mu}_k - \mu_k(c_k^T x), \bar{\mu}_{k+1} - \mu_{k+1}(-h)) \\ s.t. & x \in \hat{X}. \end{array} \tag{17}$$

where $\hat{X}$ is the feasible solution space of problem (14).
Introducing the auxiliary variable v, problem (17) changes to:

$$\begin{array}{ll} \min & v \\ s.t. & \bar{\mu}_1 - \mu_1(c_1^T x) \leq v, \\ & \quad \vdots \\ & \bar{\mu}_{k+1} - \mu_{k+1}(-h) \leq v, \\ & x \in \hat{X}. \end{array} \tag{18}$$

Setting $\hat{z}_i = z_i^1 - z_i^0, i = 1, 2, ..., k+1$; problem (18) changes to the following LP problem:

$$\begin{array}{ll} \min & \vartheta \\ s.t. & C_i^T x + \vartheta \hat{z}_i \leq z_i^0 + \hat{z}_i \bar{\mu}_i, \quad i = 1, 2, ..., k \\ & -h + \vartheta \hat{z}_i \leq z_i^0 + \hat{z}_i \bar{\mu}_i, \\ & b_i^1 + \beta_i - \delta_i x - a_i^1 x \geq F_i^{-1}(\hat{\theta}_i).(a_i^2 x - b_i^2), \quad i = 1, \ldots, m, \\ & \beta_i - \beta_i h + b_i^1 - a_i^1 x \geq F_i^{-1}(\hat{\theta}_i).(a_i^2 x - b_i^2), \quad i = 1, \ldots, m, \\ & x \geq 0, \quad 0 \leq h \leq 1. \end{array} \tag{19}$$

Remark 3.1. *We can use the IZM algorithm proposed by Safi et.al. [23], when alternative optimal solution exist.*

According to Sakawa [25], x^*, the solution to the problem (18), is an M-efficient optimal solution corresponding to $\bar{\mu}$, i.e. there is no another $x \in \hat{X}$ such that $\mu_i(c_i^T x) \geq \mu_i(c_i^T x^*)$ for all i and $\mu_j(c_j^T x) > \mu_j(c_j^T x^*)$ for at least one j.

If the DM is satisfied by the current optimal solution, stop, otherwise, the reference point $\bar{\mu}$ will be updated. For this reason, we may utilize the trade-off rates $-\dfrac{\partial \mu_i}{\partial \mu_1} = \dfrac{\pi_1}{\pi_i}$, $i = 2, \ldots, k+1$ where $\pi_i, i = 1, \ldots, k+1$ are simplex multipliers of the problem (18).

The case $\pi_l = 0, 1 \leq l \leq k$ means that l^{th} objective function is satisfied, completely. In this case the DM dont change $\bar{\mu}_l$ in the next iterations.

4 Numerical example

To evaluate the accuracy of the proposed method, consider the following problem:

$$\begin{array}{ll} \min & (c_1^T x, c_2^T x, c_3^T x) \\ s.t. & \widetilde{a}_i x \leq \widetilde{b}_i, i = 1, \ldots, 5, \\ & x \geq 0. \end{array} \tag{20}$$

The problem has 3 objectives and 5 constraints. The technological matrix and the right hand side of the constraints are fuzzy random variables. Let all of them have Gaussian (normal) distribution. For the first constraint , suppose that:

$$\bar{a}_{11}(\omega) \sim N(7,1), \quad \bar{a}_{12}(\omega) \sim N(13,4), \quad \bar{a}_{13}(\omega) \sim N(10,9), \quad \bar{a}_{14}(\omega) \sim N(17,4), \quad \bar{a}_{15}(\omega) \sim N(5,1)$$
$$\bar{b}_1(\omega) \sim N(102,25)$$

Now we can write all of $\bar{a}_{1j}(\omega)$ as $a^1_{1j} + t_1(\omega)a^2_{1j}$, $j = 1,2,...,5$ where $t_1(\omega) \sim N(\mu_1, \sigma_1^2)$. If DM has not desirable means and variances for it, we can take them as:

$$\mu_1 = \frac{7+13+10+17++5+102}{6}, \quad \sigma_1^2 = \frac{1+4+9+4+1+25}{6}$$

But, Suppose that DM takes $t_1(\omega) = N(2,1)$, thus according to (9) we have the following values for the parameters:

$$\begin{array}{llllll} a^1_{11} = 5, & a^1_{12} = 11, & a^1_{13} = 8, & a^1_{14} = 15, & a^1_{15} = 3, & b^1_1 = 100 \\ a^2_{11} = 1, & a^2_{12} = 2, & a^2_{13} = 3, & a^2_{14} = 2, & a^2_{15} = 1, & b^2_1 = 5. \end{array}$$

Simply we write $a^1_1 = (5,11,8,15,3)$,$a^2_1 = (1,2,3,2,1)$, for the other constraints, let

$$t_2(\omega) \sim N(4,4), \quad t_3(\omega) \sim N(3,4), \quad t_4(\omega) \sim N(2,1), \quad t_5(\omega) \sim N(3,4),$$

then by (9) we have:

$$\begin{array}{llll} a^1_2 = (9,6,8,3,12) & a^2_2 = (4,2,5,3,1) & b^1_2 = 107 & b^2_2 = 3 \\ a^1_3 = (3,7,13,5,15) & a^2_3 = (1,3,4,2,5) & b^1_3 = 94 & b^2_3 = 8 \\ a^1_4 = (12,10,3,10,5) & a^2_4 = (4,3,1,2,1) & b^1_4 = 98 & b^2_4 = 6 \\ a^1_5 = (8,3,15,8,10) & a^2_5 = (3,1,6,2,4) & b^1_5 = 110 & b^2_5 = 2 \end{array}$$

$$\begin{aligned} c_1 &= (-12,-9,-8,-6,-10), \\ c_2 &= (8,13,11,10,5), \\ c_3 &= (-6,-14,9,-3,10) \end{aligned}$$

Also, consider the right hand side spreads of the fuzzy numbers $\widetilde{b}_i(\omega)$ and $\widetilde{a}_i(\omega)$ as follows:

$$\begin{array}{ll} \beta = (10,17,12,9,11), & \delta_3 = (2,4,1,3,2), \\ \delta_1 = (2,4,3,1,2), & \delta_4 = (5,2,1,2,3), \\ \delta_2 = (1,3,2,2,1), & \delta_5 = (3,3,2,4,2). \end{array}$$

Trivially, if $t_i(2) \sim N(\mu_i, \sigma_i^2), i = 1,2,..,m$, then $F_i^{-1}(\theta_i) = \sigma_i \Phi_i^{-1}(\theta_i) + \mu_i$.
In this example, the probability levels are assigned as:

$$\begin{aligned} \theta &= (0.95, 0.9, 0.9, 0.8, 0.85), \\ \hat{\theta} &= (0.8, 0.85, 0.9, 0.75, 0.85). \end{aligned}$$

According to (9) and (12) the constraints of this problem are:

$$\begin{array}{l} 100 + 10 - (2,4,3,1,2)x - (5,11,8,15,3)x \geq (1.\Phi_1^{-1}(0.95) + 2).((1,2,3,2,1)x - 5), \\ 107 + 17 - (1,3,2,2,1)x - (9,6,8,3,12)x \geq (2.\Phi_2^{-1}(0.9) + 4).((4,2,5,3,1)x - 3), \\ 94 + 12 - (2,4,1,3,2)x - (3,7,13,5,15)x \geq (2.\Phi_3^{-1}(0.9) + 3).((1,3,4,2,5)x - 8), \\ 98 + 9 - (5,2,1,2,3)x - (12,10,3,10,5)x \geq (1.\Phi_4^{-1}(0.8) + 2).((4,3,1,2,1)x - 6), \\ 110 + 11 - (3,3,2,4,2)x - (8,3,15,8,10)x \geq (2.\Phi_5^{-1}(0.85) + 3)((3,1,6,2,4)x - 2), \\ 10 - 10h + 100 - (5,11,8,15,3)x \geq (1.\Phi_1^{-1}(0.8) + 2).((1,2,3,2,1)x - 5), \\ 17 - 17h + 107 - (9,6,8,3,12)x \geq (2.\Phi_2^{-1}(0.85) + 4).((4,2,5,3,1)x - 3), \\ 12 - 12h + 94 - (3,7,13,5,15)x \geq (2.\Phi_3^{-1}(0.9) + 3)((1,3,4,2,5)x - 8), \\ 9 - 9h + 98 - (12,10,3,10,5)x \geq (1.\Phi_4^{-1}(0.75) + 2)((4,3,1,2,1)x - 6), \\ 11 - 11h + 110 - (8,3,15,8,10)x \geq (2.\Phi_5^{-1}(0.85) + 3).((3,1,6,2,4)x - 2), \\ x \geq 0, \ \ h \in [0,1], \end{array} \tag{21}$$

Substituing $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ and Φ_i^{-1}, $i = 1, 2, ..., 5$ from normal distribution table lead to:

$$\begin{aligned}
&10.65x_1 + 22.3x_2 + 21.95x_3 + 23.3x_4 + 8.65x_5 \leq 128.25,\\
&36.32x_1 + 22.16x_2 + 42.9x_3 + 24.74x_4 + 19.58x_5 \leq 143.74,\\
&10.58x_1 + 27.74x_2 + 36.32x_3 + 19.16x_4 + 44.9x_5 \leq 114,\\
&28.4x_1 + 20.55x_2 + 6.85x_3 + 17.7x_4 + 10.85x_5 \leq 124.1,\\
&26.24x_1 + 11.08x_2 + 47.48x_3 + 22.16x_4 + 32.32x_5 \leq 131.16,\\
&7.85x_1 + 16.7x_2 + 16.55x_3 + 20.7x_4 + 5.85x_5 + 10h \leq 124.25,\\
&33.32x_1 + 18.16x_2 + 38.4x_3 + 21.24x_4 + 18.08x_5 + 17h \leq 142.24,\\
&8.58x_1 + 23.74x_2 + 35.32x_3 + 16.16x_4 + 42.9x_5 + 12h \leq 150.64,\\
&22.72x_1 + 18.04x_2 + 5.68x_3 + 15.36x_4 + 7.68x_5 + 9h \leq 123.08,\\
&23.24x_1 + 8.08x_2 + 45.48x_3 + 18.16x_4 + 30.32x_5 + 11h \leq 131.16.\\
&x \geq 0,\ \ h \in [0, 1].
\end{aligned} \tag{22}$$

In doing so, we transformed the fuzzy stochastic constraints into the crisp constraints, now according to the Zimmermann's method, we consider the membership functions for objective functions. These membership functions are defined by equation (15). In our example, z_i^0 and z_i^1 are calculated as:

$$\begin{array}{ll}
z_1^1 = -53.8896, & z_2^1 = 0,\\
z_1^0 = 0, & z_2^0 = 56.4532,\\
z_3^1 = -58.7825, & z_4^1 = -1,\\
z_3^0 = 1.641, & z_4^0 = 0.
\end{array}$$

Now we construct the model according to (18) with the initial reference point $\bar{\mu} = (1, 1, 1, 1)$ that is:

$$\begin{array}{ll}
\min & v\\
s.t. & -12x_1 - 9x_2 - 8x_3 - 6x_4 - 10x_5 - 53.8896v \leq -53.8896,\\
& 8x_1 + 13x_2 + 11x_3 + 10x_4 + 5x_5 - 56.4532v \leq 0,\\
& -6x_1 - 14x_2 + 9x_3 - 3x_4 + 10x_5 - 60.4235v \leq -58.4235,\\
& h + v \geq 1,\\
& 10.65x_1 + 22.3x_2 + 21.95x_3 + 23.3x_4 + 8.65x_5 \leq 128.25,\\
& 36.32x_1 + 22.16x_2 + 42.9x_3 + 24.74x_4 + 19.58x_5 \leq 143.74,\\
& 10.58x_1 + 27.74x_2 + 36.32x_3 + 19.16x_4 + 44.9x_5 \leq 114,\\
& 28.4x_1 + 20.55x_2 + 6.85x_3 + 17.7x_4 + 10.85x_5 \leq 124.1,\\
& 26.24x_1 + 11.08x_2 + 47.48x_3 + 22.16x_4 + 32.32x_5 \leq 131.16,\\
& 7.85x_1 + 16.7x_2 + 16.55x_3 + 20.7x_4 + 5.85x_5 + 10h \leq 124.25,\\
& 33.32x_1 + 18.16x_2 + 38.4x_3 + 21.24x_4 + 18.08x_5 + 17h \leq 142.24,\\
& 8.58x_1 + 23.74x_2 + 35.32x_3 + 16.16x_4 + 42.9x_5 + 12h \leq 150.64,\\
& 22.72x_1 + 18.04x_2 + 5.68x_3 + 15.36x_4 + 7.68x_5 + 9h \leq 123.08,\\
& 23.24x_1 + 8.08x_2 + 45.48x_3 + 18.16x_4 + 30.32x_5 + 11h \leq 131.16\\
& x \geq 0, \quad h \in [0, 1].
\end{array} \tag{23}$$

The obtained solution is shown at the first iteration in Table 1. Suppose that the DM is not satisfied by this solution with the satisfying levels $(\mu_1(c_1^T x), \mu_2(c_2^T x), \mu_3(c_3^T x), \mu_4(-h)) = (0.4926, 0.4926, 0.4926, 0.6316)$, therefore, the DM updates the reference point to $(1, 1, 0.8, 1)$ in order to improve $\mu_1(.)$ and $\mu_2(.)$ at the sacrifice of $\mu_3(.)$. Finally, after four interactions, the desirable solution has been arrived. The obtained M-efficient solutions also are shown in Table 1. For this problem, in all iterations, trade-off rates are equal to:

$$-\frac{\partial \mu_2}{\partial \mu_1} = 0.0025, \quad -\frac{\partial \mu_3}{\partial \mu_1} = 0.0085, \quad -\frac{\partial \mu_4}{\partial \mu_1} = 0.0063.$$

Table 1: Changes in the reference points, in order to reach the DM satisfaction

Interaction	$(\bar{\mu}_1, \bar{\mu}_2, \bar{\mu}_3, \bar{\mu}_4)$	$(c_1^T x, c_2^T x, c_3^T x, -h)$	$(\mu_1(c_1^T x), \mu_2(c_2^T x), \mu_3(c_3^T x), \mu_4(-h))$
First	(1.00,1.00,1.0,1.00)	(-26.5481, 28.6422, -28.1259, -0.631572)	(0.4926, 0.4926, 0.4926, 0.6316)
Second	(1.00,1.00,0.80,1.00)	(-30.6478, 24.3475, -20.6380, -0.7240)	(0.5687, 0.5687, 0.3687, 0.7240)
Third	(0.80,1.00,0.80,1.00)	(-21.3498, 22.7971, -22.2974, -0.7182)	(0.3962, 0.5961, 0.4189, 0.7182)
Fourth	(0.80,0.90,0.75,1.00)	(-24.9739, 24.6460, -23.3397, -0.7862)	(0.4634, 0.5634, 0.4372, 0.7862)

5 Conclusion

In this paper, a new method has been proposed for solving the fuzzy stochastic multi-objective programming problems. In the proposed method, we considered a fuzzy stochastic multi-objective programming in which the coefficients of the technological matrix and the right hand side vector are fuzzy random variables. First, we transformed the original problem into the conventional deterministic problem based on the fusion of the fuzzy inequality relation and the concept of chance constrained technique. Then, we utilized the interactive algorithm proposed by Sakawa to obtain a satisfying solution for the DM. Although, the size of obtained problem is larger than the original one, but the proposed method is simple and effective. The flexibility of our approach allows the decision maker to obtain the desired solution in the appropriate manner.

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References

[1] Z. Arjmandzadeh, A. Nazemi, M. Safi, *Solving multiobjective random interval programming problems by a capable neural network framework,* Applied Intelligence, **49**(4) (2019), 1566-1579.

[2] Z. Arjmandzadeh, M. R. Safi, *A goal programming approach for solving the random interval linear programming problem*, Turkish Journal of Mathematics, **41**(4) (2017), 775-786.

[3] Z. Arjmandzadeh, M. R. Safi, A. Nazemi, *A new neural network model for solving random interval linear programming problems*, Neural Networks, **89** (2017), 11-18.

[4] R. Benayoun, J. De Montgolfier, J. Tergny, O. Laritchev, *Linear programming with multiple objective functions: STEP Method(STEM)*, Mathematical Programming, **1** (1971), 366-375.

[5] R. Caballero, E. Cerda, M. M. Muniz, L. Rey, I. M. Stancu-Minasian, *Efficient solution concepts and their relations in stochastic multiobjective programming*, Journal of Optimization Theory and Applications, **110** (2001), 53-74.

[6] A. Charnes, W. W. Cooper, *Chance constrained programming*, Management Science, **6** (1959), 73-79.

[7] D. Dubois, H. Prade, *Fuzzy sets and systems: Theory and application*, Academic Press, New York, 1980.

[8] M. G. Iskander, *Using different dominance criteria in stochastic fuzzy linear multiobjective programming: A case of fuzzy weighted objective function*, Mathematical and Computer Modelling, **37** (2003), 167-176.

[9] M. G. Iskander, *A possibility programming approach for stochastic fuzzy multiobjective linear fractional programs*, Computers and Mathematics with Applications, **48** (2004), 1603-1609.

[10] M. G. Iskander, *A fuzzy weighted additive approach for stochactic fuzzy goal programming*, Applied Mathematics and Computation, **154** (2004), 543-553.

[11] H. Katagiri, M. Sakawa, *Interactive multiobjective fuzzy random programming through the level set-based probability model*, Information Sciences, **181**(9) (2011), 1641-1650.

[12] H. Katagiri, M. Sakawa, K. Kato, I. Nishizaki, *Interactive multiobjective fuzzy random linear programming maximization of possibility and probability*, European Journal of Operational Research, **188** (2008), 530-539.

[13] K. Kato, M. Sakawa, *An interactive fuzzy satisficing method based on variance minimization under expectation constraints for multiobjective stochastic linear programming problems*, Soft Computing, **15** (2011), 131-138.

[14] K. Kato, M. Sakawa, H. Katagiri, C. Perkgoz, *An interactive fuzzy satisficing method based on fractile criterion optimization for multiobjective stochastic integer programming problems,* Expert Systems with Applications, **37** (2010), 6012-6017

[15] M. K. Luhandjula, *Fuzziness and randomness in an optimization framework*, Fuzzy Sets and Systems, **77** (1996), 291-297.

[16] C. Mohan, H. T. Nguyen, *An interactive satisficing method for solving multiobjective mixed fuzzy-stochastic programming problems*, Fuzzy Sets and Systems, **117** (2001), 61-79.

[17] F. Rahmanniyay, A. J. Yu, J. Seif, *A multi-objective multi-stage stochastic model for project team formation under uncertainty in time requirements,* Computers and Industrial Engineering, **132** (2019), 153-165.

[18] J. Ramik, H. Rommelfanger, *Fuzzy mathematical programming based on some new inequality relations*, Fuzzy Sets and Systems, **81** (1996), 77-87.

[19] H. Rommelfanger, *A general concept for solving linear multicritera programming problems with crisp, fuzzy or stochastic values*, Fuzzy Sets and Systems, **158** (2007), 1892-1904.

[20] H. Rommelfanger, R. Hanuscheck, J. Wolf, *Linear programming with fuzzy objectives*, Fuzzy Sets and Systems, **29** (1989), 31-48.

[21] M. R. Safi, Z. Arjmandzadeh, *A generalization of variance model for solving random interval linear programming problems*, International Journal of Systems Science: Operations Logistics, (2018), 1-9.

[22] M. R. Safi, H. Maleki, E. Zaeimazad, *A geometric approach for solving fuzzy linear programming problems,* Fuzzy Optimization and Decision Making, **6**(4) (2007), 315-336.

[23] M. R. Safi, H. Maleki, E. Zaeimazad, *A note on the zimmermann method for solving fuzzy linear programming problems,* Iranian Journal of Fuzzy Systems, **4**(2) (2007), 31-45.

[24] M. R. Safi, A. Razmjoo, *Illustrating the difficulties of zimmermann method for solving the fuzzy linear programming by the geometric approach*, Proceedings of the 4th International Joint Conference on Computational Intelligence and Fuzzy Computation Theory and Application, (2012), 435-438.

[25] M. Sakawa, *Fuzzy sets and interactive multiobjective optimization*, Plenum Press, New York, 1993.

[26] M. Sakawa, H. Katagiri, K. Kato, *An interactive fuzzy satisficing method for multiobjective stochastic linear programming problems using a fractile criterion model*, IEEE International Fuzzy Systems Conference, 2001.

[27] M. Sakawa, K. Kato, H. Katagiri, *An interactive fuzzy satisficing method for multiobjective linear programming problems with random variable coefficients through a probability maximization model*, Fuzzy Sets and Systems, **146** (2004), 205-220.

[28] M. Sakawa, K. Kato, I. Nishizaki, *An interactive fuzzy satisficing method for multiobjective stochastic linear programming problems through an expectation model*, European Journal of Operational Research, **145** (2003), 665-672.

[29] L. A. Zadeh, *Fuzzy sets*, Infomation and Control, **8** (1965), 338-353.

[30] H. J. Zimmermann, *Fuzzy programming and linear programming with several objective functions*, Fuzzy Sets and Systems, **1** (1978), 45-55.