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Design of robust fuzzy Sliding-Mode control for a class of the Takagi-Sugeno uncertain fuzzy systems using scalar *Sign* function

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Abstract

This article presents a fuzzy sliding-mode control scheme for a class of Takagi-Sugeno (T-S) fuzzy which are subject to norm-bounded uncertainties in each subsystem. The proposed stabilization method can be adopted to explore T-S uncertain fuzzy systems (TSUFS) with various local control inputs. Firstly, a new design is proposed to transform TSUFS into sliding-mode dynamic systems. In addition, the proposed stabilization condition for the sliding-mode dynamic system can be formulated in terms of linear matrix inequalities (LMIs) using the Lyapunov functions. Finally, the validity of the proposed design strategy is demonstrated through the simulation of an inverted pendulum on a cart. According to simulation results, the performance indexes of the proposed method can achieve satisfactory results with minimum energy control effort, faster response, and zero steady-state error.

Keywords: Takagi-Sugeno fuzzy model, sliding mode control, linear matrix inequality (LMI), norm-bounded uncertainties, scalar *sign* function.

1 Introduction

In the last two decades, fuzzy control has generated a substantial amount of excitement in the control engineering community. Among various fuzzy control methods, Takagi-Sugeno fuzzy-model-based methods are recognized as a systematic and powerful tool for the modeling and stabilization analysis of nonlinear systems [3, 4, 5, 9, 11, 20]. The main advantage of T-S models is that they offer to the engineer the possibility to utilize linear control techniques in order to control the global nonlinear system. Moreover, T-S models can approximate every smooth nonlinear function to any degree of accuracy in a convex compact region [10].

Recently, many works based on TSUFS and Sliding Mode Control (SMC) have been studied in the literature in order to take the advantages of each approach. Indeed, the SMC is particularly useful since it can provide very robust control performance [8, 17, 18, 25]. Generally, the SMC design has a two-step called sliding step and reaching step. Step 1) Choose a switching surface, which is chosen by the desired behavior. Restricted to the intersection of the switching surface, it results in the desired behavior. Step 2) Design of control law, which will make the switching manifold attractive to the system state. The major drawback of VSC with sliding mode is the famed chattering phenomenon wherein unmodelled dynamics in the control loop are often excited by the discontinuous switching action of the sliding mode controller, leading to oscillations in the motion trajectory. Moreover, the control input matrix B_i for each subsystem in T-S fuzzy systems was assumed to be the same, i.e., $B = B_i$ as reported in [6, 7]. To overcome this drawback, Xi et *al*. [22] have partitioned the premise state space into a set of subregions and designed differential sliding surfaces for each region. However, the results are very complex and many constraint conditions for the T-S fuzzy systems must be satisfied. In addition, several methods based on FSMC have employed successfully this structure and shown its relevance [3, 4, 5, 12, 13]. Whereas, those methods do not address the problem of dynamic uncertain systems. However, in the real world, assuming that the parameters of any system during its real life are certainly known, is not

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always true since it can be subject to many sources of uncertainties such as precision errors during the experiences and modeling errors arising from assumptions due to lack of understanding of the system.

The main result of our paper is to take into consideration the uncertainties both in the design of sliding surface and the control law in the synthesis design method. The developed sliding mode controller should be able to optimally tracking and robustly staying in the user-specified sliding surface. In addition, it should be able to reduce the chattering phenomenon, a non-smooth functional behavior, so that it would not damage the control actuator and excite the undesirable unmolded dynamics. To achieve these objectives, the effectiveness of the proposed approach is applied to control the inverted pendulum on a cart.

The rest of this paper is organized as follows. In section 2, mathematical preliminaries are given. The new sliding surface formulation and the proposed robust FSMC are detailed in section 3 and 4, respectively. In Section 5, the inverted pendulum on a cart problem is used to demonstrate the merit and feasibility of the proposed method. Finally, Section 6 gives the conclusions.

Notations. $\Re^n$ denotes the real n-dimensional space; $\Re^{n\times m}$ denotes the real $n\times m$ matrix space; $\|.\|$ denotes the Euclidean norm of a vector or its induced matrix norm. For a real symmetric matrix, $M>0$ ($M<0$) means positive (negative) definite; I is used to represent an identity matrix of appropriate dimensions. $\lambda_{\max}(.)$ and $\lambda_{\min}(.)$ represent the maximum and minimum eigenvalue of a real symmetric matrix, respectively.

2 Mathematical preliminaries

2.1 Takagi and Sugeno fuzzy model (T-S fuzzy model) with uncertainty

The continuous fuzzy model, proposed by Takagi and Sugeno [19], is described by fuzzy IF-THEN rules which locally represent linear input-output relations of nonlinear systems. The i-th rule of the uncertain T-S fuzzy model is defined as follows:

$$rule_i : IF\ z_1(t)\ is\ M_{1i}\ and\ \ldots\ and\ z_p(t)\ is\ M_{pi}$$
$$THEN \begin{cases} \dot{x}(t) = (A_i + \Delta A_i(t))\, x(t) + B_i u(t), \quad i = 1,\ldots,r \\ y(t)=C_i x(t) \end{cases} \tag{1}$$

where $rule_i$ denotes the i-th IF-THEN rule, M_{ji} is the fuzzy subset, r is the number of rules, $x(t)\in\Re^n$ is the state vector, $u(t)\in\Re^m$ is the input vector, $(A_i,\Delta A_i(t))\in\Re^{n\times n}$, $B_i\in\Re^{n\times m}$ and $C_i\in\Re^{q\times n}$. Here $z(t) = [z_1(t), z_2(t), ..., z_p(t)]$ denotes the vector containing some nonlinear functions of the states variables obtained from the original nonlinear. All matrices are real with appropriate dimensions. We will assume the following to be valid:

- The pair (A_i, B_i) is controllable.
- The input matrix B_i has full rank m, $m<n$.
- $B_i = BG_i$ with $G_i\in\Re^{m\times m}$ is a known matrix function such that $G_i\neq 0$ and of a known sign (without loss of generality this sign can be taken as *positive*).
- $\Delta A_i(t)$ is the uncertainty of norm-bounded type written as:

$$\Delta A_i(t) = M_i F_i(t) N_i \tag{2}$$

where $M_i\in\Re^{n\times d}$ and $N_i\in\Re^{e\times n}$ are known real constant matrices, and is an unknown matrix function with Lebesgue-measurable elements that satistfies $F_i^T(t)F_i(t)\le I$.

By fuzzy blending, the overall fuzzy model can be inferred as:

$$\dot{x}(t) = \sum_{i=1}^{r} h_i(z(t))\left((A_i+\Delta A_i(t))\,x(t) + B_i u(t)\right) \tag{3}$$

with:

$$h_i(z(t)) = \frac{w_i(z(t))}{\sum\limits_{i=1}^{r} w_i(z(t))} \quad \text{and} \quad w_i(z(t)) = \prod_{j=1}^{p} M_{ji}(z_j(t))$$

Here $h_i(z(t))$ is regarded as the normalized weight of each model rule. $M_{ji}(z_j(t))$ denotes the grade of membership of $z_j(t)$ in M_{ji} . The membership values $h_i(z(t))$ have to satisfy the following conditions:

$$\begin{cases} \sum\limits_{i=1}^{r} h_i(z(t)) = 1 \\ 0 \le h_i(z(t)) \le 1 \quad i = 1,\ldots,r \end{cases} \tag{4}$$

2.2 Scalar/ matrix sign function

The matrix *sign* function was first introduced by [14]. It is defined as:

$$sign_{(l)}\left(X\right)=\left[\left(I_n+X\right)^l-\left(I_n-X\right)^l\right]\left[\left(I_n+X\right)^l+\left(I_n-X\right)^l\right]^{-1} \tag{5}$$

where $sign_{(l)}\in[-1\,,\,1]$ is the l th-order approximation of the matrix sign function, $X\in\Re^{n\times n}$ and I_n is the identity matrix of order n. A fast and stable algorithm for computing the matrix *sign* function (5) can be found in [16, 15].

The scalar/matrix *sign* function is used to design the nonlinear control law. The MSF has the nice properties that it is able to replace the non-smooth signum function by a smooth and differentiable continuous-time function [21, 3, 5, 4, 2]. In addition, it is able to reduce the chattering phenomenon. The scalar/matrix *sign* function is defined as [16].

$$sign_{(l)}\left(S\left(t\right)\right)=\left[\left(I_m+S\left(t\right)\right)^l-\left(I_m-S\left(t\right)\right)^l\right]\left[\left(I_m+S\left(t\right)\right)^l+\left(I_m-S\left(t\right)\right)^l\right]^{-1} \tag{6}$$

where $S\left(t\right)$ is the fuzzy sliding surface.

The scalar function with different values of "l" is given in Figure 1.

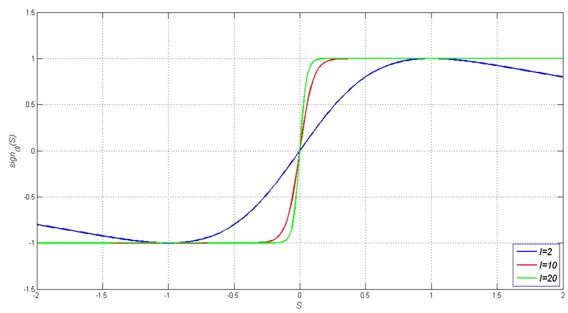

Figure 1: Scalar *sign* function

3 Sliding surface design

By assumption the matrix B_i has full rank m , there exist an $(n\times n)$ orthogonal transformation matrix T such that:

$$TB=\left[\begin{array}{c}0_{(n-m)\times m}\\ \bar{B}\end{array}\right] \tag{7}$$

where $\bar{B}\in\Re^{m\times m}$ is a nonsingular matrix.

Suppose that $\bar{B}$ satisfies the following singular value decomposition:

$$\bar{B}=U\left[\begin{array}{c}\Omega_{m\times m}\\ 0_{(n-m)\times m}\end{array}\right]V^T \tag{8}$$

$$T=col\left\{\begin{array}{cc}U_2^T & U_1^T\end{array}\right\} \tag{9}$$

where $U=\left[\begin{array}{cc}U_1 & U_2\end{array}\right]$, $U_1\in\Re^{n\times m}$ is unitary matrix, $U_2\in\Re^{n\times(n-m)}$ is unitary matrix, $V\in\Re^{m\times m}$, and $\Omega_{m\times m}$ is positive diagonal matrix.

By defining $\eta\left(t\right)=Tx\left(t\right)=\left[\begin{array}{c}\eta_1\left(t\right)\\ \eta_2\left(t\right)\end{array}\right]$, the following results are obtained from (3):

$$\dot{\eta}_1\left(t\right)=\sum_{i=1}^{r}h_i(z(t))\left[\left(\bar{A}_{11i}+\Delta\bar{A}_{11i}\right)\eta_1\left(t\right)+\left(\bar{A}_{12i}+\Delta\bar{A}_{12i}\right)\eta_2\left(t\right)\right] \tag{10}$$

$$\dot{\eta}_2\left(t\right)=\sum_{i=1}^{r}h_i(z(t))\left[\left(\bar{A}_{21i}+\Delta\bar{A}_{21i}\right)\eta_1\left(t\right)+\left(\bar{A}_{22i}+\Delta\bar{A}_{22i}\right)\eta_2\left(t\right)+\bar{B}G_iu\left(t\right)\right] \tag{11}$$

where
$\eta_1 \in \Re^{n-m}$, $\eta_2 \in \Re^{m}$, $\bar{A}_{11i} = U_2^T A_i U_2$, $\bar{A}_{12i} = U_2^T A_i U_1$, $\bar{A}_{21i} = U_1^T A_i U_2$, $\bar{A}_{22i} = U_1^T A_i U_1$, $\Delta\bar{A}_{11i} = U_2^T M_i F_i(t) N_i U_2$, $\Delta\bar{A}_{12i} = U_2^T M_i F_i(t) N_i U_1$, $\Delta\bar{A}_{21i} = U_1^T M_i F_i(t) N_i U_2$, $\Delta\bar{A}_{22i} = U_1^T M_i F_i(t) N_i U_1$ for $i = 1, ..., r$.

For (10) and (11), the corresponding sliding surface is chosen as follows:

$$S(t) = Q\eta_1(t) + \eta_2(t) = 0 \tag{12}$$

where $Q \in \Re^{m\times(n-m)}$ is a constant matrix.

It follows from (10) and (12) and then the following sliding dynamic system can be obtained:

$$\dot{\eta}_1(t) = \sum_{i=1}^{r} h_i(z(t)) \left[\left(\bar{A}_{11i} + \Delta\bar{A}_{11i}\right) - \left(\bar{A}_{12i} + \Delta\bar{A}_{12i}\right) Q\right] \eta_1(t) \tag{13}$$

Following the concept of PDC, the equivalent control law can be designed to make the trajectory reach the sliding surface, (i.e., $S(t) = 0$ and $\dot{S}(t) = 0$):

$$u_{eq}(t) = -\sum_{i=1}^{r} h_i(z(t))\left(\bar{B}G_i\right)^{-1} \left\{ \begin{array}{l} \left[Q\left\{\left(\bar{A}_{11i} + \Delta\bar{A}_{11i}\right)\eta_1(t) + \left(\bar{A}_{12i} + \Delta\bar{A}_{12i}\right)\eta_2(t)\right\}\right] \\ \quad + \left[\left(\bar{A}_{21i} + \Delta\bar{A}_{21i}\right)\eta_1(t) + \left(\bar{A}_{22i} + \Delta\bar{A}_{22i}\right)\eta_2(t)\right] \end{array} \right\} \tag{14}$$

Lemma 3.1. *Given constant matrices G and E and a symmetric constant matrix S of appropriate dimensions, the following inequality holds:*

$$W + GFE + E^T F^T G^T < 0 \tag{15}$$

where $F(t)$ satisfies $F^T(t)F(t) \leq I$ if and only if for some $\varepsilon > 0$.

$$W + \left[\begin{array}{cc} \varepsilon^{-1}E^T & \varepsilon G \end{array}\right] \left[\begin{array}{cc} I & 0 \\ 0 & I \end{array}\right] \left[\begin{array}{c} \varepsilon^{-1}E \\ \varepsilon G^T \end{array}\right] < 0 \tag{16}$$

Lemma 3.2. [24], *Given any matrices X, Y and Z with appropriate dimensions such that Z>0, we have $X^TY + XY^T \leq X^TZX + Y^TZ^{-1}Y$.*

Theorem 3.3. *For given $\bar{\varepsilon}_i > 0$, if there exist positive symmetric matrices $\bar{P}_0$ and L such that the following LMI:*

$$\left[\begin{array}{ccc} W & \left(\bar{P}_0 U_2^T - L^T U_1^T\right) N_i^T & U_2^T M_i \\ * & -\bar{\varepsilon}_i^{-1} I & 0 \\ * & * & -\bar{\varepsilon}_i I \end{array}\right] < 0 \quad i = 1, ..., r \tag{17}$$

is satisfied, then the sliding surface (12) guarantees that the sliding-mode dynamic motion in (13) is asymptotically stable.

Where

$$\begin{aligned} W &= \left(\bar{A}_{11i}\bar{P}_0 - \bar{A}_{12i}L\right) + \left(\bar{P}_0\bar{A}_{11i}^T - L^T\bar{A}_{12i}^T\right) \\ &= \left(\bar{A}_{11i}\bar{P}_0 - \bar{A}_{12i}L\right) + \left(\bar{A}_{11i}\bar{P}_0 - \bar{A}_{12i}L\right)^T \end{aligned} \tag{18}$$

$$\bar{\varepsilon}_i = \varepsilon_i^2 \tag{19}$$

and

$$L = Q\bar{P}_0 \tag{20}$$

Proof. Consider the following Lyapunov function candidate:

$$V(\eta_1) = \eta_1^T P_0 \eta_1 \tag{21}$$

where $P_0 > 0$.

The time derivative of $V(\eta_1)$ becomes:

$$\dot{V}(\eta_1) = \eta_1^T P_0 \dot{\eta}_1 + \dot{\eta}_1^T P_0 \eta_1 \tag{22}$$

with

$$\eta_1^T P_0 \dot{\eta}_1 = \eta_1^T P_0 \left[\sum_{i=1}^{r} h_i(z(t)) \left[\left(\bar{A}_{11i} + \Delta \bar{A}_{11i} \right) - \left(\bar{A}_{12i} + \Delta \bar{A}_{12i} \right) Q \right] \eta_1(t) \right] \tag{23}$$

Substitution (23) into (22) gives

$$\dot{V}(\eta_1) = \eta_1^T P_0 \left\{ \begin{array}{l} \left[\sum_{i=1}^{r} h_i(z(t)) \left[\left(\bar{A}_{11i} + \Delta \bar{A}_{11i} \right) - \left(\bar{A}_{12i} + \Delta \bar{A}_{12i} \right) Q \right] \eta_1(t) \right] \\ + \left[\sum_{i=1}^{r} h_i(z(t)) \left[\left(\bar{A}_{11i} + \Delta \bar{A}_{11i} \right) - \left(\bar{A}_{12i} + \Delta \bar{A}_{12i} \right) Q \right] \eta_1(t) \right]^T \end{array} \right\} P_0 \eta_1 < 0 \tag{24}$$

$$\dot{V}(\eta_1) = \eta_1^T \Psi \eta_1 < 0 \tag{25}$$

with $\Psi = P_0 \left(\bar{A}_{11i} - \bar{A}_{12i} Q \right) + \left(\bar{A}_{11i} - \bar{A}_{12i} Q \right)^T P_0 + P_0 \left(\Delta \bar{A}_{11i} - \Delta \bar{A}_{12i} Q \right) + \left(\Delta \bar{A}_{11i} - \Delta \bar{A}_{12i} Q \right)^T P_0$.

In order for $\dot{V}(\eta_1)$, $\forall \eta_1(t) \neq 0$, $\Psi < 0$ should be satisfied. By pre-and post-multiplying Ψ by $P_0^{-1}$and defining $\bar{P}_0 = P_0^{-1} P_0 P_0^{-1}$, the following result is obtained :

$$\Psi = \left(\bar{A}_{11i} - \bar{A}_{12i} Q \right) \bar{P}_0 + \bar{P}_0 \left(\bar{A}_{11i} - \bar{A}_{12i} Q \right)^T + \left(\Delta \bar{A}_{11i} - \Delta \bar{A}_{12i} Q \right) \bar{P}_0 + \bar{P}_0 \left(\Delta \bar{A}_{11i} - \Delta \bar{A}_{12i} Q \right)^T < 0 \tag{26}$$

$$\begin{aligned} &= \left(\bar{A}_{11i} - \bar{A}_{12i} Q \right) \bar{P}_0 + \bar{P}_0 \left(\bar{A}_{11i} - \bar{A}_{12i} Q \right)^T + \left(U_2^T M_i F_i(t) N_i U_2 - U_2^T M_i F_i(t) N_i U_1 Q \right) \bar{P}_0 \\ &+ \bar{P}_0 \left(U_2^T M_i F_i(t) N_i U_2 - U_2^T M_i F_i(t) N_i U_1 Q \right)^T < 0 \end{aligned} \tag{27}$$

$$\begin{aligned} &= \left(\bar{A}_{11i} - \bar{A}_{12i} Q \right) \bar{P}_0 + \bar{P}_0 \left(\bar{A}_{11i} - \bar{A}_{12i} Q \right)^T + U_2^T M_i F_i(t) \left[\left(\bar{P}_0 U_2^T - L^T U_1^T \right) N_i^T \right]^T \\ &+ \left[U_2^T M_i F_i(t) \left[\left(\bar{P}_0 U_2^T - L^T U_1^T \right) N_i^T \right]^T \right]^T < 0 \end{aligned} \tag{28}$$

Using lemma 3.1, the matrix inequality (28) hold for all $F_i(t)$ satisfying $F_i^T(t) F_i(t) \leq I$, if and only if there exists a constant $\varepsilon_i^{-1} > 0$ such that

$$\begin{aligned} &\left(\bar{A}_{11i} - \bar{A}_{12i} Q \right) \bar{P}_0 + \bar{P}_0 \left(\bar{A}_{11i} - \bar{A}_{12i} Q \right)^T \\ &+ \left[\begin{array}{cc} \varepsilon_i^{-1} \left(\bar{P}_0 U_2^T - L U_1^T \right) N_i^T & \varepsilon_i \left(U_2^T M_i \right) \end{array} \right] \left[\begin{array}{cc} I & 0 \\ 0 & I \end{array} \right] \left[\begin{array}{c} \varepsilon_i N_i \left(U_2 \bar{P}_0 - U_1 L \right) \\ \varepsilon_i \left(M_i^T U_2 \right) \end{array} \right] < 0 \end{aligned} \tag{29}$$

with $W = \left(\bar{A}_{11i} - \bar{A}_{12i} Q \right) \bar{P}_0 + \bar{P}_0 \left(\bar{A}_{11i} - \bar{A}_{12i} Q \right)^T$, $G = U_2^T M_i$ and $E = \left[\left(\bar{P}_0 U_2^T - L^T U_1^T \right) N_i^T \right]^T$.

Applying Schur complement to (29) yield (17). This completes the proof of the theorem 3.3.

□

4 Control law

After designing the sliding surface so that the sliding mode dynamics systems has the desired response, the next step of the SMC design procedure is to design a switching feedback control law such that the reachability of the specified sliding surface (12) is ensured. Here, the condition is designed as:

$$\dot{S}(t) < kS(t) - \sigma sign_{(l)}(S(t)), \; S(t) > 0 \tag{30}$$

$$\dot{S}(t) > -kS(t) - \sigma sign_{(l)}(S(t)), \; S(t) < 0 \tag{31}$$

Theorem 4.1. *Consider a T-S fuzzy system* (1)*. If the LMIs* (17) *are feasible and the sliding surface is given by* (12)*, where Q can be obtained from* (17)*, then all signals involved in the closed-loop T-S uncertain fuzzy system with the following control law are uniformly ultimately bounded:*

$$u(t) = u_1(t) + u_2(t) + u_3(t) \tag{32}$$

where

$$u_1(t) = -\sum_{i=1}^{r} h_i(z(t)) \left[\bar{B}^{-1}\left(Q\left(\bar{A}_{11i}\eta_1(t) + \bar{A}_{12i}\eta_2(t)\right) + \bar{A}_{21i}\eta_1(t) + \bar{A}_{22i}\eta_2(t)\right)\right] = -\sum_{i=1}^{r} h_i(z(t))\bar{B}^{-1}A_s \tag{33}$$

$$u_2(t) = -\sum_{i=1}^{r} h_i(z(t)) \left\{\bar{B}^{-1}\lambda_{\min}^{-1}\begin{bmatrix} 2\left\|QU_2^T M_i\right\|^2 + 2\left\|N_i U_2\eta_1(t)\right\|^2 \\ +2\left\|N_i U_1\eta_2(t)\right\|^2 + 2\left\|U_1^T M_i\right\|^2 \end{bmatrix}\right\} sign_{(l)}(S(t)) \tag{34}$$

$$u_3(t) = -\sum_{i=1}^{r} h_i(z(t))\bar{B}^{-1}\lambda_{\min}\left(KS(t) + \sigma sign_{(l)}(S(t))\right) \tag{35}$$

where $K > 0$, $\sigma > 0$ *and* $\left\{\begin{array}{l} SA_s \geq 0 \rightarrow \phi_i = \lambda_{\min}^{-1} \\ SA_s < 0 \rightarrow \phi_i = \lambda_{\max}^{-1} \end{array}\right\}$. *with* $\lambda_{\min}$ *is the minimum eigenvalue of* G_i *and* $\lambda_{\min}$ *is the maximum eigenvalue of* G_i.

Proof. From the reach condition $S^T(t)\dot{S}(t) < 0$ and the sliding surface (12), the following results are obtained:

$$\dot{S}(t) = \sum_{i=1}^{r} h_i(z(t)) \left[\varphi_1 + \varphi_2 - h_j(z(t))(\varphi_3 + \varphi_4 + \varphi_5)\right] = \sum_{i=1}^{r} h_i(z(t))h_j(z(t)) \left[\varphi_1 - \varphi_3 + \varphi_2 - \varphi_4 - \varphi_5\right] \tag{36}$$

where

$$\varphi_1 = Q\left(\bar{A}_{11i}\eta_1(t) + \bar{A}_{12i}\eta_2(t)\right) + \bar{A}_{21i}\eta_1(t) + \bar{A}_{22i}\eta_2(t)$$

$$\varphi_2 = Q\left(U_2^T M_i F_i(t) N_i U_2\eta_1(t) + U_2^T M_i F_i(t) N_i U_1\eta_2(t)\right) + \left(U_1^T M_i F_i(t) N_i U_2\eta_1(t) + U_1^T M_i F_i(t) N_i U_1\eta_2(t)\right)$$

$$\varphi_3 = G_i\phi_j\left(Q\left(\bar{A}_{11j}\eta_1(t) + \bar{A}_{12j}\eta_2(t)\right) + \bar{A}_{21j}\eta_1(t) + \bar{A}_{22j}\eta_2(t)\right) = G_i\phi_j\varphi_1$$

$$\varphi_4 = G_i\lambda_{\min}^{-1}\left(2\left[\left\|QU_2^T M_j\right\|^2 + \left\|N_j U_2\eta_1(t)\right\|^2 + \left\|N_j U_1\eta_2(t)\right\|^2 + \left\|U_1^T M_j\right\|^2\right]\right) sign_{(l)}(S(t)) = \bar{\varphi}_4 sign_{(l)}(S(t))$$

$$\varphi_5 = G_i\lambda_{\max}^{-1}\left(KS(t) + \sigma sign_{(l)}(S(t))\right), \ K > 0, \ \sigma > 0$$

and under the following conditions:

$$\left\{\begin{array}{l} SA_s \geq 0 \rightarrow \phi_i = \lambda_{\min}^{-1} \\ SA_s < 0 \rightarrow \phi_i = \lambda_{\max}^{-1} \end{array}\right\}$$

yield

$$\varphi_1 - \varphi_3 = \varphi_1 - G_i\phi_j\varphi_1 \leq 0, \ \ as\ S(t) > 0$$

$$\varphi_1 - \varphi_3 = \varphi_1 - G_i\phi_j\varphi_1 \geq 0, \ \ as\ S(t) < 0$$

By Lemma 3.1 and Lemma 3.2 , one can obtain $\varphi_2 \leq \bar{\varphi}_4$ and the following results:

$$\dot{S}(t) \leq \varphi_5 < 0, \ \ as\ S(t) > 0$$

$$\dot{S}(t) \geq \varphi_5 > 0, \ \ as\ S(t) < 0$$

Therefore, dynamic systems Eqs. (10) and (11) will converge to the sliding surface if the controller Eq. (32) is adapted. □

Theorem 4.2. *If Theorem 3.3 and Theorem 4.1 are satisfied, then the closed-loop T-S fuzzy system is asymptotically stable.*

Proof. If Theorem 3.3 and Theorem 4.1are satisfied, then Eqs. (10) and (11) are asymptotically stable. Therefore, by $\eta(t) = Tx(t)$, if $\eta(t) \rightarrow 0$, then $x(t) \rightarrow 0$. □

5 Numerical simulation

In this section, we are going to examine the performance and effectiveness of the proposed controller developed above.

In this paper, the reaching time (tr), rise time (tm), settling time (ts) and the control effort measures for different control methods are used as the performance indexes.

Consider the following inverted pendulum on a cart with its dynamics as follows [23]:

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = \frac{g\ \sin(x_1(t)) - aml\ x_2^2(t)\ \sin(2x_1(t))/2 - a\ \cos(x_1(t))u(t)}{4l/3 - aml\ \cos^2(x_1(t))} \end{cases}$$

where x_1 is the angle of the pendulum from the vertical, x_2 is the angular velocity, $g = 9.81 m.s^{-1}$ is the gravity constant, m is the mass of the pendulum rod, M is the mass of the cart, $2l$ is the length of the pendulum, u is the control force applied to the cart, parameter $a = \frac{1}{m+M}$ and the initial values of the state variables are $x_1(t_0) = 85°$ and $x_2(t_0) = 0$, respectively.

In this simulation, the system parameters are as follows:

$$m = 2.0Kg,\ M = 8.0Kg,\ 2l = 1.0m$$

The fuzzy model of this pendulum is obtained by linearizing the nonlinear equations over a number of operation points in the phase plane (x_1, x_2) [23]:

$$rule_1 : IF\ x_1(t)\ is\ about\ 0,\ \ x_2(t)\ is\ about\ 0$$
$$THEN\ \dot{x}(t) = A_1x(t) + B_1u(t)$$

$$rule_2 : IF\ x_1(t)\ is\ about\ 0,\ \ x_2(t)\ is\ about\ \pm 4$$
$$THEN\ \dot{x}(t) = A_2x(t) + B_2u(t)$$

$$rule_3 : IF\ x_1(t)\ is\ about\ \pm {}^{\pi}\!/_3,\ \ x_2(t)\ is\ about\ 0$$
$$THEN\ \dot{x}(t) = A_3x(t) + B_3u(t)$$

$$rule_4 : IF\ x_1(t)\ is\ about\ +{}^{\pi}\!/_3,\ \ x_2(t)\ is\ about\ +4$$
$$or\ x_1(t)\ is\ about\ -{}^{\pi}\!/_3,\ \ x_2(t)\ is\ about\ -4$$
$$THEN\ \dot{x}(t) = A_4x(t) + B_4u(t)$$

$$rule_5 : IF\ x_1(t)\ is\ about\ +{}^{\pi}\!/_3,\ \ x_2(t)\ is\ about\ -4$$
$$or\ x_1(t)\ is\ about\ -{}^{\pi}\!/_3,\ \ x_2(t)\ is\ about\ +4$$
$$THEN\ \dot{x}(t) = A_5x(t) + B_5u(t)$$

where

$$A_1 = \begin{bmatrix} 0 & 1 \\ 17.2941 & 0 \end{bmatrix}, B_1 = \begin{bmatrix} 0 \\ -0.1765 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0 & 1 \\ 14.4706 & 0 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ -0.1765 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0 & 1 \\ 5.8512 & 0 \end{bmatrix}, B_3 = \begin{bmatrix} 0 \\ -0.0779 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} 0 & 1 \\ 7.2437 & 0.5399 \end{bmatrix}, B_4 = \begin{bmatrix} 0 \\ -0.0779 \end{bmatrix},$$

$$A_5 = \begin{bmatrix} 0 & 1 \\ 7.2437 & 0.5399 \end{bmatrix}, B_5 = \begin{bmatrix} 0 \\ -0.0779 \end{bmatrix},$$

The fuzzy membership functions for state variables $x_1(t)$ and $x_2(t)$ are chosen as in Figures 2 and 3, respectively.

To show the feasibility and validity of the proposed control scheme and stabilization condition, the system parameters are randomly varied within ±20% of their nominal values. To describe the uncertain dynamic matrix, the normbounded uncertainty $(\Delta A_i(t) = M_iF_i(t)N_i)$ has been defined as follows:

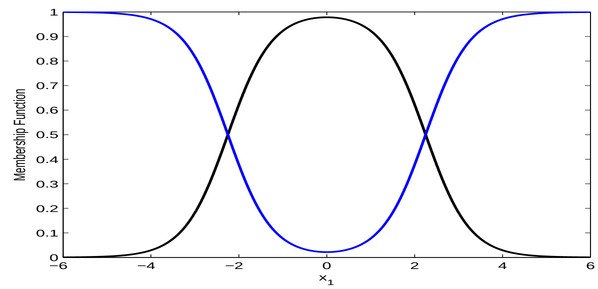

Figure 2: Membership Function of fuzzy set $x_1(rad)$.

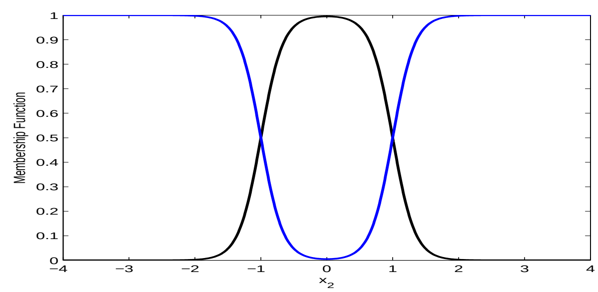

Figure 3: Membership Function of fuzzy set $x_2(rad/s)$.

$$M_1 = M_2 = \begin{bmatrix} 0 & 0 \\ 3.4 & 0 \end{bmatrix},\ M_3 = \begin{bmatrix} 0 & 0 \\ 1.2 & 0 \end{bmatrix},$$
$$M_4 = M_5 = \begin{bmatrix} 0 & 0 \\ 1.4 & 0 \end{bmatrix},\ F_i\left(t\right) = \begin{bmatrix} randn & 0 \\ 0 & randn \end{bmatrix},$$

where $randn$ is a random number taken from a normal distribution over $\begin{bmatrix} -1 & 1 \end{bmatrix}$ and $N_i = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Defining $\bar{B} = \begin{bmatrix} 0 \\ -0.1765 \end{bmatrix}$, $G_1 = G_2 = 1$, $G_3 = G_4 = G_5 = 0.4414$, $k = 5$, $\sigma = 1$, and $T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and solving the LMIs in Theorem 3.3, one can obtain $\bar{P}_0 = 0.0584$ and $Q = 6.52$.

A comparative analysis has been made between the three approaches: Conventional FSMC (CFSM) [23], NFSMC [5] and RFSMC [1].

- The CFSMC is defined as:
$$u_i = -\left(CB_i\right)^{-1} CA_i x - \left(CB_i\right)^{-1} K_i s^{\|x\|}/_{\|s\|}$$
  The parameters for the CFSMC are:
$$K_1 = 6.7543;\ K_2 = 1.4118;\ K_3 = 6.6427;$$
$$K_4 = 6.9543;\ K_5 = 6.9543;\ C = \begin{bmatrix} 5 & 1 \end{bmatrix}.$$

- The NFSMC is defined by, $u_i\left(t\right) = -L_i\, x\left(t\right) + N_i sign_{(l)}\left(S\left(t\right)\right)$, where $S\left(t\right) = \sum_{i=1}^{r} h_i(z\left(t\right))S_i\left(t\right) = \tilde{C}\, x\left(t\right)$ is the fuzzy sliding surface to be determined, $\tilde{C} \in \Re^{m \times n}$ and $\tilde{C} = \sum_{i=1}^{r} h_i(z\left(t\right))\tilde{C}_i$ is the sliding mode parameter matrix.

  The parameters for the NFSMC are:
$$L_1 = \begin{bmatrix} -481.0 & -126.4 \end{bmatrix},\ L_2 = \begin{bmatrix} -512.9 & -134.8 \end{bmatrix}$$
$$L_3 = \begin{bmatrix} -696.1 & -287.7 \end{bmatrix},\ L_4 = \begin{bmatrix} -749.4 & -307.7 \end{bmatrix},\ L_5 = \begin{bmatrix} -894.9 & -300.8 \end{bmatrix},$$

$$N_1 = -383.0,\ N_2 = -430.9,\ N_3 = -621.0,\ N_4 = -656.4,\ N_5 = -801.9,$$

$$\tilde{C}_1 = \begin{bmatrix} -1.2 & -0.27 \end{bmatrix},\ \tilde{C}_2 = \begin{bmatrix} -1.2 & -0.26 \end{bmatrix},$$

$$\tilde{C}_3 = \begin{bmatrix} -1.3 & -0.41 \end{bmatrix},\ \tilde{C}_4 = \begin{bmatrix} -1.3 & -0.41 \end{bmatrix},\ \tilde{C}_5 = \begin{bmatrix} -1.2 & -0.33 \end{bmatrix}.$$

- The RFSMC is defined as:

  $$u(t) = -\mathrm{K}_i x(t) + u_s(t)$$

  The desired closed-loop poles for each local model are chosen as -5 and -5. Thus, the feedback control gains are found as :

  $$\mathrm{K}_1 = \begin{bmatrix} -223.8 & -56.7 \end{bmatrix},\ \mathrm{K}_2 = \begin{bmatrix} -223.8 & -56.7 \end{bmatrix},$$

  $$\mathrm{K}_3 = \begin{bmatrix} -396.5.7 & -128.4 \end{bmatrix},\ \mathrm{K}_4 = \begin{bmatrix} -414.4 & -135.4 \end{bmatrix},\ \mathrm{K}_5 = \begin{bmatrix} -414.4 & -121.5 \end{bmatrix}.$$

  The parameters for the sliding mode controller $u_s(t)$ are determined by experiments and $C = \begin{bmatrix} 5 & 1 \end{bmatrix}$, $Q = 0.1$, $K = 10$.

The performance analysis shown in Figs. 4 - 7 are summarized in Table 1. The most desirable performance requires the controllers to have the smallest possible value for the tr, tm, the ts and the control effort. Table 1 compares the results of the proposed approach with those obtained in this case reported by [23, 5, 1].

For comparison, the performance results of our proposed method are better than that is reported in [1]. They applied RFSMC algorithm to drive inverted pendulum on a cart and has the tm and ts values of 0.47s and 0.86s, while the NFSMC approach proposed in [5] has 0.54s and 0.97s. However, the corresponding tm and ts values for the same problem were 0.35s and 0.67s. This indicates that the proposed controller is able to perform faster than the other methods in real application environment.

Performance	CFSMC	NFSMC	RFSMC	Proposed Controller
$tr(s)$	0.68	0.72	0.83	**0.14**
$tm(s)$	0.69	0.54	0.47	**0.35**
$ts(s)$	1.22	0.97	0.86	**0.67**
$\|u\|$	237.3	1516.2	6259.6	**316.2**

Table 1: tr, tm a ts and the control effort measures for different control methods.

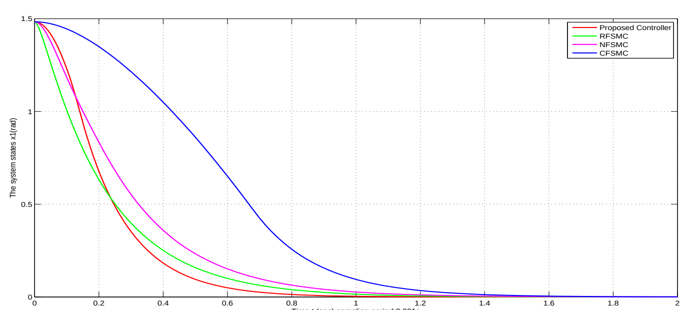

Figure 4: Response of state $x_1(t)$.

It is seen also that the proposed sliding variable $s(t)$ converges to zero in tr=0.14s and stays on it, whereas for the CFSMC, $s(t)$ takes 0.68s to converge to zero. As can be seen from figure 6, the control effort needed in the proposed method is much better, resulting in smaller control input value. So, the global control signal $u(t)$ is completely chattering-free. Indeed, this is due to the use of scalar *sign* function is helping in smoothing out the chattering of the system compared to that of a CFSMC.

These findings demonstrate the superiority of proposed controller regarding Table 1 and Figure 6.

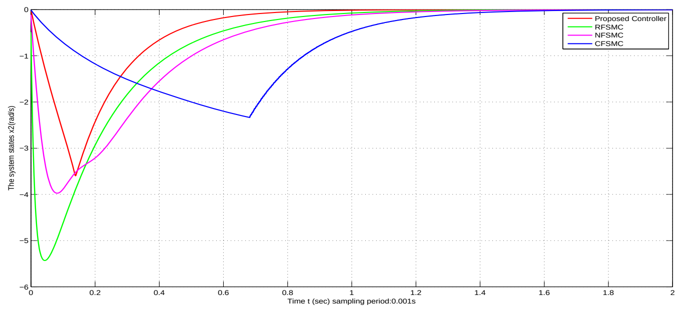

Figure 5: Response of state $x_2\,(t)$.

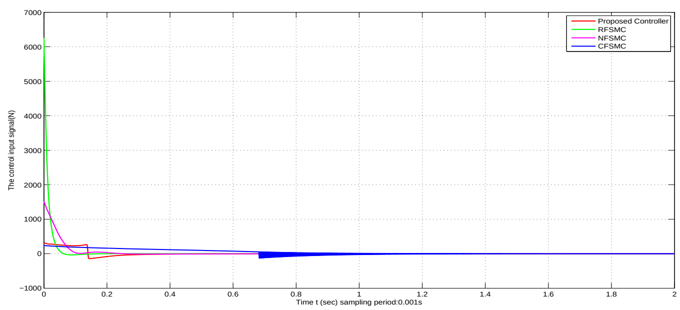

Figure 6: Control inputs.

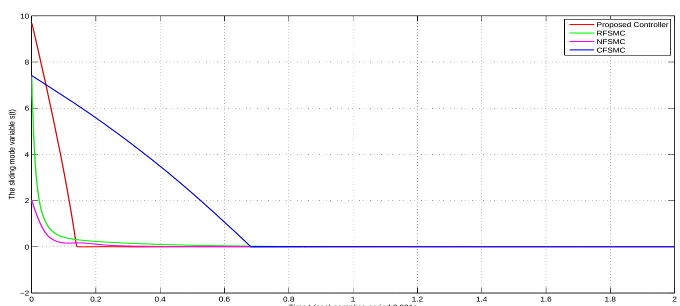

Figure 7: Sliding variable $s\,(t)$.

6 Conclusions

In this paper, stabilization problems for a class of T-S uncertain fuzzy systems were explored. The proposed stabilization method can be utilized to explore T-S uncertain fuzzy systems with various local control inputs and using scalar *sign* function. Based on the Lyapunov function, the stabilization conditions in terms of LMIs for the reachability of the sliding surface and asymptotic stability of the sliding motion were proposed. The results obtained in the present paper can be extended to a more general type of uncertainty affecting both state and input matrices. An inverted pendulum on a cart problem is used to demonstrate the validity and feasibility of the proposed fuzzy SMC approach.

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