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Risk-adjusted control charts based on LR-fuzzy data

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Abstract

Control charts are widely used in industrial processes as well as in health sciences and particularly for monitoring the performance of cardiac surgeon or a group of surgeons based on the preoperative risk of patients. Since the preoperative risk is a vague and nonprecise variable and the anesthesiologists after checking how many risk factors a patient has, determine the risk of mortality before the surgery as a linguistic term such as low, medium, high or others like that, it is better to be considered as a fuzzy number, which can be determined by using fuzzy regression models. In this condition, we need a special chart to monitor the performance of surgeons based on these fuzzy data. In this article, we propose risk-adjusted control charts based on LR-fuzzy data and then test our conclusions on real data.

Keywords: Preoperative risk, surgical outcomes, monitoring performance, risk-adjusted control charts, fuzzy logistic regression.

1 Introduction

Monitoring the outcomes of medical processes such as surgery, can be helpful to detect the changes in the performances and do corrective actions that are possible to improve them. An application of statistical process control (SPC) is increasing, which is advocated in the health science as a valuable quality improvement tool. The use of control chart in the health science related applications differs somewhat from industrial, because in this field, patients have different clinical presentation like as age, gender, hypertension, renal function and others which are explanatory variables and so have different prior risk. Therefore, risk-adjusted control charts should be used to monitor the performance of surgeons. Lovegrove et al. [12], [13] and Poloniecki et al. [20] suggested the first charting procedure based on the difference between the variable life-adjusted display (VLAD) that takes the risk of patient into account but the problem of that is about the lacking proper signaling rule.

Steiner et al. [26] proposed a risk-adjusted cumulative sum (RA-CUSUM) chart based on testing the odds of mortality. The RA-CUSUM uses optimal likelihood ratio weights and it is more efficient than methods based on predicted minus expected deaths. For further discussion, see the review papers by Grigg and Farewell [10], Rogers et al. [23] and Woodall [34].

Sego et al. [25] proposed a risk-adjusted survival time CUSUM (RAST-CUSUM) chart, which besides the preoperative risk of patients, the survival time of them (30 days after surgery) also takes into account. Afterwards, Steiner and Jones [27] proposed a RA-updating exponentially weighted moving average (RA-uEWMA) chart to monitor survival time data, updatingly. After that Cook et al. [6] introduced exponentially weighted moving average charts to compare observed and expected values for monitoring risk-adjusted hospital indicators. This one is not based on the survival time and their approach is like the VLAD control chart.

It seems among these risk-adjusted control charts just the RAST-CUSUM and the RA-uEWMA are based on the result of surgery and survival time, which make monitoring the time to event process to detect increases in mortality more sensitive.

The way of selecting a model for a risk adjustment is an important statistical issue. In all of the designed charts which are described, the preoperative risk is considered as a crisp value, whereas the risk is a vague and nonprecise variable.

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Although, there are models such as euro Score or Parsonnet score which are proposed by Albert [1] and Parsonnet [19] respectively for cardiac surgery based on characteristics of patient, but they just can help the experts to determine the risk of mortality model, like other explanatory variable,The risk in the Steiner model is modeled by the logistic regression, for which the response variable is patient survival (alive/death). But in reality, we need to evaluate the risk based on the anesthesiologist decision before the surgery and just by the knowledge on risk factors for each patient. Actually anesthesiologists after checking the number of risk factors a patient has, determine the risk of mortality, before the surgery, as a linguistic term such as low, medium, high or others like that(Miller et. al. [15]). The best description of these kinds of observations is that they are fuzzy output.

In this situation, we need a suitable model to estimate the risk when the output variable is fuzzy. There are several fuzzy regression models; the first one for fuzzy linear regression model was proposed by Tanaka et al. [28] and then some models for nonlinear fuzzy regression were introduced (See Buckly [3] and Celmins [4]). Here we have a nonlinear and fuzzy categorical output and crisp explanatory variables, so we use the fuzzy logistic regression proposed by Pourahmad et al. [21] which is based on the least squares method. There are nice properties for LR-fuzzy numbers, which make fuzzy arithmetic more comfortable, besides that there are some sort of methods for generating LR-fuzzy numbers for characteristics (Cheng [5]). Hence we assume that the risk of mortality is a LR-fuzzy number.

After risk determination, we need a special chart to monitor the performance of surgeons based on these LR-fuzzy data. There are some papers dedicated for design of control charts with linguistic data. Wang and Raz [32] proposed the representative values control charts with both the probability rule and the membership function decision rule, for which linguistic data are transformed into representative values of the fuzzy data. Wang [30] presented a CUSUM control chart with fuzzy data by using representative values that is a sum of central value of the fuzzy data with its fuzziness value. Erginel and Sentürk [7] proposed fuzzy EWMA and fuzzy CUSUM control chart. They used the fuzzy median (midrange) transformation technique, which is integrated to the $\alpha - cut$ fuzzy median (midrange).

In all of the mentioned papers, the charts are plotted with representative values of the fuzzy data. These representative values result in losing important information included in original data. The CUSUM control chart for LR-fuzzy data without using representative values methods was proposed by Wang and Hryniewicz [31], but their model is derived from the standard CUSUM, which is designed based on a normal distribution. There are no constructions risk-adjusted CUSUM or EWMA charts for fuzzy data in the literature. Motivated by Wang and Hryniewicz [31], we present RAST-CUSUM and RA-uEWMA based on LR-fuzzy data in this paper.

In Section 2, we review the RAST-CUSUM and RA-uEWMA control charts. We describe the approach in the context of monitoring surgical survival times, but the method is also applicable in other contexts with survival time data. In Section 3 we have a discussion about risk determination as a LR-fuzzy number and how to plot the charts based on these fuzzy risks. Finally, in Section 4 we apply the proposed charts to a cardiac surgery example.

2 A review of RAST-CUSUM and RA-uEWMA control charts

Control charts are one of the easiest method to control and monitor the process. The important reason of using control charts is detection of changes and improve the situation. There is a close relationship between control charts and hypotheses tests. Indeed, a control chart is a hypotheses test which is used to evaluate that the process is in control or not. If all of the points lie between upper and lower control limits, then the in-control hypotheses will not reject, otherwise we will reject that (See Montgomery [17]).

The RAST-CUSUM and the RA-uEWMA control charts are two risk-adjusted control charts which are based on the preoperative risk, the result of surgery and the survival time of patients. They are used to monitor the performance of surgeons. In this section, these charts are briefly reviewed.

2.1 RAST-CUSUM control chart

The CUSUM control chart methodology was firstly developed by Page [18] to monitor manufacturing processes. Since this chart is sensitive to small changes, it is interested by researches. An application of the CUSUM control charts to monitor surgical performance was firstly developed by William et al. [33], but in this paper, the covariate information such as patient characteristics, surgical team, and others does not take into account.

The basic form of the CUSUM is defined as follows:

$$S_0 = 0$$
$$S_n = \max\{0, S_{n-1} + W_n\}, \quad n = 1, 2, \dots, \tag{1}$$

where S_n is the CUSUM statistic and W_n is the CUSUM score.

This chart is designed to detect quickly a shift from $\theta = \theta_0$ to $\theta = \theta_1$, where θ is a vector of parameters. An alarm is

signaled if S_n is out of control limits.
Let y_n denote a measured outcome (such as mortality status) for the subgroup n (here patient) with the probability function $d(P|y_n)$, where P is the related parameter (here failure rate) corresponding to θ. Moustakides [16] showed among all scores, the optimal one to detect the shift from P_0 to P_1 is logarithm of the likelihood ratio:

$$W_n = \log(\frac{d(P_1|y_n)}{d(P_0|y_n)}).$$

Let $y_n = 1$, if the patient dies within, say, 30 days of the surgery and let $y_n = 0$, otherwise. Then $d(P|y_n) = P^{y_n}(1-P)^{1-y_n}$, and so, W_n is defined as

$$W_n = \begin{cases} \log(\frac{P_1}{P_0}) & \text{if the } n\text{th patient dies,} \\ \\ \log(\frac{1-P_1}{1-P_0}) & \text{if the } n\text{th patient survives.} \end{cases}$$

But we cannot assume that the failure rates are the same for all patients, because each patient has a different preoperative risk of mortality due to risk factors that exist prior to the surgery. As a result, the RA-CUSUM control chart based on the result of surgery (alive/death) is developed by Steiner et al. [26]
However, a monitoring scheme based on patient survival time can be more sensitive to detecting increases in mortality than a procedure that only uses binary outcomes. To this end, the RAST-CUSUM control chart is designed to monitor the right censored survival time variables. The distribution of survival time for each patient by using accelerated failure time (AFT) regression model is predicted (Klein and Moeschberger [11]). This method accounts for right censored observations, that is, those patients that survive at least until an observed censoring time.
Let a_n, c_n, and d_n denote the time of surgery, follow up time, and the time of a death, respectively. Then

$$T_n = \min\{d_n, c_n\} - a_n, \tag{2}$$

is the survival time for nth patient. Usually, $c_n = a_n + c$, where c is some constant, say, 30 days and is named as censoring time.
Next observation is δ_n, which is defined as

$$\delta_n = \begin{cases} 1, & d_n - a_n \leq c, \\ 0, & d_n - a_n > c. \end{cases} \tag{3}$$

So, the data are observed in pairs (T_n, δ_n). These values are not updated if additional information be available for the patient at a later time and the likelihood function for them is

$$L(\theta|t_n, \delta_n) = [f(t_n, \theta)]^{\delta_n}[S(t_n, \theta)]^{1-\delta_n},$$

where $f(t_n, \theta)$ is the density of survival time and $S(t_n, \theta)$ is the survival function.
Suppose that survival times have the log-logistic distribution, then

$$\begin{aligned} f_0(x) &= \frac{\alpha}{\lambda}(\frac{x}{\lambda})^{\alpha-1}(1+(\frac{x}{\lambda})^{\alpha})^{-2}, \quad x > 0, \\ S_0(x) &= (1+(\frac{x}{\lambda})^{\alpha})^{-1}, \end{aligned}$$

where $f_0(x)$ is the baseline density, $S_0(x)$ is the baseline survival function, $\alpha > 0$ is the shape parameter, and $\lambda > 0$ is the scale parameter. Using the AFT model for the log-logistic distribution, we have

$$S(x|U_n = u_n) = (1 + (\frac{x_n \exp(\beta^T u_n)}{\lambda})^{\alpha})^{-1}, \tag{4}$$

where α and $\dfrac{\lambda}{\exp(\beta^T U)}$ are the parameters of distribution of $X|U$. Hence U is the vector of covariate values that reflects the risk factors and β is the vector of regression parameters. When α and β are held fixed, the hypotheses are considered as

$$\begin{aligned} H_0(x) : \theta_0 &= (\alpha_0, \frac{\lambda_0}{\exp(\beta^T U)}), \\ H_1(x) : \theta_1 &= (\alpha_0, \frac{\rho_1 \lambda_0}{\exp(\beta^T U)}), \end{aligned} \tag{5}$$

where $\rho_1 > 0$ and $\rho_1 \neq 1$.
Therefore, the CUSUM score in (1) is:

$$\begin{aligned} W_n &= \log\left(\frac{f(x_n|\lambda=\rho_1\lambda_0, U_n=u_n)^{\delta_n} S(x_n|\lambda=\rho_1\lambda_0, U_n=u_n)^{1-\delta_n}}{f(x_n|\lambda=\lambda_0, U_n=u_n)^{\delta_n} S(x_n|\lambda=\lambda_0, U_n=u_n)^{1-\delta_n}}\right), \\ &= 2^{\delta_n}\left\{\log[1+(\frac{x_n\exp(\beta^T u_n)}{\lambda_0})^{\alpha_0}] - \log[1+(\frac{x_n\exp(\beta^T u_n)}{\rho_1\lambda_0})^{\alpha_0}]\right\} - \delta_n\alpha_0\log\rho_1. \end{aligned} \tag{6}$$

A signal is issued when $S_n > h_1$, where h_1 is an upper control limit.

Lemma 2.1. *There is a relationship between regression coefficients which are estimated from the logistic regression and AFT regression model* (Sego [24])*. Let β be the vector of the coefficients in AFT model for the log-logistic distribution, then by using* (4)*, we have*

$$Logit(1 - S_n(30|U_n = u_n)) = ln(\frac{30exp(\beta^T u_n)}{\lambda_0})^{\alpha_0} = \alpha_0 ln(\frac{30}{\lambda_0}) + \alpha_0\beta^T u_n. \tag{7}$$

So $\alpha_0 ln(\frac{30}{\lambda_0})$ and $\alpha_0\beta^T$ are equal to the intercept and the coefficients of logistic regression model, respectively.

2.2 RA-uEWMA control chart

The EWMA control chart is a well known tool for process monitoring. The chart was introduced by Roberts [22] and it was used to detect persistent shifts in a process. Both of the EWMA and CUSUM control charts are similar in the sense that both accumulate information by time passes to monitor process changes and both are sensitive to small changes and they have similar efficiency (Lucas and Saccucci [14]). The main advantage of EWMA is that it provides an ongoing local estimate of the average score. The EWMA statistic is given by:

$$E_n = \gamma W_n + (1-\gamma)E_{n-1}, \quad n = 1, 2, \ldots, \tag{8}$$

where λ is a smoothing constant, $0 < \lambda \leq 1$, and W_n is the EWMA score.
As mentioned, since in clinical field each patient has different prior risk, the chart needs to be risk-adjusted, too. Steiner and Jones [27] proposed a risk-adjusted updating EWMA (RA-uEWMA) chart, which its patient score is a function of time, and then the latest information is incorporated in the chart all the time. They showed EWMA statistic by

$$E_{n,t} = \gamma W_{n,t} + (1-\gamma)E_{n-1,t}, \quad n = 1, 2, \ldots, \tag{9}$$

to emphasis on time. Unlike RAST-CUSUM chart, this chart will be updated whenever there is a failure (death) and also at some regular time intervals say each week and at each updating time, we recalculate $E_{n,t}$ using the latest score for each patient. As time passes, more patient undergo surgery and the past patients EWMA weights will gradually decrease exponentially.
Like the previous section, we have paired observation (T_n, δ_n) in (2) and (3), with the difference that if we consider t as the present time, then

$$T_n = \min\{t, c_n, d_n\} - a_n, \tag{10}$$

where a_n, c_n, d_n are defined in section 2.1. In this situation, the nth patient is at risk if $T_n = t - a_n$ (the time between surgery and current time) and it can go to $d_n - a_n$ (the time between surgery and death) or $c_n - a_n$ (the time between surgery and follow up time).
If the AFT log-logistic distribution is considered, then the scores, $W_{n,t}$, in (9) are the same as (6).
In the RAST-CUSUM chart updated only after each patient reaches censoring time (30 days) post surgery, this approach delays the entry of each failures the CUSUM, but in the RA-uEWMA chart, the actual days of death are spread out, because of its regular updating.
If monitoring the deterioration (improvement) of surgeons is important, then $\rho_1 < 1$ ($\rho_1 > 1$). In this case, $W_{n,t}$ is a decreasing (increasing) function of risk and it is positive when nth patient dies before the censoring time (survival at least censoring time). Thereupon, when a patient with low (high) risk dies (survive), $W_{n,t}$ would be large. The conclusion is that $W_{n,t}$ can be considered as a penalty-reward score based on which of detecting deterioration or improvement in a performance of surgeon is more important, respectively. Also, since the RA-uEWMA statistic is an increasing function of $W_{n,t}$, we need a suitable upper control limit (h_2) to monitor the process and if $E_{n,t} > h_2$, then a signal is issued. However, using a reflecting lower control limit (h_L) protects against the chart inertia. The inertia problem refers to the measure of the resistance of a chart to signaling a particular process shift (Woodall and Mahmoud [35]).

3 Risk-adjusted control charts based on LR-fuzzy data

As mentioned in section 1, the risk of mortality is a linguistic term such as low, medium, high or others like that which are assigned by experts and is better to be considered as a fuzzy output. As a result we need a suitable model to estimate the risk when the output variable is fuzzy. Because the explanatory variables are crisp, we use fuzzy logistic regression based on the least squares method (Pourahmad et. al. [21]).

After risk determination, there is a necessity to monitor the performance of surgeons by a special chart, which is based on these fuzzy data. In this section after describing how the risk is determined, the RAST-CUSUM and RA-uEWMA charts based on LR-fuzzy data are discussed.

Due to the vague status of cases relative to the response categories, the probability of mortality cannot be calculated and modeled exactly based on explanatory variables, and so the odds of mortality is meaningless. In this situation, we should consider the possibility of mortality instead of probability. The possibility theory is an uncertainty theory devoted to the handling of incomplete or vague information, which was proposed by Zadeh [36], as an extension of his theory of fuzzy set. For more details see Zimmermann [29].

Let the expert assign the linguistic term as $\tilde{p}_n = \{\ldots, low, medium, high, \ldots\}$, for $n = 1, 2, 3, \ldots$, where $\tilde{p}_n$ is the possibility of mortality for the nth patient and the observed outputs. Defining a suitable membership function for that is very important. After consulting with some experts and based on Miller's Anesthesia reference book [15] in preoperative evaluation section, it is described as a triangular fuzzy number as follows:

$$Very-Low(x) = \begin{cases} 1 - \dfrac{0.004 - x}{0.004}, & 0 \leq x \leq 0.004, \\ \\ 1 - \dfrac{x - 0.004}{0.005}, & 0.004 < x \leq 0.009, \end{cases}$$

$$Low(x) = \begin{cases} 1 - \dfrac{0.009 - x}{0.005}, & 0.004 \leq x \leq 0.009, \\ \\ 1 - \dfrac{x - 0.009}{0.061}, & 0.009 < x \leq 0.07, \end{cases}$$

$$Medium(x) = \begin{cases} 1 - \dfrac{0.07 - x}{0.061}, & 0.009 \leq x \leq 0.07, \\ \\ 1 - \dfrac{x - 0.07}{0.04}, & 0.07 < x \leq 0.11, \end{cases}$$

$$High(x) = \begin{cases} 1 - \dfrac{0.11 - x}{0.04}, & 0.07 \leq x \leq 0.11, \\ \\ 1 - \dfrac{x - 0.11}{0.889}, & 0.11 < x \leq 0.999, \end{cases}$$

$$Very-High(x) = \begin{cases} 1 - \dfrac{0.999 - x}{0.889}, & 0.11 \leq x \leq 0.999, \\ \\ 1 - \dfrac{x - 0.999}{0.001}, & 0.999 < x \leq 1. \end{cases}$$

Then by using the fuzzy logistic regression when the response variable ($\tilde{p}_n$) is fuzzy, the risk can be estimated by the following model:

$$Logit(\tilde{P}_n) = \tilde{\beta}_0 + \sum_{i=1}^{m} \tilde{\beta}_i u_{ni}, \quad i = 1, 2, \ldots, m, \tag{11}$$

where u_{ni} is the ith explanatory variable for the nth patient. Thus $\tilde{\beta}_0 = (a_0, s_0)_T$ is intercept and $\tilde{\beta}_i = (a_i, s_i)_T$ is the ith coefficient of model, where $i = 1, 2, \ldots, m$. Also, $\tilde{\beta}$ is treated as a fuzzy number, and $Logit(\tilde{P}_n)$ is the estimator of logarithmic transformation of possibilistic odds and is equal to $Logit(\tilde{P}_n) = (f_n(a), f_n(s))_T$, where

$$f_n(a) = a_0 + \sum_{i=1}^{m} a_i u_{ni}, \tag{12}$$

$$f_n(s) = s_0 + \sum_{i=1}^{m} s_i u_{ni}. \tag{13}$$

Using the least squares method, the variables $a_0, a_1, \ldots, a_m$ and $s_0, s_1, \ldots, s_m$ are estimated, and then using the extension principle, $\tilde{P}_n = (m_n, l_n, r_n)_T$ in (11) as a LR-fuzzy number with l_n left width and r_n right width is obtained by:

$$\begin{aligned} m_n &= \frac{exp(f_n(a))}{1+exp(f_n(a))}, \\ l_n &= \frac{exp(f_n(a))}{1+exp(f_n(a))} - \frac{exp(f_n(a)-f_n(s))}{1+exp(f_n(a)-f_n(s))}, \\ r_n &= \frac{exp(f_n(a)+f_n(s))}{1+exp(f_n(a)+f_n(s))} - \frac{exp(f_n(a))}{1+exp(f_n(a))}. \end{aligned} \tag{14}$$

To evaluate the goodness of fit of the model, we use the model proposed by Gildeh and Gien. [9] This approach measures closeness of the observed and the estimated values and lies in the unit interval. The larger it is gained, the better goodness of fit we have. In the following section, the mentioned control charts based on LR-fuzzy data are considered.

3.1 RAST-CUSUM contol chart based on LR-fuzzy data

In this section the RAST-CUSUM control chart based on LR-fuzzy data is described. Firstly, it is necessary to mention that according to Lemma 2.1, by using (11), the coefficients in AFT model for the log-logistic distribution are fuzzy numbers as follows:

$$\tilde{\beta}_i = (\frac{a_i}{\alpha_0}, \frac{s_i}{\alpha_0})_T, \quad i = 1, 2, \ldots, m. \tag{15}$$

As a result, inserting $\frac{a_i}{\alpha_0}$ and $\frac{s_i}{\alpha_0}$ instead of a_i and s_i, respectively in (12) and (13), we can calculate $\tilde{P}_n = (m_n, l_n, r_n)_T$ in (14) by new $f_n(a)$ and $f_n(s)$. Also, the scale parameter of log-logistic distribution in (4) is a fuzzy number and the hypotheses are:

$$\begin{cases} H_0 : \theta = (\alpha_0, \dfrac{\lambda_0}{\exp(\tilde{\beta}^T U)}), \\ H_1 : \theta_1 = (\alpha_0, \dfrac{\rho_1 \lambda_0}{\exp(\tilde{\beta}^T U)}). \end{cases} \tag{16}$$

In this situation, the RAST-CUSUM statistic expresses as follows:

$$\tilde{S}_n(s) = \max\{0(a), (\tilde{s}_{n-1} \oplus \tilde{W}_n)(y)\}, \qquad s \in \mathbb{R}, \tag{17}$$

where 0 denotes a LR-fuzzy number with 0 width, that is,

$$\tilde{W}_n = \begin{cases} -\alpha_0 \log \rho_1 \oplus 2\big\{ \log(1 \oplus (\dfrac{t_n \exp(\beta^T \odot u_n)}{\lambda_0})^{\alpha_0}) \ominus \log(1 \oplus (\dfrac{t_n \exp(\beta^T \odot u_n)}{\rho_1 \lambda_0})^{\alpha_0})\big\} & \text{if the } n\text{th patient dies,} \\ \log(1 \oplus (\dfrac{t_n \exp(\beta^T \odot u_n)}{\lambda_0})^{\alpha_0}) \ominus \log(1 \oplus (\dfrac{t_n \exp(\beta^T \odot u_n)}{\rho_1 \lambda_0})^{\alpha_0}) & \text{if the } n\text{th patient survives.} \end{cases} \tag{18}$$

Then, $\tilde{W}_n$ is a fuzzy number and the following $\alpha-$cut of that for $\alpha \in [0, 1]$ is defined as:

$$W_n^-(\alpha) = \begin{cases} -\alpha_0 \log \rho_1 + 2\log(A_{1n}) - 2\{\log(A_{1n}) - \log(A_{1n} - A_{2n})\}(1-\alpha) & \text{if the } n\text{th patient dies,} \\ \log(A_{1n}) - \{\log(A_{1n}) - \log(A_{1n} - A_{2n})\}(1-\alpha) & \text{if the } n\text{th patient survives,} \end{cases}$$

$$W_n^+(\alpha) = \begin{cases} -\alpha_0 \log \rho_1 + 2\log(A_{1n}) + 2\{\log(A_{1n} + A_{3n}) - \log(A_{1n})\}(1-\alpha) & \text{if the } n\text{th patient dies,} \\ \log(A_{1n}) + \{\log(A_{1n} + A_{3n}) - \log(A_{1n})\}(1-\alpha) & \text{if the } n\text{th patient survives,} \end{cases}$$

where

$$
\begin{aligned}
A_{1n} &= \frac{1+(\frac{t_n m'_n}{\lambda_0})^{\alpha_0}}{1+(\frac{t_n m'_n}{\rho_1\lambda_0})^{\alpha_0}}, \\
A_{2n} &= \frac{\alpha_0(\frac{t_n}{\lambda_0})^{\alpha_0} m'^{\alpha_0-1}_n(\rho_1^{-\alpha_0}(1+(\frac{t_n m'_n}{\lambda_0})^{\alpha_0})r'_n+(1+(\frac{t_n m'_n}{\rho_1\lambda_0})^{\alpha_0})l'_n)}{\left(1+(\frac{t_n m'_n}{\rho_1\lambda_0})^{\alpha_0}\right)^2}, \\
A_{3n} &= \frac{\alpha_0(\frac{t_n}{\lambda_0})^{\alpha_0} m'^{\alpha_0-1}_n(\rho_1^{-\alpha_0}(1+(\frac{t_n m'_n}{\lambda_0})^{\alpha_0})l'_n+(1+(\frac{t_n m'_n}{\rho_1\lambda_0})^{\alpha_0})r'_n)}{\left(1+(\frac{t_n m'_n}{\rho_1\lambda_0})^{\alpha_0}\right)^2},
\end{aligned}
$$

where

$$
\begin{aligned}
m'_n &= exp(f_n(a)), \\
l'_n &= exp(f_n(a)) - exp(f_n(a) - f_n(s)), \\
r'_n &= exp(f_n(a) + f_n(s)) - exp(f_n(a)).
\end{aligned}
$$

Therefore, the $\alpha-$cut of $\tilde{S}_n$ is

$$S_n^-(\alpha) = \max\{0, S_{n-1}^-(\alpha) + W_n^-(\alpha)\}, \quad S_n^+(\alpha) = \max\{0, S_{n-1}^+(\alpha) + W_n^+(\alpha)\}.$$

To detect that the process is in control or not, the $\alpha - cut$ set of the control limit needs to be specified. To find out that, firstly the distribution of risk (m_n), left width (l_n), and right width (r_n) are needed and then by knowing the first type error and using simulation method, it is determined as $[h_1^-(\alpha), h_1^+(\alpha)]$.

As a result, we have two intervals $[S_n^-(\alpha), S_n^+(\alpha)]$ and $[h_1^-(\alpha), h_1^+(\alpha)]$ which are named S_n and H_1, respectively. If the interval S_n is smaller (larger) than the interval H_1 and they do not have any intersection, then the process is in (out of) control. Otherwise, having intersection makes the decision complicated. Thereupon, we need a measure that is suitable for all of the conditions which can happen.

We use the $D_{p,q}$-distance which is proposed by Gildeh and Gien [8] with some differences. The distance measures how far two fuzzy number are from each other. The analytical properties of $D_{p,q}$ depend on the parameter p, while the parameter q characterizes the subjective weight attributed to the sides of the fuzzy numbers. In the definition of $D_{p,q}$-distance, we need to integrate all the values of $\alpha \in [0,1]$, but since the chart is plotted for a specific value of α, we calculate the distance based on that. Therefore the distance between S_n and H_1 with a constant interval, for example, $C_1 = [c_1^-, c_1^+]$ is calculated by:

$$d_{S_n} = D_{p,q}(S_n, C_1) = \left[(1-q)\left|S_n^-(\alpha) - c_1^-\right|^p + q\left|S_n^+(\alpha) - c_1^+\right|^p\right]^{\frac{1}{p}}, \tag{19}$$

$$d_{H_1} = D_{p,q}(H_1, C_1) = \left[(1-q)\left|h_1^-(\alpha) - c_1^-\right|^p + q\left|h_1^+(\alpha) - c_1^+\right|^p\right]^{\frac{1}{p}}, \tag{20}$$

where $p \in [1, +\infty)$ and $q \in [0,1]$. Thus, if $d_{S_n} > d_{H_1}$, then the process is out of control.

3.2 RA-uEWMA control chart based on LR-fuzzy data

Now, we consider the RA-uEWMA control chart when the risks are LR-fuzzy. The hypotheses are the same as (16), where the coefficients are equal to (15) and the EWMA statistic is:

$$\tilde{E}_{n,t}(y) = \gamma \tilde{W}_{n,t}(y) + (1-\gamma)\tilde{E}_{n-1,t}(y), \quad y \in \mathbb{R}, \tag{21}$$

where $\tilde{W}_{n,t}$ is a fuzzy number and equals to (18). Therefore the $\alpha-$cut of $\tilde{E}_{n,t}$ for $\alpha \in [0,1]$ is:

$$
\begin{aligned}
E_{n,t}^-(\alpha) &= \gamma W_{n,t}^-(\alpha) + (1-\gamma)E_{n-1,t}^-(\alpha), \\
E_{n,t}^+(\alpha) &= \gamma W_{n,t}^+(\alpha) + (1-\gamma)E_{n-1,t}^+(\alpha),
\end{aligned}
$$

so the interval $E_{n,t} = [E^-_{n,t}(\alpha), E^+_{n,t}(\alpha)]$ should be compared by its $\alpha-$cut set of the control limit, that is, $H_2 = [h^-_2(\alpha), h^+_2(\alpha)]$.
Likewise section 3.1, we can use the $D_{p,q}$-distance to monitor the process. Let $C_2 = [c^-_2, c^+_2]$ be a constant interval; then:

$$d_{E_{n,t}} = D_{p,q}(E_{n,t}, C_2) = \left[(1-q)\left|E^-_{n,t}(\alpha) - c^-_2\right|^p + q\left|E^+_{n,t}(\alpha) - c^+_2\right|^p\right]^{\frac{1}{p}}, \tag{22}$$

$$d_{H_2} = D_{p,q}(H_2, C_2) = \left[(1-q)\left|h^-_2(\alpha) - c^-_2\right|^p + q\left|h^+_2(\alpha) - c^+_2\right|^p\right]^{\frac{1}{p}}, \tag{23}$$

where $p \in [1, +\infty)$ and $q \in [0, 1]$. Therefore if $d_{E_{n,t}} > d_{H_2}$, then the process is out of control. As mentioned in section 2.2, we use $H_L = [h^-_L(\alpha), h^+_L(\alpha)]$ as a reflecting lower control limit.

3.3 Average run length

To evaluate a control chart, the average run length (ARL) is used. The ARL is the expected number of points that are observed until a signal takes place. The ARL that results when the process remains in control, denoted by ARL_0, is used to determine the control limits. The ARL that results when the process is out of control, denoted by ARL_1, is a performance metric that can be used to compare one control chart with another. There are different ways such as the simulation or Markov chains to calculate or approximate the ARL. Both provide approximations of the ARL. Simulation may be necessary if the calculation of transition probabilities between the Markov chains states is sufficiently complicated or intractable.
For a fuzzy control chart, on each $\alpha-$cut, the ARL can be carried out based on the control limit interval. Here the calculation of ARL for the RAST-CUSUM control chart based on LR fuzzy data is explained. The calculation of that for the RA-uEWMA control chart is similar. By using simulation methods, the $ARL(\alpha)$ can be approximated by the following steps:

1. Drawing an observation u_n from its empirical distribution
2. $x_n|u_n$ is drawn from the log-logistic distribution with the α_0 shape parameter and the $\dfrac{\rho_1\lambda_0}{\exp(\beta^T U)}$ scale parameter.
3. $T_n = \min\{x_n|u_n, 30\}$ and $\delta_n = I\{x_n|u_n \leq 30\}$.
4. $W^+_n(\alpha)$ and $W^-_n(\alpha)$ are calculated which lead to $S^+_n(\alpha)$ and $S^-_n(\alpha)$, respectively.
5. Then, d_{S_n} and d_{H_1} in (19) and (20) can be calculated, respectively.
6. $ARL(\alpha) = \inf\{n|d_{S_n} > d_{H_1}\}$.

4 Application to real data

The data set contains outcomes of patient with heart surgery and is based on 6994 operations, from a single surgical center over the seven years old, 1992–1998, which is considered in Steiner et al. [26]
The data consists of some information on each patient like surgeon, type of procedure, and preoperative variable which comprises the parsonnet score (the score is based on a combination of many other information like age, gender, hypertension and others).
In this data, 461 deaths occurred with in 30 days of surgery, giving on overall mortality of 6.6%.
To identify the risk factors in phase 1, we use the first two years of data (1992–1993) and begin the monitoring from 1994, in phase 2.
In the first two years of total, 2218 surgeries were performed and 143 deaths were observed (mortality rate of 6.5%). The fuzzy logistic regression is used to estimate risk factors. As mentioned in section 3, $Logit(\tilde{P}_n) = (f_n(a), f_n(s))_T$ is the estimated output in this regression. To reach that for the data, because of not accessing to the result of experts decision as the possibility of mortality, we use the Buckley [2] approach based on the fuzzy probabilities from a confidence interval. Then by using the least squares method, $f_n(a)$ and $f_n(s)$ which are defined in (12) and (13), are obtained as follows:

$$f_n(a) = -3.528 + 0.0554u_n, \quad f_n(s) = 0.1834 + 0.000014u_n,$$

where u_n denotes the parsonnet score for patient n for $n = 1, 2, \ldots, 6994$.
To evaluate the model, the goodness of fit index which is mentioned in section 3, is used and its value was 0.72.
We are interested in designing the charts which are optimal in detecting a deterioration in performance, so we assume $\rho_1 = 0.2697$ (which is related to increasing the odds of mortality to two times of in control odds of mortality).

To plot the charts, we need to estimate the parameters of the log-logistic distribution (α_0, λ_0), using the maximum likelihood method estimated as:

$$\alpha_0 = 0.529, \quad \lambda_0 = 30606.$$

Therefore $\tilde{W}_n$ in (18) and its $\alpha-$cut for intended α can be calculated to reach d_{S_n} and $d_{E_{n,t}}$ in (19) and (22), respectively. Since there is no reason to distinguish any side of fuzzy numbers, we use $D_{p,q}$-distance when $p = 1$ and $q = 0.5$. Assuming that $ARL_0 = 10000$, the first and the end points of intervals H_1 and H_2 (the $\alpha-$cut control limit for the corresponded charts) and the $ARL_1(\alpha)$ for $\alpha \in \{0.65, 0.75, 0.85, 1\}$ can be seen in Table 1.

Table 1: The end points of α-cut control limits for the two charts and $ARL_1(\alpha)$

	RAST-CUSUM based on LR-fuzzy data			RA-uEWMA based on LR-fuzzy data				
α	$h_1^-(\alpha)$	$h_1^+(\alpha)$	$ARL_1(\alpha)$	$h_L^-(\alpha)$	$h_L^+(\alpha)$	$h_2^-(\alpha)$	$h_2^+(\alpha)$	$ARL_1(\alpha)$
0.65	3.469	25.2206	457.6	−0.0935	−0.0272	0.0085	0.0484	183.1
0.75	3.3646	10.8584	260.7	−0.0613	−0.0313	0.0128	0.0393	195.4
0.85	3.9727	6.8702	243.4	−0.062	−0.0357	0.0166	0.0314	200.3
1	4.801	4.802	224.8	−0.0442	−0.0441	0.0228	0.0231	210.6

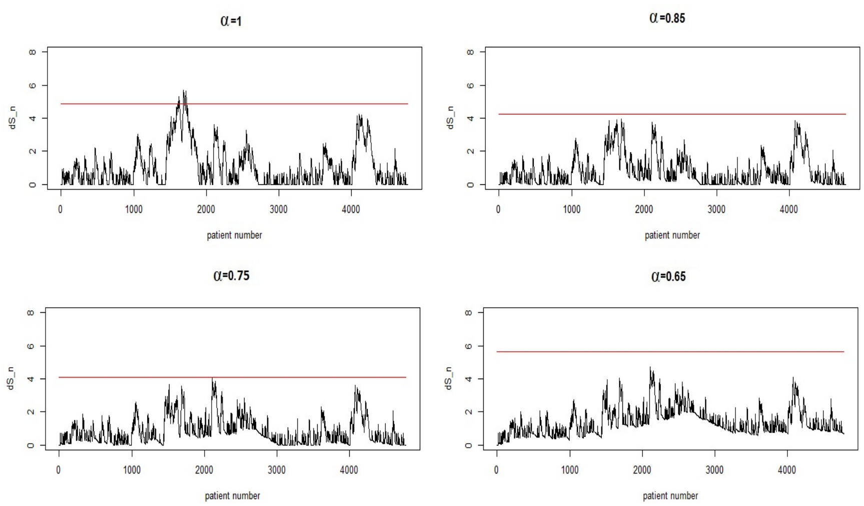

Figure 1: The RAST-CUSUM control chart based on LR-fuzzy data for $\alpha \in \{0.65, 0.75, 0.85, 1\}$.

The Figures 1 and 2 show the RAST-CUSUM and RA-uEWMA control charts based on LR-fuzzy data, respectively. When $\alpha = 1$, because of not considering the preoperative risk as a fuzzy number, both charts are exactly the same as their crisp corresponded control charts. In this case, the RA-uEWMA chart is in control, while there is some out of control points in the RAST-CUSUM chart. In general, the signal was not sustained and the process returned to their in-control level.

Decreasing α has different interpretation. For example, $\alpha = 0.85$ in Figure 1, means that we considered $\alpha-$cut set of $\tilde{W}_n$ scores with this intended α instead of considering the scores as crisp numbers (while the risks are fuzzy). In this case, the process becomes in control. Actually, there are some out of control points in crisp control chart, but by considering $\alpha < 1$, based on the position of side of fuzzy numbers and the value of $D_{p,q}$, they can be in control. So, decreasing α can make the chart more in control. However, as shown in Table 1 because the $ARL_1(\alpha)$ is a decreasing function of α, It is better not taking that too low. Hence $\alpha = 0.85$ can be a suitable choice.

Figure 2 shows the RA-uEWMA control chart based on LR-fuzzy data. Since the process is in control, the behavior of the chart and $ARL_1(\alpha)$ for every α is like each other, approximately.

The differences in two group charts are due to how the patient results are entered into the CUSUM and EWMA. In the RA-uEWMA chart, deaths are accounted for on the actual day of death, where as with the RAST-CUSUM charts, deaths enter the CUSUM statistic, 30 days after surgery and because of the cluster of deaths of patients whose surgeries were close together, we may have false alarms. By plotting the charts with $\alpha = 0.85$, the result of them is the same,

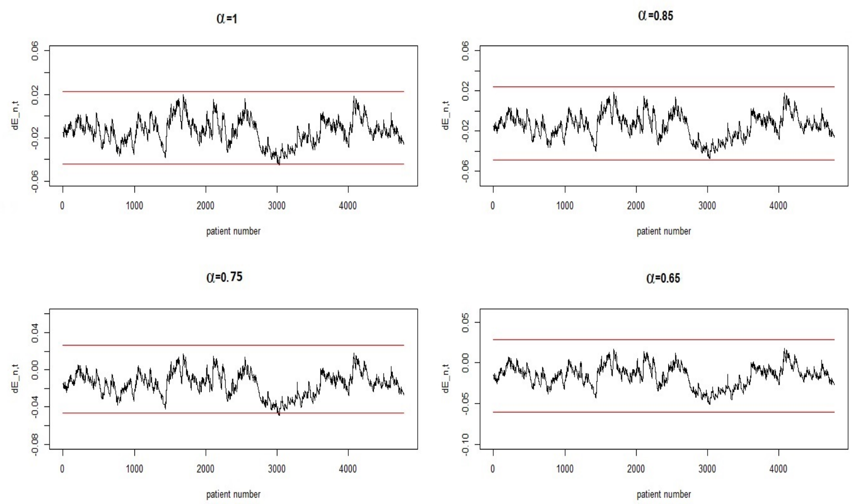

Figure 2: The RA-uEWMA control chart based on LR-fuzzy data for $\alpha \in \{0.65, 0.75, 0.85, 1\}$.

however according to Table 1, the RA-uEWMA control chart has lower $ARL_1(\alpha)$, which shows its performance is better than the RAST-CUSUM control chart.

5 Conclusion

Since the preoperative risk of surgery is a nonprecise and vague variable and a linguistic term such as low, medium, high or others like that, which are assigned by experts, it is better to be considered as a fuzzy number and determined by an appropriate membership function.

The fuzzy preoperative risk can be determined by using a fuzzy model such as the fuzzy logistic regression model. In this case, we need special charts to monitor the performance of the surgeons such that their statistics and control limits are fuzzy numbers, which are the extensions of the conventional process control charts to complicated fuzzy environment.

In this paper, we proposed the RAST-CUSUM and the RA-uEWMA control charts based on LR-fuzzy data, which by calculating the control statistics interval and their corresponded control limits interval and using the $D_{p,q}$-distance, for a specified α, can be plotted.

If $\alpha = 1$, then the behavior of the charts and ARLs are the same as their corresponded crisp charts, but decreasing α or considering the $\tilde{W}_n$ scores in more fuzzy environment, the result of the charts changes. The fuzzy control charts are more flexible than the usual (crisp) one and choosing different α allows us to have a more comprehensive monitoring.

References

[1] A. Albert, J. Walter, B. Arnrich, et al. *On-line variable live-adjusted displays with internal and external risk-adjusted mortalities. A valuable method for benchmarking and early detection of unfavourable trends in cardiac surgery*, European Journal of Cardio-Thoracic Surgery, **25** (2004), 312-319.

[2] J. J. Buckley, *Fuzzy probability and statistics*, Springer-Verlag Berlin Heidelberg, 2006.

[3] J. J. Buckley, T. Feuring, Y. Hayashi, *Multivariate non-linear fuzzy regression: An evolutionary algorithm approach*, International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems, **7**(2) (1999), 83-98.

[4] A. Celmins, *A practical approach to nonlinear fuzzy regression*, SIAM Journal on Scientific and Statistical Computing, **12**(3) (1991), 521-546.

[5] C. B. Cheng, *Fuzzy process control: Construction of control charts with fuzzy numbers*, Fuzzy Sets and Systems, **154**(2) (2005), 287-303.

[6] D. Cook, M. Coory, R. Webster, *Exponentially weighted moving average charts to compare observed and expected values for monitoring risk-adjusted hospital indicators*, BMJ Quality and Safety, **20**(6) (2011), 469-474.

[7] N. Erginel, S. Sentürk, *Fuzzy EWMA and fuzzy CUSUM control charts*, In: Springer, Cham, (2016), 281-295.

[8] B. S. Gildeh, D. Gien, *La distance-Dp, q et le coefficient de correlation entre deux variables aleatoires floues*, Encontres Francophones Sur la Logique Floue et Ses Applications LFA0 1, 2001.

[9] B. S. Gildeh, D. Gien, *A goodness of fit index to reliability analysis in fuzzy model*, In3rd WSEAS International Conference on Fuzzy Sets and Fuzzy Systems, Iterlaken, Switzerland, (2002), 11-14.

[10] O. Grigg, V. Farewell, *An overview of risk-adjusted charts*, Journal of the Royal Statistical Society: Series A (Statistics in Society), **167** (2004), 523-539.

[11] J. Klein, M. Moeschberger, *Survival analysis: Techniques for censored and truncated data*, Springer: New York, (1997), 373-400.

[12] J. Lovergrov, C. Sherlaw-Jahnson, O. Valencia, S. Gallivan, *Monitoring the performance of cardiac surgeons*, Journal Research Society, **5** (1999), 685-689.

[13] J. Lovergrov, O. Valencia, T. Treasure, C. Sherlaw-Jahnson, S. Gallivan, *Monitoring the result of cardiac surgery by variable life-adjusted display*, The Lancet, **350** (1997), 1128-1130.

[14] J. Lucas, M. Saccucci, *Exponentially weighted moving average control schemes: Properties and enhancements*, Technometrics, **32** (1990), 1-12.

[15] R. D. Miller, I. Eriksson, L. A. Fleisher, et al. *Miller's anesthesia e-book*, Elsevier Health Sciences, 2014.

[16] G. V. Moustakides, *Optimal stopping times for detecting changes in distributions*, Annals of Statistics, **14** (1986), 1379-1387.

[17] D. C. Montgomery, *Introduction to statistical quality control*, Second Edition, John Wiley and Sons, New York, 1991.

[18] E. S. Page, *Continuous inspection schemes*, Biometrika, **41** (1954), 100-115.

[19] V. Parsonnet, D. Dean, A. D. Bernstein, *A method of uniform stratification of risks for evaluating the result surgery in acquired heart disease*, Circulation, **779**(11) (1998) 1-12.

[20] J. Poloniecki, O. Valencia, P. Littlejohns, *Cumulative risk-adjusted mortality chart for detecting changes in death rate: Observational study of heart surgery*, Biritish Medical Journal, **316** (1998), 1697-1700.

[21] S. Pourahmad, S. M. T. Ayatollahi, S. M. Taheri, Z. H. Agahi, *Fuzzy logistic regression based on the least squares approach with application in clinical studies*, Computers and Mathematics with Applications, **62** (2011), 3353-3365.

[22] S. W. Roberts, *Control chart tests based on geometric moving averages*, Technometrics, **1**(3) (1959), 239-250.

[23] C. A. Rogers, B. C. Reeves, M. Caputo, J. S. Ganesh, R. S. Bonser, G. D. Angelini, *Control chart methods for monitoring cardiac surgical performance and their interpretation*, The Journal of Thoracic and Cardiovascular Surgery, **62** (2004), 811-819.

[24] L. H. Sego, *Virginia polytechnic institute and state university*, Blackburg VA, (2006). (Phd Thesis)

[25] L. H. Sego, M. J. Reynolds, W. H. Woodall, *Risk-adjusted monitoring of survival times*, The Journal of Statistics in Medicine, **28**(9) (2009), 1389-1401.

[26] S. H. Steiner, R. J. Cook, T. Treasure, *Monitoring surgical performance using risk-adjusted cumulative sum charts*, Biostatistics, **4**(1) (2000), 441-452.

[27] S. H. Steiner, M. Jones, *Risk-adjusted survival time monitoring with an updating exponentially weighted moving average (EWMA) control chart*, The Journal of Statistics in Medicine, **29** (2010), 444-454.

[28] H. Tanaka, S. Uejima, K. Asai, *Linear regression analysis with fuzzy model*, IEEE Transactions on Systems Man Cybern, **12** (1982), 903-907.

[29] H. J. Urgen Zimmermann, *Fuzzy set theory and its applications*, Springer Science and Business Media, 2011.

[30] D. Wang, *A CUSUM control chart for fuzzy quality data*, In: Springer, Berlin, Heidelberg, (2006), 357-364.

[31] D. Wang, O. Hryniewicz, *The design of a CUSUM control chart for LR-fuzzy data*, IFSA World Congress and NAFIPS Annual Meeting (IFSA/NAFIPS), Joint, IEEE, (2013), 175-180.

[32] J. H. Wang, T. Raz, *On the construction of control charts using linguistic variables*, The International Journal of Production Research, **28**(3) (1990), 477-487.

[33] S. M. William, B. R. Parry, M. Schlup, *Quality control: An application of the CUSUM*, BMJ: British Medical Journal, **304**(6838) (1992), 1359-1361.

[34] W. H. Woodall, *The use of control charts in health-care and public-health surveillance*, Journal of Quality Technology, **38** (2006), 89-104.

[35] W. H. Woodall, A. Mahmoud, *The inertial properties of quality control charts*, Technometrics, **47**(4) (2005), 425-436.

[36] L. A. Zadeh, *Fuzzy sets*, Information and Control, **8**(3) (1965), 338-353.